K-algebras on Quadripartitioned Single Valued Neutrosophic Sets

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**Abstract**

Quadripartitioned Single Valued Neutrosophic (QSVN) set is a powerful structure where we have four components: Truth-T, Falsity-F, Unknown-U and Contradiction-C. And also it generalizes the concept of fuzzy, intuitionistic and single valued neutrosophic set. In this paper we have proposed the concept of K-algebras on QSVN, level subset of QSVN and studied some of the results. In addition to this we have also investigated the characteristics of QSVN K-subalgebras under homomorphism.

**Keywords:** Quadripartitioned Single Valued Neutrosophic Set (QSVNS), K-Algebras, Homomorphism, Quadripartitioned single valued neutrosophic K-Algebras.

1 | Introduction

Dar and Akram [12] proposed a novel logical algebra known as K-algebra. The algebraic structure of a group G which K-algebra was built on should have a right identity element and satisfy the properties of non-commutative and non-associative. Furthermore this group G is of the type where each non-identity element is not of order 2 and K-algebra was built by adjoining the induced binary operation on G [11]-[13]. Zadeh’s fuzzy set theory [22] was a powerful framework which deals the concept of uncertainty, imprecision and also it represented by membership function which lies in a unit interval of \([0,1]\). Fuzzy K-algebra was introduced by Akram et al. [2], [3], [5] and also they established this in a wide-reaching way through other researchers. Later Atanassov [9] introduced the concept of intuitionistic fuzzy set in 1983. It has an additional degree called the degree of nonmembership. Intuitionistic fuzzy K-subalgebras was proposed by Akram et al. [4] and [6]. Intuitionistic fuzzy Ideals of BCK-Algebras was proposed by Jun and Kim [14].
Neutrosophic set which is a generalization of fuzzy set and intuitionistic fuzzy set was introduced by Smarandache [20] in 1998. Along with membership and non-membership function neutrosophic set has one more extra component called indeterminacy membership function. Also all the values of these three components lie in the real standard or non-standard subset of unit interval \([-0,1+\] where \(-0 = 0 - \epsilon, 1+ = 1 + \epsilon\), \(\epsilon\) is an infinitesimal number. In neutrosophic set theory algebraic structures were studied in soft topological K-algebras [7]. Agboola and Davvaz [1] presented the introduction to neutrospheric BCI/BCK algebras. Smarandache and Wang et al. [21] introduced single-valued neutrosophic set which plays a vital place in many real life problems and it takes the values from the subset of \([0,1]\. Akram et al. [8] studied K-algebras on single valued neutrosophic sets and also discussed homomorphisms between the single valued neutrosophic K-subalgebras. Belnap [10] introduced the concept of four valued logic that is the information are represented by four components \(T, F, None, Both\) which denote true, false, neither true nor false, both true and false, respectively. Based on this concept, Smarandache proposed four numerical valued neutrosophic logic where indeterminacy is splitted into two terms known as Contradiction \((C)\) and Unknown \((U)\). Chatterjee et al. [19] introduced Quadripartitioned Single Valued Neutrosophic \((QSVN)\) set in which we have four components \(T, C, U, F\), respectively, and also it lies in the real unit interval of \([0,1]\]. K. Mohana and M. Mohanasundari [15] and [17] studied the concept of Quadripartitioned Single Valued Neutrosophic Relations \((QSVNR)\) as well as some properties of quadripartitioned single valued neutrosophic rough sets and its axiomatic characterizations. Under QSVN environment multicriteria decision making problems has been discussed in [16] and [18].

In this paper Section 2 deals with the basic definitions of QSVN set and the concept of K-algebras on single valued neutrosophic set. Section 3 discusses about K-algebras on QSVN, level subset of QSVN and also studies some of the results. Section 4 defines the homomorphism of quadripartitioned single valued neutrosophic K-algebras, characteristic and fully invariant K-subalgebras. Section 5 concludes the paper.

2 | Preliminaries

This section deals with the basic definitions of QSVNS and K-algebra of single valued neutrosophic set that helps us to study the rest of the paper.

**Definition 1.** [19]. Let \(X\) be a non-empty set. A quadripartitioned neutrosophic set \(A\) over \(X\) characterizes each element \(x\) in \(X\) by a truth-membership function \(T_A\), a contradiction membership function \(C_A\), an ignorance – membership function \(U_A\), and a falsity membership function \(F_A\) such that for each \(x \in X, T_A, C_A, U_A, F_A \in [0,1]\) and \(0 \leq T_A(x) + C_A(x) + U_A(x) + F_A(x) \leq 4\). When \(X\) is discrete \(A\) is represented as, \(A = \sum_{i=1}^{n} T_A(x_i)C_A(x_i)U_A(x_i)F_A(x_i)/x, x \in X\).

However, when the universe of discourse is continuous \(A\) is represented as \(A = \int_{X} T_A(x)C_A(x)U_A(x)F_A(x)/x, x \in X\).

**Definition 2.** [19]. Consider two QSVNS \(A\) and \(B\) over \(X\). \(A\) is said to be contained in \(B\), denoted by \(A \subseteq B\) iff \(T_A(x) \leq T_B(x), C_A(x) \leq C_B(x), U_A(x) \geq U_B(x)\) and \(F_A(x) \geq F_B(x)\).
Definition 3. [19]. The complement of a QSVNS \( A \) is denoted by \( A^C \) and is defined as,
\[
A^C = \bigoplus_{i=1}^{n} F_A(x_i), U_A(x_i), C_A(x_i), T_A(x_i) / x_i \in X,
\]
\[
T_A(x_i) = F_A(x_i), C_A(x_i) = U_A(x_i), T_A(x_i) = C_A(x_i) and F_A(x_i) = T_A(x_i), x_i \in X.
\]

Definition 4. [19]. The union of two QSVNS \( A \) and \( B \) is denoted by \( A \cup B \) and is defined as,
\[
A \cup B = \bigoplus_{i=1}^{n} \left( T_A(x_i) \lor T_B(x_i), C_A(x_i) \lor C_B(x_i), U_A(x_i) \land U_B(x_i), F_A(x_i) \land F_B(x_i) \right) / x_i \in X.
\]

Definition 5. [19]. The intersection of two QSVNS \( A \) and \( B \) is denoted by \( A \cap B \) and is defined as,
\[
A \cap B = \bigoplus_{i=1}^{n} \left( T_A(x_i) \land T_B(x_i), C_A(x_i) \land C_B(x_i), U_A(x_i) \lor U_B(x_i), F_A(x_i) \lor F_B(x_i) \right) / x_i \in X.
\]

Definition 6. [12]. Let \((G, \cdot, e)\) be a group in which each non-identity element is not of order 2. Then a K-algebra is a structure \( \mathbb{K} = (G, \cdot, e) \) on a group \( G \) in which induced binary operation \( \cdot : G \times G \rightarrow G \) is defined \( b \cdot (x, y) = xy^{-1}y \) and satisfies the following axioms:

\[
(x \cdot y) \cdot (x \cdot z) = \left(x \cdot (e \cdot z \cdot (e \cdot y))\right) \cdot x,
\]
\[
x \cdot (x \cdot y) = \left(x \cdot (e \cdot y)\right) \cdot x,
\]
\[
x \cdot e = x,
\]
\[
x \cdot e = x^{-1}, \text{ for all } x, y, z \in G.
\]

Definition 7. [8]. A single-valued neutrosophic set \( A = (T_A, I_A, F_A) \) in a K-algebra \( \mathbb{K} \) is called a single-valued neutrosophic K-subalgebra of \( \mathbb{K} \) if it satisfies the following conditions:

\[
T_A(s \cdot t) \geq \min\{T_A(s), T_A(t)\},
\]
\[
I_A(s \cdot t) \geq \min\{I_A(s), I_A(t)\},
\]
\[
F_A(s \cdot t) \leq \max\{F_A(s), F_A(t)\}, \text{ for all } s, t \in G.
\]

Note that \( T_A(e) \geq T_A(s), I_A(e) \geq I_A(s), F_A(e) \leq F_A(s), \text{ for all } s \in G. \)

3 | Quadripartitioned Single Valued Neutrosophic K-Algebras

Definition 8. A quadripartitioned single valued neutrosophic set \( X = (T_X, C_X, U_X, F_X) \) in a K-algebra \( \mathbb{K} \) is called a quadripartitioned single valued neutrosophic K-subalgebra of \( \mathbb{K} \) if it satisfies the following conditions:

\[
T_X(u) \geq T_X(u), C_X(e) \geq C_X(u), U_X(e) \leq U_X(u), F_X(e) \leq F_X(u) \text{ for all } u \in G.
\]
\[ T_X(u \Theta v) \geq \min \left\{ T_X(u), T_X(v) \right\}, \]

\[ C_X(u \Theta v) \geq \min \left\{ C_X(u), C_X(v) \right\}, \]

\[ U_X(u \Theta v) \leq \max \left\{ U_X(u), U_X(v) \right\}, \]

\[ F_X(u \Theta v) \leq \max \left\{ F_X(u), F_X(v) \right\} \text{ for all } u, v \in G. \]

**Example 1.** Let \( G = \{e, g, g^2, g^3, g^4\} \) be the cyclic group of order 5 in a K-algebra \( K = (G, \cdot, \Theta, e) \).

The Cayley’s table for \( \Theta \) is given as follows.

We define a quadripartitioned single valued neutrosophic set \( X = (T_X, C_X, U_X, F_X) \) in K-algebra as follows:

\[
\begin{align*}
T_X(e) &= 0.5, & C_X(e) &= 0.7, & U_X(e) &= 0.3, & F_X(e) &= 0.5 \\
T_X(u) &= 0.2, & C_X(u) &= 0.4, & U_X(u) &= 0.5, & F_X(u) &= 0.8
\end{align*}
\]

for all \( u \neq e \in G \). Clearly, it shows that \( X = (T_X, C_X, U_X, F_X) \) is a quadripartitioned single valued neutrosophic K-algebras of \( K \).

**Proposition 1.** If \( X = (T_X, C_X, U_X, F_X) \) denotes a quadripartitioned single valued neutrosophic K-algebras of \( K \) then,

a) \( ( \forall u, v \in G ), (T_X(u \Theta v) = T_X(v) \Rightarrow T_X(u) = T_X(e)) \)

b) \( ( \forall u, v \in G ), (C_X(u \Theta v) = C_X(v) \Rightarrow C_X(u) = C_X(e)) \)

c) \( ( \forall u, v \in G ), (U_X(u \Theta v) = U_X(v) \Rightarrow U_X(u) = U_X(e)) \)

d) \( ( \forall u, v \in G ), (F_X(u \Theta v) = F_X(v) \Rightarrow F_X(u) = F_X(e)) \)

\( ( \forall u, v \in G ), (F_X(u) = F_X(e) \Rightarrow F_X(u \Theta v) \geq F_X(v)) \);
Proof. We only prove (a) and (c). (b) and (d) proved in a similar way.

(a) First we assume that $T_X(u\oplus v) = T_X(v) \forall u,v \in G$. Put $v = e$ and use (iii) of Definition 6 we get $T_X(u) = T_X(u\oplus e) = T_X(e)$. Let for $u,v \in G$ be such that $T_X(u) = T_X(e)$ then $T_X(u\oplus v) \geq \min(T_X(u), T_X(v)) = \min(T_X(e), T_X(v)) = T_X(v)$.

Now to prove (c) consider that $U_X(u\oplus v) = U_X(v) \forall u,v \in G$. Put $v = e$ and use (iii) of Definition 6, we have $U_X(u) = U_X(u\oplus e) = U_X(e)$. Let for $u,v \in G$ be such that $U_X(u) = U_X(e)$ then $U_X(u\oplus v) \leq \max(U_X(u), U_X(v)) = \max(U_X(e), U_X(v)) = U_X(v)$. Hence the proof.

Definition 9. Let $X = (T_X, C_X, U_X, F_X)$ be a quadripartitioned single valued neutrosophic set in a K-algebra of $K$ and let $(\lambda, \mu, \delta, \xi) \in [0,1] \times [0,1] \times [0,1] \times [0,1]$ with $\lambda + \mu + \delta + \xi \leq 4$. Then the sets,

$$X_{(\lambda,\mu,\delta,\xi)} = \{u \in G| T_X(u) \geq \lambda, \ C_X(u) \geq \mu, \ U_X(u) < \delta, \ F_X(u) < \xi\}$$

$$(\lambda, \mu, \delta, \xi)X_{(\lambda,\mu,\delta,\xi)} = U(T_X, \lambda) \cap U'(C_X, \mu) \cap L(U_X, \delta) \cap L'(F_X, \xi)$$

are called $(\lambda, \mu, \delta, \xi)$ level subsets of quadripartitioned single valued neutrosophic set $X$.

And also the set $X_{(\lambda,\mu,\delta,\xi)} = \{u \in G| T_X(u) > \lambda, \ C_X(u) > \mu, \ U_X(u) < \delta, \ F_X(u) < \xi\}$ is known as strong level subset of $X$.

Note. The set of all $(\lambda, \mu, \delta, \xi) \in \text{Im}(T_X) \times \text{Im}(C_X) \times \text{Im}(U_X) \times \text{Im}(F_X)$ is known as image of $X = (T_X, C_X, U_X, F_X)$.

Proposition 2. If $X = (T_X, C_X, U_X, F_X)$ is a quadripartitioned single valued neutrosophic K-algebra of $K$ then the level subsets,

$$U(T_X, \lambda) = \{u \in G| T_X(u) \geq \lambda\}, \ U'(C_X, \mu) = \{u \in G| C_X(u) \geq \mu\}$$

$$L(U_X, \delta) = \{u \in G| U_X(u) < \delta\}, \ L'(F_X, \xi) = \{u \in G| F_X(u) < \xi\}$$

are K-subalgebras of $K$ for every $(\lambda, \mu, \delta, \xi) \in \text{Im}(T_X) \times \text{Im}(C_X) \times \text{Im}(U_X) \times \text{Im}(F_X)$ \in [0,1]

where $\text{Im}(T_X), \text{Im}(C_X), \text{Im}(U_X)$ and $\text{Im}(F_X)$ are sets of values $T(X), C(X), U(X)$ and $F(X)$, respectively.

Proof. Let $X = (T_X, C_X, U_X, F_X)$ be a quadripartitioned single valued neutrosophic set in a K-algebra of $K$ and $(\lambda, \mu, \delta, \xi) \in \text{Im}(T_X) \times \text{Im}(C_X) \times \text{Im}(U_X) \times \text{Im}(F_X)$ be such that $U(T_X, \lambda) \neq \emptyset$, $U'(C_X, \mu) \neq \emptyset$, $L(U_X, \delta) \neq \emptyset$ and $L'(F_X, \xi) \neq \emptyset$. We have to show that $U, U', L$ and $L'$ are level K-subalgebras. Let for $u,v \in U(T_X, \lambda)$, $T_X(u) \geq \lambda$ and $T_X(v) \geq \lambda$. Then from Definition 8 we get $T_X(u\oplus v) \geq \min(T_X(u), T_X(v)) \geq \lambda$. It shows that $u\oplus v \in U(T_X, \lambda)$. Hence $U(T_X, \lambda)$ is a level K-subalgebra of $K$. Similarly, we can prove for $U'(C_X, \mu), L(U_X, \delta)$ and $L'(F_X, \xi)$.

Theorem 1. Let $X = (T_X, C_X, U_X, F_X)$ be a quadripartitioned single valued neutrosophic set in a K-algebra of $K$. Then $X = (T_X, C_X, U_X, F_X)$ is a quadripartitioned single valued neutrosophic K-subalgebra of $K$ if and only if $X_{(\lambda,\mu,\delta,\xi)}$ is a K-subalgebra of $K$ for every $(\lambda, \mu, \delta, \xi) \in \text{Im}(T_X) \times \text{Im}(C_X) \times \text{Im}(U_X) \times \text{Im}(F_X)$ with $\lambda + \mu + \delta + \xi \leq 4$.

Proof. First assume that $X_{(\lambda,\mu,\delta,\xi)}$ is a K-subalgebra of $K$. If the conditions in Definition 8 fail, then there exist $s, t \in G$ such that,
Theorem 2. Let $X = (T_X, C_X, U_X, F_X)$ be a quadripartitioned single valued neutrosophic $K$-subalgebra and $(\lambda_1, \mu_1, \delta_1, \xi_1), (\lambda_2, \mu_2, \delta_2, \xi_2) \in \text{Im}(T_X) \times \text{Im}(C_X) \times \text{Im}(U_X) \times \text{Im}(F_X)$ with $\lambda_i + \mu_i + \delta_i + \xi_i \leq 4$ for $i = 1, 2$. Then $X_{(\lambda_1, \mu_1, \delta_1, \xi_1)} = X_{(\lambda_2, \mu_2, \delta_2, \xi_2)}$ if $(\lambda_1, \mu_1, \delta_1, \xi_1) = (\lambda_2, \mu_2, \delta_2, \xi_2)$. This proves that the conditions of Definition 8 is true. Hence $X = (T_X, C_X, U_X, F_X)$ is a quadripartitioned single valued neutrosophic $K$-subalgebra of $K$.
Proof. When \((\lambda_1, \mu_1, \delta_1, \xi_1) = (\lambda_2, \mu_2, \delta_2, \xi_2)\) then the result is obvious for \(X((\lambda_1, \mu_1, \delta_1, \xi_1)) = X((\lambda_2, \mu_2, \delta_2, \xi_2))\). Conversely assume that \(X((\lambda_1, \mu_1, \delta_1, \xi_1)) = X((\lambda_2, \mu_2, \delta_2, \xi_2))\). Since \((\lambda_1, \mu_1, \delta_1, \xi_1) \in \text{Im}(T_X) \times \text{Im}(C_X) \times \text{Im}(U_X) \times \text{Im}(F_X)\) there exists \(u \in G\) such that \(T_X(u) = \lambda_1, C_X(u) = \mu_1, U_X(u) = \delta_1\) and \(F_X(u) = \xi_1\). This implies that \(u \in X((\lambda_1, \mu_1, \delta_1, \xi_1)) = X((\lambda_2, \mu_2, \delta_2, \xi_2))\). Hence \(\lambda_1 = T_X(u) \geq \lambda_2\) and \(\lambda_2 \geq \mu_2\), \(\delta_1 = U_X(u) \leq \delta_2\) and \(\xi_1 = F_X(u) \leq \xi_2\). Also \((\lambda_2, \mu_2, \delta_2, \xi_2) \in \text{Im}(T_X) \times \text{Im}(C_X) \times \text{Im}(U_X) \times \text{Im}(F_X)\) there exists \(v \in G\) such that \(T_X(v) = \lambda_2, C_X(v) = \mu_2, U_X(v) = \delta_2\) and \(F_X(v) = \xi_2\). This implies that \(v \in X((\lambda_2, \mu_2, \delta_2, \xi_2)) = X((\lambda_1, \mu_1, \delta_1, \xi_1))\). Hence \(\lambda_2 = T_X(v) \geq \lambda_1, \mu_2 = C_X(v) \geq \mu_1, \delta_2 = U_X(v) \leq \delta_1\) and \(\xi_2 = F_X(v) \leq \xi_1\). Hence \((\lambda_1, \mu_1, \delta_1, \xi_1) = (\lambda_2, \mu_2, \delta_2, \xi_2)\).

Theorem 3. Let \(I\) be a \(K\)-subalgebra of \(K\)-algebra \(K\). Then there exists a quadripartitioned single valued neutrosophic \(K\)-subalgebra \(X = (T_X, C_X, U_X, F_X)\) of \(K\)-algebra \(K\) such that \(X = (T_X, C_X, U_X, F_X) = I\) for some \(\lambda, \mu \in (0,1]\) and \(\delta, \xi \in [0,1]\).

Proof. Let \(X = (T_X, C_X, U_X, F_X)\) be a quadripartitioned single valued neutrosophic set in \(K\)-algebra \(K\) given by,

\[
T_X(u) = \begin{cases} 
\lambda \in (0,1], & \text{if } u \in I \\
0, & \text{otherwise}
\end{cases}
\]

\[
C_X(u) = \begin{cases} 
\mu \in (0,1], & \text{if } u \in I \\
0, & \text{otherwise}
\end{cases}
\]

\[
U_X(u) = \begin{cases} 
\delta \in [0,1], & \text{if } u \in I \\
0, & \text{otherwise}
\end{cases}
\]

\[
F_X(u) = \begin{cases} 
\xi \in [0,1], & \text{if } u \in I \\
0, & \text{otherwise}
\end{cases}
\]

Let \(u, v \in G\). If \(u, v \in I\) then \(u \triangleleft v \in I\) and so,

\[
T_X(u \triangleleft v) \geq \min\{T_X(u), T_X(v)\},
\]

\[
C_X(u \triangleleft v) \geq \min\{C_X(u), C_X(v)\},
\]

\[
U_X(u \triangleleft v) \leq \max\{U_X(u), U_X(v)\},
\]

\[
F_X(u \triangleleft v) \leq \max\{F_X(u), F_X(v)\}.
\]

Suppose \(u \notin I\) or \(v \notin I\) then,

\[
T_X(u) = 0 \text{ or } T_X(v), C_X(u) = 0 \text{ or } C_X(v), U_X(u) = 0 \text{ or } U_X(v) \text{ and } F_X(u) = 0 \text{ or } F_X(v).
\]

It implies that,

\[
T_X(u \triangleleft v) \geq \min\{T_X(u), T_X(v)\},
\]

\[
C_X(u \triangleleft v) \geq \min\{C_X(u), C_X(v)\},
\]

\[
U_X(u \triangleleft v) \leq \max\{U_X(u), U_X(v)\},
\]

\[
F_X(u \triangleleft v) \leq \max\{F_X(u), F_X(v)\}.
\]

Hence \(X = (T_X, C_X, U_X, F_X)\) is a quadripartitioned single valued neutrosophic \(K\)-subalgebra of \(K\).
Consequently $X_{(\lambda, \mu, \delta, \xi)} = I$

**Theorem 4.** Let $K$ be a $K$-algebra. Let a chain of $K$-subalgebras: $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = G$. Then the level $K$-subalgebras of the quadripartitioned single valued neutrosophic $K$-subalgebra remains same as the $K$-subalgebras of this chain.

**Proof.** Let $\{\lambda_i|i = 0, 1, \ldots, n\}, \{\mu_i|i = 0, 1, \ldots, n\}, \{\delta_i|i = 0, 1, \ldots, n\}, \{\xi_i|i = 0, 1, \ldots, n\}$ be finite decreasing sequences and $\{\delta_i|i = 0, 1, \ldots, n\}, \{\xi_i|i = 0, 1, \ldots, n\}$ be finite increasing sequences in $[0,1]$ such that $\lambda_k + \mu_k + \delta_k + \xi_k \leq 4$ for $k = 0, 1, 2, \ldots, n$. Let $X = (T_X, C_X, U_X, F_X)$ be a quadripartitioned single valued neutrosophic set in $K$, defined by $T_X(X_0) = \lambda_0, C_X(X_0) = \mu_0, U_X(X_0) = \delta_0$ and $F_X(X_0) = \xi_0$.

$T_X(X_i|X_{i-1}) = \lambda_i, C_X(X_i|X_{i-1}) = \mu_i, U_X(X_i|X_{i-1}) = \delta_i$ and $F_X(X_i|X_{i-1}) = \xi_i$, for $0 < i \leq n$.

We have $u\delta v \in X_{i-1}$, prove that $X = (T_X, C_X, U_X, F_X)$ is a quadripartitioned single valued neutrosophic $K$-subalgebra of $K$. Let $u, v \in G$. If $u, v \in X_{i-1}$ then it implies that $T_X(u) = \lambda_i = T_X(v), C_X(u) = \mu_i = C_X(v), U_X(u) = \delta_i = U_X(v)$ and $F_X(u) = \xi_i = F_X(v)$. Since each $X_i$ is a $K$-subalgebra, we get $u\delta v \in X_i$. So that either $u\delta v \in X_{i-1}$ or. In any of the above case it follows that,

$$T_X(u\delta v) \geq \lambda_i = \min[T_X(u), T_X(v)],$$

$$C_X(u\delta v) \geq \mu_i = \min[C_X(u), C_X(v)],$$

$$U_X(u\delta v) \leq \delta_i = \max[U_X(u), U_X(v)],$$

$$F_X(u\delta v) \leq \xi_i = \max[F_X(u), F_X(v)].$$

For $k > 1$ if $u \in X_k\setminus X_{k-1}$ and $v \in X_k\setminus X_{k-1}$ then,

$$T_X(u) = \lambda_k, T_X(v) = \lambda_k,$$

$$C_X(u) = \mu_k, C_X(v) = \mu_k,$$

$$U_X(u) = \delta_k, U_X(v) = \delta_k,$$

$$F_X(u) = \xi_k, F_X(v) = \xi_k,$$

and $u\delta v \in X_k$ because $X_k$ is a $K$-subalgebra and $X_i \subset X_k$. It follows that,

$$T_X(u\delta v) \geq \lambda_k = \min[T_X(u), T_X(v)],$$

$$C_X(u\delta v) \geq \mu_k = \min[C_X(u), C_X(v)],$$

$$U_X(u\delta v) \leq \delta_k = \max[U_X(u), U_X(v)],$$

$$F_X(u\delta v) \leq \xi_k = \max[F_X(u), F_X(v)].$$

Hence $X = (T_X, C_X, U_X, F_X)$ is a quadripartitioned single valued neutrosophic $K$-subalgebra of $K$ and all its non-empty level subsets are level $K$-subalgebras of $K$. Since $\text{Im}(T_X) = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}, \text{Im}(C_X) = \{\mu_0, \mu_1, \ldots, \mu_n\}, \text{Im}(U_X) = \{\delta_0, \delta_1, \ldots, \delta_n\}$ and $\text{Im}(F_X) = \{\xi_0, \xi_1, \ldots, \xi_n\}$. Therefore, the level $K$-subalgebras of $X = (T_X, C_X, U_X, F_X)$ are given by the chain of $K$-subalgebras:
\[ U(T_X, \lambda_0) \subset U(T_X, \lambda_1) \subset \cdots \subset U(T_X, \lambda_n) = G, \]
\[ U'(C_X, \mu_0) \subset U'(C_X, \mu_1) \subset \cdots \subset U'(C_X, \mu_n) = G, \]
\[ L(U_X, \delta_0) \subset L(U_X, \delta_1) \subset \cdots \subset L(U_X, \delta_n) = G, \]
\[ L'(F_X, \xi_0) \subset L'(F_X, \xi_1) \subset \cdots \subset L'(F_X, \xi_n) = G, \]
respectively. Indeed,
\[ U(T_X, \lambda_0) = \{ u \in G | T_X(u) \geq \lambda_0 \} = X_0, \]
\[ U'(C_X, \mu_0) = \{ u \in G | C_X(u) \geq \mu_0 \} = X_0, \]
\[ L(U_X, \delta_0) = \{ u \in G | U_X(u) \leq \delta_0 \} = X_0, \]
\[ L'(F_X, \xi_0) = \{ u \in G | F_X(u) \leq \xi_0 \} = X_0. \]

Now we have to prove that,
\[ U(T_X, \lambda_i) = X_0, U'(C_X, \mu_i) = X_0, L(U_X, \delta_i) = X_i \text{ and } L'(F_X, \xi_i) = X_i \text{ for } 0 < i \leq n. \] Clearly \( X_i \subseteq U(T_X, \lambda_i), X_i \subseteq U'(C_X, \mu_i), X_i \subseteq L(U_X, \delta_i) \text{ and } X_i \subseteq L'(F_X, \xi_i) \). If \( u \in U(T_X, \lambda_i) \) then \( T_X(u) \geq \lambda_i \) and so \( u \notin A_k \text{ for } k > i \). Hence \( T_X(u) \in \{ \lambda_0, \lambda_1, \ldots, \lambda_i \} \) which shows that \( u \in X_k \) for \( k \leq i \), since \( X_k \subseteq X_i \). It follows that \( u \in X_i \). Consequently \( U(T_X, \lambda_i) = X_i \text{ for some } 0 < i \leq n. \) Similarly, it is proved for \( U'(C_X, \mu_i) = X_i \). Now if \( v \in L(U_X, \delta_i) \) then \( U_X(v) \leq \delta_i \) and so \( v \notin X_k \text{ for some } k \leq i. \) Thus \( U_X(u) \in \{ \delta_0, \delta_1, \ldots, \delta_i \} \) which shows that \( u \in X_l \) for some \( l \leq i \), since \( X_l \subseteq X_i \). It follows that \( v \in X_i \). Consequently, \( L(U_X, \delta_i) = X_i \text{ for some } 0 < i \leq n. \) Similarly, it is proved for \( L'(F_X, \xi_i) = X_i \). Hence the proof.

4 | Homomorphism of Quadripartitioned Single Valued Neutrosophic K-Algebras

**Definition 10.** Consider two K-algebras \( K_1 = (G_1, \cdot, O_1, e_1) \) and \( K_2 = (G_2, \cdot, O_2, e_2) \) and \( f \) be a function from \( K_1 \) into \( K_2 \). If \( Y = (T_Y, C_Y, U_Y, F_Y) \) is a quadripartitioned single valued neutrosophic K-subalgebra of \( K_2 \), then the preimage of \( Y = (T_Y, C_Y, U_Y, F_Y) \) under \( f \) is a quadripartitioned single valued neutrosophic K-subalgebra of \( K_1 \) defined by,

\[ f^{-1}(T_Y(u)) = T_Y(f(u)), f^{-1}(C_Y(u)) = C_Y(f(u)), \]
\[ f^{-1}(U_Y(u)) = U_Y(f(u)), f^{-1}(F_Y(u)) = F_Y(f(u)), \]
for all \( u \in G. \)

**Definition 11.** A quadripartitioned single valued neutrosophic K-subalgebra \( X = (T_X, C_X, U_X, F_X) \) of a K-algebra \( K \) is called characteristic if \( T_X(f(u)) = T_X(u), C_X(f(u)) = C_X(u), U_X(f(u)) = U_X(u) \text{ and } F_X(f(u)) = F_X(u) \) for all \( u \in G \) and \( f \in Aut(K). \)

**Definition 12.** A K-subalgebra \( U \) of a K-algebra \( K \) is said to be fully invariant if \( f(U) \subseteq U \) for all \( f \in End(K) \) where \( End(K) \) is the set of all endomorphisms of a K-algebra \( K \). A quadripartitioned single valued neutrosophic K-subalgebra \( X = (T_X, C_X, U_X, F_X) \) of a K-algebra \( K \) is called fully invariant if \( T_X(f(u)) \leq T_X(u), C_X(f(u)) \leq C_X(u), U_X(f(u)) \geq U_X(u) \text{ and } F_X(f(u)) \geq F_X(u) \) for all \( u \in G \) and \( f \in End(K). \)
Definition 13. Let $X_1 = (T_{X_1}, C_{X_1}, U_{X_1}, F_{X_1})$ and $X_2 = (T_{X_2}, C_{X_2}, U_{X_2}, F_{X_2})$ be two quadripartitioned single valued neutrosophic $K$-subalgebras of $K$. Then $X_1 = (T_{X_1}, C_{X_1}, U_{X_1}, F_{X_1})$ is said to be the same type of $X_2 = (T_{X_2}, C_{X_2}, U_{X_2}, F_{X_2})$ if there exists $f \in Aut(K)$ such that $X_1 = X_2 \circ f$ i.e., $T_{X_1}(u) = T_{X_2}(f(u)), C_{X_1}(u) = C_{X_2}(f(u)), U_{X_1}(u) = U_{X_2}(f(u))$ and $F_{X_1}(u) = F_{X_2}(f(u))$ for all $u \in G$.

Theorem 5. Let $f: K_1 \rightarrow K_2$ be an epimorphism of $K$-algebras. If $Y = (T_Y, C_Y, U_Y, F_Y)$ is a quadripartitioned single valued neutrosophic $K$-subalgebra of $K_2$, then $f^{-1}(Y)$ is a quadripartitioned single valued neutrosophic $K$-subalgebra of $K_1$.

Proof. It is obvious that,

$$f^{-1}(T_Y)(e) \geq f^{-1}(T_Y)(u), f^{-1}(C_Y)(e) \geq f^{-1}(C_Y)(u),$$

$$f^{-1}(U_Y)(e) \leq f^{-1}(U_Y)(u), f^{-1}(F_Y)(e) \leq f^{-1}(F_Y)(u),$$

for all $u \in G$. Let $u, v \in G$ then,

$$f^{-1}(T_Y)(u \otimes v) = T_Y(f(u \otimes v)),$$

$$f^{-1}(T_Y)(u \otimes v) = T_Y(f(u) \otimes f(v)),$$

$$f^{-1}(T_Y)(u \otimes v) \geq \min [T_Y(f(u)), T_Y(f(v))],$$

$$f^{-1}(T_Y)(u \otimes v) \geq \min [f^{-1}(T_Y)(u), f^{-1}(T_Y)(v)];$$

$$f^{-1}(C_Y)(u \otimes v) = C_Y(f(u \otimes v)),$$

$$f^{-1}(C_Y)(u \otimes v) = C_Y(f(u) \otimes f(v)),$$

$$f^{-1}(C_Y)(u \otimes v) \geq \min \{C_Y(f(u)), C_Y(f(v))\},$$

$$f^{-1}(C_Y)(u \otimes v) \geq \min [f^{-1}(C_Y)(u), f^{-1}(C_Y)(v)];$$

$$f^{-1}(U_Y)(u \otimes v) = U_Y(f(u \otimes v)),$$

$$f^{-1}(U_Y)(u \otimes v) = U_Y(f(u) \otimes f(v)),$$

$$f^{-1}(U_Y)(u \otimes v) \leq \max \{U_Y(f(u)), U_Y(f(v))\},$$

$$f^{-1}(U_Y)(u \otimes v) \leq \max [f^{-1}(U_Y)(u), f^{-1}(U_Y)(v)];$$

$$f^{-1}(F_Y)(u \otimes v) = F_Y(f(u \otimes v)),$$

$$f^{-1}(F_Y)(u \otimes v) = F_Y(f(u) \otimes f(v)),$$

$$f^{-1}(F_Y)(u \otimes v) \leq \max \{F_Y(f(u)), F_Y(f(v))\},$$

$$f^{-1}(F_Y)(u \otimes v) \leq \max [f^{-1}(F_Y)(u), f^{-1}(F_Y)(v)].$$
Hence $f^{-1}(Y)$ is a quadripartitioned single valued neutrosophic $K$-subalgebra of $K_1$.

**Theorem 6.** Let $f: K_1 \to K_2$ be an epimorphism of $K$-algebras. If $Y = (T_Y, C_Y, U_Y, F_Y)$ is a quadripartitioned single valued neutrosophic $K$-subalgebra of $K_2$ and $X = (T_X, C_X, U_X, F_X)$ is the preimage of $Y$ under $f$. Then $X$ is a quadripartitioned single valued neutrosophic $K$-subalgebra of $K_1$.

**Proof.** It is obvious that $T_X(e) \geq T_X(u), C_X(e) \geq C_X(u), U_X(u) \leq U_X(u)$ and $F_X(u) \leq F_X(u)$ for all $u \in G_1$. Now for any $u, v \in G_1$,

\[
\begin{align*}
T_X(uOv) &= T_Y(f(uOv)), \\
T_X(uOv) &= T_Y(f(u)O(f(v))), \\
T_X(uOv) &\geq \min\{T_Y(f(u)), T_Y(f(v))\}, \\
T_X(uOv) &\geq \min\{T_X(u), T_X(v)\}; \\
C_X(uOv) &= C_Y(f(uOv)), \\
C_X(uOv) &= C_Y(f(u)O(f(v))), \\
C_X(uOv) &\geq \min\{C_Y(f(u)), C_Y(f(v))\}, \\
C_X(uOv) &\geq \min\{C_X(u), C_X(v)\}; \\
U_X(uOv) &= U_Y(f(uOv)), \\
U_X(uOv) &= U_Y(f(u)O(f(v))), \\
U_X(uOv) &\leq \max\{U_Y(f(u)), U_Y(f(v))\}, \\
U_X(uOv) &\leq \max\{U_X(u), U_X(v)\}; \\
F_X(uOv) &= F_Y(f(uOv)), \\
F_X(uOv) &= F_Y(f(u)O(f(v))), \\
F_X(uOv) &\leq \max\{F_Y(f(u)), F_Y(f(v))\}, \\
F_X(uOv) &\leq \max\{F_X(u), F_X(v)\}.
\end{align*}
\]

Hence $X$ is a quadripartitioned single valued neutrosophic $K$-subalgebra of $K_1$.

**Definition 14.** Let $f$ be a mapping from $K_1$ into $K_2$ i.e., $f: K_1 \to K_2$ of $K$-algebras and let $X = (T_X, C_X, U_X, F_X)$ be a quadripartitioned single valued neutrosophic set of $K_2$. The map $X = (T_X, C_X, U_X, F_X)$ is called the preimage of $X$ under $f$ if $T_X^f(u) = T_X(f(u)), C_X^f(u) = C_X(f(u)), U_X^f(u) = U_X(f(u))$ and $F_X^f(u) = F_X(f(u))$ for all $u \in G_1$. 
**Theorem 7.** Let \( f : K_1 \to K_2 \) be an epimorphism of \( K \)-algebras. Then \( X'^f = (T'_{X'}, C'_{X'}, U'_{X'}, F'_{X'}) \) is a quadripartitioned single valued neutrosophic \( K \)-subalgebra of \( K_1 \) if and only if \( X = (T_X, C_X, U_X, F_X) \) is a quadripartitioned single valued neutrosophic \( K \)-subalgebra of \( K_2 \).

**Proof.** Let \( f : K_1 \to K_2 \) be an epimorphism of \( K \)-algebras. First assume that \( X'^f = (T'_{X'}, C'_{X'}, U'_{X'}, F'_{X'}) \) is a quadripartitioned single valued neutrosophic \( K \)-subalgebra of \( K_1 \). Then we have to prove that \( X = (T_X, C_X, U_X, F_X) \) is a quadripartitioned single valued neutrosophic \( K \)-subalgebra of \( K_2 \). Since there exists \( u \in G_1 \) such that \( v = f(u) \) for any \( v \in G_2 \):

\[
T_X(v) = T_X(\hat{f}(u)) = T_X^{(f(u))} \leq T_X^{(e_1)} = T_X(\hat{f}(e_1)) = T_X(e_2),
\]

\[
C_X(v) = C_X(\hat{f}(u)) = C_X^{(f(u))} \leq C_X^{(e_1)} = C_X(\hat{f}(e_1)) = C_X(e_2),
\]

\[
U_X(v) = U_X(\hat{f}(u)) = U_X^{(f(u))} \geq U_X^{(e_1)} = U_X(\hat{f}(e_1)) = U_X(e_2),
\]

\[
F_X(v) = F_X(\hat{f}(u)) = F_X^{(f(u))} \geq F_X^{(e_1)} = F_X(\hat{f}(e_1)) = F_X(e_2).
\]

For any \( u, v \in G_2, s, t \in G_1 \) such that \( u = f(s) \) and \( v = f(t) \). It follows that:

\[
T_X(u \circ v) = T_X(f(s \circ t)),
\]

\[
T_X(u \circ v) = T_X(s \circ t),
\]

\[
T_X(u \circ v) \geq \min\{T_X(s), T_X(t)\},
\]

\[
T_X(u \circ v) \geq \min\{T_X(\hat{f}(s)), T_X(\hat{f}(t))\},
\]

\[
T_X(u \circ v) \geq \min\{T_X(u), T_X(v)\};
\]

\[
C_X(u \circ v) = C_X(f(s \circ t)),
\]

\[
C_X(u \circ v) = C_X(s \circ t),
\]

\[
C_X(u \circ v) \geq \min\{C_X(s), C_X(t)\},
\]

\[
C_X(u \circ v) \geq \min\{C_X(\hat{f}(s)), C_X(\hat{f}(t))\},
\]

\[
C_X(u \circ v) \geq \min\{C_X(u), C_X(v)\};
\]

\[
U_X(u \circ v) = U_X(f(s \circ t)),
\]

\[
U_X(u \circ v) = U_X(s \circ t),
\]

\[
U_X(u \circ v) \leq \max\{U_X(s), U_X(t)\},
\]

\[
U_X(u \circ v) \leq \max\{U_X(\hat{f}(s)), U_X(\hat{f}(t))\},
\]

\[
U_X(u \circ v) \leq \max\{U_X(u), U_X(v)\};
\]
F_X(u\odot v) = F_X(f(s\odot t)),
F_X(u\odot v) = F_X^f(s\odot t),
F_X(u\odot v) \leq \max\{F_X^f(s), F_X^f(t)\},
F_X(u\odot v) \leq \max\{F_X(f(s)), F_X(f(t))\},
F_X(u\odot v) \leq \max\{F_X(u), F_X(v)\}.

Hence X = (T_X, C_X, U_X, F_X) is a quadripartitioned single valued neutrosophic K-subalgebra of \( K_2 \). Conversely, assume that X = (T_X, C_X, U_X, F_X) is a quadripartitioned single valued neutrosophic K-subalgebra of \( K_2 \). Then we have to prove that \( X' = (T_X^f, C_X^f, U_X^f, F_X^f) \) is a quadripartitioned single valued neutrosophic K-subalgebra of \( K_1 \). For any \( u \in G_1 \) we have:

\[ T_X^f(e_1) = T_X(f(e_1)) = T_X(e_2) \geq T_X(f(u)) = T_X^f(u), \]
\[ C_X^f(e_1) = C_X(f(e_1)) = C_X(e_2) \geq C_X(f(u)) = C_X^f(u), \]
\[ U_X^f(e_1) = U_X(f(e_1)) = U_X(e_2) \leq U_X(f(u)) = U_X^f(u), \]
\[ F_X^f(e_1) = F_X(f(e_1)) = F_X(e_2) \leq F_X(f(u)) = F_X^f(u). \]

Since X is a quadripartitioned single valued neutrosophic K-subalgebra of \( K_2 \) and for any \( u, v \in G_1 \),

\[ T_X^f(u\odot v) = T_X(f(u\odot v)), \]
\[ T_X^f(u\odot v) = T_X(f(u)\odot f(v)), \]
\[ T_X^f(u\odot v) \geq \min\{T_X(f(u)), T_X(f(v))\}, \]
\[ T_X^f(u\odot v) \geq \min\{T_X^f(u), T_X^f(v)\}, \]
\[ C_X^f(u\odot v) = C_X(f(u\odot v)), \]
\[ C_X^f(u\odot v) = C_X(f(u)\odot f(v)), \]
\[ C_X^f(u\odot v) \geq \min\{C_X(f(u)), C_X(f(v))\}, \]
\[ C_X^f(u\odot v) \geq \min\{C_X^f(u), C_X^f(v)\}, \]
\[ U_X^f(u\odot v) = U_X(f(u\odot v)), \]
\[ U_X^f(u\odot v) = U_X(f(u)\odot f(v)), \]
\[ U_X^f(u\odot v) \leq \max\{U_X(f(u)), U_X(f(v))\}, \]
\[ U^K_X(u \vee v) \leq \max\{ U^K_X(u), U^K_X(v) \}; \]
\[ F^K_X(u \vee v) = F_X(f(u \vee v)), \]
\[ F^K_X(u \vee v) = F_X(f(u) \odot f(v)), \]
\[ F^K_X(u \vee v) \leq \max\{ F_X(f(u)), F_X(f(v)) \}, \]
\[ F^K_X(u \vee v) \leq \max\{ F^K_X(u), F^K_X(v) \}. \]

Hence \( X' = (T'_X, C'_X, U'_X, F'_X) \) is a quadripartitioned single valued neutrosophic \( K \)-subalgebra of \( K \).

**Theorem 8.** Let \( X_1 = (T_{X_1}, C_{X_1}, U_{X_1}, F_{X_1}) \) and \( X_2 = (T_{X_2}, C_{X_2}, U_{X_2}, F_{X_2}) \) be two quadripartitioned single valued neutrosophic \( K \)-subalgebras of \( K \). Then a quadripartitioned single valued neutrosophic \( K \)-subalgebra \( X_1 = (T_{X_1}, C_{X_1}, U_{X_1}, F_{X_1}) \) is of the same type of quadripartitioned single valued neutrosophic \( K \)-subalgebra \( X_2 = (T_{X_2}, C_{X_2}, U_{X_2}, F_{X_2}) \) if and only if \( X_1 \) is isomorphic to \( X_2 \).

**Proof.** It is enough to prove only the necessary condition since sufficient condition holds trivially. Let \( X_1 = (T_{X_1}, C_{X_1}, U_{X_1}, F_{X_1}) \) be quadripartitioned single valued neutrosophic \( K \)-subalgebra having same type of \( X_2 = (T_{X_2}, C_{X_2}, U_{X_2}, F_{X_2}) \). Then there exists \( f \in Aut(K) \) such that \( T_{X_1}(u) = T_{X_2}(f(u)), C_{X_1}(u) = C_{X_2}(f(u)), U_{X_1}(u) = U_{X_2}(f(u)) \) and \( F_{X_1}(u) = F_{X_2}(f(u)) \) for all \( u \in G \).

Let \( g: X_1(K) \rightarrow X_2(K) \) be a mapping defined by \( g(X_1(s)) = X_2(f(s)) \) for all \( u \in G \). i.e., \( g(T_{X_1}(u)) = T_{X_2}(f(u)), g(C_{X_1}(u)) = C_{X_2}(f(u)), g(U_{X_1}(u)) = U_{X_2}(f(u)) \) and \( g(F_{X_1}(u)) = F_{X_2}(f(u)) \) for all \( u \in G \). \( g \) is surjective obviously. And if \( g(T_{X_1}(u)) = g(T_{X_2}(v)) \) for all \( u, v \in G \) then \( T_{X_2}(f(u)) = T_{X_2}(f(v)) \) and we get \( T_{X_1}(u) = T_{X_1}(v) \). Similarly we can prove for \( C_{X_1}(u) = C_{X_2}(v), U_{X_1}(u) = U_{X_2}(v) \) and \( F_{X_1}(u) = F_{X_2}(v) \).

Hence \( g \) is injective. Therefore \( g \) is a homomorphism such that for \( u, v \in G \) we have:

\[ g(T_{X_1}(u \vee v)) = T_{X_2}(f(u \vee v)) = T_{X_2}(f(u) \odot f(v)), \]
\[ g(C_{X_1}(u \vee v)) = C_{X_2}(f(u \vee v)) = C_{X_2}(f(u) \odot f(v)), \]
\[ g(U_{X_1}(u \vee v)) = U_{X_2}(f(u \vee v)) = U_{X_2}(f(u) \odot f(v)), \]
\[ g(F_{X_1}(u \vee v)) = F_{X_2}(f(u \vee v)) = F_{X_2}(f(u) \odot f(v)). \]

Hence \( X_1 = (T_{X_1}, C_{X_1}, U_{X_1}, F_{X_1}) \) is isomorphic to \( X_2 = (T_{X_2}, C_{X_2}, U_{X_2}, F_{X_2}) \).

5 | Conclusion

In recent years, a new branch of logical algebra known as \( K \)-algebra applied in fuzzy set, intuitionistic fuzzy set and single valued neutrosophic set which helps us to extend the concept to \( K \)-algebra on quadripartitioned single valued neutrosophic sets. Quadripartitioned single valued neutrosophic set has four components truth, contradiction, unknown, false which helps to deal the concept of indeterminacy effectively. In this paper we defined \( K \)-algebras on quadripartitioned single valued neutrosophic sets and studied some of the results. Further the homomorphism of quadripartitioned single valued neutrosophic \( K \)-algebras, characteristic and fully invariant \( K \)-subalgebras also discussed in detail.
References


