Interval Valued Pythagorean Fuzzy Ideals in Semigroups

Veerappan Chinnadurai*, Arul Selvam
Department of Mathematics, Annamalai University, Chidambaram, Tamilnadu, India.

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<th>PAPER INFO</th>
<th>ABSTRACT</th>
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<td>Chronicle:</td>
<td>In this paper, we define the new notion of interval-valued Pythagorean fuzzy ideals in semigroups and established the properties of its with suitable examples. Also, we introduce the concept of interval valued Pythagorean fuzzy sub-semigroup, interval valued Pythagorean fuzzy left (resp. right) ideal, interval valued Pythagorean fuzzy bi-ideal, interval valued Pythagorean fuzzy interior ideal and homomorphism of an interval valued Pythagorean fuzzy ideal in semigroups with suitable illustration. We show that every interval valued Pythagorean fuzzy left (resp. right) ideal is an interval valued Pythagorean fuzzy bi-ideal.</td>
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Keywords
Pythagorean Fuzzy.
Fuzzy Ideals.
Homomorphism.
Semigroups.

1. Introduction


2. Preliminaries

**Definition 1.** [12]. Let $X$ be a universe of discourse, A Pythagorean fuzzy set (PFS) $P = \{w, \phi_p(w), \psi_p(w)/w \in X\}$ where $\phi:X \to [0,1]$ and $\psi:X \to [0,1]$ represent the degree of membership and non-membership of the object $w \in X$ to the set $P$ subset to the condition $0 \leq (\phi_p(w))^2 + (\psi_p(w))^2 \leq 1$ for all $w \in X$. For the sake of simplicity a PFS is denoted as $P = (\phi_p(w), \psi_p(w))$.

3. Interval-Valued Pythagorean Fuzzy Ideals in Semigroups

**Definition 2.** An Interval-Valued Pythagorean Fuzzy Set (IVPFS) $\tilde{P} = [\tilde{\phi}_p, \tilde{\psi}_p]$ on $S$ is known to be an interval-valued Pythagorean fuzzy sub-semigroup of $S$. If for all $w_1, w_2 \in S$, it holds.

$$\tilde{\phi}_p(w_1w_2) \geq \min\{\phi_p(w_1), \phi_p(w_2)\},$$

$$\tilde{\psi}_p(w_1w_2) \leq \max\{\psi_p(w_1), \psi_p(w_2)\}.$$

**Example 1.** Consider a semigroup $S = \{u, v, w, x, y\}$ with the Cayley Table.

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<tr>
<th></th>
<th>u</th>
<th>v</th>
<th>w</th>
<th>x</th>
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Define an interval-valued Pythagorean fuzzy set (IVPFS) $\tilde{P} = [\tilde{\phi}_p, \tilde{\psi}_p]$ in $S$ as follows.

<table>
<thead>
<tr>
<th></th>
<th>$[\phi_p(w_1), \psi_p(w_1)]$</th>
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<tbody>
<tr>
<td>u</td>
<td>[0.7, 0.8], [0.1, 0.2]</td>
</tr>
<tr>
<td>v</td>
<td>[0.4, 0.6], [0.4, 0.5]</td>
</tr>
<tr>
<td>w</td>
<td>[0.3, 0.5], [0.5, 0.6]</td>
</tr>
<tr>
<td>x</td>
<td>[0.1, 0.2], [0.3, 0.5]</td>
</tr>
<tr>
<td>y</td>
<td>[0.3, 0.5], [0.5, 0.6]</td>
</tr>
</tbody>
</table>
\[ \tilde{\Phi}_p(uv) \geq \min\{\tilde{\Phi}_p(u), \tilde{\Phi}_p(v)\} \]
\[ ([0.7,0.8],[0.1,0.2]) \geq [0.4,0.6],[0.1,0.2]. \]
\[ \tilde{\Psi}_p(uv) \leq \max\{\tilde{\Psi}_p(u), \tilde{\Psi}_p(v)\}. \]
\[ ([0.7,0.8],[0.1,0.2]) \leq [0.7,0.8],[0.4,0.5]. \]

Thus \( \tilde{P} = [\tilde{\Phi}_p, \tilde{\Psi}_p] \) is an Interval-Valued Pythagorean Fuzzy Sub-semigroup (IVPFSS) of \( S \).

**Definition 3.** An IVPFS \( \tilde{P} = (\tilde{\Phi}_p, \tilde{\Psi}_p) \) on semigroup \( S \), is said to be an interval-valued Pythagorean fuzzy left \((\tilde{P}_l)\) (resp. right \((\tilde{P}_r)\)) ideal of \( S \). If for all \( w_1, w_2 \in S \), it holds.

\[ \tilde{\Phi}_p(w_1w_2) \geq \tilde{\Phi}_p(w_2); \]
\[ \tilde{\Psi}_p(w_1w_2) \leq \tilde{\Psi}_p(w_2) \text{ (resp. right \((\tilde{P}_r)\));} \]
\[ \tilde{\Phi}_p(w_1w_2) \geq \tilde{\Phi}_p(w_1); \]
\[ \tilde{\Psi}_p(w_1w_2) \leq \tilde{\Psi}_p(w_1). \]

**Definition 4.** An IVPFS \( \tilde{P} = [\tilde{\Phi}_p, \tilde{\Psi}_p] \) on \( S \) is called IVPFI \((\tilde{P}_i)\) of \( S \). If for all \( w_1, w_2 \in S \), it \( \tilde{P} \) is both a left and right IVPFI of \( S \).

\[ (w_1w_2) \geq \max\{\tilde{\Phi}_p(w_1), \tilde{\Phi}_p(w_2)\}; \]
\[ \tilde{\Psi}_p(w_1w_2) \leq \min\{\tilde{\Psi}_p(w_1), \tilde{\Psi}_p(w_2)\}. \]

**Definition 5.** An IVPFS \( \tilde{P} = [\tilde{\Phi}_p, \tilde{\Psi}_p] \) on \( S \) is known to be an interval-Valued Pythagorean Fuzzy Bi-Ideal (IVPFBI) \((\tilde{P}_{bi})\) of \( S \). If for all \( a, w_1, w_2 \in S \) and satisfy.

\[ \tilde{\Phi}_p(w_1aw_2) \geq \min\{\tilde{\Phi}_p(w_1), \tilde{\Phi}_p(w_2)\}; \]
\[ \tilde{\Psi}_p(w_1aw_2) \leq \max\{\tilde{\Psi}_p(w_1), \tilde{\Psi}_p(w_2)\}. \]

**Example 2.** Consider a semigroup \( S = \{u, v, w, x, y\} \) with the Cayley Table.

Define an interval-valued Pythagorean fuzzy set \( \tilde{P} = [\tilde{\Phi}_p, \tilde{\Psi}_p] \) in \( S \) as follows.

<table>
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<tr>
<th></th>
<th>( \Phi_p(w_1), \Psi_p(w_1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u )</td>
<td>[0.8,0.9],[0.1,0.3]</td>
</tr>
<tr>
<td>( v )</td>
<td>[0.3,0.5],[0.7,0.9]</td>
</tr>
<tr>
<td>( w )</td>
<td>[0.4,0.6],[0.6,0.7]</td>
</tr>
<tr>
<td>( x )</td>
<td>[0.3,0.5],[0.7,0.9]</td>
</tr>
<tr>
<td>( y )</td>
<td>[0.7,0.8],[0.4,0.5]</td>
</tr>
</tbody>
</table>

Thus \( \tilde{P} = [\tilde{\Phi}_p, \tilde{\Psi}_p] \) is an interval valued Pythagorean fuzzy bi-ideal of \( S \).

**Definition 7.** An IVPFS \( \tilde{P} = ([\tilde{\Phi}_p, \tilde{\Psi}_p]) \) on \( S \) is known to be an interval-valued Pythagorean fuzzy interior ideal (IVPFII) \((\tilde{P}_{ii})\) of \( S \). If for all \( a, w_1, w_2 \in S \) and satisfy.

\[ \tilde{\Phi}_p(w_1aw_2) \geq \tilde{\Phi}_p(a); \]
\( \overline{\psi}_p(w_1, aw_2) \leq \overline{\psi}_p(a). \)

**Definition 8.** For any non-empty subset \( N \) of a semigroup \( S \) is defined to be a structure \( \chi_N = \{ w_1, [\overline{\psi}_X(w_1), \overline{\psi}_X(w_2)] | w_2 \in S \} \) which is briefly denoted by \( \chi_N = [\overline{\phi}_X, \overline{\psi}_X] \)

where, \( \overline{\phi}_X(w_1) = \begin{cases} 1 & \text{if } x \in N \\ 0 & \text{otherwise} \end{cases} \) \( \overline{\psi}_X(w_1) = \begin{cases} 0 & \text{if } x \in N \\ 1 & \text{otherwise} \end{cases} \)

**Theorem 1.** Let \( S \) be a semigroup. Then the following are equivalent.

- The intersection of two interval-valued Pythagorean fuzzy sub-semigroup of \( S \), is an interval-valued Pythagorean fuzzy sub-semigroup of \( S \).
- The intersection of two interval-valued Pythagorean fuzzy left (resp. right) ideal of \( S \), is IVPFLLI (resp. IVPFLLRI) of \( S \).

**Proof.** Let \( \overline{p}_1 = [\overline{\phi}_{p_1}, \overline{\psi}_{p_1}] \) and \( \overline{p}_2 = [\overline{\phi}_{p_2}, \overline{\psi}_{p_2}] \) be two interval-valued Pythagorean fuzzy sub-semigroup of \( S \). Let \( w_1, w_2 \in S \).

Then,

\[
\begin{align*}
(\overline{\phi}_{p_1} \cap \overline{\phi}_{p_2})(w_1, w_2) &= \min\{\overline{\phi}_{p_1}(w_1, w_2), \overline{\phi}_{p_2}(w_1, w_2)\} \\
&\geq \min\{\min\{\overline{\phi}_{p_1}(w_1), \overline{\phi}_{p_1}(w_2)\}, \min\{\overline{\phi}_{p_2}(w_1), \overline{\phi}_{p_2}(w_2)\}\} \\
&= \min\{\min\{\overline{\phi}_{p_1}(w_1), \overline{\phi}_{p_2}(w_1)\}, \min\{\overline{\phi}_{p_1}(w_2), \overline{\phi}_{p_2}(w_2)\}\} \\
&= \min\{\overline{\phi}_{p_1} \cap \overline{\phi}_{p_2}(w_1), \overline{\phi}_{p_1} \cap \overline{\phi}_{p_2}(w_2)\};
\end{align*}
\]

\[
(\overline{\psi}_{p_1} \cup \overline{\psi}_{p_2})(w_1, w_2) = \max\{\overline{\psi}_{p_1}(w_1, w_2), \overline{\psi}_{p_2}(w_1, w_2)\} \\ 
\leq \max\{\max\{\overline{\psi}_{p_1}(w_1), \overline{\psi}_{p_1}(w_2)\}, \max\{\overline{\psi}_{p_2}(w_1), \overline{\psi}_{p_2}(w_2)\}\} \\
= \max\{\max\{\overline{\psi}_{p_1}(w_1), \overline{\psi}_{p_2}(w_1)\}, \max\{\overline{\psi}_{p_1}(w_2), \overline{\psi}_{p_2}(w_2)\}\} \\
= \max\{\overline{\psi}_{p_1} \cup \overline{\psi}_{p_2}(w_1), \overline{\psi}_{p_1} \cup \overline{\psi}_{p_2}(w_2)\}.
\]

Therefore, \( \overline{p}_1 \cap \overline{p}_2 = \{(\overline{\phi}_{p_1} \cap \overline{\phi}_{p_2}), (\overline{\psi}_{p_1} \cup \overline{\psi}_{p_2})\} \).

**Interval-valued Pythagorean fuzzy sub-semigroup of \( S \).**

\[
(\overline{\phi}_{p_1} \cap \overline{\phi}_{p_2})(w_1, w_2) = \min\{\overline{\phi}_{p_1}(w_1, w_2), \overline{\phi}_{p_2}(w_1, w_2)\} \\
\geq \min\{\overline{\phi}_{p_1}(w_2), \overline{\phi}_{p_2}(w_2)\} \\
= (\overline{\phi}_{p_1} \cap \overline{\phi}_{p_2})(w_2);
\]

\[
(\overline{\psi}_{p_1} \cup \overline{\psi}_{p_2})(w_1, w_2) = \max\{\overline{\psi}_{p_1}(w_1, w_2), \overline{\psi}_{p_2}(w_1, w_2)\} \\ 
\leq \max\{\overline{\psi}_{p_1}(w_2), \overline{\psi}_{p_2}(w_2)\} \\
= (\overline{\psi}_{p_1} \cup \overline{\psi}_{p_2})(w_2).
\]

Therefore, \( \overline{p}_1 \cap \overline{p}_2 = \{(\overline{\phi}_{p_1} \cap \overline{\phi}_{p_2}), (\overline{\psi}_{p_1} \cup \overline{\psi}_{p_2})\} \) is an interval-valued Pythagorean fuzzy left (resp. right) ideal of \( S \).
Theorem 2. An IVPFS $\bar{P} = \{\bar{\phi}_p, \bar{\psi}_p\}$ of a semigroup $S$ is an IVPFBI of $S$, if and only if $((\phi_p^L, \phi_p^U), (\psi_p^L, \psi_p^U))$ of $S$.

Proof. Let $\bar{P} = \{\bar{\phi}_p, \bar{\psi}_p\}$ be an interval-valued Pythagorean fuzzy bi-ideal of $S$, for any $w_1, w_2 \in S$.

Then, we have membership

$$\begin{align*}
\phi_p^L(w_1, w_2) & = \bar{\phi}_p(w_1, w_2) \\
& \geq \min\{\bar{\phi}_p(w_1), \bar{\phi}_p(w_2)\} \\
& = \min\{\phi_p^L(w_1), \phi_p^L(w_2)\}, \phi_p^L(w_2)\} \\
& = \min\{\phi_p^L(w_1), \phi_p^L(w_2)\}, \phi_p^L(w_2)\}.
\end{align*}$$

It follows that $\phi_p^L(w_1, w_2) \geq \min\{\phi_p^L(w_1), \phi_p^L(w_2)\}$ and $\phi_p^L(w_1, w_2) \geq \min\{\phi_p^L(w_1), \phi_p^L(w_2)\}$ and non-membership

$$\begin{align*}
\psi_p^L(w_1, w_2) & = \bar{\psi}_p(w_1, w_2) \\
& \leq \max\{\bar{\psi}_p(w_1), \bar{\psi}_p(w_2)\} \\
& = \max\{\psi_p^L(w_1), \psi_p^L(w_2)\} \\
& = \max\{\psi_p^L(w_1), \psi_p^L(w_2)\}.
\end{align*}$$

It follows that $\psi_p^L(w_1, w_2) \leq \max\{\psi_p^L(w_1), \psi_p^L(w_2)\}$ and $\psi_p^L(w_1, w_2) \leq \max\{\psi_p^L(w_1), \psi_p^L(w_2)\}$

Therefore, $\bar{P} = ((\phi_p^L, \phi_p^U), (\psi_p^L, \psi_p^U))$ are Pythagorean fuzzy ideal of $S$.

Conversely, suppose that $((\phi_p^L, \phi_p^U), (\psi_p^L, \psi_p^U))$ are Pythagorean fuzzy ideal of $S$, let $w_1, w_2 \in S$.

$$\begin{align*}
\bar{\phi}_p(w_1, w_2) & = [\phi_p^L(w_1, w_2), \phi_p^U(w_1, w_2)] \\
& \geq [\min\{\phi_p^L(w_1), \phi_p^L(w_2)\}, \min\{\phi_p^U(w_1), \phi_p^U(w_2)\}] \\
& = \min\{\phi_p^L(w_1), \phi_p^L(w_2)\}, \phi_p^L(w_2)\} \\
& = \min\{\phi_p^L(w_1), \phi_p^L(w_2)\}; \\
\bar{\psi}_p(w_1, w_2) & = [\psi_p^L(w_1, w_2), \psi_p^U(w_1, w_2)] \\
& \leq [\max\{\psi_p^L(w_1), \psi_p^L(w_2)\}, \max\{\psi_p^U(w_1), \psi_p^U(w_2)\}] \\
& = \max\{\psi_p^L(w_1), \psi_p^L(w_2)\} \\
& = \max\{\psi_p^L(w_1), \psi_p^L(w_2)\}.
\end{align*}$$

$\bar{P} = \{\bar{\phi}_p, \bar{\psi}_p\}$ is an interval-valued Pythagorean fuzzy sub-semigroup of $S$.

$$\begin{align*}
\bar{\phi}_p(w_1, aw_2) & = [\phi_p^L(w_1, aw_2), \phi_p^U(w_1, aw_2)] \\
& \geq [\min\{\phi_p^L(w_1), \phi_p^L(w_2)\}, \min\{\phi_p^U(w_1), \phi_p^U(w_2)\}] \\
& = \min\{\phi_p^L(w_1), \phi_p^L(w_2)\}, \phi_p^L(w_2)\} \\
& = \min\{\phi_p^L(w_1), \phi_p^L(w_2)\}; \\
\bar{\psi}_p(w_1, aw_2) & = [\psi_p^L(w_1, aw_2), \psi_p^U(w_1, aw_2)] \\
& \leq [\max\{\psi_p^L(w_1), \psi_p^L(w_2)\}, \max\{\psi_p^U(w_1), \psi_p^U(w_2)\}] \\
& = \max\{\psi_p^L(w_1), \psi_p^L(w_2)\} \\
& = \max\{\psi_p^L(w_1), \psi_p^L(w_2)\}.
\end{align*}$$
\[
\max\{\tilde{\psi}_p(w_1), \tilde{\psi}_p(w_2)\}.
\]

\(\tilde{p} = [\tilde{\phi}_p, \tilde{\psi}_p]\) is an interval-valued Pythagorean fuzzy bi-ideal of \(S\).

**Theorem 3.** If \(\{P_i\}_{i \in I}\) is a family of interval-valued Pythagorean fuzzy bi-ideal of a semigroup \(S\). Then \(\bigcap P_i\) is an interval-valued Pythagorean fuzzy bi-ideal of \(S\). Where \(\bigcap P_i = (\bigcap \tilde{\phi}_p, \bigcup \tilde{\psi}_p)\).

\(\cap (\tilde{\phi}_p) = \inf\{(\tilde{\phi}_p)(w_i)/i \in I, w_i \in S\}, \cup (\tilde{\psi}_p) = \sup\{(\tilde{\psi}_p)(w_i)/i \in I, w_i \in S\}\) and \(i \in I\) is any index set.

**Proof.** Since \(\tilde{P}_i = ([\tilde{\phi}_p, \tilde{\psi}_p] | i \in I)\) is a family of interval-valued Pythagorean fuzzy bi-ideal of \(S\).

Let \(a, w_1, w_2 \in S\).

\[
\begin{align*}
\cap \tilde{\phi}_p(w_i, w_2) &= \inf\{\tilde{\phi}_p(w_1, w_2)/i \in I, w_1, w_2 \in S\} \\
&\geq \inf\\{\min\{\tilde{\phi}_p(w_1), \tilde{\phi}_p(w_2)\}\} \\
&= \min\{\inf\{\tilde{\phi}_p(w_1)\}, \inf\{\tilde{\phi}_p(w_2)\}\} \\
&= \min\{\cap \tilde{\phi}_p(w_1) \cap \tilde{\phi}_p(w_2)\}; \\
\cup \tilde{\psi}_p(w_1, w_2) &= \sup\{\tilde{\psi}_p(w_1, w_2)/i \in I, w_1, w_2 \in S\} \\
&\leq \sup\\{\max\{\tilde{\psi}_p(w_1), \tilde{\psi}_p(w_2)\}\} \\
&= \max\{\sup\{\tilde{\psi}_p(w_1)\}, \sup\{\tilde{\psi}_p(w_2)\}\} \\
&= \max\{\cup \tilde{\psi}_p(w_1) \cup \tilde{\psi}_p(w_2)\}.
\end{align*}
\]

Hence, \(\cap \tilde{P}_i = (\cap \tilde{\phi}_p, \cup \tilde{\psi}_p)\) is an interval-valued Pythagorean fuzzy sub-semigroup of \(S\).

\[
\begin{align*}
\cap \tilde{\phi}_p(w_i, aw_2) &= \inf\{\tilde{\phi}_p(w_1, aw_2)/i \in I, a, w_1, w_2 \in S\} \\
&\geq \inf\\{\min\{\tilde{\phi}_p(w_1), \tilde{\phi}_p(w_2)\}\} \\
&= \min\{\inf\{\tilde{\phi}_p(w_1)\}, \inf\{\tilde{\phi}_p(w_2)\}\} \\
&= \min\{\cap \tilde{\phi}_p(w_1) \cap \tilde{\phi}_p(w_2)\}; \\
\cup \tilde{\psi}_p(w_1, aw_2) &= \sup\{\tilde{\psi}_p(w_1, aw_2)/i \in I, a, w_1, w_2 \in S\} \\
&\leq \sup\\{\max\{\tilde{\psi}_p(w_1), \tilde{\psi}_p(w_2)\}\} \\
&= \max\{\sup\{\tilde{\psi}_p(w_1)\}, \sup\{\tilde{\psi}_p(w_2)\}\} \\
&= \max\{\cup \tilde{\psi}_p(w_1) \cup \tilde{\psi}_p(w_2)\}.
\end{align*}
\]

Hence, \(\cap P_i = (\cap \tilde{\phi}_p, \cup \tilde{\psi}_p)\) is an interval-valued Pythagorean fuzzy bi-ideals of \(S\).

**Theorem 4.** Let \(N\) be any non-empty subset of a semigroup \(S\). Then \(N\) is a bi-ideal of \(S\), if and only if the characteristic interval-valued Pythagorean fuzzy set \(\chi_N = [\tilde{\phi}_p x_N, \tilde{\psi}_p x_N]\) is IVPFBI of \(S\).

**Proof.** Assume that \(N\) is a bi-ideal of \(S\). Let \(a, w_1, w_2 \in S\).
Suppose that $\tilde{\phi}_{\text{pxn}}(w_1w_2) < \min\{\tilde{\phi}_{\text{pxn}}(w_1), \tilde{\phi}_{\text{pxn}}(w_2)\}$ and $\tilde{\psi}_{\text{pxn}}(w_1w_2) > \max\{\tilde{\psi}_{\text{pxn}}(w_1), \tilde{\psi}_{\text{pxn}}(w_2)\}$ it follows that $\tilde{\phi}_{\text{pxn}}(w_1w_2) = 0, \min\{\tilde{\phi}_{\text{pxn}}(w_1), \tilde{\phi}_{\text{pxn}}(w_2)\} = 1$

$\tilde{\psi}_{\text{pxn}}(w_1w_2) = 1, \max\{\tilde{\psi}_{\text{pxn}}(w_1), \tilde{\psi}_{\text{pxn}}(w_2)\} = 0.$

This implies that $w_1, w_2 \in N$ by $w_1, w_2 \not\in N$ a contradiction to $N$.

So $\tilde{\phi}_{\text{pxn}}(w_1w_2) \geq \min\{\tilde{\phi}_{\text{pxn}}(w_1), \tilde{\phi}_{\text{pxn}}(w_2)\}, \tilde{\psi}_{\text{pxn}}(w_1w_2) \leq \max\{\tilde{\psi}_{\text{pxn}}(w_1), \tilde{\psi}_{\text{pxn}}(w_2)\}$.

Suppose that $\tilde{\phi}_{\text{pxn}}(w_1aw_2) < \min\{\tilde{\phi}_{\text{pxn}}(w_1), \tilde{\phi}_{\text{pxn}}(w_2)\}$ and $\tilde{\psi}_{\text{pxn}}(w_1aw_2) > \max\{\tilde{\psi}_{\text{pxn}}(w_1), \tilde{\psi}_{\text{pxn}}(w_2)\}$ it follows that $\tilde{\phi}_{\text{pxn}}(w_1aw_2) = 0, \min\{\tilde{\phi}_{\text{pxn}}(w_1), \tilde{\phi}_{\text{pxn}}(w_2)\} = 1, \tilde{\psi}_{\text{pxn}}(w_1w_2) = 1, \max\{\tilde{\psi}_{\text{pxn}}(w_1), \tilde{\psi}_{\text{pxn}}(w_2)\} = 0.$

This implies that $a, w_1, w_2 \in N$ by $a, w_1, w_2 \not\in N$ a contradiction to $N$.

So $\tilde{\phi}_{\text{pxn}}(w_1aw_2) \geq \min\{\tilde{\phi}_{\text{pxn}}(w_1), \tilde{\phi}_{\text{pxn}}(w_2)\}, \tilde{\psi}_{\text{pxn}}(w_1aw_2) \leq \max\{\tilde{\psi}_{\text{pxn}}(w_1), \tilde{\psi}_{\text{pxn}}(w_2)\}$.

This shows that $\chi_N$ is an interval-valued Pythagorean fuzzy bi-ideal of $S$.

Conversely, $\chi_N = [\tilde{\phi}_{\text{pxn}}, \tilde{\psi}_{\text{pxn}}]$ is an IVPFBI of $S$ for any subset $N$ of $S$.

Let $w_1, w_2 \in N$ then $\tilde{\phi}_{\text{pxn}}(w_1) = \tilde{\phi}_{\text{pxn}}(w_2) = \tilde{1}, \tilde{\psi}_{\text{pxn}}(w_1) = \tilde{\psi}_{\text{pxn}}(w_2) = \tilde{0}$, since $\chi_N$ is an IVPFBI of $S$.

$\tilde{\phi}_{\text{pxn}}(w_1w_2) = \min\{\tilde{\phi}_{\text{pxn}}(w_1), \tilde{\phi}_{\text{pxn}}(w_2)\} = \min\{\tilde{1}, \tilde{1}\} = \tilde{1}, \tilde{\psi}_{\text{pxn}}(w_1w_2) = \max\{\tilde{\psi}_{\text{pxn}}(w_1), \tilde{\psi}_{\text{pxn}}(w_2)\} \leq \max\{\tilde{0}, \tilde{0}\} = \tilde{0}.$

This implies that $w_1w_2 \in N$.

Let $a, w_1, w_2 \in N$ then $\tilde{\phi}_{\text{pxn}}(w_1) = \tilde{\phi}_{\text{pxn}}(a) = \tilde{\phi}_{\text{pxn}}(w_2) = \tilde{1}$,

$\tilde{\psi}_{\text{pxn}}(w_1) = \tilde{\psi}_{\text{pxn}}(a) = \tilde{\psi}_{\text{pxn}}(w_2) = \tilde{0}$, since $\chi_N$ is an IVPFBI of $S$.

$\tilde{\phi}_{\text{pxn}}(w_1w_2) = \min\{\tilde{\phi}_{\text{pxn}}(w_1), \tilde{\phi}_{\text{pxn}}(w_2)\} = \min\{\tilde{1}, \tilde{1}\} = \tilde{1}, \tilde{\psi}_{\text{pxn}}(w_1w_2) = \max\{\tilde{\psi}_{\text{pxn}}(w_1), \tilde{\psi}_{\text{pxn}}(w_2)\} \leq \max\{\tilde{0}, \tilde{0}\} = \tilde{0}.$

Which implies that $w_1, w_2 \in N$. Hence $N$ is a bi-ideal of $S$.

**Theorem 5.** If $\{\tilde{P}_i\}_{i \in I}$ is a family of interval-valued Pythagorean fuzzy interior ideal of a semigroup $S$. Then $\cap \; \tilde{P}_i$ is an interval-valued Pythagorean fuzzy interior ideal (IVPFI) of $S$.

Where $\cap \; \tilde{P}_i = (\cap \; \tilde{\phi}_{p_i}) \cup \tilde{\psi}_{p_i}$;

$\cap \; (\tilde{\phi}_{p_i}) = \inf\{(\tilde{\phi}_{p_i})(w_i)/i \in I, w_i \in S\}, \cup \; (\tilde{\psi}_{p_i}) = \sup\{(\tilde{\psi}_{p_i})(w_i)/i \in I, w_i \in S\}$ and $i \in I$ is any index set.
Theorem 6. Let $N$ be any non-empty subset of a semigroup $S$. Then $N$ is a interior ideal of $S$, if and only if the characteristic interval-valued Pythagorean fuzzy set $\chi_N = [\tilde{\phi}_{p\chi N}, \tilde{\psi}_{p\chi N}]$ is IVFHI of $S$.

4. Homomorphism of Interval-Valued Pythagorean Fuzzy Ideals in Semigroups

Let $R$ and $T$ be two non-empty sets of semigroup $S$. A mapping $f: R \to T$ is called a homomorphism if $(rt) = f(r)f(t) \ \forall r, t \in R$.

Definition 9. Let $f$ be a mapping from a set $R$ to a set $T$ and $\bar{P} = [\bar{\phi}_r, \bar{\psi}_r]$ be an interval-valued Pythagorean fuzzy set $R$ the image of $R$ (i.e.) $f(\bar{P}) = (f(\bar{\phi}_r), f(\bar{\psi}_r))$ is an interval-valued Pythagorean fuzzy set of $T$ is defined by

$$f(\bar{P})(r) = \begin{cases} 
\sup_{t \in f^{-1}(r)} (\overline{\Phi}_{P}(t)), & \text{iff } r^{-1} = 0 \\
[0,0] & \text{otherwise}
\end{cases}$$

$$f(\bar{\psi}_r)(r) = \inf_{t \in f^{-1}(r)} (\overline{\Psi}_{P}(t)), \text{iff } r^{-1} = 0$$

$$f(\bar{\psi}_r)(r) = \begin{cases} 
\sup_{t \in f^{-1}(r)} (\overline{\Phi}_{P}(t)), & \text{iff } r^{-1} = 0 \\
[0,0] & \text{otherwise}
\end{cases}$$

Let $f$ be a mapping from a set $R$ to $T$ and $\bar{P} = [\bar{\phi}_r, \bar{\psi}_r]$ be an interval-valued Pythagorean fuzzy set $T$ then the preimage of $T$ (i.e.) $f^{-1}(\bar{P}) = ([f^{-1}(\bar{\phi}_r), f^{-1}(\bar{\psi}_r)])$ is an interval-valued Pythagorean fuzzy set of $R$ is defined as

$$f^{-1}(\bar{P})(r) = \begin{cases} 
f^{-1}(\overline{\Phi}_{P})(r) = \overline{\Phi}_{P}(f(r)) \\
f^{-1}(\overline{\psi}_{P})(r) = \overline{\psi}_{P}(f(r))
\end{cases}$$

Theorem 7. Let $R, T$ be a semigroups, $f: R \to T$ be a homomorphism of semigroups.

If $\bar{P} = [\bar{\phi}_r, \bar{\psi}_r]$ is an interval-valued Pythagorean fuzzy sub-semigroup of $T$ the the preimage $f^{-1}(\bar{P}) = (f^{-1}(\bar{\phi}_r), f^{-1}(\bar{\psi}_r))$ is an interval-valued Pythagorean fuzzy sub-semigroup of $R$.

If $\bar{P} = [\bar{\phi}_r, \bar{\psi}_r]$ is an interval-valued Pythagorean fuzzy left (resp. right) ideal of $T$ the the preimage $f^{-1}(\bar{P}) = (f^{-1}(\bar{\phi}_r), f^{-1}(\bar{\psi}_r))$ is an interval-valued Pythagorean fuzzy left ideal (resp. right ideal) of $R$.

Proof. Assume that $\bar{P} = [\bar{\phi}_r, \bar{\psi}_r]$ is an interval-valued Pythagorean fuzzy sub-semigroup of $T$ and $r, t \in R$. Then

$$f^{-1}(\overline{\phi}_r)(rt) = \overline{\phi}_r(f(rt))$$

$$= \overline{\phi}_r(f(r)f(t))$$

$$\geq \min\{\overline{\phi}_r(f(r)), \overline{\phi}_r(f(t))\}$$

$$= \min\{f^{-1}(\overline{\phi}_r)(r), f^{-1}(\overline{\phi}_r)(t))\};$$

$$f^{-1}(\overline{\psi}_r)(rt) = \overline{\psi}_r(f(rt))$$
\[
R = \bar{\psi}_p(f(r)f(t)) \\
\leq \max\{\bar{\psi}_p(f(r)), \bar{\psi}_p(f(t))\} \\
= \max\{f^{-1}(\bar{\psi}_p)(r), f^{-1}(\bar{\psi}_p)(f(t))\}.
\]

Hence, \(f^{-1}(\bar{P}) = (f^{-1}(\bar{\phi}_p), f^{-1}(\bar{\psi}_p))\) is an interval-valued Pythagorean fuzzy sub-semigroup of \(R\).

\[
f^{-1}(\bar{\phi}_p)(rt) = \bar{\phi}_p(f(rt)) \\
= \bar{\phi}_p(f(r)f(t)) \\
\geq \bar{\phi}_p(f(t)) \\
= f^{-1}(\bar{\phi}_p)(f(t));
\]

\[
f^{-1}(\bar{\psi}_p)(rt) = \bar{\psi}_p(f(rt)) \\
= \bar{\psi}_p(f(r)f(t)) \\
\leq \bar{\psi}_p(f(t)) \\
= f^{-1}(\bar{\psi}_p)(f(t)).
\]

Hence, \(f^{-1}(\bar{P}) = (f^{-1}(\bar{\phi}_p), f^{-1}(\bar{\psi}_p))\) is an interval-valued Pythagorean fuzzy left (resp. right) ideal of \(R\).

**Theorem 8.** Let \(R, T\) be a semigroups, \(f: R \rightarrow T\) be a homomorphism of semigroups. If \(\bar{P} = [\bar{\phi}_p, \bar{\psi}_p]\) is an interval-valued Pythagorean fuzzy bi-ideal of \(T\) the the preimage \(f^{-1}(\bar{P}) = (f^{-1}(\bar{\phi}_p), f^{-1}(\bar{\psi}_p))\) is an interval-valued Pythagorean fuzzy bi-ideal of \(R\).

**Proof.** Assume that \(\bar{P} = [\bar{\phi}_p, \bar{\psi}_p]\) is an interval-valued Pythagorean fuzzy sub-semigroup of \(T\) and \(a, r, t \in R\). Then

\[
f^{-1}(\bar{\phi}_p)(rat) = \bar{\phi}_p(f(rat)) \\
= \bar{\phi}_p(f(r)f(a)f(t)) \\
\geq \min\{\bar{\phi}_p(f(r)), \bar{\phi}_p(f(t))\} \\
= \min\{f^{-1}(\bar{\phi}_p)(r), f^{-1}(\bar{\phi}_p)(f(t))\};
\]

\[
f^{-1}(\bar{\psi}_p)(rat) = \bar{\psi}_p(f(rat)) \\
= \bar{\psi}_p(f(r)f(a)f(t)) \\
\leq \max\{\bar{\psi}_p(f(r)), \bar{\psi}_p(f(t))\} \\
= \max\{f^{-1}(\bar{\psi}_p)(r), f^{-1}(\bar{\psi}_p)(f(t))\}.
\]

Hence \(f^{-1}(\bar{P}) = (f^{-1}(\bar{\phi}_p), f^{-1}(\bar{\psi}_p))\) is an interval-valued Pythagorean fuzzy bi-ideal of \(R\).

**Theorem 9.** Let \(R, T\) be a semigroups, \(f: R \rightarrow T\) be a homomorphism of semigroups. If \(\bar{P} = [\bar{\phi}_p, \bar{\psi}_p]\) is an interval-valued Pythagorean fuzzy interior ideal of \(T\) the preimage \(f^{-1}(\bar{P}) = (f^{-1}(\bar{\phi}_p), f^{-1}(\bar{\psi}_p))\) is an interval-valued Pythagorean fuzzy interior ideal of \(R\).
5. Conclusion

In this paper interval valued Pythagorean fuzzy sub-semigroup, interval valued Pythagorean fuzzy left (resp. right) ideal, interval valued Pythagorean fuzzy ideal, interval valued Pythagorean fuzzy bi-ideal, interval valued Pythagorean fuzzy interior ideal and Homomorphism of interval valued Pythagorean fuzzy ideal in semigroups are studied and investigated some properties with suitable examples.

References