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New View of Fuzzy Aggregations. Part III: Extensions of the FPOWA Operator in the Problem of Political Management

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Abstract

The Ordered Weighted Averaging (OWA) operator was introduced by Yager [34] to provide a method for aggregating inputs that lie between the max and min operators. In this article we continue to present some extensions of OWA-type aggregation operators. Several variants of the generalizations of the fuzzy-probabilistic OWA operator-FPOWA (introduced by Merigo [13], [14]) are presented in the environment of fuzzy uncertainty, where different monotone measures (fuzzy measure) are used as uncertainty measures. The considered monotone measures are: possibility measure, Sugeno $\lambda -$ additive measure, monotone measure associated with Belief Structure and Choquet capacity of order two. New aggregation operators are introduced: AsFPOWA and SA-AsFPOWA. Some properties of new aggregation operators and their information measures are proved. Concrete faces of new operators are presented with respect to different monotone measures and mean operators. Concrete operators are induced by the Monotone Expectation (Choquet integral) or Fuzzy Expected Value (Sugeno Integral) and the Associated Probability Class (APC) of a monotone measure. New aggregation operators belong to the Information Structure I6 (see Part I, Section 3). For the illustration of new constructions of AsFPOWA and SA-AsFPOWA operators an example of a fuzzy decision-making problem regarding the political management with possibility uncertainty is considered. Several aggregation operators ("classic" and new operators) are used for the comparing of the results of decision making.

Keywords: Mean aggregation operators, Fuzzy aggregations, Fuzzy measure, Fuzzy numbers, Fuzzy decision making.

1 | Introduction

In this paper we continue the research concerned with quantitative-information analysis of the complex uncertainty and its use for modelling of more precise decisions with minimal decision risks from the point of view of systems approach. We continue the construction of new generalizations of OWA-type operators in fuzzy-probabilistic uncertainty [13], [14], which condense both characteristics of incomplete information - an uncertainty measure and an imprecision variable in the scalar ranking values of possible alternatives in the decision-making system. In the Part I of this work the definition of the OWA operator ([15]-[17], [25]-[30], [34]-[36] and others) and some of its extensions–POWA and FPOWA operators were presented. In this work our focus is directed to the construction of new generalizations of the FPOWA operator described in the Section 1 of Part I (Definition 5).

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In Section 2 new generalizations of the FPOWA operator are presented with respect to different monotone measures (instead of the probability measure) and different mean operators. New versions of the FPOWA operator are defined: AsFPOWA operators are induced by the Monotone Expectation (ME) ([1], [2], [5], [10], [11], [20]-[23] and others) and SA-AsFPOWA operators are induced by the Fuzzy Expected Value (FEV) ([3], [5], [11], [12], [18], [19], [21], [23], [24] and others). All generalizations are constructed with respect to different monotone measures ([1]-[12], [18]-[24], [31]-[33] and others). Some properties of new operators and their information measures [13], [14] are proved. For the illustration of the applicability of the new generalizations of the FPOWA operator an example of the fuzzy decision-making problem regarding political management is considered (Section 3), where we study a country that is planning its fiscal policy for the next year analogously to the example considered by Merigo [14]. But we use the possibility distribution (possibility uncertainty) on the states of nature of decision-making system instead of probability distribution (probability uncertainty) as considered in [14]. We think our approach is more natural and applicable then the case presented in [14]. In this example several aggregation operators are used for the comparing of the results in decision making:

- *SEV (Shapely Expected Value) operator, introduced be Yager [29].*
- *A new operator SEV-FOWA as a weighted combination of SEV and FOWA operators.*
- *New operators – AsFPOWAmin, AsFPOWAmean, AsFPOWAmax, SA-AsFPOWAmin, SA-AsFPOWAmean and Sa-AsFPOWAmax operators introduced in Section 2. The resulting table (see Table 8) is presented for ordering of the policies. The values of Orness parameter are calculated for all presented aggregation operators.*

2| Associated Probabilities' Aggregations in the FPOWA Operator

In this section we construct new aggregations in the FPOWA operator (Definition 5, Part I) by monotone measure's associated probabilities (Definition 3, Part II [37]) when the imprecision variable is presented by the fuzzy triangular numbers, FTNs, (Definition 2, Part I). So, we consider the Information Structure I6 (Definition 7, Part I).

Let on the states of nature of General Decision-Making System (Definition 7, Part I) be given some monotone measure as a uncertainty measure of incomplete information and on defined some payoffs (utilities and so on) which are presented by triangular fuzzy numbers as expert reflections on possible alternatives. i.e., for every alternative and for every state of nature there exists- positive triangular fuzzy number as some payoff. So, vector is imprecision values of expert reflections on states of nature with respect to alternatives.

Using the arithmetic operations on the triangular fuzzy numbers [6], [8], presented in Section 1, Part I, we may define new aggregations in the FPOWA operator with respect to monotone measures' associated probabilities.

2.1| AsFPOWA Operators Induced by the ME

Let $M : \Psi^{+k} \Rightarrow \Psi^{+}$ $\left(k = m!\right)$ be some deterministic mean aggregation function with symmetricity, boundedness, monotonicity and idempotency properties ([29] and Section 1, Part I), where Ψ^{+} denotes the set of all positive TNF.

Definition 1. An associated FPOWA operator AsFPOWA of dimension m is mapping $AsFPOWA : \Psi^{+m} \Rightarrow \Psi^{+}$, that has an associated objective weighted vector W of dimension m such that

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$w_j \in (0,1)$ and $\sum_{j=1}^{m} w_j = 1$, and some uncertainty measure – monotone measure $g: 2^S \Rightarrow [0,1]$ with associated probability class $\{P_\sigma\}_{\sigma \in S_m}$ and is defined according to the following formula:

$$\begin{aligned} &\mathrm{AsFPOWA}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m) = \\ &= \beta \sum_{j=1}^{m} w_j \tilde{b}_j + (1-\beta) M \left\{ \sum_{i=1}^{m} \tilde{a}_i P_\sigma(s_i) \middle/ \sigma \in S_m \right\} = \\ &= \beta \sum_{j=1}^{m} w_j \tilde{b}_j + (1-\beta) M \left(E_{P_{\sigma_1}}(\tilde{a}), E_{P_{\sigma_2}}(\tilde{a}), ..., E_{P_{\sigma_k}}(\tilde{a}) \right) \end{aligned} \tag{1}$$

where $\tilde{b}_j$ is the jth largest of the $\{\tilde{a}_i\}, i = 1,...,m$.

Now we consider concrete AsFPOWA operators for concrete mean functions M and induced by the ME.

Definition 2.

I. Let be the-operator dimension of k=m! then

$$\begin{aligned} &\mathrm{AsFPOWA\,min}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m) = \\ &= \beta \sum_{j=1}^{m} w_j \tilde{b}_j + (1-\beta) \underset{\sigma \in S_m}{\mathrm{Min}} \left\{ \sum_{i=1}^{m} \tilde{a}_i P_\sigma(s_i) \right\}. \end{aligned} \tag{2}$$

II. Let be the -operator dimension of k=m! then

$$\begin{aligned} &\mathrm{AsFPOWA\,max}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m) = \\ &= \beta \sum_{j=1}^{m} w_j \tilde{b}_j + (1-\beta) \underset{\sigma \in S_m}{\mathrm{Max}} \left\{ \sum_{i=1}^{m} \tilde{a}_i P_\sigma(s_i) \right\}. \end{aligned} \tag{3}$$

III. Let be the averaging operator dimension of k=m!, then

$$\begin{aligned} &\mathrm{AsFPOWAmean}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m) = \\ &= \beta \sum_{j=1}^{m} w_j \tilde{b}_j + (1-\beta) \left\{ \frac{1}{m!} \sum_{\sigma \in S_m} \sum_{i=1}^{m} \tilde{a}_i P_\sigma(s_i) \right\}. \end{aligned} \tag{4}$$

IV. Let be the -averaging operator dimension of k=m!, then

$$\begin{aligned} &\mathrm{AsFPOWAmean}\alpha(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m) = \\ &= \beta \sum_{j=1}^{m} w_j \tilde{b}_j + (1-\beta) \left\{ \frac{1}{m!} \sum_{\sigma \in S_m} \{\sum_{i=1}^{m} \tilde{a}_i P_\sigma(s_i)\}^\alpha \right\}^{\frac{1}{\alpha}}. \end{aligned} \tag{5}$$

The propositions analogous to Propositions 9-12, Part II [37], are true (we omitted this proposition here).

Now we define concrete AsFPOWA operators for concrete monotone measures analogously to Section 3, Part II [37]. Consider AsFPOWAmax for Sugeno λ -additive monotone measure - g_λ. Analogously to *Eq. (37)*, Part II [37], we have:

$$\text{AsFPOWA}\max\left(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_m\right)= \\ =\beta\sum_{j=1}^{m}\tilde{b}_j w_j+(1-\beta)\cdot \\ \cdot \operatorname*{Max}_{\sigma\in S_m}\left\{\sum_{i=1}^{m}\left[g_\lambda\left(\left\{s_{\sigma(i)}\right\}\right)\cdot\prod_{j=1}^{i-1}\left(1+\lambda g_\lambda\left(\left\{s_{\sigma(j)}\right\}\right)\right)\right]\cdot\tilde{a}_{\sigma(i)}\right\}. \quad (6)$$

Analogously we may construct the face of the AsFPOWAmin:

$$\text{AsFPOWA}\min\left(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_m\right)= \\ =\beta\sum_{j=1}^{m}\tilde{b}_j w_j+(1-\beta)\cdot \\ \min_{\sigma\in S_m}\left\{\sum_{i=1}^{m}\left[g_\lambda\left(\left\{s_{\sigma(i)}\right\}\right)\cdot\prod_{j=1}^{i-1}\left(1+\lambda g_\lambda\left(\left\{s_{\sigma(j)}\right\}\right)\right)\right]\cdot\tilde{a}_{\sigma(i)}\right\}. \quad (7)$$

Analogously to Section 3, Part II [37] (*Eqs. (38)-(39)*) we may construct AsFPOWAmin and AsFPOWAmax operators induced by the belief structure's associated monotone measure (omitted here). We also may define some other combinations of different monotone measures and averaging operator M. So, there exist many cases of Information Structures on the level I6 for the constructions of the AsFPOWA operator. For example - *AsFPOWAmeanα* with respect to the belief structure:

$$\text{AsFPOWA}\min\left(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_m\right)= \\ =\beta\sum_{j=1}^{m}\tilde{b}_j w_j+(1-\beta)\cdot \\ \min_{\sigma\in S_m}\left\{\sum_{i=1}^{m}\left[g_\lambda\left(\left\{s_{\sigma(i)}\right\}\right)\cdot\prod_{j=1}^{i-1}\left(1+\lambda g_\lambda\left(\left\{s_{\sigma(j)}\right\}\right)\right)\right]\cdot\tilde{a}_{\sigma(i)}\right\}. \quad (8)$$

Note the information measures of the AsFPOWA operator - Orness, Entropy, Div and Bal ([13], [14], [26] and others) are defined analogously to Subsection 3.3, Part II [37] (omitted here). We may add the proposition concerning the dual monotone measures g_* and g^* [1], [20], [23] which is general for the AsPOWA (see definition 7, Part II [37]) and AsFPOWA operators.

Proposition 1. Let g_* and g^* be dual monotone measures on $2^S \Rightarrow \left[0,1\right]$; let AsPOWA* and AsPOWA* (or AsFPOWA* and AsFPOWA*) be AsFPOWA (or AsFPOWA) operators constructed on the basis of the measures g_* and g^* respectively. Then corresponding information measures coincide:

$\alpha_* = \alpha^*$; $H_* = H^*$; $Div_* = Div^*$; and $Bal_* = Bal^*$.

Proof. We prove the equality $\alpha_* = \alpha^*$. Other proofs are analogous.

Consider

$$\alpha_* = \beta\sum_{j=1}^{m} w_j\left(\frac{m-j}{m-1}\right) +$$
$$+(1-\beta)M\left[\sum_{j=1}^{m} P_{*\sigma(j)}\left(\frac{m-\sigma(j)}{m-1}\right)\Big/\sigma\in S_m\right] =$$
$$=\beta\sum_{j=1}^{m} w_j\left(\frac{m-j}{m-1}\right)+(1-\beta)\cdot$$
$$\cdot\ M\left[\sum_{j=1}^{m} P^*_{\sigma_*(m-j+1)}\left(\frac{m-\sigma_*(m-j+1)}{m-1}\right)\Big/\sigma_*\in S_m\right] =$$
$$=\beta\sum_{j=1}^{m} w_j\left(\frac{m-j}{m-1}\right)+(1-\beta)\cdot$$
$$\cdot\ M\left[\sum_{j=1}^{m} P^*_{\sigma_*(j)}\left(\frac{m-\sigma_*(j)}{m-1}\right)\Big/\sigma_*\in S_m\right] = \alpha^*.$$

In this proof we use the property of symmetry of the function M ; the fact, that Associated Probability Classes of g_* and g^* coincide $\left\{P_{*\sigma}\right\}_{\sigma\in S_m} \equiv \left\{P^*_{\sigma_*}\right\}_{\sigma_*\in S_m}$ (see Proposition 2, Part II [37]) and $P_{*\sigma(j)} \equiv P^*_{\sigma_*(m-j+1)}$, where σ and σ_* are dual permutations (Section 2, Part II [37]).

2.2| AsFPOWA Operators Induced by the FEV

Now we define new generalizations of the FPOWA operator induced by the $FEV_P(\ )$. The values of imprecision of the incomplete information on S are presented by the fuzzy variable

$$\tilde{a}\in \mathrm{TFN},\ \tilde{a}: S \Rightarrow \Psi^+,$$
$$(\text{or}\ \ \tilde{a}_i = \tilde{a}(s_i)\in\Psi^+\ \text{ for every }\ i=1,2,..,m).$$

Definition 3. A Sugeno Averaging FPOWA operator SA-FPOWA of dimension m is mapping $SA-FPOWA: \Psi^{+m} \Rightarrow \Psi^+$, that has an associated weighting vector W of dimension m such that $w_j \in \left[0,1\right]$, $\sum_{j=1}^{m} w_j = 1$ and is defined according to the following formula:

$$\begin{aligned} &\mathrm{SA-FPOWA}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m) = \\ &=\beta\sum_{j=1}^{m} w_j\tilde{b}_j + (1-\beta)\mathrm{FEV_P}\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m\right) = \\ &=\beta\sum_{j=1}^{m} w_j\tilde{b}_j + (1-\beta)\underset{l=1,m}{\max}\left\{a_l\right\}\underset{j=1,m}{\max}[\min\{\tilde{b}'_j, w^P_j\}]. \end{aligned} \quad (9)$$

Where $\tilde{b}_j$ is the jth largest of the $\left\{\tilde{a}_i\right\}$, $i=1,...,m$;

$\tilde{b}'_j = \dfrac{\tilde{b}_j}{\max\limits_{l}\{\tilde{a}_l\}}$; on the S there exist a probability distribution $p_i = P\{s_i\}$, $i=1,...,m$ with

$\sum\limits_{i=1}^{m} p_i = 1$, $0 \le p_i \le 1$ and $w_j^P \overset{\Delta}{\equiv} P\left\{ s_{i(1)},...,s_{i(j)} \right\} = \sum\limits_{i=1}^{j} p_{i(j)}$.

On the basis of Definition 8, Part II [37], and analogously to definition 1 we present a definition of the AsFPOWA operator induced by the FEV with respect to some monotone measure $g: 2^S \Rightarrow \left[0,1\right]$.

Definition 4. A Sugeno Averaging AsFPOWA operator SA-AsFPOWA of dimension m is mapping $SA-AsFPOWA: \Psi^{+m} \Rightarrow \Psi^{+}$, that has an associated objective weighted vector W of dimension m such that $w_j \in \left[0,1\right]$ and $\sum\limits_{j=1}^{m} w_j = 1$; some uncertain measure – monotone measure $g: 2^S \Rightarrow \left[0,1\right]$ with associated probability class $\left\{P_\sigma\right\}_{\sigma \in S_m}$ defined according the following formula:

$$\begin{aligned} & \mathrm{SA-AsFPOWA}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m) = \\ & = \beta \sum_{j=1}^{m} w_j \tilde{a}_{i(j)} + \\ & +(1-\beta) M\left\{ \mathrm{FEV}_{P_{\sigma_1}}\left(\tilde{a}'\right),...,\mathrm{FEV}_{P_{\sigma_k}}\left(\tilde{a}'\right) \right\}. \end{aligned} \tag{10}$$

Where

$$\begin{aligned} & \mathrm{FEV}_{P_\sigma}\left(\tilde{a}\right) \equiv \mathrm{FEV}_{P_\sigma}\left(\tilde{a}_1,...,\tilde{a}_m\right) = \\ & = \max_{l=1,m}\left\{\tilde{a}_l\right\} \max_{j=1,m}\left[\min\left\{ \tilde{a}'_{i(j)}; w_j^{P_\sigma} \right\} \right]. \end{aligned}$$

$$\tilde{a}'_{i(j)} = \frac{\tilde{a}_{i(j)}}{\max\limits_{l=1,m}\left\{a_l\right\}}.$$

And

$$w_j^{P_\sigma} = P_\sigma\left(\left\{s_{i(1)},...,s_{i(j)}\right\}\right) = \sum_{l=1}^{j} P_\sigma\left(s_{i(l)}\right), \ \forall \sigma \in S_m, \ j=1,2,..,m$$

M is some averaging operator.

Analogously to Subjection 3.2, Part II [37] (Formulas (44)-(45)) we may define new SA-AsFPOWA operators induced by the FEV with respect to concrete monotone measures: Sugeno λ-additive measure, possibility measure, believe structure's associated monotone measure and others (but these procedures are omitted here).

3| Example

Analogously to [14] we analyze an illustrative example on the use of new AsFPOWA and SA-AsFPOWA operators in a fuzzy decision-making problem regarding political management. We study a country that is planning its fiscal policy for the next year.

Assume that government of a country has to decide on the type of optimal fiscal policy for the next year. They consider five alternatives:

– d_1: *"Development a strong expansive fiscal policy".*
– d_2: *"Development an expansive fiscal policy".*
– d_3: *"Do not make any changes in the fiscal policy".*
– d_4: *"Development of a contractive fiscal policy".*
– d_5: *"Development a strong contractive fiscal policy".*

In order to analyze these fiscal policies, the government has brought together a group of experts. This group considers that the key factors are the economic situations of the world (external) and country (internal) economy for the next period. They consider 3 possible states of nature that in whole could occur in the future.

– s_1: *"Bad economic situation".*
– s_2: *"Regular economic situation".*
– s_3: *"Good economic situation".*

As a result, the group of experts gives us their opinions and results. The results depending on the state of nature s_i and alternative d_k that the government selects are presented in the *Table 1*:

Table 1. Expert's valuations in TFNs.

D \ S	s_1	s_2	s_3
d_1	(60,70,80)	(40,50,60)	(50,60,70)
d_2	(30,40,50)	(60,70,80)	(70,80,90)
d_3	(50,60,70)	(50,60,70)	(60,70,80)
d_4	(70,80,90)	(40,50,60)	(40,50,60)
d_5	(60,70,80)	(70,80,90)	(50,60,70)

Following the expert's knowledge on the world economy for the next period, experts decided that the objective weights (as an external factor) of states of nature must be $W = \left(0,5;0,3;0,2\right)$, while for the economy of the country for the next period the occurrence of presented states of nature is defined by some possibilities (as an internal factor). So, there exist some possibilities (internal levels), as an uncertainty measure, of the occurrence of states of nature in the country. This decision-making model (Information Structure I6) is more detailed than the model (Information Structure I4) presented in [14]. In another words in decision model, we cannot define the objective probabilities $p_i = P(s_i)$ for the future events, but we can define subjective possibilities $\pi_i = Pos(s_i)$ based on the experts' knowledge ([4], [8], [20] and Section 2, Part II [37]). Based on some fuzzy terms of internal factor – country economy experts define the possibility levels of states of nature:

$$poss(s_1) \equiv \pi_1 = 0,7;$$
$$poss(s_2) \equiv \pi_2 = 1;$$
$$poss(s_3) \equiv \pi_3 = 0,5.$$

So, we have the Information Structure I6 of general decision-making system (Definition 7, Part I), where

$$g := Pos(.) : 2^S \Rightarrow \left[0,1\right],$$

$$Pos(A) = \max_{s_i \in A} \pi_i, \ \forall A \subseteq S;$$

(a monotone measure is a possibility measure).

In this model as in [14] $\beta \equiv 0,3$. Decision procedure is equivalent to the detalization of GDMS as the Information Structure I6 (but in [14] the author had the IS as I4). So, for every decision d payoffs' values are the column from *Table 2*.

$g := Pos;\ W = (0,5;0,3;0,2)\ ; I = I6;\ F = AsFPOWA$ or $F = SA - AsFPOWA$ and others.

Im is the quadruple structure (Definition 7, Part I). For ranking of alternatives $\{d_1,..,d_5\}$ we must calculate its AsFOWA or other operators. For $\tilde{a} = \left(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3\right)$ we have:

$$AsFPOWA\left(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3\right) = \beta \sum_{j=1}^{3} \tilde{b}_j w_j + (1-\beta) \cdot M\left(E_{P_{\sigma_1}}\left(\tilde{a}\right), E_{P_{\sigma_2}}\left(\tilde{a}\right), \ldots, E_{P_{\sigma_6}}\left(\tilde{a}\right)\right).$$

It is clear that $k=m!=3!=6$ and for calculation of the AsFPOWA operator we firstly define the associated probability class $\{P_\sigma\}_{\sigma \in S_3}$ for the $Pos : 2^S \Rightarrow \left[0,1\right]$. For every $\sigma = \left\{\sigma(1), \sigma(2), \sigma(3)\right\} \in S_3$

$$E_{P_\sigma}\left(d\right) = E_{P_\sigma}\left(\tilde{a}\right) = \sum_{i=1}^{3} P_{\sigma(i)} \cdot \tilde{a}_{\sigma(i)},$$

where

$$P_\sigma\left(s_{\sigma(i)}\right) = Poss(\left\{s_{\sigma(1)}, \ldots, s_{\sigma(i)}\right\}) - Poss(\left\{s_{\sigma(1)}, \ldots, s_{\sigma(i-1)}\right\}) = \max_{j=1,i} \pi_{\sigma(j)} - \max_{j=1,i-1} \pi_{\sigma(j)}, \quad \pi_{\sigma(0)} \equiv 0.$$

The results are presented in the *Table 2*.

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Table 2. Associated probability class - $\{P_\sigma\}_{\sigma \in S_3}$.

$\sigma = (\sigma(1), \sigma(2), \sigma(3))$	$P_{\sigma(1)}$	$P_{\sigma(2)}$	$P_{\sigma(3)}$
$(1,2,3) = \sigma_1$	$P_1 = 0,7$	$P_2 = 0,3$	$P_3 = 0$
$(1,3,2) = \sigma_2$	$P_1 = 0,7$	$P_3 = 0$	$P_2 = 0,3$
$(2,1,3) = \sigma_3$	$P_2 = 1$	$P_1 = 0$	$P_3 = 0$
$(2,3,1) = \sigma_4$	$P_2 = 1$	$P_3 = 0$	$P_1 = 0$
$(3,1,2) = \sigma_5$	$P_3 = 0,5$	$P_1 = 0,2$	$P_2 = 0,3$
$(3,2,1) = \sigma_6$	$P_3 = 0,5$	$P_2 = 0,5$	$P_1 = 0$

Following the *Table 2* we calculate Mathematical Expectations -$\{E_{P_\sigma}(\cdot)\}_{\sigma \in S_3}$ (*Table 3*) and Fuzzy Expected Values- $\{FEV_{P_\sigma}(.)\}_{\sigma \in S_m}$ (*Table 4*).

Table 3. Mathematical expectations - $\{E_{P_\sigma}(\cdot)\}_{\sigma \in S_3}$.

$E_{P_\sigma}(\cdot)$ \ σ	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
$E_{P_\sigma}(d_1)$	(54,64,74)	(54,64,74)	(40,50,60)	(40,50,60)	(49,59,69)	(45,55,65)
$E_{P_\sigma}(d_2)$	(39,49,59)	(39,49,59)	(60,70,80)	(60,70,80)	(59,69,79)	(65,75,85)
$E_{P_\sigma}(d_3)$	(50,60,70)	(50,60,70)	(50,60,70)	(50,60,70)	(55,65,75)	(55,65,75)
$E_{P_\sigma}(d_4)$	(61,71,81)	(61,71,81)	(40,50,60)	(40,50,60)	(46,56,66)	(40,50,60)
$E_{P_\sigma}(d_5)$	(63,73,83)	(63,73,83)	(70,80,90)	(70,80,90)	(58,68,78)	(60,70,80)

Table 4. Fuzzy expected values - $\{FEV_{P_\sigma}(.)\}_{\sigma \in S_m}$.

$E_{P_\sigma}(\cdot)$ \ σ	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
$E_{P_\sigma}(d_1)$	(70,70,70)	(70,70,70)	(50,60,70)	(50,60,70)	(40,50,60)	(40,50,60)
$E_{P_\sigma}(d_2)$	(30,40,50)	(30,40,50)	(60,70,80)	(60,70,80)	(60,70,80)	(60,70,80)
$E_{P_\sigma}(d_3)$	(50,60,70)	(50,60,70)	(50,60,70)	(50,60,70)	(50,60,70)	(50,60,70)
$E_{P_\sigma}(d_4)$	(40,50,60)	(40,50,60)	(40,50,60)	(40,50,60)	(40,50,60)	(40,50,60)
$E_{P_\sigma}(d_5)$	(60,70,80)	(60,70,80)	(70,80,90)	(70,80,90)	(40,50,60)	(40,50,60)

Now we may calculate the values of different variants of the AsFPOWA and SA-AsFPOWA operators with respect to different averaging operators M (*Tables 5* and *6*):

Table 5. Aggregation results.

D/Ag. Op.	FOWA	SEV	SEV-FOWA	AsFPOWAmin	AsFPOWAmax	AsFPOWAmean
d1	(53,63,73)	(46,57,68)	(48,59,70)	(44,54,64)	(54,64,74)	(49,59,69)
d2	(59,69,73)	(53,64,75)	(55,66,77)	(45,55,65)	(64,74,84)	(57,66,75)
d3	(55,65,75)	(51,62,73)	(52,63,74)	(52,62,72)	(56,66,76)	(53,63,73)
d4	(63,73,83)	(47,58,69)	(52,63,74)	(45,55,65)	(60,70,80)	(51,61,71)
d5	(63,73,83)	(63,74,85)	(63,74,85)	(60,70,80)	(68,78,88)	(64,74,84)

Table 6. Aggregation results.

D/Ag. Op.	SA-AsFPOWAmin	SA-AsFPOWAmax	SA-AsFPOWAmean
d1	(44,54,64)	(65,68,71)	(55,61,69)
d2	(37,47,57)	(58,68,77)	(51,61,70)
d3	(51,61,71)	(51,61,71)	(51,61,71)
d4	(44,54,64)	(44,54,64)	(44,54,64)
d5	(44,54,64)	(65,75,85)	(56,66,76)

For possibility distribution $\{\pi_i\}_{i=1}^{m}$ and payoff vector $\tilde{a}=\left(\tilde{a}_1,\ldots,\tilde{a}_m\right)$ Yager in [29] defined the aggregation mean operator-Shapely Expected Value (SEV) for possibility uncertainty:

$$\mathrm{SEV}\left(\tilde{a}_1,\ldots,\tilde{a}_m\right)=\sum_{i=1}^{m} a_{\sigma(i)} P^{\pi}_{\sigma(i)}. \tag{11}$$

where $\{P^{\pi}_{\sigma(i)}\}_{i=1}^{m}$ is the probability distribution on $S=\left(s_1,\ldots,s_m\right)$ induced by possibility distribution $\{\pi_i\}_{i=1}^{m}$:

$$P^{\pi}_{\sigma(i)}=\sum_{j=1}^{\sigma(i)} \frac{\pi_{\sigma(j)}-\pi_{\sigma(j-1)}}{m+1-j}.$$

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And $\sigma=\{\sigma(1),...,\sigma(m)\}$ is some permutation from S_m form which $0=\pi_{\sigma(0)}\le\pi_{\sigma(1)}\le...\le\pi_{\sigma(m)}=1.$ On the other hand these values are Shapley Indexes of a possibility measure with a possibility distribution $\{\pi_i\}_{i=1}^{m}$.

It was proved [29] that the $SEV(\cdot)$ coincides with the ME for possibility measure:

$$\mathrm{SEV}\left(\tilde{a}_1,...,\tilde{a}_m\right)=\mathrm{ME}_{\mathrm{Poss}}\left(\tilde{a}_1,...,\tilde{a}_m\right)=$$
$$=\int_0^1 \mathrm{Poss}\left(\tilde{a}_i\ge\alpha/i=1,...,m\right)d\alpha.$$

On the basis of definition SEV we connect the SEV operator to the OWA operator as weighted sum. So we consider new generalization of the FOWA operator in Information Structure I6:

$$\mathrm{SEV-FOWA}(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_m)=\beta\sum_{j=1}^{m}w_j\tilde{b}_j+$$
$$+(1-\beta)\sum_{i=1}^{m}\tilde{a}_{\sigma(i)}\left[\sum_{j=1}^{\sigma(i)}\frac{\pi_{\sigma(j)}-\pi_{\sigma(j-1)}}{m+1-j}\right].$$

Calculating numerical values of the FOWA ([13], [14] and Definition 3, Part I), SEV, SEV-FOWA, AsFPOWAmin, AsFPOWAmax, AsFPOWAmean, SA-AsFPOWAmin, SA-AsFPOWAmax, SA-AsFPOWAmean operators we constructed the Decision Comparing Matrix (*Table 8*). Firstly, we calculated Shapely Indexes - $\{P_i^{\pi}\},j=\overline{1,3}$ for the possibility measure (*Table 7*).

Table 7. Shapley Indexes of the possibility distribution.

P_i^{π}	4/15	17/30	1/6
s_i	s_1	s_2	s_3

According to the information received in this Section, we can rank the alternatives from the most preferred to the less preferred. The results are shown in *Table 8*.

Table 8. Ordering of the policies.

N	Aggreg. Operator	Ordering	Information Structure
1	FOWA	$d_5=d_4\succ d_2\succ d_3\succ d_1$	I2
2	SEV	$d_5\succ d_2\succ d_3\succ d_4=d_1$	I6 (without weights)
3	SEV-FOWA	$d_5\succ d_2\succ d_4=d_3\succ d_1$	I6
4	AsFPOWAmin	$d_5\succ d_3\succ d_2=d_4\succ d_1$	I6
5	AsFPOWAmax	$d_5\succ d_2\succ d_4\succ d_3\succ d_1$	I6
6	AsFPOWAmean	$d_5\succ d_2\succ d_3\succ d_4\succ d_1$	I6
7	SA-AsFPOWAmin	$d_3\succ d_5=d_4=d_1\succ d_2$	I6
8	SA-AsFPOWAmax	$d_5\succ d_2\succ d_1\succ d_3\succ d_4$	I6
9	SA-AsFPOWAmean	$d_5\succ d_3\succ d_2\succ d_1\succ d_4$	I6

We also calculated values of the Orness parameter of the aggregation operators presented in *Table 9*.

Table 9. Orness values.

α \Ag.Op.	FOWA	SEV	SEV-FOWA	$AsFPOWA_{min}$	$AsFPOWA_{max}$	$AsFPOWA_{mean}$	$SA\text{-}AsFPOWA_{min}$	$SA\text{-}AsFPOWA_{max}$	$SA\text{-}AsFPOWA_{mean}$
α	0,65	0,55	0,58	0,37	0,79	0,58	0,68	0,89	0,79

Following *Table 9* we see that for the nearer of SA-AsFPOWA operators is to on *or*, the closer its measure is to one, while AsFPOWAmin operator is to on *and*, the closer is to zero. Calculations of other information measures are omitted here. More on these measures of new aggregation operators we will present in our future papers.

4| Conclusion

New generalizations of the FPOWA operator were presented with respect to monotone measure's associated probability class and induced by the Choquet and Sugeno integrals (finite cases). There exist many combinatorial variants to construct faces or expressions of generalized operators: AsFPOWA and SA-AsFPOWA for concrete mean operators (Mean, Max, Min and so on) and concrete monotone measures (Choquet capacity of order two, monotone measure associated with belief structure, possibility measure and Sugeno λ – additive measure). Some properties of new operators and their information measures (Orness, Enropy, Divergence and Balance) are proved. But only some variants (AsFPOWAmax, AsFPOWAmin and others) are presented, the list of which may be longer than it is presented in the paper. So, other presentations of new operators and properties of information measures will be considered in our future research. An example was constructed for the illustration of the properties of generalized operators in the problems of political management.

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