E-ISSN: 2717-3453 | P-ISSN: 2783-1442

Journal of Fuzzy Extension and Applications

www.journal-fea.com

J. Fuzzy. Ext. Appl. Vol. 2, No. 2 (2021) 120–129

Paper Type: Research Paper

Regular T_0-type Separations in Fuzzy Topological Spaces in the Sense of Quasi-Coincidence

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Citation:

Saravanakumar, G., Tamilselvan, S., & Vadivel, A. (2021). Regular T_0-type separations in fuzzy topological spaces in the sense of quasi-coincidence. *Journal of fuzzy extension and applications, 2*(2), 120-129.

Received: 02/02/2021 Reviewed: 09/04/2021 Revised: 03/05/2021 Accept: 16/05/2021

Abstract

In this paper, we introduce and study three notions of RT_0 property in fuzzy topological spaces using quasi-coincidence sense, and we relate to other such notions. Then, we show that all these notions satisfy good extension property. These concepts also satisfy hereditary, productive and projective properties. We note that all these concepts are preserved under one-one, onto, fuzzy regular open and fuzzy regular continuous mappings. Finally, we discuss initial and final fuzzy topological spaces on our concepts.

Keywords: Quasi-coincidence, regular fuzzy t_0-type separations, fuzzy topological spaces, initial and final fuzzy topology.

1 | Introduction

Chang [5] introduced the notion of fuzzy topology in 1968 by using the concept of fuzzy sets introduced by Zadeh [23] in 1965. Since then, an extensive work on fuzzy topological spaces has been carried out by many researchers like Goguen [6], Wong [20], Lowen [14], Warren [19], Hutton [8] and others. Separation axioms are important parts in fuzzy topological spaces. Many works on separation axioms have been done by researchers. Among those axioms, fuzzy T_0 type is one and it has been already introduced in the literature. There are many articles on fuzzy T_0 topological space which are created by many authors [1], [2], [7], [18], and [22]. The purpose of this paper is to further contribute to the development of fuzzy topological spaces specially on fuzzy regular T_0 topological spaces. In the present paper, fuzzy regular T_0 topological spaces are defined by using quasi-coincidence sense and relation among the given and other such notions

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http://dx.doi.org/10.22105/jfea.2021.275958.1085

are shown here. It is showed that the good extension property is satisfied on our notions. It is also showed that the hereditary, order preserving, productive, and projective properties hold on the new concepts.

In the last section of this paper initial and final fuzzy topological spaces are discussed on author's concept.

2 | Preliminaries

In this present paper, X and Y always denote non empty sets and $= [0,1]$, $I_1 = [0,1)$. The class of all fuzzy sets on a non empty set X is denoted by I^X and fuzzy sets on X are denoted as λ, μ, γ etc. Crisp subsets of X are denoted by capital letters U, V, W etc. throughout this paper.

Definition 1. [23]. A function λ from X into the unit interval I is called a fuzzy set in X. For every $x \in X$, $\lambda(x) \in I$ is called the grade of membership of x in λ. Some authors say that λ is a fuzzy subset of X instead of saying that λ is a fuzzy set in X. The class of all fuzzy sets from X into the closed unit interval I will be denoted by I^X.

Definition 2. [14]. A fuzzy set λ in X is called a fuzzy singleton if and only if $(x) = r$, $0 < r \leq 1$, for a certain $x \in X$ and $\lambda(y) = 0$ for all points y of X except x. The fuzzy singleton is denoted by x_r and x is its support. The class of all fuzzy singletons in X will be denoted by (X) . If $\lambda \in I^X$ and $x_r \in S(X)$, then we say that $x_r \in \lambda$ if and only if $r \leq \lambda(x)$.

Definition 3. [21]. A fuzzy set λ in X is called a fuzzy point if and only if $(x) = r$, $0 < r < 1$, for a certain $x \in X$ and $\lambda(y) = 0$ for all points y of X except x. The fuzzy point is denoted by x_r and x is its support.

Definition 4. [9]. A fuzzy singleton x_r is said to be quasi - coincidence with λ, denoted by $x_r q\lambda$ if and only if $\lambda(x) + r > 1$. If x_r is not quasi - coincidence with λ, we write $x_r \bar{q}\lambda$ and defned as $\lambda(x) + r \leq 1$.

Definition 5. [5]. Let f be a mapping from a set X into a set Y and λ be a fuzzy subset of X. Then f and λ induce a fuzzy subset μ of Y defined by

$$\mu(y) = \begin{cases} \sup\{\lambda(x)\} & \text{if } \in f^{-1}[\{y\}] \neq \phi, x \in X, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 6. [5]. Let f be a mapping from a set X into a set Y and μ be a fuzzy subset of Y. Then the inverse of μ written as f^{-1} (μ) is a fuzzy subset of X defined by $f^{-1}(\mu)(x) = \mu(f(x))$, for $x \in X$.

Definition 7. [5]. Let $I = [0,1]$, X be a non-empty set and I^X be the collection of all mappings from X into I, i.e., the class of all fuzzy sets in X. A fuzzy topology on X is defined as a family τ of members of I^X, satisfying the following conditions:

1) $1, 0 \in \tau$.
2) If $\lambda \in \tau$ for each $i \in \Lambda$, then $\bigcup_{i\in\Lambda} \lambda_i \in \tau$, where Λ is an index set.
3) If $\lambda, \mu \in \tau$ then $\lambda \cap \mu \in \tau$.

The pair (X, τ) is called a fuzzy topological space (in short fts) and members of τ are called τ - open fuzzy sets. A fuzzy set μ is called a τ -closed fuzzy set if $1 - \mu \in \tau$.

Definition 8. [4]. A fuzzy subset λ of a space X is called fuzzy regular open (resp. fuzzy regular closed) if $\lambda = intcl(A)$ (resp. $\lambda = cl(int(\lambda))$.

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The set of all fuzzy regular open sets (resp. fuzzy regular closed sets) of X is denoted by FRO(X) (resp. $FRC(X)$).

Definition 9. [15]. The function: $(X, \tau) \to (Y, \sigma)$ is called a

1) Fuzzy continuous if for every $\mu \in \sigma$, $f^{-1}(\mu) \in \tau$.
2) Fuzzy homeomorphic if f is bijective and both f and f^{-1} are fuzzy continuous.
3) Fuzzy regular continuous if for every $\mu \in \sigma$, $f^{-1}(\mu) \in FRO(X)$.

Definition 10. [13]. The function: $(X, \tau) \to (Y, \sigma)$ is called a

1) Fuzzy open if for every fuzzy open set λ in (X, τ), $f(\lambda)$ is fuzzy open set in (Y, σ).
2) Fuzzy closed if for every closed fuzzy set λ in (X, τ), $f(\lambda)$ is closed fuzzy set in (Y, σ).
3) Fuzzy regular open if for every fuzzy open set λ in (X, τ), $f(\lambda)$ is fuzzy regular open set in (X, σ).
4) Fuzzy regular closed if for every fuzzy closed set λ in (X, τ) $f(\lambda)$ is fuzzy regular closed set in (Y, σ).

Definition 11. [4]. If λ and μ are two fuzzy subsets of X and Y respectively then the Cartesian product $\lambda \times \mu$ of two fuzzy subsets λ and μ is a fuzzy subset of $X \times Y$ defined by $(\lambda \times \mu)(x, y) = min\,(\lambda(x), \mu(y))$, for each pair $(x, y) \in X \times Y$.

Definition 12. [10]. Let $\{X_i, i \in \Lambda\}$, be any class of sets and let X denotes the Cartesian product of these sets, i.e., $X = \Pi_{i\in\Lambda} X_i$. Note that X consists of all points $=< a_i, i \in \Lambda >$, where $a_i \in X_i$. Recall that, for each $j_0 \in \Lambda$, we define the projection π_{j_0} from the product set X to the coordinate space X_{j_0}, i.e. $\pi_{j_0} : X \to X_{j_0}$ by $\pi_{j_0}(< a_i, i \in \Lambda >) = a_{j_0}$. These projections are used to define the product topology.

Definition 13. [20]. Let $\{X_i, i \in \Lambda\}$ be a family of non-empty sets. Let $X = \Pi_{i\in\Lambda} X_i$ be the usual product of X_i s and let π_i be the projection from X into X_i. Further assume that each X_i is an fuzzy topological space with fuzzy topology τ_i. Now the fuzzy topology generated by $\{\pi_i^{-1}(b_i) : b_i \in \tau_i, i \in \Lambda\}$ as a sub basis, is called the product fuzzy topology on X. Clearly if w is a basis element in the product, then there exist i_1, i_2, i_3, ..., $\in \Lambda$ such that $w(x) = min\,\{b_i(x_i) : i = 1,2,3, \dots, n\}$, where $x = (x_i)_{i\in\Lambda} \in X$.

Definition 14. [16]. Let f be a real valued function on a topological space. If $\{x : f(x) > \alpha\}$ is open for every real α, then f is called lower semi continuous function.

Definition 15. [11]. Let X be a non-empty set and T be a topology on X. Let $\tau = \omega(T)$ be the set of all lower semi continuous functions from (X, τ) to I (with usual topology). Thus $\omega(T) = \{u \in I^X : u^{-1}(\alpha, 1] \in T\}$ for each $\alpha \in I_1$. It can be shown that $\omega(T)$ is a fuzzy topology on X.

Let P be the property of a topological space (X, τ) and FP be its fuzzy topological analogue. Then FP is called a 'good extension' of P if and only if the statement (X, τ) has P if and only if $(X, \omega(T))$ has FP holds good for every topological space (X, τ).

Definition 16. [12]. The initial fuzzy topology on a set X for the family of fuzzy topological space $\{(X_i, \tau_i)_{i\in\Lambda}\}$ and the family of functions $\{f_i : X \to (X_i, \tau_i)\}_{i\in\Lambda}$ is the smallest fuzzy topology on X making each f_i fuzzy continuous. It is easily seen that it is generated by the family $\{f_i^{-1}(\lambda_i) : \lambda_i \in \tau_i\}_{i\in\Lambda}$.

Definition 17. [12]. The final fuzzy topology on a set X for the family of fuzzy topological spaces $\{(X_i, \tau_i)_{i\in\Lambda}\}$ and the family of functions $\{f_i : (X_i, \tau_i) \to X\}_{i\in\Lambda}$ is the finest fuzzy topology on X making each f_i fuzzy continuous.

Theorem 1. [3]. A bijective mapping from an fts (X, τ) to an fts (Y, σ) preserves the value of a fuzzy singleton (fuzzy point).

Definition 18. [17]. A fuzzy topological space (X, τ) is called

1) Fuzzy $T_0(i)$ (briefly, $T_0(i)$) if for any pair x_t, $y_{t'} \in S(X)$ with $x \neq y$, there exists $\lambda \in \tau$ such that $x_t q\lambda$, $y_{t'}\bar{q}\lambda$ or there exists $\mu \in \tau$ such that $y_{t'}q\mu$, $x_t\bar{q}\mu$.
2) Fuzzy $T_0(ii)$ (briefly, $FT_0(ii)$) if for any pair x_t, $y_{t'} \in S(X)$ with $x \neq y$, there exists $\lambda \in \tau$ such that $x_t q\lambda$, $y_{t'} \wedge \lambda = 0$ or there exists $\mu \in \tau$ such that $y_{t'}q\mu$, $x_t \wedge \mu = 0$.
3) Fuzzy $T_0(iii)$ (briefly, $FT_0(iii)$) if for any pair $x, y \in X$ with $x \neq y$, there exists $\lambda \in \tau$ such that $\lambda(x) = 1, \lambda(y) = 0$ or there exists $\mu \in \tau$ such that $\mu(y) = 1, \mu(x) = 0$.

Note: Preimage of any fuzzy singleton (fuzzy point) under bijective mapping preserves its value.

3 | Regular Fuzzy T_0-Type Separation Axiom

In this section, we introduce regular fuzzy T_0-type separation axiom and some well-known properties are discussed.

Definition 19. A fuzzy topological space (X, τ) is called

1) Regular fuzzy $T_0(i)$ (briefy, $T_0(i)$) if for any pair x_t, $y_{t'} \in S(X)$ with $x \neq y$, there exists $\lambda \in FRO(X)$ such that $x_t q\lambda$, $y_{t'}\bar{q}\lambda$ or there exists $\mu \in FRO(X)$ such that $y_{t'}q\mu$, $x_t\bar{q}\mu$.
2) Regular fuzzy $T_0(ii)$ (briefy, $RFT_0(ii)$) if for any pair x_t, $y_{t'} \in S(X)$ with $x \neq y$, there exists $\lambda \in FRO(X)$ such that $x_t q\lambda$, $y_{t'} \wedge \lambda = 0$ or there exists $\mu \in FRO(X)$ such that $y_{t'}q\mu$, $x_t \wedge \mu = 0$.
3) Regular fuzzy $T_0(iii)$ (briefy, $RFT_0(iii)$) if for any pair $x, y \in X$ with $x \neq y$, there exists $\lambda \in FRO(X)$ such that $\lambda(x) = 1, \lambda(y) = 0$ or there exists $\mu \in FRO(X)$ such that $\mu(y) = 1, \mu(x) = 0$.

Example 1. Let $X = \{a, b\}$, $\lambda_1 \in FRO(X)$, where $\lambda_1(a) = 0.9$, $\lambda_1(b) = 0$, $\lambda_2(a) = 0$, $\lambda_2(b) = 1$, $\lambda_3(a) = 0.9$, $\lambda_3(b) = 1$. Consider the fuzzy topology τ on X generated by $\{0, \lambda_1, \lambda_2, \lambda_3, 1\}$. Let x_t, $y_{t'}$ be fuzzy points in X with $a \neq b$. Then $\lambda_1(a) + t > 1$ and $\lambda_1(b) + t' \leq 1$ for $0.\ 2 < t \leq 1, 0 < t' \leq 1$. Therefore $x_t q\lambda_1$, $y_{t'}\bar{q}\lambda_1$. This shows that (X, τ) is $T_0(i)$. Also, as $\lambda_1(b) = 0$, $y_{t'} \wedge \lambda_1 = 0$. Thus, (X, τ) is $RFT_0(ii)$.

Theorem 2. For a fuzzy topological space (X, τ) the following implications are true: $RFT_0(ii) \Rightarrow RFT_0(i)$, $RFT_0(iii) \Rightarrow RFT_0(i)$, $RFT_0(iii) \Rightarrow RFT_0(ii)$. But, in general the converse is not true.

Proof. $RFT_0(ii) \Rightarrow RFT_0(i)$: Let (X, τ) be a fuzzy topological space and (X, τ) is $RFT_0(ii)$ we have to prove that (X, τ) is $T_0(i)$. Let x_t, $y_{t'}$ be fuzzy points in X with $x \neq y$. Since (X, τ) is $RFT_0(ii)$ fuzzy topological space, we have, there exist $\lambda \in FRO(X)$ such that $x_t q\lambda$, $y_{t'} \wedge \lambda = 0$ or there exists $\mu \in FRO(X)$ such that $y_{t'}q\mu$, $x_t \wedge \mu = 0$. To prove (X, τ) is $T_0(i)$, it is only needed to prove that $y_{t'}\bar{q}\lambda$.

Now $y_{t'} \wedge \lambda = 0 \Rightarrow \lambda(y) = 0 \Rightarrow \lambda(y) + t' \leq 1 \Rightarrow y_{t'}\bar{q}\lambda$.

It follows that there exists $\lambda \in FRO(X)$ such that $x_t q\lambda$, $y_{t'}\bar{q}\lambda$. Hence (X, τ) is $T_0(i)$.

To Show (X, τ) is $RFT_0(i) \Rightarrow (X, \tau)$ is $T_0(ii)$, we give a counter example.

Counter example (a). Let $X = \{a, b\}$, $\lambda_1 \in$ FRO (X), where $\lambda_1(a) = 0.9$, $\lambda_1(b) = 0.1$, $\lambda_2(a) = 0$, $\lambda_2(b) = 1$, $\lambda_3(a) = 0.9$, $\lambda_3(b) = 1$. Consider the fuzzy topology τ on X generated by $\{0, \lambda_1, \lambda_2, \lambda_3, 1\}$. Let x_t, $y_{t'}$ be fuzzy points in X with $a \neq b$. Then $\lambda_1(a) + t > 1$ and $\lambda_1(b) + t' \leq 1$ for $0.2 < t \leq 1, 0 < t' \leq 1$. Therefore $x_t q\lambda_1$, $y_{t'}\bar{q}\lambda_1$. This shows that (X, τ) is $RFT_0(i)$. But $\lambda_1(b) \neq 0 \Rightarrow y_{t'} \wedge \lambda_1 \neq 0$. Hence (X, τ) is not $RFT_0(ii)$.

$RFT_0(iii) \Rightarrow RFT_0(i)$: Let (X, τ) be a fuzzy topological space and (X, τ) is $RFT_0(iii)$ .We have to prove that (X, τ) is $RFT_0(i)$ .Let x_t, $y_{t'}$ be fuzzy points in X with $x \neq y$. Since (X, τ) is $RFT_0(iii)$ fuzzy

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topological space, we have, there exists $\lambda \in FRO(X)$ such that $\lambda(x) = 1$, $\lambda(y) = 0$ or there exists $\mu \in FRO(X)$ such that $(y) = 1$, $\mu(x) = 0$. To prove (X,τ) is $T_0(i)$, it is needed to prove that $x_t q \lambda$, $y_{t'} \bar{q} \lambda$.

Now, $\lambda(x) = 1 \Rightarrow \lambda(x) + t > 1$, for any $t \in (0,1] \Rightarrow x_t q \lambda$ and $\lambda(y) = 0 \Rightarrow \lambda(y) + t' \leq 1$, for any $t' \in (0,1]$ $\Rightarrow y_{t'} \bar{q} \lambda$.

It follows that there exist $\lambda \in FRO(X)$ such that $x_t q \lambda$, $y_{t'} \bar{q} \lambda$. Hence (X,τ) is $T_0(i)$.

To show (X,τ) is $RFT_0(i) \Rightarrow (X,\tau)$ is $T_0(iii)$, we give a counter example.

Counter example (b). Consider the Counter example (a), $\lambda_1(a) \neq 1$ which implies (X,τ) is not $T_0(iii)$. $RFT_0(iii) \Rightarrow RFT_0(ii)$: Let (X,τ) be a fuzzy topological space and (X,τ) is $T_0(iii)$. We have to prove that (X,τ) is $T_0(ii)$. Let x_t, $y_{t'}$ be fuzzy points in X with $x \neq y$. Since (X,τ) is $RFT_0(iii)$ fuzzy topological space, we have, there exist $\lambda \in FRO(X)$ such that $\lambda(x) = 1$, $\lambda(y) = 0$ or there exists $\mu \in FRO(X)$ such that $(y) = 1$, $\mu(x) = 0$. To prove (X,τ) is $T_0(ii)$, it is needed to prove that $x_t q \lambda$, $y_{t'} \wedge \lambda = 0$.

Now, $\lambda(x) = 1 \Rightarrow \lambda(x) + t > 1$, for any $t \in (0,1] \Rightarrow x_t q \lambda$ and $\lambda(y) = 0 \Rightarrow y_{t'} \wedge \lambda = 0$.

It follows that there exist $\lambda \in FRO(X)$ such that $x_t q \lambda$, $y_{t'} \wedge \lambda = 0$. Hence (X,τ) is $T_0(ii)$.

To show (X,τ) is $RFT_0(ii) \Rightarrow (X,\tau)$ is $T_0(iii)$, we give a counter example.

Counter example (c). In *Example 1*, $\lambda_1(a) \neq 1$, $\lambda_1(b) = 0$, therefore (X,τ) is not $RFT_0(iii)$.

From the above definitions and examples, it is clear that the following implications is true but the converse of the implications is not true as shown by the following example.

$$\begin{tikzcd}
RFT_0(iii) \arrow[r] \arrow[d] & RFT_0(ii) \arrow[r] \arrow[d] & R\,FT_0(i) \arrow[d] \\
FT_0(iii) \arrow[r] & FT_0(ii) \arrow[r] & FT_0(i)
\end{tikzcd}$$

Example 2. Let $= \{a, b\}$, $\lambda \in \tau$, where $\lambda(a) = 1$, $\lambda(b) = 0$. Consider the fuzzy topology τ on X generated by $\{0, \lambda, 1\}$. Let x_t, $y_{t'}$ be fuzzy points in X with $a \neq b$. Then $\lambda(a) + t > 1$ and $\lambda(b) + t' \leq 1$ for , $t' \in (0,1]$. Therefore $x_t q \lambda$, $y_{t'} \bar{q} \lambda$. (i) This shows that (X,τ) is $FT_0(i)$ but it is not $RFT_0(i)$ since λ is not regular open. (ii) as $\lambda(b) = 0$, $y_{t'} \wedge \lambda = 0$ is $FT_0(ii)$ but it is not $RFT_0(ii)$. Also (iii) as $\lambda(a) = 1$, $\lambda(b) = 0$ is $FT_0(iii)$ but it is not $T_0(iii)$.

Now, we shall show that our notions satisfy the good extension property.

Theorem 3. Let (X, T) be a topological space. Consider the following statements:

1) (X, T) be a RT_0-topological space.
2) $(X, \omega(T))$ be an $RFT_0(i)$ space.
3) $(X, \omega(T))$ be an $RFT_0(ii)$ space.

Then the following implications are true. (1) ⇔ (2) and (1) ⇔ (3) .

Proof.

(1) ⇒ (2). Let (X,T) be a topological space and (X,T) is RT_0. We have to prove that $(X,\omega(T))$ is $T_0(i)$. Let $x_t, y_{t'}$ be fuzzy points in X with $x \neq y$. Since (X,T) is RT_0 topological space, we have, there exists $U \in T$ such that $\in U,\ y \notin U$. From the definition of lower semi continuous we have $1_U \in \omega(T)$ and $1_U(x) = 1, 1_U(y) = 0$. Then $1_U(x) + t > 1 \Rightarrow x_t q 1_U$ and $1_U(y) + t' \leq 1 \Rightarrow y_{t'} \bar{q} 1_U$. It follows that there exists $1_U \in \omega(T)$ such that $x_t q 1_U, y_{t'} \bar{q} 1_U$. Hence $(X,\omega(T))$ is $T_0(i)$. Thus (1) ⇒ (2) holds.

(2) ⇒ (1). Let $(X,\omega(T))$ be a fuzzy topological space and $(X,\omega(T))$ is $T_0(i)$. We have to prove that (X,T) is RT_0. Let x, y be points in X with $x \neq y$. Since $(X,\omega(T))$ is $RFT_0(i)$ topological space, we have, for any fuzzy points $x_t, y_{t'}$ in X, there exists $\lambda \in FRO(X)$ such that $x_t q\lambda, y_{t'} \bar{q}\lambda$ or there exist $\mu \in FRO(X)$ such that $y_{t'} q\mu, x_t \bar{q}\mu$.

Now, $x_t q\lambda \Rightarrow \lambda(x) + t > 1 \Rightarrow \lambda(x) > 1 - t = \alpha \Rightarrow x \in \lambda^{-1}(\alpha, 1]$ and $y_{t'} \bar{q}\lambda \Rightarrow \lambda(y) + t \leq 1 \Rightarrow \lambda(y) \leq 1 - t = \alpha \Rightarrow \lambda(y) \leq \alpha \Rightarrow y \notin \lambda^{-1}(\alpha, 1]$.

Also, $\lambda^{-1}(\alpha, 1] \in T$. It follows that $\exists \lambda^{-1}(\alpha, 1] \in T$ such that $x \in \lambda^{-1}(\alpha, 1], y \notin \lambda^{-1}(\alpha, 1]$. Thus (2) ⇒ (1) holds. Similarly, we can prove that (1) ⇔ (3).

Now, we shall show that the hereditary property is satisfed on our notions.

Theorem 4. Let (X,τ) be a fuzzy topological space, $B \subseteq X$, $\tau_B = \{\lambda/B : \lambda \in \tau\}$, then

1) (X,τ) is $RFT_0(i) \Rightarrow (B,\tau_B)$ is $RFT_0(i)$.
2) (X,τ) is $RFT_0(ii) \Rightarrow (B,\tau_B)$ is $RFT_0(ii)$.

Proof (1). Let (X,τ) be a fuzzy topological space and (X,τ) is $T_0(i)$. We have to prove that (B,τ_B) is $T_0(i)$. Let $x_t, y_{t'}$ be fuzzy points in B with $x \neq y$. Since $B \subseteq X$, these fuzzy points are also fuzzy points in X. Also, since (X,τ) is $RFT_0(i)$ fuzzy topological space, we have, there exists $\lambda \in FRO(X)$ such that $x_t q\lambda$, $y_{t'} \bar{q}\lambda$ or there exists $\mu \in FRO(X)$ such that $y_{t'} q\mu, x_t \bar{q}\mu$. For $B \subseteq X$, we have $\lambda/B \in FRO(X)$.

Now, $x_t q\lambda \Rightarrow \lambda(x) + t > 1,\ x \in X \Rightarrow \lambda/B(x) + t > 1,\ x \in B \subseteq X \Rightarrow x_t q\lambda/B$ and $y_{t'} \bar{q}\lambda \Rightarrow \lambda(y) + t' \leq 1$, $y \in X \Rightarrow \lambda/B(y) + t' \leq q, y \in B \subseteq X \Rightarrow y_{t'} \bar{q}\lambda/B$.

Hence, (B,τ_B) is $T_0(i)$. Proof of (2) is similar to proof of (1).

Then, we discuss the productive and projective properties on our concepts.

Theorem 5. Let (X_i,τ_i), $i \in \Lambda$ be fuzzy topological spaces and $X = \Pi_{i\in\Lambda} X_i$ and τ be the product topology on X, then

3) for all $i \in \Lambda$, (X_i,τ_i) is $RFT_0(i)$ if and only if (X,τ) is $RFT_0(i)$.
4) for all $i \in \Lambda$, (X_i,τ_i) is $RFT_0(ii)$ if and only if (X,τ) is $RFT_0(ii)$.

Proof 2. Let for all $i \in \Lambda$, (X_i,τ_i) is $RFT_0(ii)$ space. We have to prove that (X,τ) is $T_0(ii)$. Let $x_t, y_{t'}$ be fuzzy points in X with $x \neq y$. Then $(x_i)_t, (y_i)_{t'}$ are fuzzy points with $x_i \neq y_i$ for some $i \in \Lambda$. Since (X_i,τ_i) is $T_0(ii)$, there exists $\lambda_i \in FRO(X_i)$ such that $(x_i)_t q\lambda_i, (y_i)_{t'} \wedge \lambda_i = 0$ or there exists $\mu_i \in FRO(X_i)$ such that $(y_i)_{t'} q\mu_i, (x_i)_t \wedge \mu_i = 0$. But we have $\pi_i(x) = x_i$ and $\pi_i(y) = y_i$.

Now, $(x_i)_t q\lambda_i \Rightarrow \lambda_i(x_i) + t > 1,\ x \in X \Rightarrow \lambda_i(\pi_i(x)) + t > 1 \Rightarrow (\lambda_i \circ \pi_i)(x) + t > 1 \Rightarrow x_t q(u_i \circ \pi_i)$ and $(y_i)_{t'} \wedge \lambda_i = 0 \Rightarrow \lambda_i(y_i) = 0, y \in X \Rightarrow \lambda_i(\pi_i(y)) = 0, y \in X \Rightarrow (\lambda_i \circ \pi_i)(y) = 0 \Rightarrow y_{t'} \wedge (\lambda_i \circ \pi_i) = 0$.

It follows that there exists $(\lambda_i \circ \pi_i) \in t_i$ such that $x_t q(\lambda_i \circ \pi_i), y_{t'} \wedge (\lambda_i \circ \pi_i) = 0$. Hence, (X,τ) is $T_0(ii)$.

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Conversely, let (X,τ) be a fuzzy topological space and (X,τ) is $T_0(ii)$. We have to prove that (X_i,τ_i), $i \in \Lambda$ is $T_0(ii)$. Let a_i be a fxed element in X_i. Let $A_i = \{x \in X = \Pi_{i\in\Lambda} X_i : x_j = a_j$ for some $i \neq j\}$. Then A_i is a subset of X, and hence (A_i, τ_{A_i}) is a subspace of (X,τ) . Since (X,τ) is $T_0(ii)$, so (A_i,τ_{A_i}) is $RFT_0(ii)$. Now we have A_i is homeomorphic image of X_i. Hence it is clear that for all $i \in \Lambda$, (X_i,τ_i) is $RFT_0(ii)$ space. Thus (2) holds. Proof of (1) is similar to proof of (2).

Now, we shall show that our notions satisfy the order preserving property.

Theorem 6. Let (X,τ) and (Y,σ) be two fuzzy topological spaces and: $X \to Y$ be a one - one, onto and regular open map then,

1) (X,τ) is $RFT_0(i) \Rightarrow (Y,\sigma)$ is $RFT_0(i)$.
2) (X,τ) is $RFT_0(ii) \Rightarrow (Y,\sigma)$ is $RFT_0(ii)$.

Proof 1. Let (X,τ) be a fuzzy topological space, and (X,τ) is $T_0(i)$. We have to prove that (Y,σ) is $T_0(i)$. Let x_t, y_t, be fuzzy points in Y with $x' \neq y'$. Since f is onto then there exist x, $y \in X$ with $f(x) = x'$, $f(y) = y'$ and $x_t, y_{t'}$ are fuzzy points in X with $x \neq y$ as f is one - one. Again since (X,τ) is $RFT_0(i)$ space, there exists $\lambda \in FRO(X)$ such that $x_t q\lambda$, $y_{t'}\bar{q}\lambda$ or there exists $\mu \in FRO(X)$ such that $y_{t'}q\mu$, $x_t\bar{q}\mu$.

Now, $x_t q\lambda \Rightarrow \lambda(x) + t > 1$ and $y_{t'}\bar{q}\lambda \Rightarrow \lambda(y) + t' \leq 1$.

Now, $(\lambda)(x') = \{ sup\ \lambda(x) : f(x) = x'\} \Rightarrow f(\lambda)(x') = \lambda(x)$, for some x and $(\lambda)(y') = \{ sup\ \lambda(y) : f(y) = y'\} \Rightarrow f(\lambda)(y') = \lambda(y)$, for some y.

Also, since f is regular open map then $f(\lambda) \in FRO(Y)$ as $\lambda \in \tau$.

Again, $\lambda(x) + t > 1 \Rightarrow f(\lambda)(x') + t > 1 \Rightarrow x'_t qf(\lambda)$ and, $\lambda(y) + t' \leq 1 \Rightarrow f(\lambda)(y') + t' \leq 1 \Rightarrow y'_{t'}\bar{q}f(\lambda)$.

It follows that there exists $f(\lambda) \in FRO(Y)$ such that $x'_t qf(\lambda)$, $y'_{t'}\bar{q}f(\lambda)'$. Hence, it is clear that (Y,σ) is $RFT_0(i)$ space. Proof of (2) is similar to proof of (1).

Theorem 7. Let (X,τ) and (Y,σ) be two fuzzy topological spaces and: $X \to Y$ be a one - one, onto and regular continuous map then,

1) (Y,σ) is $RFT_0(i) \Rightarrow (X,\tau)$ is $RFT_0(i)$.
2) (Y,σ) is $RFT_0(ii) \Rightarrow (X,\tau)$ is $RFT_0(ii)$.

Proof 2. Let (Y,σ) be a fuzzy topological space and (Y,σ) is $T_0(ii)$. We have to prove that (X,τ) is $T_0(ii)$. Let x_t, $y_{t'}$ be fuzzy points in X with $x \neq y$. Then $(f(x))_t$, $(f(y))_{t'}$ are fuzzy points in Y with $f(x) \neq f(y)$ as f is one - one. Again, since (Y,σ) is $RFT_0(ii)$ space, there exists $\lambda \in FRO(Y)$ such that $(f(x))_t q\lambda$, $(f(y))_{t'} \wedge \lambda = 0$ or there exist $\mu \in FRO(Y)$ such that $(f(y))_{t'}q\mu$, $(f(x))_t \wedge \mu = 0$.

Now, $(f(x))_t q\lambda \Rightarrow \lambda(f(x)) + t > 1 \Rightarrow f^{-1}(\lambda(x)) + t > 1 \quad \Rightarrow (f^{-1}(\lambda))(x) + t > 1 \Rightarrow x_t qf^{-1}(\lambda)$ and, $(f(y))_{t'} \wedge \lambda = 0 \Rightarrow \lambda(f(y)) = 0 \Rightarrow f^{-1}(\lambda(y)) = 0 \Rightarrow (f^{-1}(\lambda))(y) = 0 \Rightarrow y_{t'} \wedge f^{-1}(\lambda) = 0$.

Now, since f is regular continuous map and $\lambda \in \sigma$ then $f^{-1}(\lambda) \in FRO(X)$. It follows that there exist $f^{-1}(\lambda) \in FRO(X)$ such that $x_t qf^{-1}(\lambda)$, $y_{t'} \wedge f^{-1}(\lambda) = 0$. Hence it is clear that (X,τ) is $RFT_0(ii)$ space. Proof of (1) is similar to proof of (2).

Theorem 8. If $\{(X_i,\tau_i)\}_{i\in\Lambda}$ is a family of $RFT_0(ii)$ fts and $\{f_i : X \to (X_i,\tau_i)\}_{i\in\Lambda}$, a family of one - one and fuzzy regular continuous functions, then the initial fuzzy topology on X for the family $\{f_i\}_{i\in\Lambda}$ is $RFT_0(ii)$.

Proof. Let τ be the initial fuzzy topology on X for the family $\{f_i\}_{i\in\Lambda}$. Let x_t, $y_{t'}$ be fuzzy points in X with $x \neq y$. Then $f_i(x)$, $f_i(y) \in X_i$ and $f_i(x) \neq f_i(y)$ as f_i is one - one. Since (X_i, τ_i) is $T_0(ii)$, then for every two distinct fuzzy points $(f_i(x))_t$, $(f_i(y))_{t'}$ in X_i, there exists regular fuzzy sets λ_i or $\mu_i \in FRO(X_i)$ such that $(f_i(x))_t q\lambda_i$, $(f_i(y))_{t'} \wedge \lambda_i = 0$ or $(f_i(y))_{t'} q\mu_i$, $(f_i(x))_t \wedge \mu_i = 0$.

Now, $(f_i(x))_t q\lambda_i$ and $(f_i(y))_{t'} \wedge \lambda_i = 0$. That is $\lambda_i(f_i(x)) + t > 1$ and $\lambda_i(f_i(y)) = 0$. That is $f_i^{-1}(\lambda_i)(x) + t > 1$ and $f_i^{-1}(\lambda_i)(y) = 0$.

This is true for every $i \in \Lambda$. So, $\inf f_i^{-1}(\lambda_i)(x) + t > 1$ and $\inf f_i^{-1}(\lambda_i)(y) = 0$. Let $= \inf f_i^{-1}(\lambda_i)$. Then $\lambda \in FRO(X)$ as f_i is fuzzy regular continuous. So, $\lambda(x) + t > 1$ and $\lambda(y) = 0$. Hence, $x_t q\lambda$ and $y_{t'} \wedge \lambda = 0$. Therefore, (X, τ) is must $T_0(ii)$. Thus, the proof is complete.

Theorem 9. If $\{(X_i, \tau_i)\}_{i\in\Lambda}$ is a family of $RFT_0(ii)$ fts and $\{f_i(X_i, \tau_i) \to X\}_{i\in\Lambda}$, a family of fuzzy regular open and bijective function, then the fnal fuzzy topology on X for the family $\{f_i\}_{i\in\Lambda}$ is $RFT_0(ii)$.

Proof. Let τ be the fnal fuzzy topology on X for the family $\{f_i\}_{i\in\Lambda}$. Let x_t, $y_{t'}$ be fuzzy points in X with $x \neq y$. Then $f_i^{-1}(x)$, $f_i^{-1}(y) \in X_i$ and $f_i^{-1}(x) \neq f_i^{-1}(y)$ as f_i is bijective. Since (X_i, τ_i) is $T_0(ii)$, then for every two distinct fuzzy points $(f_i^{-1}(x))_t$, $(f_i^{-1}(y))_{t'}$ in X_i, there exists regular fuzzy sets λ_i or $\mu_i \in FRO(X_i)$ such that $(f_i^{-1}(x))_t q\lambda_i$, $(f_i^{-1}(y))_{t'} \wedge \lambda_i = 0$ or $(f_i^{-1}(y))_{t'} q\mu_i$, $(f_i^{-1}(x))_t \wedge \mu_i = 0$.

Now, $(f_i^{-1}(x))_t q\lambda_i$ and $(f_i^{-1}(y))_{t'} \wedge \lambda_i = 0$. That is $\lambda_i(f_i^{-1}(x)) + t > 1$ and $\lambda_i\left(f_i^{-1}(y)\right) = 0$. That is $f_i(\lambda_i)(x) + t > 1$ and $f_i(\lambda_i)(y) = 0$.

This is true for every $i \in \Lambda$. So, $\inf f_i(\lambda_i)(x) + t > 1$ and $\inf f_i(\lambda_i)(y) = 0$. Let $= \inf f_i(\lambda_i)$. Then $\lambda \in FRO$ (X) as f_i is fuzzy regular open. So, $\lambda(x) + t > 1$ and $\lambda(y) = 0$. Hence, $x_t q\lambda$ and $y_{t'} \wedge \lambda = 0$. Therefore, (X, τ) is must $T_0(ii)$. Thus, the proof is complete.

4 | Conclusion

In this paper, we introduce and study notion of RT_0 separation axiom in fts in quasi-coincidence sense. We have shown that all of our concepts are good extension of their counterparts and are stronger than other such notion. Further, we have shown that hereditary, productive, projective and other preserving properties hold on our concepts. Finally, initial and final fuzzy topologies are studied on one of our notions. We hope that the results of this paper will aid researchers in developing a general structure for fuzzy mathematics expansion.

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