E-ISSN: 2717-3453 | P-ISSN: 2783-1442

Journal of Fuzzy Extension and Applications

www.journal-fea.com

J. Fuzzy. Ext. Appl. Vol. 2, No. 2 (2021) 106–119

Paper Type: Research Paper

New Characterization Theorems of the mp-Quantales

George Georgescu *

Department of Computer Science, Faculty of Mathematics and Computer Science, University of Bucharest, Bucharest, Romania; georgescu.capreni@yahoo.com

Citation:

Georgescu, G. (2021). New characterization theorems of the mp-Quantales. *Journal of fuzzy extension and applications*, 2(2), 106-119.

Received: 07/03/2021 Reviewed: 30/03/2021 Revised: 15/04/2021 Accept: 29/04/2021

Abstract

The mp-quantales were introduced in a previous paper as an abstraction of the lattices of ideals in mp-rings and the lattices of ideals in conormal lattices. Several properties of m-rings and conormal lattices were generalized to mp-quantales. In this paper we shall prove new characterization theorems for mp-quantales and for semiprime mp-quantales (these last structures coincide with the P F - quantales). Some proofs reflect the way in which the reticulation functor (from coherent quantales to bounded distributive lattices) allows us to export some properties from conormal lattices to mp-quantales.

Keywords: Coherent quantale. reticulation of a quantale, mp–quantales, PF-quantales, transfer properties.

1 | Introduction

The mp-quantales were introduced in [13] as an abstraction of the lattice of ideals of an mp - ring [1] and the lattice of ideals of a conormal lattice (see [7], [17], [23]). In [13] we proved some characterization theorems for mp - quantales that extend some results of [1] and [7] that describe the mp - rings, respectively the conormal lattices. The P F - quantales constitute an important class of mp - quantales (cf. [13]). They generalize the lattices of ideals in P F - rings. In fact, the P F - quantales are the semiprime P F - quantales. The paper [13] also contains several characterizations of a P F - quantale.

An important tool in proving the mentioned results was the reticulation of a coherent quantale [6] and [12] (the reticulation of a coherent quantale A is a bounded distributive lattice L(A) whose

Corresponding Author: georgescu.capreni@yahoo.com

https://dx.doi.org/10.22105/jfea.2021.279892.1090

Stone prime spectrum SpecId, Z(L(A)) is homeomorphic with the Zariski prime spectrum SpecZ(A) of A).

The retic1 ulation construction provides a covariant functor from the category of coherent quantales to the category of bounded distributive lattices [6].

In this paper we shall obtain new characterization theorem for mp - quantales and P F - quantales. Some of these theorems contain properties expressed in terms of equations or pure and w-pure elements (see *Theorems 5 and 6*), while others (see *Theorems 7 and 8*) extend some conditions existing in some results of [27].

Now we give a short description of the content of this paper. Section 2 contains some notions and basic results in quantale theory [22] and [10]: residuation and negation operation, m - prime and minimal m - prime elements, Zariski and flat topologies on the spectra of a quantale, radical elements, etc. In Section 3 we recall from [6] and [12] the construction of the reticulation L(A) of a coherent quantale A and we present some results that describe how the reticulation functor preserves the m - prime elements, the annihilators, the pure and the w - pure elements, etc. In Section 4 we discuss the way in which the mp - quantales (defined in [13]) generalize the mp - rings [1] and the conormal lattices [7] and [23]. Some properties that characterize the mp - quantale are recalled from [13]. The main results of the paper are placed in Section 5. We prove three theorems with new algebraic and topological characterizations of the mp-quantales. Some characterization results of the P F-quantales (= the semiprime mp-quantales) are obtained as corollaries. Some of proofs reflect the way in which the reticulation functor transfer some properties of conormal lattices to mp-quantales.

2 | Preliminaries on Quantales

This section contains some basic notions and results in quantale theory [22] and [10]. Let (A, W, ∧, ·, 0, 1) be a quantale and K(A) the set of its compact elements. A is said to be integral if (A, ·, 1) is a monoid and commutative, if the multiplication is commutative. A frame is a quantale in which the multiplication coincides with the meet [17]. The quantale A is algebraic if any a ∈ A has the form a = W X for some subset X of K(A). An algebraic quantale A is coherent if 1 ∈ K(A) and K(A) is closed under the multiplication. The set Id(R) of ideals of a (unital) commutative ring R is a coherent quantale and the set Id(L) of ideals of a bounded distributive lattice L is a coherent frame.

Throughout this paper, the quantales are assumed to be integral and commutative. We shall write ab instead of a · b. We fix a quantale A.

On each quantale A one can consider a residuation operation (= implication) a → b = W {x | ax ≤ b} and a negation operation a ⊥ = a ⊥A, defined by a ⊥ = a → 0 = W {x ∈ A | ax = 0} (extending the terminology from ring theory [2], a ⊥ is also called the annihilator of a). Then for all a, b, c ∈ A the following residuation rule holds: a ≤ b → c if and only if ab ≤ c, so (A, ∨, ∧, ·, →, 0, 1) becomes a (commutative) residuated lattice. Particularly, we have a ≤ b ⊥ if and only if ab = 0. In this paper we shall use without mention the basic arithmetical 2 properties of a residuated lattice [11].

An element p < 1 of A is m-prime if for all a, b ∈ A, ab ≤ p implies a ≤ p or b ≤ p. If A is an algebraic quantale, then p < 1 is m-prime if and only if for all c, d ∈ K(A), cd ≤ p implies c ≤ p or d ≤ p. Let us introduce the following notations: Spec(A) is the set of m-prime elements and M ax(A) is the set of maximal elements of A. If 1 ∈ K(A) then for any a < 1 there exists m ∈ M ax(A) such that a ≤ m. The same hypothesis 1 ∈ K(A) implies that M ax(A) ⊆ Spec(A). We remark that the set Spec(R) of prime ideals in R is the prime spectrum of the quantale Id(R) and the set of prime ideals in L is the prime spectrum of the frame Id(L).

Recall from [22] that the radical ϱ(a) of an element a of A is defined by ϱ(a) = V {p ∈ Spec(A) | a ≤ p}. If a = ϱ(a) then a is said to be a radical element of A. The set R(A) of the radical elements of A is a frame [22] and [23]. In [6] it is proven that Spec(A) = Spec(R(A)) and M ax(A) = M ax(R(A)).

Lemma 1. [19]. Let A be a coherent quantale and a ∈ A. Then

ϱ(a) = W {c ∈ K(A) | c k ≤ a for some integer k ≥ 1}.

For any c ∈ K(A), c ≤ ϱ(a) iff c k ≤ a for some k ≥ 1.

A is semiprime if and only if for any integer k ≥ 1, c k = 0 implies c = 0.

Let A be a quantale such that 1 ∈ K(A). For any a ∈ A, denote DA(a) = D(a) = {p ∈ Spec(A) | a 6≤ p} and VA(a) = V (a) = {p ∈ Spec(A) | a ≤ p}. Then Spec(A) is endowed with a topology whose closed sets are (V (a))a∈A. If the quantale A is algebraic then the family (D(c))c∈K(A) is a basis of open sets for this topology. The topology introduced here generalizes the Zariski topology (defined on the prime spectrum Spec(R) of a commutative ring R [2]) and the Stone topology (defined on the prime spectrum SpecId(L) of a bounded distributive lattice L [3]). Then this topology will be also called the Zariski topology of Spec(A) and the corresponding topological space will be denoted by SpecZ(A). According to [13], SpecZ(A) is a spectral space in the sense of [15]. The flat topology associated with this spectral space has as basis the family of the complements of compact open subsets of SpecZ(A)(cf.[8] and [17]). Recall from [13] that the family {V (c) | c ∈ K(A)} is a basis of open sets for the flat topology on Spec(A). We shall denote by SpecF (A) this topological space. For any p ∈ Spec(A), let us denote Λ(p) = {q ∈ Spec(A) | q ≤ p}. According to Proposition 5.6 of [13], the flat closure clF ({p}) of the set {p} is equal to Λ(p).

Let L be a bounded distributive lattice. For any x ∈ L, denote DId(x) = {P ∈ SpecId(L) | x 6∈ P} and VId(x) = {P ∈ SpecId,Z(L) | x ∈ P}. The family (DId(x))x∈L is a basis of open sets for the Stone topology on SpecId(L); this topological space will be denoted by SpecId,Z(L). We will denote by SpecId,F (L) the prime spectrum SpecId(L) endowed with the flat topology; the family (VId(x))x∈L is a basis of open sets for the flat topology.

If A is a quantale then we denote by M in(A) the set of minimal m - prime elements of A; M in(A) is called the minimal prime spectrum of A. If 1 ∈ K(A) then for any p ∈ Spec(A) there exists q ∈ M in(A) such that q ≤ p.

Proposition 1. If A is a coherent quantale and p ∈ Spec(A) then p ∈ M in(A) if and only if for all c ∈ K(A), the following equivalence holds: c ≤ p iff c → ϱ(0) 6≤ p.

Corollary 1. [18]. If A is a semiprime coherent quantale and p ∈ Spec(A) then p ∈ M in(A) if and only if for all c ∈ K(A), the following equivalence holds: c ≤ p iff c ⊥ 6≤ p.

An element e of the quantale A is a complemented element if there exists f ∈ A such that e∨f = 1 and e∧f = 0. The set B(A) of complemented elements of A is a Boolean algebra (cf. [5] and [16]). B(A) will be called the Boolean center of the quantale A.

Proposition 2. If a ∈ A then a ⊥ = V (V (a ⊥) T M in(A)).

Georgescu | J. Fuzzy. Ext. Appl. 2(2) (2021) 106-119

3 | Reticulation of a Coherent Quantale

Let A be a coherent quantale and K(A) the set of its compact elements. We define the following equivalence relation on the set K(A): for all c, d ∈ K(A), c ≡ d iff ϱ(c) = ϱ(d). The quotient set L(A) = K(A)/ ≡ is a bounded distributive lattice. For any c ∈ K(A) denote by c/ ≡ its equivalence class. Consider the canonical surjection λA : K(A) → L(A) defined by λA(c) = c/ ≡, for any c ∈ K(A). The pair (L(A), λA : K(A) → L(A)) (or shortly L(A)) will be called the reticulation of A. In [6] and [12] an axiomatic definition of the reticulation of a coherent quantale was given. We remark that the reticulation L(R) of a commutative ring R (defined in [17] and [23]) is isomorphic with the reticulation L(Id(R)) of the quantale Id(R).

For any a ∈ A and I ∈ Id(L(A)) let us denote a ∗ = {λA(c) | c ∈ K(A), c ≤ a} and I∗ = W {c ∈ K(A) | λA(c) ∈ I}. The assignments a 7→ a ∗ and I 7→ I∗ define two order - preserving maps (·) ∗ : A → Id(L(A) and (·)∗ : Id(L(A)) → A. The following lemma collects the main properties of the maps (·) ∗ and (·)∗.

Lemma 2. [6]. The following assertions hold

If a ∈ A then a ∗ is an ideal of L(A) and a ≤ (a ∗)∗.

If I ∈ Id(L(A)) then (I∗) ∗ = I.

If p ∈ Spec(A) then (p ∗)∗ = p and p ∗ ∈ SpecId(L(A)).

If P ∈ SpecId((L(A)) then P∗ ∈ Spec(A).

If p ∈ K(A) then c ∗ = (λA(c)].

If c ∈ K(A) and I ∈ Id(L(A)) then c ≤ I∗ iff λA(c) ∈ I.

If a ∈ A and I ∈ Id(L(A)) then ϱ(a) = (a ∗)∗, a ∗ = (ϱ(a))∗ and ϱ(I∗) = I∗.

If c ∈ K(A) and p ∈ Spec(A) then c ≤ p iff λA(c) ∈ p ∗.

By the previous lemma one can consider the maps δA : Spec(A) → SpecId(L(A)) and A : SpecId(L(A)) → Spec(A), defined by δA(p) = p ∗ and A(I) = I∗, for all p ∈ Spec(A) and I ∈ SpecId(L(A)).

Lemma 3. [6] and [13]. The functions δA and A are homeomorphisms w.r.t. the Zariski and the flat topologies, inverse to one another.

We also observe that δA and A are order - isomorphisms. In particular, for any m - prime element p of A, we have p ∈ M in(A) if and only in p ∗ ∈ M inId(L(A)).

We denote by M inZ(A) (resp. M inF (A)) the topological space obtained by restricting the topology of SpecZ(A) (resp. SpecF (A)) to M in(A). Then M inZ(A) is homeomorphic to the space M inId,Z(L(A)) of minimal prime ideals in L(A) with the Stone topology and M inF (A) is homeomorphic to the space M inId,F (L(A)) of minimal prime ideals in L(A) with the flat topology (cf. *Lemma 3*). By [13], M inZ(A) is a zero - dimensional Hausdorff space and M inF (A) is a compact T1 space.

For a bounded distributive lattice L we shall denote by B(L) the Boolean algebra of the complemented elements of L. It is well-known that B(L) is isomorphic to the Boolean center B(Id(L)) of the frame Id(L) (see [5] and [17]). By [6], the function λA | B(A) : B(A) → B(L(A)) is a Boolean isomorphism.

If L is a bounded distributive lattice and I ∈ Id(L) then the annihilator of I is the following ideal of L(A): AnnL(I) = Ann(I) = {x ∈ I | x ∧ y = 0, for all y ∈ L}.

Let us fix a coherent quantale A.

Lemma 4. [13]. If c ∈ K(A) and p ∈ Spec(A) then Ann(λA(c)) ⊆ p ∗ if and only if c → ϱ(0) ≤ p.

Proposition 3. [13]. If a is an element of a coherent quantale then Ann(a ∗) = (a → ϱ(0))∗ ; if A is semiprime then Ann(a ∗) = (a ⊥) ∗.

Particularly, for any c ∈ K(A), we have Ann(λA(c)) = (c → ϱ(0))∗.

Proposition 4. [13]. Assume that A is a coherent quantale. If I is an ideal of L(A) then (Ann(I))∗ = I∗ → ϱ(0); if A is semiprime then (Ann(I))∗ = (I∗) ⊥.

An ideal I of a commutative ring R is said to be pure if for any x ∈ I we have I ∨ Ann(x) = R. An ideal I of a bounded distributive lattice L is said to be a σ -ideal if for any x ∈ I we have I ∨ Ann(x) = L. These two notions can be generalized to quantale theory: an element a of an algebraic quantale A is 5 said to be pure if for any c ∈ K(A) we have a ∨ c ⊥ = 1. We note that the σ - ideals of a bounded distributive lattice L coincide with the pure elements of the frame Id(L).

An element a of an algebraic quantale A is said to be w - pure [14] if for any c ∈ K(A) we have a ∨ (c → ϱ(0)) = 1. It is easy to see that any pure element of A is w - pure.

Lemma 5. [14]. If an element a of a coherent quantale A is w - pure then a ∗ is a σ - ideal of the lattice L(A). Particulary, if a is pure then a ∗ is a σ - ideal.

Lemma 6. [14]. Let A be a coherent quantale and J a σ - ideal of L(A). Then J∗ is a w - pure element of A.

For any p ∈ Spec(A) let us denote O(p) = W {c ∈ K(A) | c ⊥ 6≤ p}.

Lemma 7. [14]. Let A be a coherent quantale. If p ∈ Spec(A) and c ∈ K(A) then c ≤ O(p) if and only if c ⊥ 6≤ p.

4 | From mp-Rings and Conormal Lattices to the mp–Quantales

Recall from [1] that a commutative ring R is an mp-ring if each prime ideal of R contains a unique minimal prime ideal. The following theorem of [1], that collects several characterizations of mp-rings, emphasizes some of their algebraic and topological properties.

Theorem 1. [1]. If R is a commutative ring then the following assertions are equivalent R is an mp – ring.

If P and Q are distinct minimal prime ideals of the ring R then P +Q = R.

R/n(R) is an mp - ring, whenever n(A) is the nil - ideal of R.

SpecF (R) is a normal space.

The inclusion M inF (R) ⊆ SpecF (R) has a flat continuous retraction.

If P is a minimal prime ideal of R then VR(P) is a flat closed subset of SpecF (R).

For all a, b ∈ R, ab = 0 implies Ann(a n) + Ann(b n) = R, for some integer n ≥ 1.

Any minimal prime ideal of A is the radical of a unique pure ideal of A.

The conormal lattices were introduced by Cornish in [7] under the name of normal lattices (a discussion of the terminology can be found in [23] and [17]). According to [23], a bounded distributive lattice L is conormal if for all a, b ∈ L such that a∧b = 0 there exist x, y ∈ L such that a∧x = b∧y = 0 and x∨y = 1.

Theorem 2. [7]. If A is a conormal lattice then the following assertions are equivalent

A is a conormal lattice.

If P and Q are distinct minimal prime ideals of the lattice L then P ∨ Q = L.

Any minimal prime ideal of L is a σ – ideal.

If x, y ∈ L and x ∧ y = 0 then Ann(x) ∨ Ann(y) = L.

If x, y ∈ L then Ann(x ∧ y) = Ann(x) ∨ Ann(y) = L.

Any prime ideal of L contains a unique minimal prime ideal.

For each x ∈ L, Ann(x) is a σ-ideal.

From the previous two theorems we observe that the mp - rings and the conormal lattices have similar characterizations. This remark allows us to extend these notions to quantale theory. A quantale A is said to be an mp - quantale if for any m-prime element p of A there exists a unique minimal m - prime element q such that q ≤ p. The mp-frames are defined in a similar manner.

The following theorem establishes a strong connection between mp - quantales, mp - frames and conormal lattices.

Theorem 3. [13]. For any coherent quantale A the following assertions are equivalent

A is an mp–quantale.

R(A) is an mp–frame.

L(A) is a conormal lattice.

Proof. We know that Spec(A) = Spec(R(A)) and M in(A) = M in(R(A)), so the equivalence (1) ⇔ (2) is clear. The equivalence (1) ⇔ (3) was established in [13].

The following theorem is a generalization of some parts of *Theorems 1* and *2*.

Theorem 4. [13]. For any coherent quantale A the following are equivalent

A is an mp–quantale.

For any distinct p, q ∈ M in(A) we have p ∨ q = 1.

Georgescu | J. Fuzzy. Ext. Appl. 2(2) (2021) 106-119

The inclusion M inF (A) ⊆ SpecF (A) has a flat continuous retraction.

SpecF (A) is a normal space.

If p ∈ Spec(A) then V (p) is a closed subset of SpecF (A).

Recall from [13] that a quantale A is a P F - quantale if for each c ∈ K(A), c ⊥ is a pure element. From [13] we know that a quantale A is a P F - quantale if and only if it is a semiprime mp-quantale.

5 | New Characterization Theorems

This section concerns the new characterization theorems for mp-quantales and P F - quantales. Our results extend some characterization theorems of mp -rings and P F - rings proven in [1], [26], [27]. Let us fix a coherent quantale A.

Recall from Section 3 that the maps δA : Spec(A) → SpecId(L(A)) and A : SpecId(L(A)) → Spec(A) are order - isomorphisms and homeomorphisms w.r.t. the Zariski and the flat topologies, inverse to one another. Then the following lemma is immediate.

Lemma 8. [13]. The functions δA|M in(A) : M in(A) → M inId(L(A)) and A|M inId(L(A)) : M inId(L(A)) → M in(A) are homeomorphisms w.r.t. the Zariski and the flat topologies, inverse to one another.

The previous lemma allows us to transfer some topological results from M inId(L(A)) to M in(A). Often we shall apply this lemma and its direct consequences without mention.

Theorem 5. The following assertions are equivalent:

A is an mp–quantale.

Any minimal m-prime element of A is w - pure.

Proof.

(1) ⇒ (2) Let p be a minimal m - prime element of A, hence p ∗ ∈ M inId(L(A)) (cf. *Lemma 8*). According to Theorem 3, L(A) is a conormal lattice, hence, by using Theorem 2, it follows that p ∗ is a σ - ideal of the lattice L(A). By *Lemmas 2* and 6, p = (p ∗)∗ is a w - pure element of A.

(2) ⇒ (1) Let P be a minimal prime ideal of L(A), so P = p ∗ , for some minimal m - prime element p of A (cf. Lemma 8). By hypothesis (2), p is a w-pure element of A. According to *Lemma 5*, P = p ∗ is a σ - ideal of L(A). Applying the implication (3) ⇒ (1) of Theorem 2, it follows that L(A) is a conormal lattice, so A is an mp - quantale (cf. *Theorem 3*).

Corollary 2. If A is semiprime then the following assertions are equivalent

A is a P F – quantale.

Any minimal m-prime element of A is pure.

Proof.

We know from [13] that A is a P F - quantale if and only if it is a semiprime mp - quantale. In a semiprime quantale the pure and the w - pure elements coincide, so the corollary follows from *Theorem 5*.

Theorem 6. The following are equivalent

A is an mp–quantale.

For all c, d ∈ K(A), cd ≤ ϱ(0) implies (c → ϱ(0)) ∨ (d → ϱ(0)) = 1.

For all c, d ∈ K(A), cd → ϱ(0) = (c → ϱ(0)) ∨ (d → ϱ(0)).

For any c ∈ K(A), c → ϱ(0) is a w - pure element of A.

For all c, d ∈ K(A), cd = 0 implies (c n) ⊥ ∨ (d n) ⊥ = 1.

For any minimal m - prime element p of A there exists a unique pure element q such that p = ϱ(q).

Proof.

(1) ⇒ (2) Assume by absurdum that there exist c, d ∈ K(A), such that cd ≤ ϱ(0) and (c → ϱ(0)) ∨ (c → ϱ(0)) < 1, so (c → ϱ(0)) ∨ (c → ϱ(0)) ≤ m, for some maximal element m of A. Consider a minimal m - prime element p of A such that p ≤ m. By *Theorem 5*, p is a w - pure element of A. From cd ≤ ϱ(0) ≤ p we get c ≤ p or d ≤ p (because p is m - prime). Assuming that c ≤ p we get p ∨ (c → ϱ(0) = 1 (because p is a w - pure element), contradicting that p ≤ m and c → ϱ(0) ≤ m. Thus the implication (1) ⇒ (2) is verified.

(2) ⇒ (1) Let p, q be two distinct minimal m - prime elements of A, hence there exists d ∈ K(A) such that d ≤ p and d 6≤ q. By Proposition 1, from d ≤ p it follows that d → ϱ(0) 6≤ p, so there exists c ∈ K(A) such that c ≤ d → ϱ(0) and c 6≤ p. Then cd ≤ ϱ(0), hence (c → ϱ(0)) ∨ (d → ϱ(0)) = 1 (by hypothesis (2). The last equality implies that there exist e, f ∈ K(A) such that e ≤ (c → ϱ(0)), f ≤ (d → ϱ(0)) and e ∨ f = 1. From ce ≤ ϱ(0) ≤ p and c 6≤ p we obtain e ≤ p. Similarly, we can prove that f ≤ q, so p ∨ q = 1. By applying the implication (2) → (1) of *Theorem 4*, it results that A is an mp-quantale.

(2) ⇒ (3) Firstly we shall establish the inequality cd → ϱ(0) ≤ (c → ϱ(0)) ∨ (d → ϱ(0)). Let e be a compact element of A such that e ≤ cd → ϱ(0), hence we get cde ≤ ϱ(0). In accordance with the hypothesis (2), it follows that (ce → ϱ(0))∨(d → ϱ(0)) = 1, so there exist x, y ∈ K(A) such that x ≤ ce → ϱ(0) and y ≤ d → ϱ(0) and x∨y = 1. From x ≤ ce → ϱ(0) we obtain ex ≤ c → ϱ(0).

Then e = e(x ∨ y) = ex ∨ ey ≤ ex ∨ y ≤ (c → ϱ(0)) ∨ (d → ϱ(0)), so we have proven that cd → ϱ(0) ≤ (c → ϱ(0)) ∨ (d → ϱ(0)).

From cd ≤ c and cd ≤ d it results that c → ϱ(0) ≤ cd → ϱ(0) and d → ϱ(0) ≤ cd → ϱ(0), hence the converse inequality (c → ϱ(0)) ∨ (d → ϱ(0)) ≤ cd → ϱ(0) follows.

(3) ⇒ (2) If cd ≤ ϱ(0) then, by using the property (3), it follows that (c → ϱ(0)) ∨ (d → ϱ(0)) = cd → ϱ(0) = 1.

(2) ⇒ (4) Let c be a compact element of A. Assume that d is a compact element of A such that d ≤ c → ϱ(0), so cd ≤ ϱ(0), hence (c → ϱ(0)) ∨ (d → ϱ(0)) = 1 (by the condition (2)). It results that c → ϱ(0) is a

Georgescu | J. Fuzzy. Ext. Appl. 2(2) (2021) 106-119

w-pure element of A.

(4) ⇒ (2) Assume that c, d ∈ K(A) and cd ≤ ϱ(0), so d ≤ c → ϱ(0). Since c → ϱ(0) is a w - pure element of A, the equality (c → ϱ(0)) ∨ (d → ϱ(0)) = 1 follows.

(1) ⇒ (5) By *Theorem 3*, the reticulation L(A) is a conormal lattice. Assume that c, d ∈ K(A) and cd = 0, hence λA(c)∧λA(d) = λA(cd) = λA(0) = 0 (by Definition 3(ii) of [6]). In accordance with Lemma 4.4 of [14], we know that the function (·) ∗ : A → Id(L(A)) preserves the joins. According to *Theorem 2*, we have Ann(λA(c)) ∨ Ann(λA(d)) = L(A), hence, by using *Proposition 3*, it follows that ((c → ϱ(0)) ∨ (c → ϱ(0)))∗ = ((c → ϱ(0))∗ ∨ (c → ϱ(0))∗ = (c → ϱ(0))⊥ ∨ (c → ϱ(0))⊥ = Ann(λA(c)) ∨ Ann(λA(d)) = L(A).

By applying *Lemma 2*, it follows that ϱ((c → ϱ(0)) ∨ (c → ϱ(0))) = (((c → ϱ(0)) ∨ (c → ϱ(0)))∗)∗ = 1, so (c → ϱ(0)) ∨ (c → ϱ(0)) = 1 (cf. Lemma 5(3) of [6]). Since 1 ∈ K(A), there exist x, y ∈ K(A) such that x ≤ c → ϱ(0), y ≤ d → ϱ(0) and x ∨ y = 1. From xc ≤ ϱ(0), yd ≤ ϱ(0) we get x nc n = 0, y nd n = 0, for some integer n ≥ 1 (cf. Lemma 1), hence x n ≤ (c n) ⊥ and y n ≤ (d n) ⊥. By Lemma 2(4) of [6], from x ∨ y = 1 it results that x n ∨ y n = 1, therefore (c n) ⊥ ∨ (d n) ⊥ = 1.

(5) ⇒ (1) According to *Theorem 3*, it suffices to show that L(A) is a conormal lattice. Let x, y ∈ L(A) such that x ∧ y = 0, so x = λA(c), y = λA(d), for some c, d ∈ K(A), hence λA(cd) = x ∧ y = 0. By Lemma 1, there exists an integer n ≥ 1 such that c nd n = 0, so (c nk) ⊥ ∨ (d nk) ⊥ = 1, for some integer k ≥ 1 (according to hypothesis (5)). Since (c nk) ⊥ ≤ c nk → ϱ(0) and (d nk) ⊥ ≤ d nk → ϱ(0), it follows that (c nk → ϱ(0)) ∨ (d nk → ϱ(0)) = 1. We know from Lemma 9(6) of [6] that λA(c nk) = λA(c) and λA(c nk) = λA(c). We recall that the map (·) ∗ preserves the joins.

In accordance with *Proposition 3*, the following equalities hold: Ann(x)∨Ann(y) = Ann(λA(c))∨Ann(λA(d)) = Ann(λA(c nk))∨Ann(λA(d nk)) = (c nk → ϱ(0))∗∨(d nk → ϱ(0))∗ = ((c nk → ϱ(0))∨((d nk → ϱ(0)))∗ = 1∗ = L(A). By applying the implication (4) ⇒ (1) of *Theorem 2*, it follows that L(A) is a conormal lattice.

(1) ⇒ (6) Let us denote by V ir(A) the set of pure elements of the quantale A. Recall that the pure elements of the frame Id(L(A)) are exactly the σ - ideals of the lattice L(A), so V ir(Id(L(A)) will be the frame of σ - ideals the lattice L(A). We recall from [12] that the map w: V ir(A) → V ir(Id(L(A)), defined by w(a) = a ∗ for any a ∈ V ir(A), is a frame isomorphism.

Let p be a minimal m - prime element of A, so p ∗ is a minimal prime ideal of the lattice L(A). Since L(A) is a conormal lattice, p ∗ is a σ-ideal of L(A) (cf. *Theorem 2*. But w is a frame isomorphism, so there exists a unique pure element q of A such that p ∗ = w(q) = q ∗. By using *Lemma 2*, it follows that p = (p ∗)∗ = (q ∗)∗ = ϱ(q).

Assume that q1, q2 are pure elements of A such that p = ϱ(q1) = ϱ(q2). By *Lemma 2*, it follows that p ∗ = (ϱ(qi))∗ = q ∗ i , for i = 1, 2. Thus w(q1) = w(q2), hence q1 = q2 (because w is a bijection).

(6) ⇒ (1) Let p, q be two distinct minimal m - prime elements of A, so there exists c ∈ K(A) such that c ≤ p and c 6≤ q. By the hypothesis (6), there exists a unique pure element r such that p = ϱ(r). From c ≤ ϱ(r) we get c n ≤ r, for some integer n ≥ 1 (cf. *Lemma 1*). But r is pure, so we get r ∨ (c n) ⊥ = 1, therefore we obtain p ∨ (c n) ⊥ = 1 (because r ≤ p). Since q is m - prime, c 6≤ q implies c n 6≤ q, hence (c n) ⊥ ≤ q. It follows that p ∨ q = 1. According to *Theorem 4*, it results that A is an mp-quantale.

The *properties (5)* and *(6)* of the previous theorem are the quantale versions of the conditions (5) and (6) of *Theorem 1*.

Corollary 3. [13]. The following are equivalent

A is a P F – quantale.

A is a semiprime mp – quantale.

For all c, d ∈ K(A), cd = 0 implies $c\perp \vee d\perp = 1$.

For all c, d ∈ K(A), $(cd)\perp = c\perp \vee d\perp$.

Proof.

(1) ⇒ (2) By Lemma 8.17 of [13], A is semiprime, so for any c ∈ K(A), $c \to \varrho(0) = c\perp$ is a pure element, so $c \to \varrho(0)$ is w - pure. By implication (4) ⇔ (1) of *Theorem 6*, A is an mp-quantale.

(2) ⇒ (1) Let c be a compact element of A. Since A is a semiprime mp - quantale, $c\perp = c \to \varrho(0)$ is a w - pure element (cf. implication (1) ⇒ (4) of *Theorem 6*). But in a semiprime quantale the pure and w - pure elements concide, so $c\perp$ is a pure element of A. Thus A is a P F - quantale.

(1) ⇔ (3) The condition (3) says that for any c ∈ K(A), $c\perp$ is a pure element of A.

(2) ⇒ (4) Since A is semiprime we have $\varrho(0) = 0$, so (4) follows by using the implication (1) ⇒ (3) of *Theorem 6*.

(4) ⇒ (2) Assume that c n = 0, where c ∈ K(A) and n ≥ 1 is a natural number. By the hypothesis (3), from c n = 0 we obtain $c\perp = (c\ n)\perp = 1$, so $c \leq c\perp\perp = 0$. It result that c = 0, so the quantale A is semiprime (by *Lemma 1*). In this case we have $cd \to \varrho(0) = (cd)\perp = c\perp \vee d\perp = (c \to \varrho(0)) \vee (d \to \varrho(0))$.

By using the implication (3) → (1) of *Theorem 6*, we conclude that A is an mp - quantale.

Lemma 9. If p ∈ Spec(A) then $\varrho(O(p)) \leq p$.

If p ∈ Spec(A) then p ∈ M in(A) if and only if $\varrho(O(p)) = p$.

Proof.

(1) If p ∈ Spec(A) then $\varrho(O(p)) \leq \varrho(p) \leq p$.

(2) Let p be an m - prime element of A. Assume that p ∈ M in(A). Firstly, from (1) we know that $\varrho(O(p)) \leq p$. In order to show the converse inequality $p \leq \varrho(O(p))$, suppose that c ∈ K(A) and c ≤ p. According to *Proposition 1*, c ≤ p implies $c \to \varrho(0)$ 6≤ p, so there exists d ∈ K(A) such that $d \leq c \to \varrho(0)$ and d 6≤ p. Applying *Lemma 1*, from $cd \leq \varrho(0)$ we get c nd n = 0, for some integer n ≥ 1, hence $d\ n \leq (c\ n)\perp$. Since d 6≤ p and p ∈ Spec(A) we have d n 6≤ p, so $(c\ n)\perp \leq p$. By using *Lemma 7*, it follows that c n ≤ O(p), therefore $c \leq \varrho(O(p))$ (cf. *Lemma 1*). Conclude that $p \leq \varrho(O(p))$.

Now assume that $\varrho(O(p)) = p$. Let us consider a minimal m - prime element q of A such that q ≤ p. We want to prove that O(p) ≤ q. For any c ∈ K(A), by using *Lemma 7*, the following implications hold: c ≤ O(p) ⇒ $c\perp$ 6≤ p ⇒ c ≤ q. It follows that O(p) ≤ q, so $p = \varrho(O(p)) \leq O(q) \leq q$. Thus p = q, therefore p is a minimal m - prime element of A.

Georgescu | J. Fuzzy. Ext. Appl. 2(2) (2021) 106-119

The following result generalizes Theorem 3.2 of [27] to the framework of quantale theory.

Theorem 7. The following are equivalent

A is an mp–quantale.

If p and q are distinct minimal m - prime elements of A then we have O(p) ∨ O(q) = 1.

For any p ∈ Spec(A), ϱ(O(p)) is m – prime.

For any m ∈ M ax(A), ϱ(O(p)) is m – prime.

If p, q ∈ Spec(A) and p ≤ q then ϱ(O(p)) = ϱ(O(q)).

Proof.

(1) ⇒ (2) Assume that p, q ∈ M in(A) and p 6= q, so p ∨ q = 1 (cf. *Theorem 4*). By using *Lemma 9*, we have p = ϱ(O(p)) and q = ϱ(O(q)), hence ϱ(O(p))∨ϱ(O(q)) = 1. In accordance with Lemma 2.2 of [13], it follows that O(p) ∨ O(q) = 1.

(2) ⇒ (1) Assume that p, q ∈ M in(A) and p 6= q, so O(p) ∨ O(q) = 1. Since O(p) ≤ p and O(q) ≤ q we get p ∨ q = 1. By the implication (2) ⇒ (1) of *Theorem 4* it follows that A is an mp-quantale.

(1) ⇒ (3) Suppose that p ∈ Spec(A). In order to show that ϱ(O(p)) is m - prime, let c, d be two compact elements of A such that cd ≤ ϱ(O(p)), so there exists an integer n ≥ 1 such that c nd n ≤ O(p) (cf. *Lemma 1*). By *Lemma 7* we have (c nd n) ⊥ 6≤ p, so there exists e ∈ K(A) such that e ≤ (c nd n) ⊥ and e 6≤ p. Thus ecnedn = 0, hence (ecn → ϱ(0)) ∨ (edn → ϱ(0)) = 1 (by *Theorem 6*). Since 1 ∈ K(A) there exist two compact elements c and d such that x ≤ ecn → ϱ(0), y ≤ edn → ϱ(0) and x ∨ y = 1.

Then xecn ≤ ϱ(0) and yedn ≤ ϱ(0), so there exists an integer k ≥ 1 such that x k e k c nk = 0 and y k e kd nk = 0. By Lemma 2.1of [13], we have x k ∨y k = 1, so x k 6≤ p or y k 6≤ p. Let us assume that x k 6≤ p, hence x k e k 6≤ p (because e 6≤ p and p ∈ Spec(A) implies e k 6≤ p). From x k e k c nk = 0 we get x k e k ≤ (c nk) ⊥, hence (c nk) ⊥ 6≤ p. In virtue of *Lemma 7*, it follows that c nk ≤ O(p), so c ≤ ϱ(O(p)) (by *Lemma 1*). Similarly, y k 6≤ p implies d ≤ ϱ(O(p)). Conclude that ϱ(O(p) is m - prime.

(3) ⇒ (4) Obviously.

(4) ⇒ (1) Suppose that p ∈ Spec(A) and fix a maximal element m such that p ≤ m. Let q be a minimal m - prime element such that q ≤ p ≤ m. For any c ∈ K(A) such that c ≤ O(m) we have c ⊥ 6≤ m (by *Lemma 7*), hence c ⊥ 6≤ q. Since q is m - prime it follows that c ≤ q, so we conclude that O(m) ≤ q, so ϱ(O(m)) ≤ ϱ(q) = q. According to the hypothesis (4), ϱ(O(m)) is m - prime, therefore q = ϱ(O(m)). We have proven that there exists a unique minimal m - prime element q such that q ≤ p, so A is an mp - quantale.

(1) ⇒ (5) Assume that p, q ∈ Spec(A) and p ≤ q. Let us consider m ∈ M ax(A) and r ∈ M in(A) such that r ≤ p ≤ q ≤ m, hence, by using *Lemma 9*, we get r = ϱ(O(r)) ≤ ϱ(O(p)) ≤ ϱ(O(q)) ≤ ϱ(O(m)). According to the proof of the implication (4) ⇒ (1) we have r = ϱ(O(m)), so ϱ(O(p)) = ϱ(O(q)).

(5) ⇒ (1) Suppose that p ∈ Spec(A) and q1, q2 ∈ M in(A) such that q1 ≤ p and q2 ≤ p. Let m be a maximal element of A such that p ≤ m. Applying the hypothesis (5) and *Lemma 9* we get q1 = ϱ(O(q1)) = ϱ(O(q2)) = q2, hence there exists a unique minimal m-prime element q of A such that q ≤ p.

Now we shall present some consequences of Theorem 7 that extend Corollaries 3.3 of 3.5 of [27].

Lemma 10. V {O(m) | m ∈ M ax(A)} = 0.

Proof. Assume that c ∈ K(A) and c ≤ V {O(m) | m ∈ M ax(A)}. If c ⊥ < 1 then c ⊥ ≤ n, for some n ∈ M ax(A). By Lemma 7 we have c 6≤ O(n), contradicting the assumption that c ≤ V {O(m) | m ∈ M ax(A)}. Then c ⊥ = 1, hence c ≤ c ⊥⊥ = 0. Thus c = 0, so we conclude that V {O(m) | m ∈ M ax(A)} = 0.

Theorem 8. The following are equivalent

A is a P F – quantale.

For any p ∈ Spec(A), O(p) is m – prime.

For any m ∈ M ax(A), O(p) is m – prime.

Proof.

(1) ⇒ (2) Assume that p ∈ K(A) and c, d ∈ K(A) such that cd ≤ O(p), so (cd) ⊥ 6≤ p (cf. *Lemma 10*). By *Corollary 3* we have (cd) ⊥ = c ⊥ ∨ d ⊥, so c ⊥ 6≤ p or d ⊥ 6≤ p. By applying again *Lemma 10* it follows that c ≤ O(p) or d ≤ O(p), so O(p) is m-prime.

(2) ⇒ (3) Obviously.

(3) ⇒ (1) By the hypothesis (3), we have ϱ(O(m)) = O(m), for any m ∈ M ax(A). By using *Theorem 7* it results that A is an mp - quantale.

We shall prove that A is semiprime. Assume c ∈ K(A) and c ≤ ϱ(0), hence c n = 0 for some integer n ≥ 1 (cf. *Lemma 1*). Then for each m ∈ M ax(A) we have c n ≤ O(m), hence c ≤ O(m) (because O(m) is m - prime). Thus c ≤ V {O(m) | m ∈ M ax**(A)} = 0 (b**y *Lemma 10*), so c = 0. Thus A is a semiprime mp-quantale, so A is a P F - quantale.

By *Theorem 4*, we know that for any mp - quantale A, the inclusion M inF (A) ⊆ SpecF (A) has a flat continuous retraction. The following result establishes the form of a continuous retraction γ: SpecF (A) → M inF (A) of the inclusion M inF (A) ⊆ SpecF (A) (whenever such retraction exists).

Proposition 6. If the inclusion M inF (A) ⊆ SpecF (A) has a continuous retraction γ: SpecF (A) → M inF (A) then γ(p) = ϱ(O(p)), for all P ∈ Spec(A).

Proof. Let p be an m-prime element of A. Recall from Proposition 5.6 of [13] that the flat closure of the set {p} is given by clF ({p}) = Λ(p), where Λ(p) = {s ∈ Spec(A) | q ≤ p}. Assume that q ∈ M in(A) and q ≤ p, so q ∈ Λ(p) = clF ({p}) (cf. Proposition 5.6 of [13]). Since the map γ is a continuous retraction of the inclusion M inF (A) ⊆ SpecF (A) we have q = γ(q) ∈ clF ({γ(p)}) = Λ(γ(p)) = {γ(p)}, so q = γ(p). Thus γ(p) is the unique minimal m - prime element q such that q ≤ p. In particular, we have proven that A is an mp - quantale. According to *Lemma 9* and *Theorem 7*, from γ(p) ≤ q it follows that γ(p) = ϱ(O(γ(p))) = ϱ(O(p)).

From the previous proposition we get the uniqueness of the continuous retraction γ of the inclusion M inF (A) ⊆ SpecF (A) (whenever the retraction γ exists).

Georgescu | J. Fuzzy. Ext. Appl. 2(2) (2021) 106-119

Reference

[1] Aghajani, M., & Tarizadeh, A. (2020). Characterizations of Gelfand rings specially clean rings and their dual rings. *Results in mathematics*, *75*(3), 1-24.

[2] Atiyah, M. F., & Macdonald, I. G. (2018). *Introduction to commutative algebra*. CRC Press.

[3] Balbes, R., & Dwinger, Ph. (2011). *Distributive lattices*. Abstract Space Publishing

[4] Bhattacharjee, P. (2009). *Minimal prime element space of an algebraic frame* (Doctoral dissertation, Bowling Green State University). Available at http://rave.ohiolink.edu/etdc/view?acc_num=bgsu1243364652

[5] Birkhoff, G. (1940). *Lattice theory* (Vol. 25). American mathematical Soc.

[6] Cheptea, D., & Georgescu, G. (2020). Boolean lifting properties in quantales. *Soft computing*, *24*, 6119-6181.

[7] Cornish, W. H. (1972). Normal lattices. *Journal of the australian mathematical society*, *14*(2), 200-215. DOI: https://doi.org/10.1017/S1446788700010041

[8] Dickmann, M., Schwartz, N., & Tressl, M. (2019). *Spectral spaces* (Vol. 35). Cambridge University Press.

[9] Dobbs, D. E., Fontana, M., & Papick, I. J. (1981). On certain distinguished spectral sets. *Annali di matematica pura ed applicata*, *128*(1), 227-240. (In Italian). DOI: https://doi.org/10.1007/BF01789475

[10] Eklund, P., García, J. G., Höhle, U., & Kortelainen, J. (2018). Semigroups in complete lattices. In *Developments in mathematics* (Vol. 54). Cham: Springer.

[11] Galatos, N., Jipsen, P., Kowalski, T., & Ono, H. (2007). *Residuated lattices: an algebraic glimpse at substructural logics*, volum *151*. Elsevier.

[12] Georgescu, G. (1995). The reticulation of a quantale. *Revue roumaine de mathematiques pures et appliquees*, *40*(7), 619-632.

[13] Georgescu, G. (2021). Flat topology on the spectra of quantales. *Fuzzy sets and systems*, *406*, 22-41. https://doi.org/10.1016/j.fss.2020.08.009

[14] Georgescu, G. (2020). Reticulation of a quantale, pure elements and new transfer properties. *Fuzzy sets and systems*. https://doi.org/10.1016/j.fss.2021.06.005

[15] Hochster, M. (1969). Prime ideal structure in commutative rings. *Transactions of the american mathematical society*, *142*, 43-60.

[16] Jipsen, P. (2009). Generalizations of Boolean products for lattice-ordered algebras. *Annals of pure and applied logic*, *161*(2), 228-234. https://doi.org/10.1016/j.apal.2009.05.005

[17] Johnstone, P. T. (1982). *Stone spaces* (Vol. 3). Cambridge university press.

[18] Keimel, K. (1972). A unified theory of minimal prime ideals. *Acta mathematica academiae scientiarum hungarica*, *23*(1-2), 51-69. https://doi.org/10.1007/BF01889903

[19] Powell, W. B. (1985). *Ordered algebraic structures (Vol. 99)*. CRC Press.

[20] Matlis, E. (1983). The minimal prime spectrum of a reduced ring. *Illinois journal of athematics*, *27*(3), 353-391.

[21] Paseka, J., & Rosický, J. (2000). Quantales. In *Current research in operational quantum logic* (pp. 245-262). Dordrecht: Springer. https://doi.org/10.1007/978-94-017-1201-9_10

[22] Rosenthal, K. I., & Niefield, S. B. (1989). Connections between ideal theory and the theory of locales a. *Annals of the new york academy of sciences*, *552*(1), 138-151.

[23] Simmons, H. (1980). Reticulated rings. *Journal of algebra*, *66*(1), 169-192.

[24] Speed, T. P. (1974). Spaces of ideals of distributive lattices II. Minimal prime ideals. *Journal of the australian mathematical society*, *18*(1), 54-72.

[25] Tarizadeh, A. (2019). Flat topology and its dual aspects. *Communications in algebra*, *47*(1), 195-205. https://doi.org/10.1080/00927872.2018.1469637

[26] Tarizadeh, A. (2019). Zariski compactness of minimal spectrum and flat compactness of maximal spectrum. *Journal of algebra and its applications*, *18*(11). https://doi.org/10.1142/S0219498819502025

[27] Tarizadeh, A., & Aghajani, M. (2021). Structural results on harmonic rings and lessened rings. *Beiträge zur algebra und geometrie/contributions to algebra and geometry*, 1-17. https://doi.org/10.1007/s13366-020-00556-x