A Study of Maximal and Minimal Ideals of n-Refined Neutrosophic Rings

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<tr>
<th>PAPER INFO</th>
<th>ABSTRACT</th>
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</thead>
<tbody>
<tr>
<td>Chronicle:</td>
<td>If ( R_n(I) ) is a ring, then ( R_n(I) ) is called a refined neutrosophic ring. Every AH-subset of ( R_n(I) ) has the form ( P = \sum_{i=0}^{n} P_i I_i = { a_0 + a_1 I + \cdots + a_n I_n : a_i \in P_i } ), where ( P_i ) are subsets of the classical ring ( R ). The objective of this paper is to determine the necessary and sufficient conditions on ( P_i ) which make ( P ) be an ideal of ( R_n(I) ). Also, this work introduces a full description of the algebraic structure and form for AH-maximal and minimal ideals in ( R_n(I) ).</td>
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| Keywords: | n-Refined Neutrosophic Ring. n-Refined AH-Ideal. Maximal Ideal. Minimal Ideal. |

1. Introduction

Neutrosophy is a new kind of generalized logic proposed by Smarandache [12]. It becomes a useful tool in many areas of science such as number theory [16, 20], solving equations [18, 21], and medical studies [11, 15]. Also, there are many applications of neutrosophic structures in statistics [14], optimization [8], and decision making [7]. On the other hand, neutrosophic algebra began in [4], Smarandache and Kandasamy defined concepts such as neutrosophic groups and neutrosophic rings. These notions were handled widely by Agboola et al. in [6, 10], where homomorphisms and AH-substructures were studied [3, 13, 17].

Recently, there is an arising interest by the generalizations of neutrosophic algebraic structures. Authors proposed n-refined neutrosophic groups [9], rings [1], modules [2, 22], and spaces [5, 19].
If $R$ is a classical ring, then the corresponding refined neutrosophic ring is defined as follows:

$$R_n(I) = \{ a_0 + a_1 I + \cdots + a_n I_n : a_i \in R \}.$$ 

Addition and multiplication on $R_n(I)$ are defined as:

$$\sum_{i=0}^{n} x_i I_i + \sum_{i=0}^{n} y_i I_i = \sum_{i=0}^{n} (x_i + y_i) I_i, \quad \sum_{i=0}^{n} x_i I_i \times \sum_{i=0}^{n} y_i I_i = \sum_{i,j=0}^{n} (x_i \times y_j) I_i I_j.$$ 

Where $\times$ is the multiplication defined on the ring $R$ and $I_i I_j = I_{\min(i,j)}$.

Every AH-subset of $R_n(I)$ has the form $P = \sum_{i=0}^{n} P_i I_i = \{ a_0 + a_1 I + \cdots + a_n I_n : a_i \in P_i \}$. There is an important question arises here. This question can be asked as follows:

What are the necessary and sufficient conditions on the subsets $P_i$ which make $P$ be an ideal of $R_n(I)$? On the other hand, can we determine the structure of all AH-maximal and minimal ideals in the $n$-refined neutrosophic ring $R_n(I)$?

Through this paper, we try to answer the previous questions in the case of $n$-refined neutrosophic rings with unity. All rings through this paper are considered with unity.

2. Preliminaries

**Definition 1.** [1]. Let $(R, +, \times)$ be a ring and $I_k; 1 \leq k \leq n$ be $n$ indeterminacies. We define $R_n(I) = \{ a_0 + a_1 I + \cdots + a_n I_n : a_i \in R \}$ to be $n$-refined neutrosophic ring. If $n=2$ we get a ring which is isomorphic to 2-refined neutrosophic ring $R(I_1, I_2)$.

Addition and multiplication on $R_n(I)$ are defined as:

$$\sum_{i=0}^{n} x_i I_i + \sum_{i=0}^{n} y_i I_i = \sum_{i=0}^{n} (x_i + y_i) I_i, \quad \sum_{i=0}^{n} x_i I_i \times \sum_{i=0}^{n} y_i I_i = \sum_{i,j=0}^{n} (x_i \times y_j) I_i I_j.$$ 

Where $\times$ is the multiplication defined on the ring $R$.

It is easy to see that $R_n(I)$ is a ring in the classical concept and contains a proper ring $R$.

**Definition 2.** [1]. Let $R_n(I)$ be an $n$-refined neutrosophic ring, it is said to be commutative if $x y = y x$ for each $x, y \in R_n(I)$, if there is $I \in R_n(I)$ such 1. $x = x. 1 = x$, then it is called an $n$-refined neutrosophic ring with unity.

**Theorem 1.** [1]. Let $R_n(I)$ be an $n$-refined neutrosophic ring. Then (a) $R$ is commutative if and only if $R_n(I)$ is commutative, (b) $R$ has unity if and only if $R_n(I)$ has unity, and (c) $R_n(I) = \sum_{i=0}^{n} P_i I_i = \{ \sum_{i=0}^{n} x_i I_i : x_i \in R \}$.

**Definition 3.** [1]. (a) Let $R_n(I)$ be an $n$-refined neutrosophic ring and $P = \sum_{i=0}^{n} P_i I_i = \{ a_0 + a_1 I + \cdots + a_n I_n : a_i \in P_i \}$ where $P_i$ is a subset of $R$. We define $P$ to be an AH-subring if $P_i$ is a subring of $R$ for all $i$. AHS-
subring is defined by the condition \( P_i = P_j \) for all \( i, j \). (b) \( P \) is an AH-ideal if \( P_i \) is an two sides ideal of \( R \) for all \( i \), the AHS-ideal is defined by the condition \( P_i = P_j \) for all \( i, j \). (c) The AH-ideal \( P \) is said to be null if \( P_i = R \) or \( P_i = \{0\} \) for all \( i \).

**Definition 4.** [1]. Let \( R_n(I) \) be an \( n \)-refined neutrosophic ring and \( P = \sum_{i=0}^{n} P_i I_i \) be an AH-ideal, we define AH-factor \( R(I)/P = \sum_{i=0}^{n} (R/P_i) I_i = \sum_{i=0}^{n} (x_i + P_i) I_i \); \( x_i \in R \).

**Theorem 2.** [1]. Let \( R_n(I) \) be an \( n \)-refined neutrosophic ring and \( P = \sum_{i=0}^{n} P_i I_i \) be an AH-ideal; \( R_n(I)/P \) is a ring with the following two binary operations:

\[
\begin{align*}
\sum_{i=0}^{n} (x_i + P_i) I_i + \sum_{i=0}^{n} (y_i + P_i) I_i &= \sum_{i=0}^{n} (x_i + y_i + P_i) I_i, \\
\sum_{i=0}^{n} (x_i + P_i) I_i \times \sum_{i=0}^{n} (y_i + P_i) I_i &= \sum_{i=0}^{n} (x_i \times y_i + P_i) I_i.
\end{align*}
\]

**Definition 5.** [1]. (a) Let \( R_n(I), T_n(I) \) be two \( n \)-refined neutrosophic rings respectively, and \( f_{AB}: R \rightarrow T \) be a ring homomorphism. We define \( n \)-refined neutrosophic AHS-homomorphism as \( f: R_n(I) \rightarrow T_n(I); f(\sum_{i=0}^{n} x_i I_i) = \sum_{i=0}^{n} f(x_i) I_i \), (b) \( f \) is an \( n \)-refined neutrosophic AHS-isomorphism if it is a bijective \( n \)-refined neutrosophic AHS-homomorphism, and (c) \( AH-Ker f = \sum_{i=0}^{n} Ker(f_{AB}) I_i = \sum_{i=0}^{n} x_i I_i \); \( x_i \in Ker f_{AB} \).

3. Main Discussion

**Theorem 3.** Let \( R_n(I) = \{a_0 + a_1 I + \cdots + a_n I_n ; a_i \in R\} \) be any \( n \)-refined neutrosophic ring with unity \( 1 \). Let \( P = \sum_{i=0}^{n} P_i I_i = \{a_0 + a_1 I + \cdots + a_n I_n ; a_i \in P_i \} \) be any AH-subset of \( R_n(I) \), where \( P_i \) are subsets of \( R \). Then \( P \) is an ideal of \( R_n(I) \) if and only if (a) \( P_i \) are classical ideals of \( R \) for all \( I \) and (b) \( P_0 \leq P_k \leq P_{k-1} \). For all \( 0 < k \leq n \).

**Proof.** First of all, we assume that (a), (b) are true. We should prove that \( P \) is an ideal. Since \( P_i \) are classical ideals of \( R \), then they are subgroups of \( (R, +) \), hence \( P \) is a subgroup of \( (R_n(I), +) \). Let \( r = r_0 + r_1 I_1 + \cdots + r_n I_n \) be any element of \( R_n(I) \), \( x = x_0 + x_1 I_1 + \cdots + x_n I_n \) be an arbitrary element of \( P \), where \( x_i \in P_i \). We have For \( n = 0 \), the statement \( r \cdot x \in P \) is true clearly. We assume that it is true for \( n = k \), we must prove it for \( k + 1 \).

\[
r \cdot x = (r_0 + r_1 I_1 + \cdots + r_k I_k + r_{k+1} I_{k+1})(x_0 + x_1 I_1 + \cdots + x_k I_k + x_{k+1} I_{k+1}) = \\
(r_0 + r_1 I_1 + \cdots + r_k I_k)(x_0 + x_1 I_1 + \cdots + x_k I_k) + r_{k+1} I_{k+1}(x_0 + \cdots + x_k I_k + 1) + (r_0 + \cdots + r_k I_k) x_{k+1} I_{k+1}.
\]

We remark 

\[
(r_0 + r_1 I_1 + \cdots + r_k I_k)(x_0 + x_1 I_1 + \cdots + x_k I_k) \in P_0 + P_1 I_1 + \cdots + P_k I_k \quad \text{(by induction hypothesis)}.
\]

On the other hand, we have

\[
r_{k+1} I_{k+1}(x_0 + \cdots + x_{k+1} I_{k+1}) = (r_{k+1} x_0 + r_{k+1} x_{k+1}) I_{k+1} + r_{k+1} x_1 I_1 + \cdots + r_{k+1} x_k I_k.
\]
Since all $P_i$ are ideals and $P_0 \leq P_{k+1}$, we have $r_{k+1}x_i \in P_i$ and $r_{k+1}x_0 + r_{k+1}x_{k+1} \in P_{k+1}$, hence $r_{k+1}x_{k+1}(x_0 + \cdots + x_{k+1}I_{k+1}) \in P$. Also, $(r_0 + \cdots + r_k)x_{k+1}I_{k+1} = r_0x_{k+1}I_{k+1} + r_1x_{k+1}I_{k+1} + \cdots + r_kx_{k+1}I_{k+1}$. Under the assumption of theorem, we have $r_0x_{k+1} \in P_{k+1}$ and $r_1x_{k+1} \in P_{k+1} \leq P_i$.

For all $1 \leq i \leq k$. Thus $P$ is an ideal.

For the converse, we assume that $P$ is an ideal of $R_n(I)$. We should prove (a) and (b).

It is easy to check that if $P = P_0 + \cdots + P_nI_n$ is a subgroup of $(R_n(I), +)$, then every $P_i$ is a subgroup of $(R, +)$. Now we show that (b) is true.

For every $1 \leq i \leq n$, we have an element $l_i$, that is because $R$ is a ring with unity, hence. Let $x_0$ be any element of $p_0$, we have $x_0 \in P$, and $x_0l_i \in P$.

Thus $x_0 \in P_i$, which means that $P_0 \leq P_i$ for all $1 \leq i \leq n$.

Also, for every $x_i \in P_i$, we have $x_i l_i \in P_i$, thus $x_i l_i^j l_{i-1} = x_i l_{i-1} \in P_i$, so that $x_i \in P_{i-1}$, which means that $P_i \leq P_{i-1}$ and (b) holds.

Example 1. Let $Z$ be the ring of integers, $Z_3(I) = \{a + bl_1 + cl_2 + dl_3; a, b, c, d \in Z\}$ be the corresponding 3-refined neutrosophic ring, we have:

$$P = \langle 16 > + \langle 2 > 1 \rangle + \langle 4 > 1 \rangle + \langle 8 > 1 \rangle = \{16x + 2y l_1 + 4z l_2 + 8tl_3; x, y, z, t \in Z\}$$

is an ideal of $Z_3(I)$, that is because, $< 16 > \leq < 8 > \leq < 4 > \leq < 2 >$.

Now, we are able to describe all AH-maximal and minimal ideals in $R_n(I)$.

Theorem 4. Let $R_n(I) = [a_0 + a_1I + \cdots + a_nI_n; a_i \in R]$ be any $n$-refined neutrosophic ring with unity $1$.

Let $P = \sum_{i=0}^n P_i l_i = [a_0 + a_1I + \cdots + a_nI_n; a_i \in P_i]$ be any ideal of $R_n(I)$. Then (a) non trivial AH-maximal ideals in $R_n(I)$ have the form $P_0 + RI_1 + \cdots + RI_n$, where $P_0$ is maximal in $R$ and (b) non trivial AH-minimal ideals in $R_n(I)$ have the form $P_i l_1$, where $P_i$ is minimal in $R$.

Proof. (a) assume that $P$ is an AH-maximal ideal on the refined neutrosophic ring $R_n(I)$, hence for every ideal $M = (M_0 + M_1I_1 + \cdots + M_nI_n)$ with property $P \leq M \leq R_n(I)$, we have $M = P$ or $M = R_n(I)$. This implies that $M_i = R$ or $M_i = P_i$, which means that $P_0$ is maximal in $R$. On the other hand, we have $P_0 \leq P_k \leq P_{k-1}$.

For all $0 < k \leq n$, thus $P_i \in \{P_0, R\}$ for all $1 \leq i \leq n$. Now suppose that there is at least $j$ such that $P_j = P_0$, we get that $P_0 + \cdots + P_j l_j + \cdots + RI_n \leq P_0 + RI_1 + \cdots + RI_j + \cdots + RI_n$, hence $P$ is not maximal. This means that $P_0 + RI_1 + \cdots + RI_n$, where $P_0$ is maximal in $R$ is the unique form of AH-maximal ideals.

For the converse, we suppose that $P_0$ is maximal in $R$ and $P_i = R$. For all $1 \leq i \leq n$. Consider $M = (M_0 + M_1I_1 + \cdots + M_nI_n)$ as an arbitrary ideal of $R_n(I)$ with AH-structure. If $P \leq M \leq R_n(I)$, then $P_i \leq M_i \leq R$ and, this means that $P_0 = M_0$ or $M_0 = R$, that is because $P_0$ is maximal.

According to Theorem 3, we have $M_0 \leq M_i \leq M_{i-1}$. Now if $M_0 = R$, we get $M_i = R$, thus $M = R_n(I)$.
If $M_0 = P_0$, we get $M = P$. This implies that $P$ is maximal.

(b) It is clear that if $P_i$ is minimal in $R$, then $P_i I_1$ is minimal in $R_n(I)$. For the converse, we assume that $P = P_0 + P_1 I_1 + \cdots + P_n I_n$ is minimal in $R_n(I)$, consider an arbitrary ideal with AH-structure $M = (M_0 + M_1 I_1 + \cdots + M_n I_n)$ of $R_n(I)$ with the property $M \leq P$, we have: $M = \{0\}$ or $M = P$ which means that $M_1 = P_1$ or $M_1 = \{0\}$. Hence $P_1$ is minimal.

According to Theorem 3, we have $M_0 \leq M_k \leq M_{k-1}$ for all $k$. Now, suppose that there is at least $\# 1$ such that $P_j \neq \{0\}$, we get $P_j I_j \leq P_0 + P_1 I_1 + \cdots + P_n I_n$. Thus $P$ is not minimal, which is a contradiction with respect to assumption. Hence any non trivial minimal ideal has the form $P_1 I_1$, where $P_1$ is minimal in $R$.

**Example 2.** Let $R = \mathbb{Z}$ be the ring of integers, $Z_n(I) = \{a_0 + a_1 I_1 + \cdots + a_n I_n; a_i \in \mathbb{Z}\}$ be the corresponding $n$-refined neutrosophic ring, we have

(a) the ideal $P = \langle 2 \rangle + Z I_1 + \cdots + Z I_n$ is AH-maximal, that is because $\langle 2 \rangle$ is maximal in $R$ and (b) there is no AH-minimal ideals in $Z_n(I)$, that is because $R$ has no minimal ideals.

**Example 3.** Let $R = \mathbb{Z}_{12}$ be the ring of integers modulo 12, $Z_{12n}(I)$ be the corresponding $n$-refined neutrosophic ring, we have

(a) the ideal $P = \langle 6 \rangle I_1 = \{0,6 I_1\}$ is AH-minimal, that is because $\langle 6 \rangle$ is minimal in $R$.

(b) the ideal $Q = \langle 2 \rangle + Z_{12} I_1 + \cdots + Z_{12} I_n$ is maximal, that is because $\langle 2 \rangle$ is maximal in $R$.

Now, we show that Theorem 4 is not available if the ring $R$ has no unity, we construct the following example.

**Example 4.** Consider $2\mathbb{Z}_2(I) = \{(2a + 2bl_1 + 2cl_2); a,b,c \in \mathbb{Z}\}$ the 2-refined neutrosophic ring of even integers, let $P = (2\mathbb{Z} + 4Z I_1 + 4Z I_2) = \{(2a + 4bl_1 + 4cl_2); a,b,c \in \mathbb{Z}\}$ be an AH-subset of it. First of all, we show that $P$ is an ideal of $2\mathbb{Z}_2(I)$. It is easy to see that $(P, \cdot)$ is a subgroup. Let $x = (2m + 4nl_1 + 4tI_2)$ be any element of $P$, $r = (2a + 2b l_1 + 2c l_2)$ be any element of $2\mathbb{Z}_2(I)$, we have $rx = (4am + 8an + 4bm + 8bn + 8bt + 8cm) + I_2[8at + 8ct + 4cm]) \in P$. Thus $P$ is an ideal and the inclusion’s condition is not available, that is because $2\mathbb{Z}$ is not contained in $4\mathbb{Z}$.

4. Conclusion

In this article, we have found a necessary and sufficient condition for any subset to be an ideal of any $n$-refined neutrosophic ring with unity. On the other hand, we have characterized the form of maximal and minimal ideals in this class of neutrosophic rings. As a future research direction, we aim to study Köthe’s Conjecture on $n$-refined neutrosophic rings about the structure of nil ideals and the maximality/minimality conditions if $R$ has no unity.
Open Problems

According to our work, we find two interesting open problems.

- Describe the algebraic structure of the group of units of any n-refined neutrosophic ring.
- What are the conditions of AH-maximal and minimal ideals if R has no unity?

References

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