Abstract

In this paper we introduce the concept of the spherical interval-valued fuzzy bi-ideal of gamma near-ring $\mathcal{R}$ and its some results. The union and intersection of the spherical interval-valued fuzzy bi-ideal of gamma near-ring $\mathcal{R}$ is also a spherical interval-valued fuzzy bi-ideal of gamma near-ring $\mathcal{R}$. Further we discuss about the relationship between bi-ideal and spherical interval-valued fuzzy bi-ideal of gamma near-ring $\mathcal{R}$.

Keywords: Spherical fuzzy set, Interval-Valued fuzzy set, $\Gamma$-Near-Rings.

1 | Introduction

The Fuzzy Set (FS) was introduced by Zadeh [14] in 1965. It is identified as better tool for the scientific study of uncertainty, and came as a boost to the researchers working in the field of uncertainty. Many extensions and generalizations of FS was conceived by a number of researchers and a large number of real-life applications were developed in a variety of areas. In addition to this, parallel analysis of the classical results of many branches of Mathematics was also carried out in the fuzzy settings. Properties of fuzzy ideals in near-rings was studied by Hong et al. [3]. The monograph by Chinnadurai [1] gives a detailed discussion on fuzzy ideals in algebraic structures. Fuzzy ideals in Gamma near-ring $\mathcal{R}$ was discussed by Jun et al. [6] and [7] and Satyanarayana [8]. Thillaigovindan et al. [13] studied the interval valued fuzzy quasi-ideals of semigroups. Meenakumari and Tamizh chelvam [9] have defined fuzzy bi-ideal in $\mathcal{R}$ and established some properties of this structure. Srinivas and Nagaiah [11] have proved some results on $T$-fuzzy ideals of $\Gamma$-near-rings.

In this research work, we introduce the notion of Spherical Interval-Valued Fuzzy Bi-Ideal (SIVFBI) of gamma near-ring $\mathcal{R}$ as a generalization of spherical fuzzy bi-ideals of gamma near-rings $\mathcal{R}$. We will discuss some of the properties of spherical interval-valued fuzzy bi-ideal of gamma near-ring $\mathcal{R}$.

2 | Preliminaries

In this section we present some definitions which are used for this research. Let $\mathcal{R}$ be a near-ring and $\Gamma$ be a non-empty set such that $\mathcal{R}$ is a Gamma near-ring. A subgroup $H$ of $(\mathcal{R}, +)$ is a Bi-Ideal (BI) if and only if $H^+ \mathcal{R}H \subseteq H$. A Spherical Fuzzy Set (SFS) $\tilde{A}_s$ of the universe of discourse $U$ is given by, $\tilde{A}_s = \{u, (\tilde{\mu}(u), \tilde{v}(u), \tilde{\xi}(u))|u \in U\}$ where $\tilde{\mu}(u): U \rightarrow [0,1]$, $\tilde{v}(u): U \rightarrow [0,1]$ and $\tilde{\xi}(u): U \rightarrow [0,1]$ and $0 \leq \tilde{\mu}^2(u) + \tilde{v}^2(u) + \tilde{\xi}^2(u) \leq 1, u \in U$.

For each $u$, the numbers $\tilde{\mu}(u), \tilde{v}(u)$ and $\tilde{\xi}(u)$ are the degree of membership, non-membership and hesitancy of $u$ to $\tilde{A}_s$, respectively.

A SFS $A_s = (\mu, v, \xi)$, where $\mu: \mathcal{R} \rightarrow [0,1], v: \mathcal{R} \rightarrow [0,1]$ and $\xi: \mathcal{R} \rightarrow [0,1]$ of $\mathcal{R}$ is said to be a Spherical Fuzzy Bi-Ideal (SFBII) of $\mathcal{R}$ if the following conditions are satisfied

$$\mu(u - v) \geq \min(\mu(u), \mu(v)),$$

$$v(u - v) \geq \min(v(u), v(v)),$$

$$\xi(u - v) \leq \max(\xi(u), \xi(v)),$$

$$\mu(uav\beta w) \geq \min(\mu(u), \mu(w)),$$

$$v(uav\beta w) \geq \min(v(u), v(w)),$$

$$\xi(uav\beta w) \leq \max(\xi(u), \xi(w)),$$

for all $u, v, w \in \mathcal{R}$ and $a, \beta \in \Gamma$.

3 | Spherical Interval-Valued Fuzzy Bi-Ideals of Gamma Near-Rings

In this section we define SIVFBI of $\mathcal{R}$ and study some of its properties. We obtain the condition for an arbitrary fuzzy subset of $\mathcal{R}$ is said to be SIVFBI.

**Definition 1.** A spherical fuzzy set $\tilde{A}_s = (\tilde{\mu}, \tilde{v}, \tilde{\xi})$ of $\mathcal{R}$ is to be SIVFBI of $\mathcal{R}$ if the following conditions are satisfied

$$\tilde{\mu}(u - v) \geq \min^l(\tilde{\mu}(u), \tilde{\mu}(v)),$$

$$\tilde{v}(u - v) \geq \min^l(\tilde{v}(u), \tilde{v}(v)),$$

$$\tilde{\xi}(u - v) \leq \max^l(\tilde{\xi}(u), \tilde{\xi}(v)),$$

$$\tilde{\mu}(uav\beta w) \geq \min^l(\tilde{\mu}(u), \tilde{\mu}(w)),$$
\[ \forall (u, v, w) \in \mathcal{R} \text{ and } \alpha, \beta \in \Gamma, \text{ where } \bar{\mu} : \mathcal{R} \to D[0,1], \bar{\nu} : \mathcal{R} \to D[0,1] \text{ and } \bar{\xi} : \mathcal{R} \to D[0,1]. \text{ Here } D[0,1] \text{ denotes the family of closed subintervals of } [0,1]. \]

**Example 1.** Let \( \mathcal{R} = \{0,1,2,3\} \) with binary operation \( + \) on \( \Gamma = \{0,1\} \) and \( \mathcal{R} \times \Gamma \times \mathcal{R} \to \mathcal{R} \) be a mapping. From the cayley table,

\[
\begin{array}{cccc}
+ & 0 & 1 & 2 & 3 \\
0 & 0 & 1 & 2 & 3 \\
1 & 1 & 0 & 3 & 2 \\
2 & 2 & 3 & 1 & 0 \\
3 & 3 & 2 & 0 & 1 \\
\end{array}
\]

Define SFS \( \bar{\mu} : \mathcal{R} \to D[0,1] \) by \( \bar{\mu}(0) = [0.2,0.3], \bar{\mu}(1) = [0.3,0.6], \bar{\mu}(2) = [0.7,0.9], \bar{\mu}(3) = [0.5,0.9]; \)
\( \bar{\nu} : \mathcal{R} \to D[0,1] \) by \( \bar{\nu}(0) = [0.2,0.4], \bar{\nu}(1) = [0.5,0.6], \bar{\nu}(2) = [0.6,0.7], \bar{\nu}(3) = [0.7,0.9]; \)
\( \bar{\xi} : \mathcal{R} \to D[0,1] \) by \( \bar{\xi}(0) = [0.1,0.3], \bar{\xi}(1) = [0.4,0.6], \bar{\xi}(2) = [0.8,0.9], \bar{\xi}(3) = [0.5,0.7]. \) Then \( \bar{A}_s \) is SIVFBI of \( \mathcal{R} \).

**Theorem 1.** Let \( \bar{A}_s = [A^-_s; A^+_s] \) be a Spherical interval-valued fuzzy subset of a gamma near-ring \( \mathcal{R} \), then \( \bar{A}_s \) is a SIVFBI of \( \mathcal{R} \) if and only if \( A^-_s, A^+_s \) are SFBI of \( \mathcal{R} \).

**Proof.** If \( \bar{A}_s \) is a SIVFBI of \( \mathcal{R} \). For any \( u, v, w \in \mathcal{R} \). Now,

\[
[\mu^-(u - v), \mu^+(u - v)] = \bar{\mu}(u - v)
\]

\[
\geq \min^1[\bar{\mu}(u), \bar{\mu}(v)]
\]

\[
= \min^1[[\mu^-(u), \mu^+(u)],[\mu^-(v), \mu^+(v)]]
\]

\[
= \min^1[[\mu^-(u), \mu^-(v)]], \min^1[[\mu^+(u), \mu^+(v)]],
\]

\[
[v^-(u - v), v^+(u - v)] = \bar{\nu}(x - y)
\]

\[
\geq \min^1[\bar{\nu}(u), \bar{\nu}(v)]
\]

\[
= \min^1[[v^-(u), v^+(u)],[v^-(v), v^+(v)]]
\]

\[
= \min^1[[v^-(u), v^-(v)]], \min^1[[v^+(u), v^+(v)]], \text{ and}
\]

\[
[\xi^-(u - v), \xi^+(u - v)] = \bar{\xi}(u - v)
\]
\[ \leq \max^1[\xi(u), \xi(v)] \]
\[ = \max^1[[\xi^-(u), \xi^+(u)], [\xi^-(v), \xi^+(v)]] \]
\[ = \max^1[[\xi^-(u), \xi^-(v)], \max^1[[\xi^+(u), \xi^+(v)]]; \]
\[ [\mu^-(uvbw), \mu^+(uvbw)] = \tilde{\mu}(uvbw) \]
\[ \geq \min^1[\tilde{\mu}(u), \tilde{\mu}(w)] \]
\[ = \min^1[[\mu^-(u), \mu^+(u)], [\mu^-(w), \mu^+(w)]] \]
\[ = \min^1[[\mu^-(u), \mu^-(w)], \min^1[[\mu^+(u), \mu^+(w)]]; \]
\[ [v^-(uvbw), v^+(uvbw)] = \tilde{v}(uvbw) \]
\[ \geq \min^1[\tilde{v}(u), \tilde{v}(w)] \]
\[ = \min^1[[v^-(u), v^+(u)], [v^-(w), v^+(w)]] \]
\[ = \min^1[[v^-(u), v^-(w)], \min^1[[v^+(u), v^+(w)], \text{and} \]
\[ [\xi^-(uvbw), \xi^+(uvbw)] = \tilde{\xi}(uvbw) \]
\[ \leq \max^1[\xi(u), \xi(w)] \]
\[ = \max^1[[\xi^-(u), \xi^+(u)], [\xi^-(w), \xi^+(w)]] \]
\[ = \max^1[[\xi^-(u), \xi^-(w)], \max^1[[\xi^+(u), \xi^+(w)]]. \]

Therefore \( A^-_x, A^+_x \) are SFBI o \( R \) f.

Conversely let \( A^-_x, A^+_x \) are SFBI of \( R \). Let \( u, v, w \in R \). Now,

\[ \tilde{\mu}(u - v) = [\mu^-(u - v), \mu^+(u - v)] \]
\[ \geq [\min^1[\mu^-(u), \mu^-(v)], \min^1[\mu^+(u), \mu^+(v)]] \]
\[ = \min^1[\mu^-(u), \mu^+(u)], \min^1[\mu^-(v), \mu^+(v)] \]
\[ = \min^1[\tilde{\mu}(u), \tilde{\mu}(v)]; \]
\[ \tilde{v}(u - v) = [v^-(u - v), v^+(u - v)] \]
\[ \geq [\min^1[v^-(u), v^-(v)], \min^1[v^+(u), v^+(v)]] \]
\[ = \min^1[v^-(u), v^+(u)], \min^1[v^-(v), v^+(v)] \]
\[ = \min^1[\tilde{v}(u), \tilde{v}(v)], \text{and} \]
\[ \tilde{\xi}(u - v) = [\xi^-(u - v), \xi^+(u - v)] \]
\[ \leq [\max^{1}([\xi^{-}(u), \xi^{-}(v)], \max^{1}([\xi^{+}(u), \xi^{+}(v)])] \\
= \max^{1}([\xi^{-}(u), \xi^{+}(u)], \max^{1}([\xi^{-}(v), \xi^{+}(v)])] \\
= \max^{1}([\xi^{-}(u), \xi^{-}(v)]; \\
\bar{\mu}(\alpha \land \beta) = [\mu^{-}(\alpha \land \beta), \mu^{+}(\alpha \land \beta)] \\
\geq [\min^{1}([\mu^{-}(u), \mu^{-}(w)], \min^{1}([\mu^{+}(u), \mu^{+}(w)])] \\
= \min^{1}([\mu^{-}(u), \mu^{+}(u)], \min^{1}([\mu^{-}(w), \mu^{+}(w)])] \\
= \min^{1}([\bar{\mu}(u), \bar{\mu}(w)]; \\
\bar{\nu}(\alpha \land \beta) = [\nu^{-}(\alpha \land \beta), \nu^{+}(\alpha \land \beta)] \\
\geq [\min^{1}([\nu^{-}(u), \nu^{-}(w)], \min^{1}([\nu^{+}(u), \nu^{+}(w)])] \\
= \min^{1}([\nu^{-}(u), \nu^{+}(u)], \min^{1}([\nu^{-}(w), \nu^{+}(w)])] \\
= \min^{1}([\bar{\nu}(u), \bar{\nu}(w)], \text{ and} \\
\tilde{\xi}(\alpha \land \beta) = [\xi^{-}(\alpha \land \beta), \xi^{+}(\alpha \land \beta)] \\
\leq [\max^{1}([\xi^{-}(u), \xi^{-}(w)], \max^{1}([\xi^{+}(u), \xi^{+}(w)])] \\
= \max^{1}([\xi^{-}(u), \xi^{+}(u)], \max^{1}([\xi^{-}(w), \xi^{+}(w)])] \\
= \max^{1}([\tilde{\xi}(u), \tilde{\xi}(w)].
\]

So \( \tilde{A}_i \) is a SIVFBI of \( R \).

Hence the proof.

**Theorem 2.** If \( \{\tilde{A}_i; i \in I\} \) be a family of SIVFBI of a gamma near-ring \( R \), then \( \bigcap_{i \in I} \tilde{A}_i \) is also SIVFBI of \( R \), where \( I \) is an index set.

**Proof.** Let \( \{\tilde{A}_i; i \in I\} \) be a family of SIVFBI of a gamma near-ring \( R \). For any \( u, v, w \in R \) and \( \alpha, \beta \in \Gamma \).

\[
\bigcap_{i \in I} \bar{\mu}_i(u - v) = \inf_{i \in I} \bar{\mu}_i(u - v) \\
\geq \inf_{i \in I} \min^{1}([\bar{\mu}_i(u), \bar{\mu}_i(v)]) \\
= \min^{1}([\inf_{i \in I} \bar{\mu}_i(u), \inf_{i \in I} \bar{\mu}_i(v)]) \\
= \min^{1}([\bigcap_{i \in I} \bar{\mu}_i(u), \bigcap_{i \in I} \bar{\mu}_i(v)]; \\
\bigcap_{i \in I} \bar{\nu}_i(u - v) = \inf_{i \in I} \bar{\nu}_i(u - v)
\]
\[ \geq \inf_{i \in I} \min_{1, \mu i} |\tilde{v}_i(u), \tilde{v}_i(v)| \]
\[ = \min_{1, \mu i} \{ \inf_{i \in I} \tilde{v}_i(u), \inf_{i \in I} \tilde{v}_i(v) \} \]
\[ = \min_{1, \mu i} |\bigcap_{i \in I} \tilde{v}_i(u), \bigcap_{i \in I} \tilde{v}_i(v)|; \]
\[ \bigcap_{i \in I} \tilde{v}_i(u - v) = \inf_{i \in I} \tilde{v}_i(u - v) \]
\[ \leq \inf_{i \in I} \max_{1, \mu i} |\tilde{v}_i(u), \tilde{v}_i(v)| \]
\[ = \max_{1, \mu i} \{ \inf_{i \in I} \tilde{v}_i(u), \inf_{i \in I} \tilde{v}_i(v) \} \]
\[ = \max_{1, \mu i} |\bigcap_{i \in I} \tilde{v}_i(u), \bigcap_{i \in I} \tilde{v}_i(v)|; \]
\[ \bigcap_{i \in I} \tilde{v}_i(u \alpha \beta w) = \inf_{i \in I} \tilde{v}_i(u \alpha \beta w) \]
\[ \geq \inf_{i \in I} \min_{1, \mu i} |\tilde{v}_i(u), \tilde{v}_i(w)| \]
\[ = \min_{1, \mu i} \{ \inf_{i \in I} \tilde{v}_i(u), \inf_{i \in I} \tilde{v}_i(w) \} \]
\[ = \min_{1, \mu i} |\bigcap_{i \in I} \tilde{v}_i(u), \bigcap_{i \in I} \tilde{v}_i(w)|; \]
\[ \bigcap_{i \in I} \tilde{v}_i(u \alpha \beta w) = \inf_{i \in I} \tilde{v}_i(u \alpha \beta w) \]
\[ \leq \inf_{i \in I} \max_{1, \mu i} |\tilde{v}_i(u), \tilde{v}_i(w)| \]
\[ = \max_{1, \mu i} \{ \inf_{i \in I} \tilde{v}_i(u), \inf_{i \in I} \tilde{v}_i(w) \} \]
\[ = \max_{1, \mu i} |\bigcap_{i \in I} \tilde{v}_i(u), \bigcap_{i \in I} \tilde{v}_i(w)|. \]

Hence the proof.

**Theorem 3.** If \( \{ \tilde{A}_{i i}; i \in I \} \) be a family of SIVFBI of a gamma near-ring \( \mathcal{R} \), then \( \bigcup_{i \in I} \tilde{A}_{i i} \) is also SIVFBI of \( \mathcal{R} \), where \( I \) is an index set.

**Proof.** Let \( \{ \tilde{A}_{i i}; i \in I \} \) be a family of SIVFBI of a gamma near-ring \( \mathcal{R} \). For any \( u, v, w \in \mathcal{R} \) and \( \alpha, \beta \in I \).
\[
\bigcup_{i \in I} \tilde{\mu}_i(u - v) = \sup_{i \in I} \tilde{\mu}_i(u - v) \\
\geq \sup_{i \in I} \min^i \{\tilde{\mu}_i(u), \tilde{\mu}_i(v)\} \\
= \min^i \{\sup_{i \in I} \tilde{\mu}_i(u), \sup_{i \in I} \tilde{\mu}_i(v)\} \\
= \min^i \{\bigcup_{i \in I} \tilde{\mu}_i(u), \bigcup_{i \in I} \tilde{\mu}_i(v)\} \\
\bigcup_{i \in I} \tilde{\nu}_i(u - v) = \sup_{i \in I} \tilde{\nu}_i(u - v) \\
\geq \sup_{i \in I} \min^i \{\tilde{\nu}_i(u), \tilde{\nu}_i(v)\} \\
= \min^i \{\sup_{i \in I} \tilde{\nu}_i(u), \sup_{i \in I} \tilde{\nu}_i(v)\} \\
= \min^i \{\bigcup_{i \in I} \tilde{\nu}_i(u), \bigcup_{i \in I} \tilde{\nu}_i(v)\} \\
\bigcup_{i \in I} \tilde{\xi}_i(u - v) = \inf_{i \in I} \tilde{\xi}_i(u - v) \\
\leq \sup_{i \in I} \max^i \{\bar{\xi}_i(u), \bar{\xi}_i(v)\} \\
= \max^i \{\sup_{i \in I} \bar{\xi}_i(u), \sup_{i \in I} \bar{\xi}_i(v)\} \\
= \max^i \{\bigcup_{i \in I} \bar{\xi}_i(u), \bigcup_{i \in I} \bar{\xi}_i(v)\} \\
\bigcup_{i \in I} \tilde{\mu}_i(u \alpha v \beta w) = \sup_{i \in I} \tilde{\mu}_i(u \alpha v \beta w) \\
\geq \sup_{i \in I} \min^i \{\tilde{\mu}_i(u), \tilde{\mu}_i(w)\} \\
= \min^i \{\sup_{i \in I} \tilde{\mu}_i(u), \sup_{i \in I} \tilde{\mu}_i(w)\} \\
= \min^i \{\bigcup_{i \in I} \tilde{\mu}_i(u), \bigcup_{i \in I} \tilde{\mu}_i(w)\} \\
\bigcup_{i \in I} \tilde{\nu}_i(u \alpha v \beta w) = \sup_{i \in I} \tilde{\nu}_i(u \alpha v \beta w) \\
\geq \sup_{i \in I} \min^i \{\tilde{\nu}_i(u), \tilde{\nu}_i(w)\} \\
= \min^i \{\sup_{i \in I} \tilde{\nu}_i(u), \sup_{i \in I} \tilde{\nu}_i(w)\} \\
= \min^i \{\bigcup_{i \in I} \tilde{\nu}_i(u), \bigcup_{i \in I} \tilde{\nu}_i(w)\}, and \\
\bigcup_{i \in I} \tilde{\xi}_i(u \alpha v \beta w) = \sup_{i \in I} \tilde{\xi}_i(u \alpha v \beta w) \\
\leq \inf_{i \in I} \max^i \{\bar{\xi}_i(u), \bar{\xi}_i(w)\} \\
= \max^i \{\sup_{i \in I} \bar{\xi}_i(u), \sup_{i \in I} \bar{\xi}_i(w)\}
\[= \max \left\{ \bigcup_{i \in I} \xi_i(u), \bigcup_{i \in I} \eta_i(w) \right\}.\]

Hence the proof.

**Theorem 4.** If \(\tilde{A}_s\) and \(\tilde{\sigma}_s\) are SIVFBIs of \(R\), then \(\tilde{A}_s \land \tilde{\sigma}_s\) is SIVFBi of \(R\).

**Proof.** Let \(\tilde{A}_s\) and \(\tilde{\sigma}_s\) be SFBIs of \(R\). Let \(u, v, w \in R\) and \(\alpha, \beta \in \Gamma\). Then,

\[(\bar{\mu} \land \bar{\sigma})(u - v) = \min'[(\bar{\mu}(u - v), \bar{\sigma}(u - v))], \text{ since by } (\bar{\mu} \land \bar{\sigma})(u) = \min'[(\bar{\mu}(u), \bar{\sigma}(u))]\]

\[\geq \min^1[\min^1[\bar{\mu}(u), \bar{\nu}(v)], \min^1[\bar{\sigma}(u), \bar{\sigma}(v)]]\]

\[= \min^1[\min^1[\min^1[\bar{\mu}(u), \bar{\nu}(v)], \bar{\sigma}(u)], \bar{\sigma}(v)]\]

\[= \min^1[\min^1[\bar{\mu}(u), \bar{\sigma}(u)], \min^1[\bar{\nu}(v), \bar{\sigma}(v)]]\]

\[= \min^1[\bar{\mu} \land \bar{\sigma}(u)], (\bar{\nu} \land \bar{\sigma}(v))].\]

Also \((\bar{\nu} \land \bar{\sigma})(u - v) \geq \min^1[(\bar{\nu} \land \bar{\sigma})(u), (\bar{\nu} \land \bar{\sigma})(v)]\) and \((\bar{\xi} \land \bar{\sigma})(u - v) \leq \max^1[(\bar{\xi} \land \bar{\sigma})(u), (\bar{\xi} \land \bar{\sigma})(v)]\).

Since \((\bar{\mu}(u \alpha v \beta w) \geq \min^1[\bar{\mu}(u), \bar{\nu}(v)].\)

\[(\bar{\mu} \land \bar{\sigma})(u \alpha v \beta w) = \min^1[\bar{\mu}(u \alpha v \beta w), \bar{\sigma}(u \alpha v \beta w)]\]

\[\geq \min^1[\min^1[\bar{\mu}(u), \bar{A}_s(w)], \min^1[\bar{\sigma}(u), \bar{\sigma}(w)]]\]

\[= \min^1[\min^1[\bar{\mu}(u), \bar{\sigma}(u)], \min^1[\bar{\mu}(w)], \bar{\sigma}(w))]\]

\[= \min^1[(\bar{\mu} \land \bar{\sigma}_s)(u), (\bar{\mu} \land \bar{\sigma}_s)(w)].\]

Also \((\bar{\nu} \land \bar{\sigma})(u \alpha v \beta w) \geq \min^1[(\bar{\nu} \land \bar{\sigma})(u), (\bar{\nu} \land \bar{\sigma})(w)]\) and \((\bar{\xi} \land \bar{\sigma})(u \alpha v \beta w) \leq \max^1[(\bar{\xi} \land \bar{\sigma})(u), (\bar{\xi} \land \bar{\sigma})(w)].\)

Hence \(\tilde{A}_s \land \tilde{\sigma}_s\) is a SIVFBi of \(R\).

**Lemma 1.** Let \(A\) be BI of \(R\). For any \(0 < m < 1\), there exists a SIVFBi \(\tilde{A}_{sm}\) of \(R\) such that \(\tilde{A}_{sm} = A\).

**Proof.** Let \(A\) be BI of \(R\). Define \(\tilde{A}_s: R \rightarrow [0, 1]\) by

\[
\tilde{A}_s(u) = \begin{cases} 
  m, & \text{if } u \in A \\
  0, & \text{if } u \notin A.
\end{cases}
\]

where \(m\) be a constant in \((0, 1)\). Clearly \(\tilde{A}_{sm} = A\). Let \(u, v \in R\). If \(u, v \in A\), then \(\bar{\mu}(u - v) = m \geq \min^1[\bar{\mu}(u), \bar{\nu}(v)], \bar{\nu}(u - v) = m \geq \min^1[\bar{\nu}(u), \bar{\nu}(v)]\) and \(\bar{\xi}(u - v) = m \leq \max^1[\bar{\xi}(u), \bar{\xi}(v)].\)

If at least one of \(u\) and \(v\) is not in \(A\), then \(u - v \notin A\) and so \(\bar{\mu}(u - v) = 0 = \min^1[\bar{\mu}(u), \bar{\nu}(v)], \bar{\nu}(u - v) = 0 = \min^1[\bar{\nu}(u), \bar{\nu}(v)]\) and \(\bar{\xi}(u - v) = 0 = \max^1[\bar{\xi}(u), \bar{\xi}(v)].\)
Let $u, v, w \in \mathcal{R}$ and $\alpha, \beta \in \Gamma$. If $u, w \in A$, then $\tilde{\mu}(u), \tilde{\nu}(u), \tilde{\xi}(u) = m; \tilde{\mu}(w), \tilde{\nu}(w), \tilde{\xi}(w) = m$. Also $
abla(\mu(\alpha \vee \beta \vee \omega)) = m \geq \min\{\mu(u), \mu(w)\}, \nabla(\nu(\alpha \vee \beta \vee \omega)) = m \geq \min\{\nu(u), \nu(w)\}$ and $
abla(\xi(\alpha \vee \beta \vee \omega)) = m \leq \max\{\xi(u), \xi(w)\}$.

If at least one of $u$ and $w$ is not in $A$, then $\tilde{\mu}(\alpha \vee \beta \vee \omega) \geq 0 = \min\{\tilde{\mu}(u), \tilde{\mu}(w)\}, \tilde{\nu}(\alpha \vee \beta \vee \omega) \geq 0 = \min\{\tilde{\nu}(u), \tilde{\nu}(w)\}$ and $
abla(\xi(\alpha \vee \beta \vee \omega)) \leq 0 = \max\{\nabla(\xi(u)), \nabla(\xi(w))\}$.

Thus $\tilde{A}_s$ is SIVFBI of $\mathcal{R}$.

**Theorem 5.** If $\tilde{A}_s$ be SIVFBI of $\mathcal{R}$, then the complement $\tilde{A}_s$ is also SIVFBI of $\mathcal{R}$.

**Proof.** For $u, v, w \in \mathcal{R}$ and $\alpha, \beta \in \Gamma$, we have

$$\tilde{\nu}(u - v) = 1 - \tilde{\nu}(u - v) \geq 1 - \min\{\tilde{\nu}(u), \tilde{\nu}(v)\} = \min\{1 - \tilde{\nu}(u), 1 - \tilde{\nu}(v)\} = \min\{\tilde{\nu}(u), \tilde{\nu}(v)\},$$

and also $\tilde{\xi}(u - v) \leq \max\{\tilde{\xi}(u), \tilde{\xi}(v)\}$.

$$\nu(\alpha \vee \beta \vee \omega) = 1 - \nu(\alpha \vee \beta \vee \omega) \geq 1 - \min\{\nu(u), \nu(w)\} = \min\{1 - \nu(\alpha \vee \beta \vee \omega), 1 - \nu(u)\} = \min\{\nu(u), \nu(w)\},$$

and also $\xi(\alpha \vee \beta \vee \omega) \leq \max\{\xi(u), \xi(w)\}$.

Hence $\tilde{A}_s$ is also SIVFBI of $\mathcal{R}$.

**Lemma 6.** Let $U$ is fuzzy subset of $\mathcal{R}$. Then $U$ is BI of $\mathcal{R}$ if and only if $\tilde{A}_U$ is SIVFBI of $\mathcal{R}$.

**Proof.** Let $U$ be BI of $\mathcal{R}$. For $u, v \in U, u - v \in U$.

Let $u, v \in \mathcal{R}$.

case(a): If $u, v \in U$, then $\tilde{\mu}_U(u) = 1$ and $\tilde{\nu}_U(v) = 1$. Thus $\tilde{\mu}_U(u - v) = 1 \geq \min\{\tilde{\mu}_U(u), \tilde{\mu}_U(v)\}$.

case(b): If $u \in U$ and $v \notin U$, then $\tilde{\mu}_U(u) = 1$ and $\tilde{\nu}_U(v) = 0$. Thus $\tilde{\mu}_U(u - v) = 0 \geq \min\{\tilde{\mu}_U(u), \tilde{\mu}_U(v)\}$.

case(c): If $u \notin U$ and $v \in U$, then $\tilde{\mu}_U(u) = 0$ and $\tilde{\nu}_U(v) = 1$. Thus $\tilde{\mu}_U(u - v) = 0 \geq \min\{\tilde{\mu}_U(u), \tilde{\mu}_U(v)\}$.

case(d): If $u \notin U$ and $v \notin U$, then $\tilde{\mu}_U(u) = 0$ and $\tilde{\nu}_U(v) \leq \max\{\tilde{\xi}_U(u), \tilde{\xi}_U(v)\}) = 0$. Thus $\tilde{\nu}_U(u - v) = 0 \geq \min\{\tilde{\nu}_U(u), \tilde{\nu}_U(v)\}$.

In the above four cases $\tilde{\nu}_U(u - v) \geq \min\{\tilde{\nu}_U(u), \tilde{\nu}_U(v)\}$.

Let $u, v, w \in \mathcal{R}$.

case(a): If $u \in U$ and $w \in U$, then $\tilde{\mu}_U(u) = 1$ and $\tilde{\mu}_U(w) = 1$. Thus $\tilde{\mu}_U(\alpha \vee \beta \vee \omega) = 1 \geq \min\{\tilde{\mu}_U(u), \tilde{\mu}_U(w)\}$.

case(b): If $u \notin U$ and $w \notin U$, then $\tilde{\mu}_U(u) = 1$ and $\tilde{\mu}_U(w) = 0$. Thus $\tilde{\mu}_U(\alpha \vee \beta \vee \omega) = 0 \geq \min\{\tilde{\mu}_U(u), \tilde{\mu}_U(w)\}$.

case(c): If $u \in U$ and $w \in U$, then $\tilde{\mu}_U(u) = 0$ and $\tilde{\mu}_U(w) = 1$. Thus $\tilde{\mu}_U(\alpha \vee \beta \vee \omega) = 0 \geq \min\{\tilde{\mu}_U(u), \tilde{\mu}_U(w)\}$.

case(d): If $u \notin U$ and $w \notin U$, then $\tilde{\mu}_U(u) = 0$ and $\tilde{\mu}_U(w) = 0$. Thus $\tilde{\mu}_U(\alpha \vee \beta \vee \omega) = 0 \geq \min\{\tilde{\mu}_U(u), \tilde{\mu}_U(w)\}$.

Also $\vartheta(\alpha \vee \beta \vee \omega) \geq \min\{\vartheta(u), \vartheta(w)\}$ and $\xi(\alpha \vee \beta \vee \omega) \leq \max\{\xi(u), \xi(w)\}$. Thus $\tilde{A}_U$ is a SIVFBI of $\mathcal{R}$.

Conversely, suppose is a SIVFBI of $\mathcal{R}$. Then by Lemma 4 $\tilde{A}_U$ has only two elements.
Hence $U$ is BI of $R$.

**Theorem 7.** If $R$ be a gamma near-ring and $\bar{A}_s$ be SIVFBI of $R$, then the set $R_{\bar{A}_s} = \{u \in R | \bar{A}_s(u) = \bar{A}_s(0)\}$ is BI of $R$.

**Proof.** Let $\bar{A}_s$ be SIVFBI of and le $u, v, w \in R$ t. Then

\[
\bar{\mu}(u - v) \geq \min^i[\bar{\mu}(u), \bar{\mu}(v)] = \min^i[\bar{\mu}(0), \bar{\mu}(0)] = \bar{\mu}(0). \text{ So } \bar{\mu}(u - v) = \bar{\mu}(0), \text{ then } u-v \in R_{\bar{A}_s}.
\]

\[
\bar{\nu}(u - v) \geq \min^i[\bar{\nu}(u), \bar{\nu}(v)] = \min^i[\bar{\nu}(0), \bar{\nu}(0)] = \bar{\nu}(0). \text{ So } \bar{\nu}(u - v) = \bar{\nu}(0), \text{ then } u-v \in R_{\bar{A}_s}.
\]

\[
\bar{\xi}(u - v) \leq \max^i[\bar{\xi}(u), \bar{\xi}(v)] = \max^i[\bar{\xi}(0), \bar{\xi}(0)] = \bar{\xi}(0). \text{ So } \bar{\xi}(u - v) = \bar{\xi}(0), \text{ then } u-v \in R_{\bar{A}_s}.
\]

\[
\bar{\mu}(u \alpha \nu \beta \omega) \geq \min^i[\bar{\mu}(u), \bar{\mu}(w)] = \min^i[\bar{\mu}(0), \bar{\mu}(0)] = \bar{\mu}(0). \text{ So } \bar{\mu}(u \alpha \nu \beta \omega) = \bar{\mu}(0), \text{ then } u \alpha \nu \beta \omega \in R_{\bar{A}_s}.
\]

\[
\bar{\nu}(u \alpha \nu \beta \omega) \geq \min^i[\bar{\nu}(u), \bar{\nu}(w)] = \min^i[\bar{\nu}(0), \bar{\nu}(0)] = \bar{\nu}(0). \text{ So } \bar{\nu}(u \alpha \nu \beta \omega) = \bar{\nu}(0), \text{ then } u \alpha \nu \beta \omega \in R_{\bar{A}_s}.
\]

\[
\bar{\xi}(u \alpha \nu \beta \omega) \leq \max^i[\bar{\xi}(u), \bar{\xi}(w)] = \max^i[\bar{\xi}(0), \bar{\xi}(0)] = \bar{\xi}(0). \text{ So } \bar{\xi}(u \alpha \nu \beta \omega) = \bar{\xi}(0), \text{ then } u \alpha \nu \beta \omega \in R_{\bar{A}_s}.
\]

Then $R_{\bar{A}_s}$ is BI of $R$.

**Theorem 8.** If $B$ be a non-empty subset of $R$ and $\bar{A}_{sg}$ be a Spherical Interval-Valued Fuzzy Set (SIVFS) $R$ defined by

\[
\bar{A}_{sg}(u) = \begin{cases} \bar{p}, & \text{if } u \in B \\ \bar{q}, & \text{otherw}\bar{p} \geq \bar{q} \text{ise.} \end{cases}
\]

for $u \in R$, $p, q \in D[0,1]$ and. Then $\bar{A}_{sg}(u)$ is a SIVFBI of $R$ if and only if $B$ is a BI of $R$. Als $R_{\bar{A}_{sg}} = B o$.

**Proof.** Let $\bar{A}_{sg}^t$ be a SIVFBS $R$ and le $u, v, w \in B$ t. Then $\bar{A}_{sg}(u) = \bar{p} = \bar{A}_{sg}(v) = \bar{A}_{sg}(w)$.

Now,

\[
\bar{A}_{sg}(u - v) \geq \min^i[\bar{A}_{sg}(u), \bar{A}_{sg}(v)]
\]

\[= \min^i[\bar{p}, \bar{p}] = \bar{p}. \]

\[
\bar{A}_{sg}(u - v) = \bar{p}, \text{ so } u - v \in B.
\]

\[
\bar{A}_{sg}(u \alpha \nu \beta \omega) \geq \min^i[\bar{A}_{sg}(u), \bar{A}_{sg}(w)]
\]

\[= \min^i[\bar{p}, \bar{p}] = \bar{p}. \]

\[
\bar{A}_{sg} B(u \alpha \nu \beta \omega) = \bar{p}, \text{ so } u \alpha \nu \beta \omega \in B.
\]

Then is a BI of $R$.

Conversely let $B$ be a BI of $R$ and le $u, v, w \in B$ t.

If at $\bar{A}_{sg}(u)$ least one $u, v$ is not in $B$, then $u - v \notin B$ and so $\bar{A}_{sg}(u - v) \geq \min^i[\bar{A}_{sg}(u), \bar{A}_{sg}(v)] = \bar{q}$. If at least one $u, w$ is not in $B$, then $u \alpha \nu \beta \omega \notin B$ and so $\bar{A}_{sg}(u \alpha \nu \beta \omega) \geq \min^i[\bar{A}_{sg}(u), \bar{A}_{sg}(w)] = \bar{q}$. 

Thus is a SIVFBI of $\mathcal{R}$.

4 | Conclusion

We obtained the union and intersection of the spherical interval-valued fuzzy bi-ideal of gamma near-ring $\mathcal{R}$ is also a spherical interval-valued fuzzy bi-ideal of gamma near-ring. And for that condition, $\tilde{A}_m = A$, for any $0 < m < 1$, bi-ideal of gamma near-ring $\mathcal{R}$ becomes spherical interval-valued fuzzy bi-ideal of gamma near-ring $\mathcal{R}$. In future we will discuss the spherical fuzzy sets in some other algebraic structures.

References


