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New View of Fuzzy Aggregations. Part I: General Information Structure for Decision-Making Models

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Abstract

The Ordered Weighted Averaging (OWA) operator was introduced by Yager [57] to provide a method for aggregating inputs that lie between the max and min operators. In this article two variants of probabilistic extensions the OWA operator-POWA and FPOWA (introduced by Merigo [26] and [27]) are considered as a basis of our generalizations in the environment of fuzzy uncertainty (parts II and III of this work), where different monotone measures (fuzzy measure) are used as uncertainty measures instead of the probability measure. For the identification of “classic” OWA and new operators (presented in parts II and III) of aggregations, the Information Structure is introduced where the incomplete available information in the general decision-making system is presented as a condensation of uncertainty measure, imprecision variable and objective function of weights.

Keywords: Mean aggregation operators, Fuzzy aggregations, Fuzzy measure, Fuzzy numbers, Fuzzy decision making.

1 | Introduction

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It is well recognized that intelligent decision support systems and technologies have been playing an important role in improving almost every aspect of human society. Intensive study over the past several years has resulted in significant progress in both the theory and applications of optimization and decision sciences.

Optimization and decision-making problems are traditionally handled by either the deterministic or the probabilistic approach. When working with complex systems in parallel with classical approaches of their modelling, the most important matter is to assume fuzziness ([3], [6], [13], [15]-[32], [35]-[43], [49]-[62] and others). All this is connected to the complexity of study of complex and vague processes and events in nature and society, which are caused by lack or

shortage of objective information and when expert data are essential for construction of credible decisions. With the growth of complexity of information our ability to make credible decisions from possible alternatives with complex states of nature reduces to some level, below which some dual characteristics such as precision and certainty become mutually conflicting ([3], [11], [20]-[22], [36]-[38], [41], [49], [51], [54], [55] and others). When working on real, complex decision systems using an exact or some stochastic quantitative analysis is often less convenient, concluding that the use of fuzzy methods is necessary, because systems approach for development of information structure of investigated decision system [20], [36], [37] with combined fuzzy-stochastic uncertainty enables us to construct convenient intelligent decision support instruments. Obviously, the source for obtaining combined objective + fuzzy + stochastic samplings is the populations of fuzzy-characteristics of expert's knowledge ([22], [36], [38], [42], [51] and others). Our research is concerned with quantitative-information analysis of the complex uncertainty and its use for modelling of more precise decisions with minimal decision risks from the point of view of systems research. The main objects of our attention are 1) the analysis of Information Structures of expert's knowledge, its uncertainty measure and imprecision variable and 2) the construction of instruments of aggregation operators, which condense both characteristics of incomplete information - an uncertainty measure and an imprecision variable in the scalar ranking values of possible alternatives in the decision-making system. The first problem is considered in this paper. The second problem will be presented in the Parts II and III of this work.

Making decisions under uncertainty is a pervasive task faced by many Decision-Making Persons (DMP), experts, investigators or others. The main difficulty is that a selection must be made between alternatives in which the choice of alternative doesn't necessarily lead to well determined payoffs (experts' valuations, utilities and so on) to be received as a result of selecting an alternative. In this case DMP is faced with the problem of comparing multifaceted objects whose complexity often exceeds his/her ability to compare of uncertain alternatives. One approach to addressing this problem is to use valuation functions (or aggregation operators). These valuation functions convert the multifaceted uncertain outcome associated with an alternative into a single (scalar) value. This value provides a characterization of the DMP or expert perception of the worth the possible uncertain alternative being evaluated. The problems of Decision Making Under Uncertainty (DMUU) [51] were discussed and investigated by many well-known authors ([1]-[6], [9], [10], [13], [15]-[18], [23]-[60], [62] and others). In this work our focus is directed on the construction of new generalizations of the aggregation Ordered Weighted Averaging (OWA) operator in the fuzzy-probabilistic uncertainty environment.

In Section 2 some preliminary concepts are presented on the OWA operator; on the arithmetic of the triangular fuzzy numbers; on some extensions of the OWA operator – POWA and FPOWA operators in the probabilistic uncertainty (developed by Merigo [26] and [27]) and their information measures (see Section 3). In Section 4 a new conceptual Information Structure (IS) of a General Decision-Making System (GDMS) with fuzzy-probabilistic uncertainty is defined. This IS classifying some aggregation operators and new generalizations of the OWA operator defined in the parts II and III of this work.

2 | On the OWA Operator and Its Some Fuzzy-Probabilistic Generalizations

In this type of problem, the DMP has a collection $D = \{d_1, d_2, \dots, d_n\}$ of possible uncertain alternatives from which he must select one or some ranking of decisions by some expert's preference relation values. Associated with this problem is a variable of characteristics, activities, symptoms and so on, which acts on the decision procedure. This variable is normally called the state of nature, which affects the payoff, utilities, valuations and others to the DMP's preferences or subjective activities. This variable is assumed to take its values (states of nature) from some set $S = \{s_1, s_2, \dots, s_m\}$. As a result, the DMP knows that if he selects d_i and the state of nature assumes the value s_j then his payoff (valuation, utility and so on) is a_{ij} . The objective of the decision is to select the "best" alternative, get the biggest payoff (valuation, utility and so on). But in DMUU [51] the selection procedure becomes more difficult. In this case each alternative can be seen as

corresponding to a row vector of possible payoffs. To make a choice the DMP must compare these vectors, a problem which generally doesn't lead to a compelling solution. Assume d_i and d_k are two alternatives such that for all $j, j = 1, 2, \dots, m$ $a_{ij} \geq a_{kj}$ (Table 1). In this case there is no reason to select d_k . In this situation we shall say d_i dominates d_k ($d_i \succeq d_k$). Furthermore, if there exists one alternative that dominates all the alternatives then it will be optimal solution and as a result, we call this the Pareto optimal. Faced with the general difficulty of comparing vector payoffs we must provide some means of comparing these vectors. Our focus in this work is on the construction of valuation function (aggregation operator) F that can take a collection of m values and convert it into a single value,

$$F : R^m \Rightarrow R^1.$$

Once we apply this function to each of the alternatives, we select the alternative with the *largest scalar value*. The construction of F involves considerations of two aspects. The first being the satisfaction of some rational, objective properties naturally required of any function used to convert (aggregate) a vector of payoffs (valuations, utilities and so on) into an equivalent scalar value. The second aspect being the inclusion of characteristics particular to the DMP's subjective properties or preferences, dependences with respect to risks and other main external factors.

Table 1. Decision matrix.

S \ D	s_1	s_2	\dots	s_k	\dots	s_m
d_1	a_{11}	a_{12}	\dots	a_{1k}	\dots	a_{1m}
d_2	a_{21}	a_{22}	\dots	a_{2k}	\dots	a_{2m}
\dots	\dots	\dots	\dots	\dots	\dots	\dots
d_3	a_{i1}	a_{i2}	\dots	a_{ik}	\dots	a_{im}
\dots	\dots	\dots	\dots	\dots	\dots	\dots
d_n	a_{n1}	a_{n2}	\dots	a_{nk}	\dots	a_{nm}

First, we shall consider the objective properties required of the valuation function (aggregation operator) F [51].

- 1) The first property is the satisfaction of Pareto optimality. To insure this, we require that if for $j=1,2,\dots,m$, then

$$F(a_{i1}, a_{i2}, \dots, a_{im}) \geq F(a_{k1}, a_{k2}, \dots, a_{km}). \tag{1}$$

An aggregation operator satisfying this condition is said to be *monotonic*.

- 2) A second condition is that the value of an alternative should be bounded by its best payoffs (valuations, utilities) and worst possible one. $\forall i = 1, 2, \dots, n$.

$$\min\{a_{ij}\} \leq F(a_{i1}, a_{i2}, \dots, a_{im}) \leq \max\{a_{ij}\} \quad \min_{j=1,m}\{a_{ij}\} \leq F(a_{i1}, a_{i2}, \dots, a_{im}) \leq \max_{j=1,m}\{a_{ij}\} \tag{2}$$

This condition is said to be *bounded*.

- 3) Remark: if $a_{ij} \equiv a_i$ for all j , then from Eq. (2)

$$\min_{j=1,m} \min\{a_{ij}\} = \max_{j=1,m}\{a_{ij}\} \text{ and } F(a_{i1}, a_{i2}, \dots, a_{im}) = a_i.$$

This condition is said to be idempotent.

- 4) The final objective condition is that the indexing of the states of nature shouldn't affect the answer:

$$F(a_{i_1}, a_{i_2}, \dots, a_{i_m}) = F(\text{Permutation}(a_{i_1}, a_{i_2}, \dots, a_{i_m})), \quad (3)$$

where *permutatio n(.)* is some permutation of the set $\{a_{i_1}, a_{i_2}, \dots, a_{i_m}\}$. An aggregation function satisfying this is said to be symmetric (or commutative).

Finally, we have required that our aggregation function satisfy four conditions: monotonicity, boundedness, idempotency and symmetry. Such functions are called mean or averaging operators [51].

In determining which of the many possible aggregation operators to select as our valuation function, we need some guidance from the DMP. The choice of a valuation function, from among the aggregation operators is essentially a "subjective" act reflecting the preferences of the DMP for one vector of payoffs over another. What is needed are tools and procedure to enable a DMP to reflect their subjective preferences into valuations. There are important problems in expert knowledge engineering for which we often use such intelligent technologies as neural networks, machine learning, fuzzy logic control systems, knowledge representations and others.

These problems may be solved by introducing information measures of aggregation operators ([1], [2], [4], [12], [13], [15], [16], [26]-[33], [35], [38], [40]-[42], [45]-[60], [62] and others). In this paper we will present new extensions of information measures of operators constructed below.

As an example, we present some mean aggregation operators. Assume we have an m -tuple of values $\{a_1, a_2, \dots, a_m\}$.

Then $F(a_1, a_2, \dots, a_m) = \min_{i=1, \dots, m} \{a_i\}$ is one mean aggregation operator. The use of the operator *Min* corresponds to a pessimistic attitude, one in which the DMP assumes the worst thing will happen. Another example of a mean aggregation operator is $F(a_1, a_2, \dots, a_m) = \text{Max}\{a_i\}$.

Here we have very optimistic valuations. Another example is the simple average:

$$\text{Mean}(a_1, a_2, \dots, a_m) = \frac{1}{m} \sum_{i=1}^m a_i.$$

In [58] Yager introduced a class of mean operators called OWA operator.

Definition 1. [57]. An OWA operator of dimension m is mapping $OWA : R^m \Rightarrow R^1$ that has an associated weighting vector W of dimension m with $w_j \in [0; 1]$ and $\sum_{j=1}^m w_j = 1$, such that

$$OWA(a_1, \dots, a_m) = \sum_{j=1}^m w_j b_j, \quad (4)$$

where b_j is the j th largest of the $\{a_i\}$, $i=1, 2, \dots, m$.

Note that different properties could be studied such as the distinction between descending and ascending orders, different measures for characterizing the weighting vector and different families of the OWA operator ([1], [4], [26]-[33], [45], [47]-[52], [56], [57], [59], [60], [62] and others).

The OWA operator and its modifications are among the most known mean aggregation operators to the construction of DMUU valuation functions. These aggregations are generalizations of known instrument

as Choquet Integral ([5], [7], [23], [38], [41], [51], [53], [54], [57] and others), Sugeno integral ([14], [17], [24], [25], [36], [42], [44] and others) or induced mean functions ([2], [12], [60], [62] and others).

The Fuzzy Numbers (FN) have been studied by many authors ([11] and [19] and others). It can represent in a more complete way as an imprecision variable of the incomplete information because it can consider the maximum and minimum and the possibility that the interval values may occur.

Definition 2. [19]. $\tilde{a}(t) : R' \Rightarrow [0; 1]$ is called the FN which can be considered as a generalization of the interval number:

$$\tilde{a}(t) = \begin{cases} 1 & \text{if } t \in [a'_2, a''_2] \\ \frac{t - a_1}{a'_2 - a_1} & \text{if } t \in [a_1, a'_2] \\ \frac{a_3 - t}{a_3 - a''_2} & \text{if } t \in [a''_2, a_3] \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

where $a_1 \leq a'_2 \leq a''_2 \leq a_3 \in R'$.

In the following, we are going to review the triangular FN (TFN) [20] arithmetic operation as follows (in Eq. (5) $a'_2 = a''_2$). Let \tilde{a} and \tilde{b} be two TFNs, where $\tilde{a} = (a_1, a_2, a_3)$ and $\tilde{b} = (b_1, b_2, b_3)$. Then

- 1: $\tilde{a} + \tilde{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$.
- 2: $\tilde{a} - \tilde{b} = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$. $\tilde{a} - \tilde{b} = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$.
- 3: $\tilde{a} \times k = (ka_1, ka_2, ka_3)$, $k > 0$.
- 4: $\tilde{a}^k = (a_1^k, a_2^k, a_3^k)$, $k > 0, a_i > 0$.
- 5: $\tilde{a} \cdot \tilde{b} = (a_1 b_1, a_2 b_2, a_3 b_3)$, $a_i > 0, b_i > 0$. (6)
- 6: $\tilde{b}^{-1} = \{\frac{1}{b_3}, \frac{1}{b_2}, \frac{1}{b_1}\}$, $b_i > 0$.
- 7: $\tilde{a} > \tilde{b}$ if $a_2 > b_2$ and if $a_2 = b_2$ than $\tilde{a} > \tilde{b}$ if $\frac{a_1 + a_3}{2} > \frac{b_1 + b_3}{2}$ otherwise $\tilde{a} = \tilde{b}$.
 $\tilde{a} > \tilde{b}$ if $a_2 > b_2$ and if $a_2 = b_2$ then $\tilde{a} > \tilde{b}$ if $\frac{a_1 + a_3}{2} > \frac{b_1 + b_3}{2}$ otherwise $\tilde{a} = \tilde{b}$.

The set of all TFNs is denoted by ψ and positive TFNs ($a_i > 0$) by ψ^+ .

Note that other operations and ranking methods could be studied ([19] and others).

Now we consider some extensions of the OWA operator, mainly developed by [26], [27], and [29], because our future investigations concern with extensions of Merigo's aggregation operators constructed on the basis of the OWA operator.

Definition 3. [29]. Let ψ be the set of TFNs. A fuzzy OWA operator - FOWA of dimension m is a mapping $FOWA : \psi^m \Rightarrow \psi$ that has an associated weighting vector w of dimension m with $w_j \in [0, 1]$,

$$\sum_{j=1}^m w_j = 1 \text{ and}$$

$$FOWA(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_m) = \sum_{j=1}^m w_j \tilde{b}_j \tag{7}$$

where \tilde{b}_j is the j th largest of the $\{\tilde{a}_i\}_{i=1}^m$, and $a_i \in \psi, i=1,2,\dots,m$.

The FOWA operator is an extension of the OWA operator that uses imprecision information in the arguments represented in the form of TFNs. The reason for using this aggregation operator is that sometimes the available information presented by the DMP and formalized in payoffs (valuations, utilities and others) can't be assessed with exact numbers and it is necessary to use other techniques such as TFNs. So, in this aggregation incomplete information is presented by imprecision variable of expert's reflections and formalized in TFNs. Sometimes the available information presented by the DMP (or expert) also has an uncertain character, which is presented by the probability distribution on the states of nature consequents on the payoffs of the DMP.

The fuzzy-probability aggregations based on the OWA operator was constructed by Merigo and others. One of the variants we present here:

$$POWA(a_1, a_2, \dots, a_m) = \sum_{j=1}^m \hat{p}_j b_j \tag{8}$$

where b_j is the j th largest of the $\{a_i\}, i=1,2,\dots,m$; each argument a_i has an associated probability p_i with $\sum_{i=1}^m p_i = 1, 0 \leq p_i \leq 1, \hat{p}_j = \beta w_j + (1-\beta)p_j$ with $\beta \in [0,1]$ and p_j is the probability p_i ordered according to b_j , that is according to the j th largest of the a_i .

Note that if $\beta = 0$, we get the usual probabilistic mean aggregation (mathematical expectation - E_p with respect to probability distribution $\{p_i\}_{i=1}^m$), and if $\beta = 1$, we get the OWA operator. Equivalent representation of Eq. (8) may be defined as:

$$\begin{aligned} POWA(a_1, a_2, \dots, a_m) &= \beta \sum_{j=1}^m w_j b_j + \\ &+ (1-\beta) \sum_{i=1}^m p_i a_i = \beta \cdot OWA(a_1, a_2, \dots, a_m) + \\ &+ (1-\beta) \cdot E_p(a_1, a_2, \dots, a_m). \end{aligned} \tag{9}$$

We often use probabilistic information in the decision-making systems and consequently in their aggregation operators. Many fuzzy-probabilistic aggregations have been researched in OWA and other operators ([5], [17], [18], [26]-[32], [35]-[42], [49]-[53], [59], [60], [62] and others). In the following we present one of them defined in [27]:

Definition 4. [27]. Let ψ be the set of TFNs. A fuzzy-probabilistic OWA operator - FPOWA of dimension m is a mapping $FPOWA: \psi^m \Rightarrow \psi$ that associated a weighting vector w of dimension m such that $w_j \in [0,1], \sum_{j=1}^m w_j = 1$, according to the following formula:

$$FPOWA(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_m) = \sum_{j=1}^m \hat{p}_j \tilde{b}_j \tag{10}$$

where \tilde{b}_j is the j th largest of the $\{a_i\}_{i=1}^m$ are TFNs and each one has an associated probability $p_i \equiv P(\tilde{a} = \tilde{a}_i)$, with $\sum_{j=1}^m \hat{p}_j = 1, 0 \leq \hat{p}_j \leq 1, \hat{p}_j = \beta w_j + (1-\beta)p_j', \beta \in [0,1]$ and p_j' is the probability ordered according to $\tilde{b}_j (p_j' = P(\tilde{a} = \tilde{b}_j))$ that is according to the j th largest of the $\{\tilde{a}_i\}_{i=1}^m$.

Analogously to Eq. (9) we present the equivalent form of the FPOWA operator as a weighted sum of the OWA operator and the mathematical expectation - E_p :

$$\begin{aligned} \text{FPOWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_m) &= \beta \sum_{j=1}^n w_j b_j + \\ &+ (1-\beta) \sum_{i=1}^m p_i \tilde{a}_i = \beta \cdot \text{OWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_m) + \\ &+ (1-\beta) \cdot E_p(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_m). \end{aligned} \tag{11}$$

In [27] the Semi-boundary condition of the *aggregation operator* (11) was proved. Semi-boundary condition of some operator F if defined as:

$$\begin{aligned} \beta \times \min_i \{\tilde{a}_i\} + (1-\beta) \cdot E_p(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_m) &\leq \\ \leq F(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_m) &\leq \beta \times \max_i \{\tilde{a}_i\} + \\ + (1-\beta) \cdot E_p(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_m). \end{aligned} \tag{12}$$

So, the FPOWA operator is monotonic, bounded, idempotent, symmetric and semi-bounded.

3 | on the Information Measures of the POWA and FPOWA Operators

As preliminary concepts of our investigation we present four probabilistic information measures of the POWA and FPOWA operators defined in [27] following similar methodology developed for the OWA operator ([1], [2], [3], [6], [47], [48], [50], [52] and others):

- The Orness parameter classifies the POWA and FPOWA operators in regard to their location between and and *or*:

$$\begin{aligned} \hat{\alpha}(p_1, p_2, \dots, p_m) &= \\ &= \beta \sum_{j=1}^m w_j \left(\frac{m-j}{m-1} \right) + \\ &+ (1-\beta) \sum_{j=1}^m p_j \left(\frac{m-j}{m-1} \right). \end{aligned} \tag{13}$$

- The Entropy (dispersion) measures the amount of information being used in the aggregation:

$$\begin{aligned} \hat{H}(p_1, p_2, \dots, p_m) &= \\ &= - \left\{ \beta \sum_{j=1}^m w_j \ln w_j + (1-\beta) \sum_{i=1}^m p_i \ln p_i \right\}. \end{aligned} \tag{14}$$

- The divergence of weighted vector w measures the divergence of the weights against the degree of Orness:

$$\begin{aligned} \text{Div}(\hat{p}_1, \hat{p}_2, \dots, \hat{p}_m) &= \\ &= \beta \left\{ \sum_{j=1}^m w_j \left(\frac{m-j}{m-1} - \alpha(W) \right)^2 \right\} + \\ &+ (1-\beta) \left\{ \sum_{j=1}^m p'_j \left(\frac{m-j}{m-1} - \alpha(P) \right)^2 \right\} \end{aligned} \quad (15)$$

where $a(W)$ is an *Orness* measure of the OWA or FOWA operators ($\beta = 1$):

$$\alpha(W) = \sum_{j=1}^m w_j \left(\frac{m-j}{m-1} \right), \quad (16)$$

and $a(p)$ is an *Orness* measure of the fuzzy-probabilistic aggregation ($\beta = 0$):

$$\alpha(P) = \sum_{j=1}^m p'_j \cdot \left(\frac{m-j}{m-1} \right). \quad (17)$$

– The balance parameter measures the balance of the weights against the Orness or the andness

$$\begin{aligned} \text{Bal}(\hat{p}_1, \hat{p}_2, \dots, \hat{p}_m) &= \\ &= \beta \left\{ \sum_{j=1}^m w_j \left(\frac{m+1-2j}{m-1} \right) \right\} + \\ &+ (1-\beta) \left\{ \sum_{j=1}^m p'_j \left(\frac{m+1-2j}{m-1} \right) \right\}. \end{aligned} \quad (18)$$

4 | General Decision-Making System (GDMS) and Its Information Structure (IS)

In the parts II and III of this work we will focus on the construction of new generalizations of the POWA and FPOWA fuzzy-probabilistic aggregation operators induced by the ME (Choquet Integral [5], [7], [23], [38], [41], [51], [53], [54], [57] and others), or the FEV (Sugeno integral [14], [17], [24], [25], [36], [42], [44] and others) with respect to different monotone measures (fuzzy measure [8], [14], [21], [22], [36]-[38], [43], [44], [53]-[55], [61] and others). When trying to functionally describe insufficient expert data, in many real situations the property of additivity remains unrevealed for a measurable representation of a set and this creates an additional restriction. Hence, to study such data, it is better to use monotone measures (estimators) instead of additive ones. So, we will construct new generalizations of the POWA and FPOWA operators with respect to different monotone measures (instead of the probability measure) and different mean operators.

We introduce the definition of a monotone measure (fuzzy measure) [44] adapted to the case of a finite referential.

Definition 5. Let $S = \{s_1, s_2, \dots, s_m\}$ be a finite set and g be a set function $g: 2^S \Rightarrow [0, 1]$. We say g is a monotone measure on S if it satisfies

- (i) $g(\emptyset) = 0; g(S) = 1;$
- (ii) $\forall A, B \subseteq S, \text{ if } A \subseteq B, \text{ then } g(A) \leq g(B).$

A monotone measure is a normalized and monotone set function. It can be considered as an extension of the probability concept, where additivity is replaced by the weaker condition of monotonicity. Non-additive but monotone measures were first used in the fuzzy analysis in the 1980s [44] and are well

investigated ([8], [14], [21], [22], [36]-[38], [43], [44], [53]-[55], [61] and others). Therefore, in order to classify OWA-type aggregation operators with probabilistic (POWA, FPOWA operators and others) or fuzzy uncertainty (defined in parts II and III) it is necessary to define an information structure of these operators. The different cases of incompleteness (uncertainty measure + imprecision variable) and objectivity (objective weighted function) will be considered in our new aggregation operators. Therefore, from the point of view of systems approach it is necessary to describe and formally present the scheme of GDMS in uncertain – objective environment. GDMS gives us the possibility to identify the different cases of levels of incompleteness and objectivity of available information which in whole defines the aggregation procedure.

Now we define the general decision-making system and its information structure which will be considered in the aggregation problems of parts II and III.

Definition 6. The GDMS that will combine decision-making technologies and methods of construction of decision functions (aggregation operators) may be presented by the following 8-tuple

$$\langle D, S, a, g, W, I, F, Im \rangle, \tag{19}$$

where $D = \{d_1, d_2, \dots, d_n\}$ is a set of all possible alternatives (decisions, diagnosis and so on) that are made by a Decision-Making Person (DMP).

$S = \{s_1, s_2, \dots, s_m\}$ is a set of systems states of nature (actions, activities, factors, symptoms and so on) that are act on the possible alternatives in the decision procedure?

a - is an imprecision on precision variable of payoffs (utilities, valuations, some degrees of satisfaction to a fuzzy set, prices and so on), which will by defined by DMP's subjective properties of preferences, dependences with respect to risks and other external factors. As a result, variable a constructs some decision matrix (binary relation) on $D \times S$.

g is an uncertainty measure on $2^S (g : 2^S \Rightarrow [0,1])$. In our case it may be some monotone measure. W is an objective weighted function (or vector) on the states of nature - S

I is the Information Structure on the data of states of nature. Cases of different levels of information incompleteness (uncertainty measure + imprecision variable) and objectivity (objective weighted function) on the states of nature will be considered as:

$I =$ Information Structure (on S): =imprecision (on S) + uncertainty (on S) + objectivity (on S), where:

- Imprecision on S may be presented by some inexact (stochastic, fuzzy, fuzzy-stochastic or other) variable.
- Uncertainty on S may be presented by the levels of belief, credibility, probability, possibility and other monotone measures on 2^S . These levels identify the possibility of occurrence of some groups (events, focal elements and others) on the states of nature.
- Objectivity on S is defined by the objective importance of states of nature in the procedure of decision making. As usual the objective function is presented by a weighted function (vector) $W : S \Rightarrow R_0^+$.

Now we may classify cases of the Information Structure – I:

I1: The case:

- Imprecision is presented by some exact variable $a : S \Rightarrow R^l$.
- The measure of uncertainty does not exist.

- Objectivity is presented by the weights $W = \{w_1, w_2, \dots, w_m\}$.

Examples: OWA and MEAN operators belong to I1.

I2: The case:

- Imprecision is presented by some fuzzy variable: $\tilde{a} \in \psi; \tilde{a} : S \Rightarrow [0, 1]$.
- The measure of uncertainty does not exist.
- Objectivity is presented by the weights $W = \{w_1, w_2, \dots, w_m\}$.

Examples: FOWA operator belongs to I2.

I3: The case:

- Imprecision is presented by some stochastic variable: $a : S \Rightarrow \mathbb{R}^l$.
- The measure of uncertainty is presented by concerning probability distribution on S ($P : 2^S \Rightarrow [0, 1]$)
 $p_i = P\{s_i\}, i = 1, 2, \dots, m$.
- Objectivity is presented by the weights $W = \{w_1, w_2, \dots, w_m\}$.

Example: POWA operator belongs to I3.

I4: The case:

- Imprecision is presented by the some fuzzy-stochastic variable:
 $\tilde{a} \in \psi; \tilde{a} : S \Rightarrow [0, 1]$.
- Uncertainty measure is presented by the concerning probability distribution on S ($P : 2^S \Rightarrow [0, 1]$)
 $p_i = P\{s_i\}, i = 1, 2, \dots, m$.
- Objectivity is presented by the weights $W = \{w_1, w_2, \dots, w_m\}$.

Example: FPOWA operator belongs to I4.

I5: The case:

- Imprecision is presented by some exact variable: $a : S \Rightarrow \mathbb{R}^l$.
- The measure of uncertainty defined by some monotone measure (possibility measure [11], [14], [21], [22], λ -additive measure ([44] and so on) $g : 2^S \Rightarrow [0, 1]$.
- Objectivity is presented by the weights $W = \{w_1, w_2, \dots, w_m\}$.

Examples: SEV (Yager [51]) operator belongs to I5; SEV-POWA, AsPOWA, SA-POWA, SA-AsPOWA (will be defined in the part II of this work) operators belong to I5.

I6: The case:

- Imprecision is presented by some fuzzy variable: $\tilde{a} \in \psi; \tilde{a} : S \Rightarrow [0, 1]$.
- The measure of uncertainty is presented by some monotone measure $g : 2^S \Rightarrow [0, 1]$.
- Objectivity is presented by the weights $W = \{w_1, w_2, \dots, w_m\}$.

Examples. SEV-FOWA, AsFPOWA, and SA-AsFPOWA operators (will be defined in the part III of this work) belong to I6.

Note that some other cases may be considered in the Information Structure – I (for an example, the cases when the weights in structure are not present and others).

- 7) F – is an aggregation (in our case OWA-type) operator for ranking of possible alternatives by its outcome values calculated by the F . Following the Information Structure I on the states of nature for all possible alternatives $d \in D, F(d)$ is a ranking value. In general, $F(d)$ is defined as converted (or condensed) information of imprecision values plus uncertainty measure and objective weights.

$$F(d) = \text{aggregation}(a(d), g, w).$$

We say – that alternative d_j is more preferred (dominated) than

$$d_k, d_j \succ d_k, \text{ if } F(d_j) > F(d_k),$$

and d_j is equivalent to $d_k, d_j \equiv d_k, \text{ if } F(d_j) = F(d_k)$. So, the aggregation operator F induces some preference binary relation \succeq on the all-possible alternatives - D .

- 8) Im is a set of information measures of an aggregation operator F :

$$Im = \{Orness, Dispersion, Divergence, Balance\}. \tag{20}$$

In order to classify OWA-type aggregation operators $\{F\}$ it is necessary to investigate information measures (Eq.(20)). This analysis also gives us some information on the inherent subjectivity of the choice of the decision aggregation operator by DMP [6].

5 | Conclusion

This paper has a conceptual and introductory character. The main preliminary concepts were presented. Definitions of the OWA operator and the POWA and FPOWA operators as some fuzzy-probabilistic extensions of the OWA operator were introduced. Their information measures as - *Orness, Entropy, Divergence* and *Balance* were considered. From the point of view of systems approach the scheme of GDMS in uncertain – objective environment and its Information Structure was described and formally presented. New GDMS gives us the possibility to identify the different cases of levels of incompleteness and objectivity of available information which in whole defines the aggregation procedure. The main results on the constructions of new generalizations of the POWA and FPOWA operators will be presented in Parts II and III of this work.

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