E-ISSN: 2717-3453 | P-ISSN: 2783-1442

Journal of Fuzzy Extension and Applications

www.journal-fea.com

J. Fuzzy. Ext. Appl. Vol. 2, No. 3 (2021) 191–203.

Paper Type: Research Paper

A Novel Mathematical Approach for Fuzzy Multi-Period Multi-Objective Portfolio Optimization Problem under Uncertain Environment and Practical Constraints

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Citation:

Peykani, P., Nouri, M., Eshghi, F., Khamechian, M., & Farrokhi-Asl, M. (2021). A novel mathematical approach for fuzzy multi-period multi-objective portfolio optimization problem under uncertain environment and practical constraints. *Journal of fuzzy extension and application, 2* (3), 191-203.

Received: 22/05/2021 Reviewed: 30/06/2021 Revised: 31/08/2021 Accept: 03/09/2021

Abstract

Investment Portfolio Optimization (IPO) is one of the most important problems in the financial area. Also, one of the most important features of financial markets is their embedded uncertainty. Thus, it is essential to propose a novel IPO model that can be employed in the presence of uncertain data. Accordingly, the main goal of this paper is to propose a novel Fuzzy Multi-Period Multi-Objective Portfolio Optimization (FMPMOPO) model that is capable to be used under data ambiguity and practical constraints including budget constraint, cardinality constraint, and bound constraint. It should be noted that three objectives including terminal wealth, risk, and liquidity as well as practical constraints are considered in proposed FMPMOPO model. Also, the alpha-cut method is employed to deal with fuzzy data. Finally, the proposed Fuzzy Multi-Period Wealth-Risk-Liquidity (FMPWRL) model is implemented in real-world case study from Tehran Stock Exchange (TSE). The experimental results show the applicability and efficacy of the proposed FMPWRL model for fuzzy multi-period multi-objective investment portfolio optimization problem under fuzzy environment.

Keywords: Portfolio optimization, Multi-Period problem, Fuzzy optimization, Alpha-Cut method, Data ambiguity, Investment problem, Stock market.

1 | Introduction

Portfolio optimization problem is selection of the best combination of financial assets to achieve the maximum return and minimum risk for portfolio as much as possible. So far, many models, approaches, and algorithms have been proposed by different researchers to achieve the optimal portfolio [1]-[6]. The most important and influential researches in this field have been done by Markowitz [7] and Sharp [8]. Please note that one of the most important points that should be considered in the proposed approach for portfolio optimization problem is the uncertainty of data in financial markets [9]-[13].

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http://dx.doi.org/10.22105/jfea.2021.287429.1150

Because, in real cases, financial data such as rate of return and liquidity can be tainted by uncertainty. Thus, it is essential to propose a new portfolio optimization model that can be applied under ambiguity and uncertainty. Fuzzy Mathematical Programming (FMP) is one of the popular and effective approaches to deal with data uncertainty and linguistic variables [14]-[20]. Sadjadi et al. [21] proposed Fuzzy Multi-Period Portfolio Selection (FMPPS) model with different rates for borrowing and lending.

Liu et al. [22] introduced FMPPS approach by considering return, transaction cost, risk and skewness of portfolio. Zhang et al. [23] presented possibilistic mean-semi variance-entropy model for FMPPS problem, and designed hybrid intelligent algorithm for solving the presented model. Zhang et al. [24] applied Particle Swarm Optimization (PSO) algorithm for solving fuzzy multi-period portfolio optimization model under possibility measures. Zhang and Zhang [25] proposed fuzzy multi-period mean-absolute deviation (MAD) for portfolio selection problem by considering risk control and cardinality constraints. Zhang et al. [26] presented a new FMP approach for multi-period portfolio optimization with return demand and risk control. Mehlawat [27] proposed credibilistic mean-entropy models for FMPPS with multi-choice aspiration levels by considering wealth, risk, transaction cost, liquidity, and cardinality constraint. Wang et al. [28] introduced a new uncertain multi-period portfolio selection model with dynamic risk/expected-return level that is capable to be used in the presence of fuzzy random uncertainty.

Liu and Zhang [29] proposed fuzzy multi-period portfolio selection optimization model based on possibility theory and applied Genetic Algorithm (GA) to solve the proposed model. Liagkouras and Metaxiotis [30] discussed the multi-period mean–variance portfolio optimization problem with transaction costs and fuzzy variables to count for the uncertainty of future returns and liquidities on assets. Cao [31] employed PSO algorithm for solving multi-objective portfolio optimization problem under fuzzy environment, in which the return rates and the turnover rates are characterized by fuzzy variables. Liu & Zhang [32] examined possibilistic moment models for FMPPS with fuzzy returns by taking into account some realistic constraints including budget constraint, higher moments, cardinality constraint, round-lot constraint, and bound constraint. Gupta et al. [33] proposed intuitionistic fuzzy optimistic and pessimistic multi-period portfolio optimization models. Last but not the least, Gupta et al. [34] used coherent fuzzy numbers to model the returns and the investor attitude in credibilistic multi-period multi-objective portfolio optimization problem.

The goal of this paper is to propose a new Fuzzy Multi-Period Multi-Objective Portfolio Optimization (FMPMOPO) model by considering three objectives including wealth, risk, and liquidity. Notably, to reach this goal, the alpha-cut technique and goal programming approach are applied. Additionally, to show the applicability and efficacy of proposed FMPMOPO model, a real-world case study from Tehran stock market is utilized. The rest of this paper is organized as follows. The research background of the paper will be explained in Section 2. The mathematical modeling of fuzzy multi-period multi-objective portfolio optimization approach will be proposed in Section 3. Then, the implementation of the proposed FMPMOPO model in Tehran Stock Exchange (TSE) will be discussed in Section 4. Finally, conclusions as well as some directions for future research will be introduced in Section 5.

2 | Research Background

In this section, the research background of the paper to propose FMPMOPO model including alpha-cut method and goal programming technique as well as required equations, formulations, and explanations will be introduced.

2.1 | Alpha-Cut Method

An alpha-cut operation is one of the important solution methods that widely applied in literature to solve Fuzzy Linear Programming (FLP) problem [35]. For more details, let $\tilde{\beta}$ be a triangular fuzzy number that is determined by $\tilde{\beta}=(\beta_1,\beta_2,\beta_3), \beta_1 \le \beta_2 \le \beta_3$. The membership function of $\tilde{\beta}$ is defined as *Eq. (1)*:

$$\mu_{\tilde{\beta}}(x)=\begin{cases} 0, & \text{if } x \le \beta_1; \\ \dfrac{x-\beta_1}{\beta_2-\beta_1}, & \text{if } \beta_1 \le x \le \beta_2; \\ \dfrac{\beta_3-x}{\beta_3-\beta_2}, & \text{if } \beta_2 \le x \le \beta_3; \\ 0, & \text{if } x \ge \beta_3. \end{cases} \tag{1}$$

The alpha-cut of the fuzzy number $\tilde{\beta}$ is defined as $\beta_\alpha=\{x \in R / \mu_{\tilde{\beta}}(x) \ge \alpha\}$ where α is the confidence level [36]. Accordingly, the alpha-cut of $\tilde{\beta}$ is actually a close interval of the real number field as follows:

$$\beta_\alpha=[\beta_L(\alpha),\beta_R(\alpha)]=[(\beta_2-\beta_1)\alpha+\beta_1\ ,\ \beta_3-(\beta_3-\beta_2)\alpha]. \quad \alpha \in [0,1] \tag{2}$$

The graphical presentation of alpha-cut method for triangular fuzzy number is shown in *Fig. 1*:

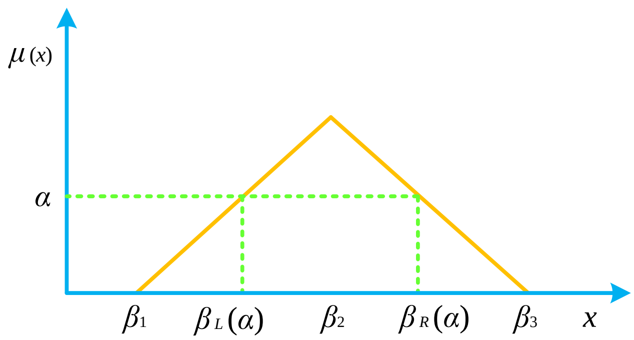

Fig. 1. Alpha-Cut of triangular fuzzy number.

Notably, by applying the alpha-cut method, FLP can be transformed to Interval-Parameter Linear Programming (IPLP). Finally, the resulting IPLP problem can be solved as a crisp linear programming (CLP) model for a given α with some variable substitutions.

2.2 | Goal Programming Technique

So far, several approaches and algorithms have been proposed to solve Multiple-Objectives Decision Making (MODM) problems in which some objectives are conflicting and non-commensurable. Goal Programming (GP) is one of the popular, powerful, and effective solution methods for MODM problems. The major variants of GP in terms of underlying distance metric are lexicographic, weighted, and Chebyshev goal programming. Also, GP in terms of the mathematical nature of the decision variables and/or goals can be categorized into fuzzy, integer, binary, and fractional goal programming [37]-[39]. Please consider following multiple objective linear programming (MOLP) problem that c, a, and b, are the objective function coefficient, the technological coefficient, and the right-hand-side, respectively:

$$\text{Max} \quad \sum_{j=1}^{J} c_{j1} x_j \tag{3}$$

$$\vdots$$

$$\text{Max} \quad \sum_{j=1}^{J} c_{jk} x_j$$

$$\vdots$$

$$\text{Max} \quad \sum_{j=1}^{J} c_{jK} x_j$$

$$\text{S.t.} \quad \sum_{j=1}^{J} a_{ij} x_j \le b_i, \quad \forall i$$

$$x_j \ge 0, \quad \forall j.$$

Assume that a set of K goals $\{\Theta_1, \dots, \Theta_k, \dots, \Theta_K\}$ is specified by the Decision Maker (DM) for objective functions. Goal programming try to achieve an optimal solution "as close as possible" to the set of specified goals which may not be simultaneously attainable. The equivalent weighted GP mathematical formulation to the above MOLP is written as follows:

$$\text{Min} \quad \sum_{k=1}^{K} \left(\lambda_k^- \delta_k^- + \lambda_k^+ \delta_k^+ \right) \tag{4}$$

$$\text{S.t.} \quad \sum_{j=1}^{J} c_{jk} x_j + \delta_k^- - \delta_k^+ = \Theta_k, \quad \forall k$$

$$\sum_{j=1}^{J} a_{ij} x_j \le b_i, \quad \forall i$$

$$x_j, \delta_k^-, \delta_k^+ \ge 0. \quad \forall j, k$$

It should be explained that non-negative variables δ_k^- and δ_k^+ are deviational variables of goal k. Also, λ_k^- and λ_k^+ are weights assigned to the deviational variables of goal k that determined by the DM. Notably, the weighted GP mathematical formulation can be extended to handle the objectives (goals) at different priority levels and classes.

3 | The Proposed FMPMOPO Model

In this section, the fuzzy multi-period multi-objective portfolio optimization model will be introduced. It should be noted that three objectives including wealth, risk, and liquidity as well as practical constraints are considered in FMPMOPO model. The indices, parameters, and decision variables that will be used in this study are described as follows:

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i : the indices of risky assets $i = 1,...,I$.

t : the indices of investment periods $t = 1,...,T$.

$\tilde{\gamma}_{it}$: the return of i th risky asset in t th investment period (triangular fuzzy number).

$\tilde{\eta}_{it}$: The liquidity of i th risky asset in t th investment period (triangular fuzzy number).

l_{it} : The lower bound of budget allocation for i th risky asset in t th investment period.

u_{it} : The upper bound of budget allocation for i th risky asset in t th investment period.

φ_{it} : The transaction cost rate of i th risky asset in t th investment period.

Φ_t : The expected return of portfolio x_t in t th investment period.

Υ_t : The total transaction cost of portfolio x_t in t th investment period.

Γ_t : The return of portfolio x_t in t th investment period

Ω_t : The maximum number of risky assets of portfolio x_t in t th investment period.

Ψ_t : The expected value of wealth at the beginning of investment period t.

Δ_t : the absolute deviation of portfolio in t th investment period.

ω_{it} : The weight of i th risky asset in portfolio in t th investment period.

ξ_{it} : A binary variable which will be one if i th risky asset is selected in t th investment period and zero otherwise.

Assume that the investor acquires his initial wealth at the beginning of the first period and the terminal wealth at the end of period T. The investor can reinvest the wealth among these risky assets at the beginning of each $T-1$ sub-periods. The investor does not invest any additional wealth in the entire investment horizon. Also, the return and the liquidity of risky assets have a triangular fuzzy distribution $\tilde{\gamma}(\gamma^1, \gamma^2, \gamma^3)$ and $\tilde{\eta}(\eta^1, \eta^2, \eta^3)$ in which $\gamma^1 \le \gamma^2 \le \gamma^3$ and $\eta^1 \le \eta^2 \le \eta^3$.

In the following, the objective functions and constraints of FMPMOPO model will be described. Notably, the terminal wealth is the investor's wealth in the last period of his investment. To find the terminal wealth, it is necessary to obtain the general relation of the wealth gained in each period. The wealth earned in each period consists of two components: the expected return of portfolio, and the transaction cost. The expected return of portfolio is calculated as follows:

$$\Phi_t = \sum_{i=1}^{I} \omega_{it} \tilde{\gamma}_{it}. \quad \forall t \tag{5}$$

In order to calculate the transaction cost, the V-shape function is used, which is the difference between two consecutive portfolios:

$$\Upsilon_t = \sum_{i=1}^{I} \varphi_{it} \left| \omega_{it} - \omega_{it-1} \right|. \quad \forall t \tag{6}$$

As a result, the return of portfolio in t th investment period is defined as follows:

$$\Gamma_t = \Phi_t - \Upsilon_t. \quad \forall t \tag{7}$$

Then, the wealth of investor in t^{th} investment period is calculated as follows:

$$\Psi_t = \Psi_{t-1}\left(1+\Gamma_t\right). \quad \forall t \tag{8}$$

By replacing *Eqs. (5) - (7)* in *Eq. (8)*, the wealth of investor in each investment period is described as follows:

$$\Psi_t = \Psi_{t-1}\left(1+\sum_{i=1}^{I}\omega_{it}\tilde{\gamma}_{it} - \sum_{i=1}^{I}\varphi_{it}\left|\omega_{it}-\omega_{it-1}\right|\right). \quad \forall t \tag{9}$$

Accordingly, the fuzzy multi-objective wealth-risk-liquidity model for multi-period portfolio optimization problem under fuzzy environment is proposed as *Model (10)*:

$$\text{Max} \quad \Psi_T, \tag{10}$$

$$\text{Min} \quad \frac{1}{T}\sum_{t=1}^{T}\Delta_t,$$

$$\text{Max} \quad \frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{I}\omega_{it}\tilde{\eta}_{it},$$

$$\text{S.t.} \quad \Delta_t \geq \left(\frac{1}{T}\sum_{t=1}^{T}\Psi_t\right) - \Psi_t, \quad \forall t$$

$$\Delta_t \geq \Psi_t - \left(\frac{1}{T}\sum_{t=1}^{T}\Psi_t\right), \quad \forall t$$

$$\Psi_t = \Psi_{t-1}\left(1+\sum_{i=1}^{I}\omega_{it}\tilde{\gamma}_{it} - \sum_{i=1}^{I}\varphi_{it}\left|\omega_{it}-\omega_{it-1}\right|\right), \quad \forall t$$

$$\sum_{i=1}^{I}\omega_{it} = 1, \quad \forall t$$

$$\sum_{i=1}^{I}\xi_{it} = \Omega_t, \quad \forall t$$

$$\omega_{it} \leq \xi_{it}u_{it}, \quad \forall i, t$$

$$\omega_{it} \geq \xi_{it}l_{it}, \quad \forall i, t$$

$$\xi_{it} \in \{0,1\}, \quad \forall i, t$$

$$\omega_{it} \geq 0. \quad \forall i, t$$

It should be explained that some practical constraints including budget constraint, cardinality constraint, and bound constraint are taking into account in proposed FMPMOPO model. Finally, the FMPMOPO model is transformed to single objective model by applying goal programming technique. Also, alpha-cut method is employed to deal with fuzzy data of return and liquidity of risky assets.

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$$\text{Min} \ \left(\lambda_1^- \delta_1^- + \lambda_2^- \delta_2^- + \lambda_3^- \delta_3^-\right) \tag{11}$$

$$\text{s.t.} \ \ \Psi_T + \delta_1^- = \Theta_1$$

$$\frac{1}{T}\sum_{t=1}^{T}\Delta_t + \delta_2^- = \Theta_2,$$

$$\frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{I}\omega_{it}\left[(\eta_{it}^2 - \eta_{it}^1)\alpha + \eta_{it}^1, \ \eta_{it}^3 - (\eta_{it}^3 - \eta_{it}^2)\alpha\right] + \delta_3^- = \Theta_3,$$

$$\Delta_t \geq \left(\frac{1}{T}\sum_{t=1}^{T}\Psi_t\right) - \Psi_t, \ \ \forall t$$

$$\Delta_t \geq \Psi_t - \left(\frac{1}{T}\sum_{t=1}^{T}\Psi_t\right), \ \ \forall t$$

$$\Psi_t = \Psi_{t-1}\left(1 + \sum_{i=1}^{I}\omega_{it}\left[(\gamma_{it}^2 - \gamma_{it}^1)\alpha + \gamma_{it}^1, \ \gamma_{it}^3 - (\gamma_{it}^3 - \gamma_{it}^2)\alpha\right] - \sum_{i=1}^{I}\varphi_{it}\left(P_{it} + Q_{it}\right)\right), \ \ \forall t$$

$$\sum_{i=1}^{I}\omega_{it} = 1, \ \ \forall t$$

$$\sum_{i=1}^{I}\xi_{it} = \Omega_t, \ \ \forall t$$

$$\omega_{it} \leq \xi_{it} u_{it}, \ \ \forall i, t$$

$$\omega_{it} \geq \xi_{it} l_{it}, \ \ \forall i, t$$

$$\omega_{it} - \omega_{it-1} = P_{it} - Q_{it}, \ \ \forall i, t$$

$$\xi_{it} \in \{0,1\}, \ \ \forall i, t$$

$$\omega_{it}, \delta_k^-, \delta_k^+, P_{it}, Q_{it} \geq 0. \ \ \forall i, t, k$$

To solve the proposed FMPMOPO model, *Model (11)* is run once for the lower bound of the alpha-cut interval and again for the upper bound of the alpha-cut interval.

4 | Case Study and Experimental Results

In this section, the proposed fuzzy multi-period multi-objective portfolio optimization model will be implemented for a real-world case study from the Tehran Stock Exchange (TSE). TSE, with a history of nearly half a century, is one of the most attractive financial markets in the Middle East region. Accordingly, the data set of 30 stocks for 5 periods are extracted from TSE. *Tables (1)* and *(2)* show the data set for return and liquidity of 30 stocks for 5 periods under triangular fuzzy distribution:

Table 1. Fuzzy data set for return.

Stocks	First Period	Second Period	Third Period	Fourth Period	Fifth Period
Stock 01	(-0.042, 0.000, 0.049)	(-0.036, -0.004, 0.050)	(-0.049, 0.002, 0.040)	(-0.036, 0.001, 0.040)	(-0.049, -0.004, 0.050)
Stock 02	(-0.022, 0.001, 0.001)	(-0.002, 0.000, 0.002)	(-0.049, -0.002, 0.050)	(-0.002, 0.004, 0.012)	(-0.026, 0.045, 0.048)
Stock 03	(-0.046, 0.001, 0.040)	(-0.049, -0.004, 0.050)	(-0.034, -0.030, -0.004)	(-0.049, 0.001, 0.054)	(-0.048, 0.032, 0.049)
Stock 04	(-0.025, -0.006, 0.047)	(-0.026, -0.003, 0.015)	(-0.010, -0.003, 0.007)	(-0.026, 0.001, 0.048)	(-0.046, 0.040, 0.047)
Stock 05	(-0.005, 0.004, 0.008)	(-0.003, 0.000, 0.003)	(-0.026, -0.003, 0.048)	(-0.003, -0.001, 0.040)	(-0.211, 0.000, 0.049)
Stock 06	(-0.183, 0.004, 0.012)	(-0.026, 0.045, 0.048)	(-0.002, 0.002, 0.045)	(-0.026, -0.002, 0.049)	(-0.046, 0.043, 0.060)
Stock 07	(-0.047, 0.008, 0.049)	(-0.041, -0.002, 0.028)	(-0.046, 0.014, 0.032)	(-0.041, 0.011, 0.048)	(-0.030, 0.013, 0.049)
Stock 08	(-0.021, 0.001, 0.031)	(-0.013, -0.006, 0.001)	(-0.048, 0.002, 0.049)	(-0.013, -0.006, 0.064)	(-0.050, 0.000, 0.066)
Stock 09	(-0.045, 0.001, 0.054)	(-0.048, 0.032, 0.049)	(-0.009, 0.032, 0.045)	(-0.048, -0.003, 0.050)	(-0.025, 0.002, 0.101)
Stock 10	(-0.046, 0.000, 0.038)	(-0.017, 0.005, 0.015)	(-0.018, -0.003, 0.021)	(-0.017, -0.002, 0.028)	(-0.043, 0.006, 0.019)
Stock 11	(-0.016, 0.001, 0.048)	(-0.007, -0.002, 0.000)	(-0.046, 0.004, 0.047)	(-0.007, -0.004, 0.020)	(-0.033, 0.000, 0.034)
Stock 12	(-0.022, 0.001, 0.048)	(-0.046, 0.040, 0.047)	(-0.013, 0.037, 0.040)	(-0.046, 0.000, 0.023)	(-0.050, 0.000, 0.049)
Stock 13	(-0.211, -0.003, 0.032)	(-0.022, -0.005, 0.031)	(-0.029, 0.006, 0.029)	(-0.022, 0.000, 0.050)	(-0.040, 0.012, 0.047)
Stock 14	(0.003, -0.001, 0.040)	(-0.001, -0.003, 0.004)	(-0.211, -0.001, 0.049)	(-0.001, 0.000, 0.044)	(-0.048, 0.039, 0.069)
Stock 15	(-0.019, -0.001, 0.040)	(-0.211, 0.000, 0.049)	(-0.003, -0.002, 0.000)	(-0.211, -0.017, 0.044)	(-0.049, 0.018, 0.058)
Stock 16	(-0.046, 0.003, 0.041)	(-0.028, 0.000, 0.021)	(-0.028, -0.010, 0.041)	(-0.028, -0.046, 0.048)	(-0.047, 0.004, 0.065)
Stock 17	(-0.019, -0.002, 0.018)	(-0.014, -0.008, 0.003)	(-0.046, 0.012, 0.060)	(-0.014, -0.003, 0.046)	(-0.049, 0.008, 0.049)
Stock 18	(-0.022, -0.002, 0.049)	(-0.046, 0.043, 0.060)	(0.024, 0.043, 0.049)	(-0.046, 0.039, 0.048)	(-0.049, -0.012, 0.095)
Stock 19	(-0.022, -0.003, 0.035)	(-0.026, 0.011, 0.028)	(-0.027, 0.002, 0.034)	(-0.026, -0.032, 0.042)	(-0.138, 0.007, 0.049)
Stock 20	(-0.015, 0.011, 0.022)	(-0.017, -0.005, -0.001)	(-0.030, 0.005, 0.049)	(-0.017, -0.004, 0.050)	(-0.075, 0.001, 0.047)
Stock 21	(-0.033, 0.011, 0.048)	(-0.030, 0.013, 0.049)	(-0.018, 0.009, 0.013)	(-0.030, 0.045, 0.048)	(-0.183, 0.004, 0.014)
Stock 22	(-0.024, 0.001, 0.035)	(-0.040, -0.001, 0.048)	(-0.050, -0.014, 0.050)	(-0.040, 0.032, 0.049)	(-0.048, 0.001, 0.054)
Stock 23	(-0.020, -0.006, 0.022)	(-0.001, -0.001, 0.006)	(-0.050, -0.002, 0.066)	(-0.001, 0.040, 0.047)	(-0.126, 0.001, 0.056)
Stock 24	(-0.020, -0.006, 0.064)	(-0.050, 0.000, 0.066)	(-0.003, -0.002, 0.000)	(-0.050, 0.000, 0.049)	(-0.173, -0.001, 0.067)
Stock 25	(-0.022, -0.005, 0.050)	(-0.025, -0.008, 0.101)	(-0.008, 0.014, 0.050)	(-0.025, 0.043, 0.060)	(-0.082, -0.002, 0.050)
Stock 26	(-0.004, -0.003, 0.018)	(-0.015, 0.036, 0.049)	(-0.025, -0.001, 0.101)	(-0.015, 0.013, 0.049)	(-0.050, 0.011, 0.106)
Stock 27	(-0.042, -0.003, 0.050)	(-0.025, 0.002, 0.101)	(-0.001, 0.002, 0.002)	(-0.025, 0.000, 0.066)	(-0.053, -0.006, 0.064)
Stock 28	(-0.043, 0.002, 0.012)	(-0.010, -0.009, 0.017)	(-0.015, -0.008, 0.019)	(-0.010, 0.002, 0.101)	(-0.130, -0.003, 0.050)
Stock 29	(-0.005, -0.002, 0.023)	(0.011, -0.010, 0.000)	(-0.043, 0.000, 0.019)	(0.011, 0.006, 0.019)	(-0.034, -0.002, 0.028)
Stock 30	(-0.034, -0.002, 0.028)	(-0.043, 0.006, 0.019)	(-0.020, 0.000, 0.006)	(-0.043, 0.000, 0.034)	(-0.014, -0.004, 0.020)

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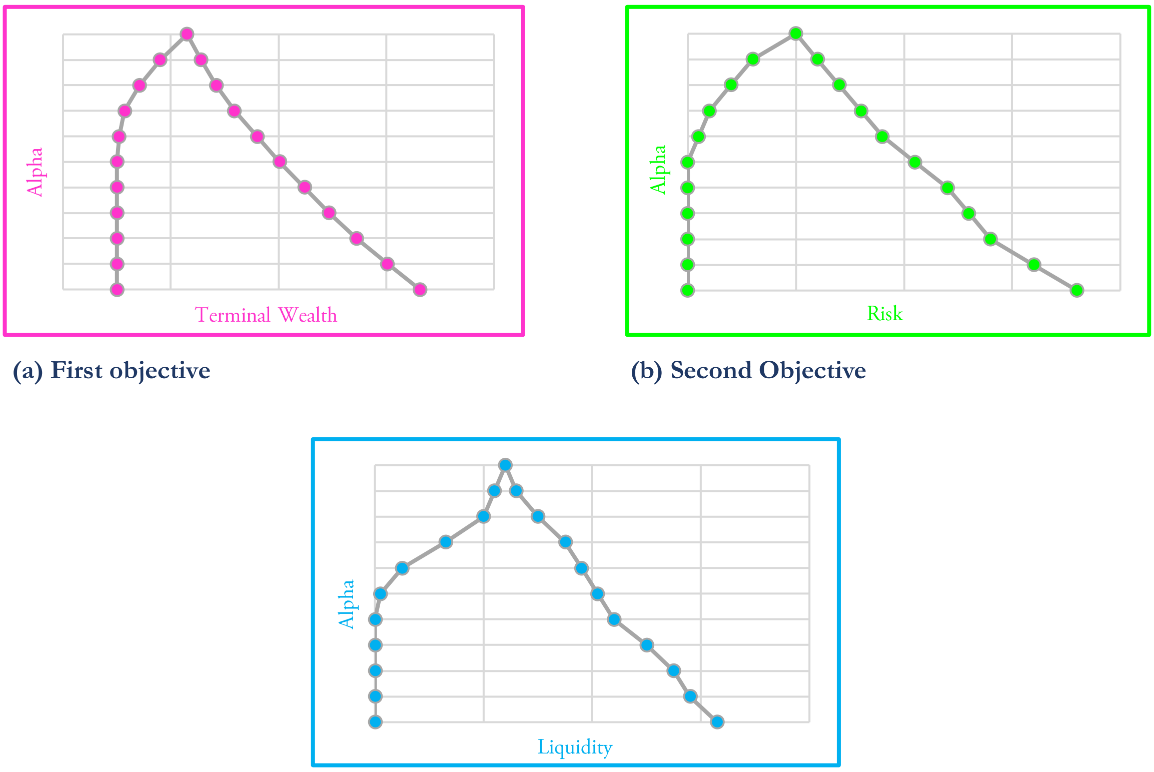
Table 2. Fuzzy data set for liquidity.

Stocks	First Period	Second Period	Third Period	Fourth Period	Fifth Period
Stock 01	(0.068, 0.024, 0.068)	(0.014, 0.021, 0.078)	(0.008, 0.038, 0.049)	(0.008, 0.038, 0.049)	(0.014, 0.021, 0.078)
Stock 02	(0.007, 0.024, 0.070)	(0.000, 0.023, 0.078)	(0.003, 0.022, 0.064)	(0.008, 0.021, 0.029)	(0.013, 0.015, 0.023)
Stock 03	(0.000, 0.025, 0.089)	(0.022, 0.023, 0.052)	(0.000, 0.015, 0.078)	(0.006, 0.024, 0.044)	(0.014, 0.018, 0.050)
Stock 04	(0.039, 0.032, 0.039)	(0.013, 0.015, 0.023)	(0.008, 0.021, 0.029)	(0.009, 0.034, 0.053)	(0.007, 0.025, 0.041)
Stock 05	(0.012, 0.032, 0.039)	(0.008, 0.018, 0.047)	(0.011, 0.030, 0.062)	(0.010, 0.016, 0.063)	(0.000, 0.023, 0.042)
Stock 06	(0.000, 0.022, 0.056)	(0.021, 0.025, 0.030)	(0.008, 0.043, 0.062)	(0.009, 0.025, 0.036)	(0.008, 0.012, 0.023)
Stock 07	(0.056, 0.030, 0.056)	(0.014, 0.018, 0.050)	(0.006, 0.024, 0.044)	(0.000, 0.024, 0.033)	(0.012, 0.014, 0.035)
Stock 08	(0.015, 0.030, 0.056)	(0.000, 0.014, 0.047)	(0.013, 0.034, 0.060)	(0.004, 0.012, 0.049)	(0.006, 0.030, 0.043)
Stock 09	(0.017, 0.033, 0.066)	(0.025, 0.034, 0.035)	(0.000, 0.028, 0.060)	(0.012, 0.033, 0.084)	(0.019, 0.040, 0.066)
Stock 10	(0.038, 0.013, 0.038)	(0.007, 0.025, 0.041)	(0.009, 0.034, 0.053)	(0.000,0.0000, 0.019)	(0.000, 0.000, 0.013)
Stock 11	(0.007, 0.013, 0.040)	(0.000, 0.029, 0.053)	(0.013, 0.025, 0.069)	(0.000, 0.000, 0.053)	(0.000, 0.005, 0.016)
Stock 12	(0.008, 0.007, 0.054)	(0.019, 0.025, 0.040)	(0.000, 0.039, 0.069)	(0.000, 0.000, 0.008)	(0.000, 0.000, 0.020)
Stock 13	(0.053, 0.021, 0.053)	(0.000, 0.023, 0.042)	(0.010, 0.016, 0.063)	(0.007, 0.020, 0.032)	(0.010, 0.016, 0.040)
Stock 14	(0.015, 0.021, 0.053)	(0.000, 0.023, 0.054)	(0.016, 0.051, 0.070)	(0.020, 0.053, 0.053)	(0.016, 0.026, 0.033)
Stock 15	(0.005, 0.019, 0.062)	(0.017, 0.030, 0.051)	(0.000, 0.023, 0.070)	(0.031, 0.064, 0.063)	(0.026, 0.048, 0.075)
Stock 16	(0.048, 0.028, 0.048)	(0.008, 0.012, 0.023)	(0.009, 0.025, 0.036)	(0.013, 0.030, 0.084)	(0.033, 0.059, 0.099)
Stock 17	(0.009, 0.028, 0.048)	(0.000, 0.022, 0.046)	(0.008, 0.016, 0.072)	(0.000, 0.034, 0.040)	(0.012, 0.027, 0.074)
Stock 18	(0.013, 0.010, 0.054)	(0.016, 0.017, 0.020)	(0.000, 0.037, 0.072)	(0.000, 0.020, 0.054)	(0.034, 0.082, 0.092)
Stock 19	(0.066, 0.026, 0.066)	(0.012, 0.014, 0.035)	(0.000, 0.024, 0.033)	(0.025, 0.037, 0.075)	(0.000, 0.034, 0.052)
Stock 20	(0.012, 0.026, 0.066)	(0.000, 0.014, 0.047)	(0.000, 0.024, 0.062)	(0.007, 0.024, 0.068)	(0.000, 0.000, 0.019)
Stock 21	(0.011, 0.021, 0.069)	(0.017, 0.024, 0.036)	(0.000, 0.032, 0.066)	(0.012, 0.032, 0.039)	(0.018, 0.038, 0.076)
Stock 22	(0.078, 0.041, 0.078)	(0.006, 0.030, 0.043)	(0.004, 0.012, 0.049)	(0.020, 0.030, 0.056)	(0.015, 0.042, 0.063)
Stock 23	(0.008, 0.041, 0.078)	(0.000, 0.030, 0.059)	(0.000, 0.011, 0.052)	(0.009, 0.013, 0.038)	(0.000, 0.029, 0.067)
Stock 24	(0.000, 0.041, 0.072)	(0.009, 0.011, 0.030)	(0.000, 0.024, 0.089)	(0.000, 0.021, 0.053)	(0.009, 0.037, 0.085)
Stock 25	(0.084, 0.055, 0.084)	(0.019, 0.040, 0.066)	(0.012, 0.033, 0.084)	(0.009, 0.028, 0.048)	(0.016, 0.031, 0.073)
Stock 26	(0.022, 0.055, 0.084)	(0.000, 0.025, 0.060)	(0.001, 0.032, 0.066)	(0.011, 0.026, 0.066)	(0.028, 0.060, 0.085)
Stock 27	(0.000, 0.024, 0.063)	(0.032, 0.035, 0.063)	(0.000, 0.021, 0.084)	(0.008, 0.041, 0.078)	(0.000, 0.046, 0.080)
Stock 28	(0.019, 0.000, 0.019)	(0.000, 0.000, 0.013)	(0.000, 0.000, 0.019)	(0.000, 0.055, 0.084)	(0.068, 0.024, 0.068)
Stock 29	(0.000, 0.000, 0.049)	(0.000, 0.005, 0.017)	(0.000, 0.000, 0.036)	(0.000, 0.000, 0.019)	(0.007, 0.024, 0.070)
Stock 30	(0.000, 0.000, 0.025)	(0.000, 0.001, 0.049)	(0.000, 0.011, 0.049)	(0.000, 0.000, 0.039)	(0.000, 0.025, 0.089)

Finally, after collecting required data, the results of proposed FMPMOPO model are presented in *Table (3)* and *Fig. (2)* as follows:

Table 3. The results of FMPMOPO model under different Alpha-Cuts.

Alpha (α)	Terminal Wealth	Risk	Liquidity
0.00	(1.000, 1.281)	(0.000, 0.036)	(0.000, 0.063)
0.10	(1.000, 1.251)	(0.000, 0.032)	(0.000, 0.058)
0.20	(1.000, 1.222)	(0.000, 0.028)	(0.000, 0.055)
0.30	(1.000, 1.197)	(0.000, 0.026)	(0.000, 0.050)
0.40	(1.000, 1.174)	(0.000, 0.024)	(0.000, 0.044)
0.50	(1.000, 1.151)	(0.000, 0.021)	(0.001, 0.041)
0.60	(1.002, 1.130)	(0.001, 0.018)	(0.005, 0.038)
0.70	(1.007, 1.109)	(0.002, 0.016)	(0.013, 0.035)
0.80	(1.021, 1.092)	(0.004, 0.014)	(0.020, 0.030)
0.90	(1.040, 1.078)	(0.006, 0.012)	(0.022, 0.026)
1.00	(1.065, 1.065)	(0.010, 0.010)	(0.024, 0.024)



(a) First objective **(b) Second Objective**

(C) Third objective

Fig. 2. The presentation of objective functions under different alpha-cuts.

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It should be explained that the parameters of the proposed FMPMOPO model including u_{it}, l_{it}, φ_{it}, Ω_t, and Ψ_1, are set equal to 0.1, 0, 0.1%., 10, and 1, respectively. Also, ideal goal of three objectives including terminal wealth, risk, and liquidity, are set equal to 2, 0, and 0.5, respectively. Notably, the results indicate on applicability and efficacy of the FMPMOPO model for multi-period portfolio optimization problem under ambiguity.

5 | Conclusions and Future Research Directions

In this paper, a new fuzzy multi-period multi-objective portfolio optimization model in the context of fuzzy uncertainty is proposed. Notably, three objectives including wealth, risk, and liquidity as well as practical investment constraints are considered to propose FMPMOPO model. Also, the proposed fuzzy multi-period wealth-risk-liquidity model is implemented in real-world case study from Tehran stock market. The experimental results show the applicability of the proposed FMPMOPO model. For future studies, Robust Convex Programming (RCP) and Scenario-Based Robust Optimization (SBRO) approach can be employed in order to deal with uncertainty of financial data [40]-[50]. Moreover, Data Envelopment Analysis (DEA) approach can be applied for stock selection [51]-[62].

Conflicts of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper

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