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# A Survey on Different Definitions of Soft Points: Limitations, Comparisons and Challenges

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## Abstract

The aim of this paper is to investigate different definitions of soft points in the existing literature on soft set theory and its extensions in different directions. Then limitations of these definitions are illustrated with the help of examples. Moreover, the definition of soft point in the setup of fuzzy soft set, intervalvalued fuzzy soft set, hesitant fuzzy soft set and intuitionistic soft set are also discussed. We also suggest an approach to unify the definitions of soft point which is more applicable than the existing notions.

**Keywords:** Soft point, Fuzzy soft point, Interval-Valued fuzzy soft point, Hesitant fuzzy soft point, Intuitionistic fuzzy soft point, Neutrosophic soft points, Hypersoft points.

## 1 | Introduction

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Molodtsov [4] initiated soft set theory as an extension of fuzzy set theory to deal with uncertainties occurring in natural and social sciences. It attracted the attention of mathematicians as well as social scientists due to its potential to unify certain mathematical aspects and applications in decision making processes (see [5]). In an attempt to study different existing mathematical structures in the context of soft set theory, the notion of a soft point plays a significant role. A careful formulation of the notion of a soft point is required to define the notion of a soft mapping; an important ingredient of a soft set theory. The study of different mathematical structures such as metric spaces, topological spaces, vectors spaces, normed spaces and inner product spaces in the setup of soft set theory also require appropriate definitions of a soft point and soft mapping.

One may find different definitions of soft points and hence of soft mappings in the existing literature on soft set theory. The reader interested in different definitions of soft points and soft mappings is referred to [7], [8], [12]-[18] and references mentioned therein.



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These definitions have their own limitations and merits. Some authors have attempted to address these limitations. In this direction, Murtaza et al. [1] defined the notion of interval valued neutrosophic soft points and discussed their properties. Moreover, Smarandache [3] extended the notion of a soft set to the hypersoft set by replacing the function  $F$  with a multi-argument function defined on the Cartesian product of  $n$  different set of parameters. After that, Abbas et al. [2] introduced hypersoft points in different setups such as fuzzy hypersoft set, intuitionistic fuzzy hypersoft set, neutrosophic hypersoft, plithogenic hypersoft set, and studied some basic properties of hypersoft points in these frameworks.

The objective of this article is to provide a survey of these definitions and to present their limitations. Moreover, we have discussed the inherited flaws in the definitions of soft points in the framework of fuzzy soft set, intervalvalued fuzzy soft set, hesitant fuzzy soft set and intuitionistic soft set.

First we recall some basic definitions that will be necessary for the environment.

**Definition 1. [4].** Let  $U$  be a universe and  $A$  a nonempty subset of a set  $E$  of parameters. A soft set  $(\mathcal{S}, A)$  over  $U$  is characterized by a set valued mapping  $\mathcal{S}: A \rightarrow \mathcal{P}(U)$ , where  $\mathcal{P}(U)$  denotes the power set of  $U$ . In other words, a soft set over  $U$  is a parameterized family of subsets of the universe  $U$ . For  $\varepsilon \in A$ ,  $\mathcal{S}(\varepsilon)$  may be considered as the set of  $\varepsilon$ - approximate elements of the soft set  $(\mathcal{S}, A)$ .

For the sake of brevity, we denote the soft set  $(\mathcal{S}, A)$  by  $\mathcal{S}_A$  and  $S(U)$  by the collection of all soft sets over  $U$ .

**Definition 2. [5].** Let  $\mathcal{S}_A, \mathcal{T}_B \in S(U)$ . Then  $\mathcal{S}_A$  is called a soft subset of  $\mathcal{T}_B$  if  $A \subseteq B$  and for all  $e \in A$ ,  $\mathcal{S}_A(e) \subseteq \mathcal{T}_B(e)$ . We denote it as  $\mathcal{S}_A \subseteq \mathcal{T}_B$ .

**Definition 3. [5].** The complement of  $\mathcal{S}_A$  denoted by  $\mathcal{S}_A^c$  is identified by a set valued mapping  $\mathcal{S}^c: A \rightarrow \mathcal{P}(U)$  given by  $\mathcal{S}^c(\alpha) = U - \mathcal{S}(\alpha)$ , for all  $\alpha \in A$ .

**Definition 4. [5].** Let  $\mathcal{S}_E \in S(U)$ . Then  $\mathcal{S}_E$  is said to be an absolute soft set, denoted by  $\tilde{U}$ , if for all  $\varepsilon \in E$ ,  $\mathcal{S}(\varepsilon) = U$  whereas a soft set  $\mathcal{S}_E$  over  $U$  is said to be a null soft set denoted by  $\Phi$  if for all  $\varepsilon \in E$ ,  $\mathcal{S}(\varepsilon) = \emptyset$ .

**Definition 5. [5].** Let  $\mathcal{S}_A, \mathcal{T}_B \in S(U)$ . The union of  $\mathcal{S}_A$  and  $\mathcal{T}_B$  is a soft set  $\mathcal{V}_C$ , where  $C = A \cup B$  and for all  $e \in C$ , the mapping  $\mathcal{V}: C \rightarrow \mathcal{P}(U)$  is given by

$$\mathcal{V}(e) = \begin{cases} \mathcal{S}(e), & \text{if } e \in A - B \\ \mathcal{T}(e), & \text{if } e \in B - A \\ \mathcal{S}(e) \cup \mathcal{T}(e), & \text{if } e \in A \cap B \end{cases}$$

We express it as  $\mathcal{S}_A \tilde{\cup} \mathcal{T}_B = \mathcal{V}_C$ .

**Definition 6. [5].** Let  $\mathcal{S}_A, \mathcal{T}_B \in S(U)$ . The intersection of  $\mathcal{S}_A$  and  $\mathcal{T}_B$  is a soft set  $\mathcal{V}_C$ , where  $C = A \cap B$  and for all  $e \in C$ , the mapping  $\mathcal{V}: C \rightarrow \mathcal{P}(U)$  is given by  $\mathcal{V}(e) = \mathcal{S}(e) \cap \mathcal{T}(e)$ . We write it as  $\mathcal{S}_A \tilde{\cap} \mathcal{T}_B = \mathcal{V}_C$ .

**Definition 7. [6].** Let  $\mathcal{S}_A, \mathcal{T}_A \in S(U)$ . The difference  $\mathcal{V}_A$  of  $\mathcal{S}_A$  and  $\mathcal{T}_A$  denoted by  $\mathcal{S}_A \tilde{\setminus} \mathcal{T}_A$  is defined by the mapping given by  $\mathcal{V}(e) = \mathcal{S}(e) \setminus \mathcal{T}(e)$  for all  $e \in A$ .

## 2 | Definitions of Soft Point and Their Limitations

In this section, we state different existing definitions of soft point in the literature and discuss the limitations of these definitions by providing suitable examples. There are two important issues that are related with definition of a soft point: one is to define the concept of a soft point on its own and other is to define the concept in relation with a soft set, that is, the concept of belonging of a soft point to the soft set.

In this section, both issues are discussed and points are highlighted that what should be kept in mind while using these definitions.

We start with the definition in [16], where elements of universal set  $U$  were taken as "soft points" and their belonging to the soft sets was given in the following manner.

**Definition 8. [16].** Let  $\mathcal{S}_A \in S(U)$ . A point  $u \in U$  is said to be in  $\mathcal{S}_A$  denoted by  $u \tilde{\in} \mathcal{S}_A$  if  $u \in \mathcal{S}(e)$  for all  $e \in A$ .

In case of the above definition, we may face the following situations:

If  $\mathcal{S}_A, \mathcal{T}_A \in S(U)$  then there may be an  $u \tilde{\in} \mathcal{S}_A \tilde{\cup} \mathcal{T}_A$  such that  $u \not\tilde{\in} \mathcal{S}_A$  and  $u \not\tilde{\in} \mathcal{T}_A$ . Thus there may exists an  $u \in U$  and a soft set  $\mathcal{S}_A$  such that  $u \not\tilde{\in} \mathcal{S}_A$  and  $u \not\tilde{\in} \mathcal{S}_A^c$ .

To observe this fact see the following example.

**Example 1.** Let  $U = \{u_1, u_2\}$  and  $E = A = \{e_1, e_2\}$ . Define the soft sets  $\mathcal{S}_A$  and  $\mathcal{T}_A$  by

$$\mathcal{S}_A = \{(e_1, \{u_1\}), (e_2, \{u_2\})\},$$

$$\mathcal{T}_A = \{(e_1, \{u_2\}), (e_2, \{u_1\})\}.$$

Note that the union  $\mathcal{S}_A \tilde{\cup} \mathcal{T}_A$  is given by

$$\{(e_1, \{u_1, u_2\}), (e_2, \{u_1, u_2\})\}.$$

Clearly  $u_1, u_2 \tilde{\in} \mathcal{S}_A \tilde{\cup} \mathcal{T}_A$  but  $u_1, u_2 \not\tilde{\in} \mathcal{S}_A, \mathcal{T}_A$ . Note that  $\mathcal{T}_A$  is the soft complement of  $\mathcal{S}_A$ , and hence we have arrived at the situation where  $u \in U$  but  $u \not\tilde{\in} \mathcal{S}_A$  and  $u \not\tilde{\in} \mathcal{S}_A^c$ .

We may also face the following problem:

- I. If  $\mathcal{S}_A, \mathcal{T}_A \in S(U)$  then there may exist a non null soft subset  $\mathcal{T}_A$  of a given soft set  $\mathcal{S}_A$  such that none of  $u \tilde{\in} \mathcal{S}_A$  belongs to  $\mathcal{T}_A$ .

We now give the following example which illustrate the above situation.

**Example 2.** Let  $U = \{u_1, u_2\}$  and  $E = A = \{e_1, e_2\}$ . Define soft sets  $\mathcal{S}_A$  and  $\mathcal{T}_A$  by

$$\mathcal{S}_A = \{(e_1, \{u_1, u_2\}), (e_2, \{u_1, u_2\})\},$$

$$\mathcal{T}_A = \{(e_1, \{u_2\}), (e_2, \{u_1\})\}.$$

Clearly  $u_1, u_2 \tilde{\in} \mathcal{S}_A$  but there is no  $u$  such that  $u \tilde{\in} \mathcal{T}_A$ . Moreover,  $\mathcal{T}_A$  is a non null soft subset of  $\mathcal{S}_A$ .

There may arise the following situation as well.

- II. If  $\mathcal{S}_A, \mathcal{T}_A \in S(U)$  then every  $u \tilde{\in} \mathcal{S}_A$  implies that  $u \tilde{\in} \mathcal{T}_A$  but it does not imply  $\mathcal{S}_A \tilde{\subseteq} \mathcal{T}_A$ .

We present the following example where the above statement is valid.

**Example 3.** Let  $U = \{u_1, u_2, u_3\}$  and  $E = A = \{e_1, e_2\}$ . If we define soft sets  $\mathcal{S}_A$  and  $\mathcal{T}_A$  by

$$\mathcal{S}_A = \{(e_1, \{u_1, u_2\}), (e_2, \{u_1\})\},$$

$$\mathcal{T}_A = \{(e_1, \{u_1, u_3\}), (e_2, \{u_1, u_2\})\}.$$

Then every  $u \in \mathcal{S}_A$  implies that  $u \in \mathcal{T}_A$  but  $\mathcal{S}_A \not\subseteq \mathcal{T}_A$ . Also, note that  $\mathcal{T}_A \not\subseteq \mathcal{S}_A$

El-Shafei et al. [14] relaxed the *Definition 8* in the following manner.

**Definition 9.** If  $\mathcal{S}_A \in S(U)$ . A point  $u \in U$  is said to be partially in  $\mathcal{S}_A$  denoted by  $u \in \mathcal{S}_A$ , if  $u \in \mathcal{S}(e)$  for some  $e \in A$ .

El-Shafei et al. [14] discussed the limitation (I) on the *Definition 8* and considered the limitations ((I) and (III) in the following list) of the above definition as well.

Here we give the list of those limitations and present an example (different from the example given in [14]) to illustrate the limitations.

- I. If  $\mathcal{S}_A, \mathcal{T}_A \in S(U)$  then there may be a  $u \in \mathcal{S}_A$  and  $u \in \mathcal{T}_A$  such that  $u$  does not partially belong to  $\mathcal{S}_A \tilde{\cap} \mathcal{T}_A$ .
- II. If  $\mathcal{S}_A$  and  $\mathcal{T}_A$  are soft complement of each other, then there may be an  $u \in U$  such that  $u \in \mathcal{S}_A$  and  $u \in \mathcal{T}_A$ .
- III. Each  $u \in \mathcal{S}_A$  implies that  $u \in \mathcal{T}_A$  but it does not imply that  $\mathcal{S}_A \subseteq \mathcal{T}_A$ .

**Example 4.** Let  $U = \{u_1, u_2\}$  and  $E = A = \{e_1, e_2\}$ . Define soft sets  $\mathcal{S}_A$  and  $\mathcal{T}_A$  by

$$\mathcal{S}_A = \{(e_1, \{u_1\}), (e_2, \{u_2\})\},$$

$$\mathcal{T}_A = \{(e_1, \{u_2\}), (e_2, \{u_1\})\}.$$

Note that

$$\mathcal{S}_A \tilde{\cap} \mathcal{T}_A = \{(e_1, \emptyset), (e_2, \emptyset)\}.$$

We have the following observations:

- I. Clearly  $u_1, u_2$  do not belong to  $\mathcal{S}_A \tilde{\cap} \mathcal{T}_A$  (even partially) although  $u_1, u_2 \in \mathcal{S}_A$  and  $u_1, u_2 \in \mathcal{T}_A$ .
- II. Although  $\mathcal{S}_A$  and  $\mathcal{T}_A$  are soft complements of each other but  $u_1, u_2 \in \mathcal{S}_A$  and  $u_1, u_2 \in \mathcal{T}_A$ .
- III. Here each  $u \in \mathcal{S}_A$  also implies that  $u \in \mathcal{T}_A$ , but  $\mathcal{S}_A$  is not a soft subset of  $\mathcal{T}_A$ .

Now we discuss those definitions of soft points for which soft point is itself a soft set and its belonging to a soft set is defined as a soft subset.

**Definition 10. [18].** A soft set  $\mathcal{S}_A$  is called a soft point if for the element  $e \in A$ ,  $\mathcal{S}_A(e) \neq \emptyset$ , and  $\mathcal{S}_A(e') = \emptyset$ , for all  $e' \in A - \{e\}$ . A soft point is denoted by  $e_s$ .

A soft point  $e_s$  is said to belong to a soft set  $\mathcal{T}_A$  if  $\mathcal{S}(e) \subseteq \mathcal{T}(e)$  for every  $e \in A$ . We denote this by  $e_s \in \mathcal{T}_A$ .

In the above definition, we find the following problem:

If  $\mathcal{S}_A, \mathcal{T}_A \in S(U)$  then  $e_s \in \mathcal{S}_A \tilde{\cap} \mathcal{T}_A$  does not imply that  $e_s \in \mathcal{S}_A$  or  $e_s \in \mathcal{T}_A$ . Thus there may exist a soft point  $e_s \in \tilde{U}$  and a soft set  $\mathcal{S}_A$  such that  $e_s \notin \mathcal{S}_A$  and  $e_s \notin \mathcal{S}_A^c$ .

This can be observed by the following example.

**Example 5.** Let  $U = \{u_0, u_1, u_2\}$  and  $E = A = \{e_1, e_2, e_3\}$ . Let us consider the following soft sets

$$\mathcal{S}_A = \{(e_1, \{u_0, u_1\}), (e_2, \{u_0\}), (e_3, \{u_2\})\},$$

$$\mathcal{T}_A = \{(e_1, \{u_2\}), (e_2, \{u_1, u_2\}), (e_3, \{u_0, u_1\})\}.$$

Note that

$$\mathcal{S}_A \tilde{\cup} \mathcal{T}_A = \{(e_1, \{u_0, u_1, u_2\}), (e_2, \{u_0, u_1, u_2\}), (e_3, \{u_0, u_1, u_2\})\} = \tilde{U}.$$

Also, the soft point  $e_{1_s} = \{(e_1, \{u_0, u_1, u_2\})\} \tilde{\in} \mathcal{S}_A \tilde{\cup} \mathcal{T}_A$  but  $e_{1_s} \not\tilde{\in} \mathcal{S}_A$  and  $e_{1_s} \not\tilde{\in} \mathcal{T}_A$ . Moreover,  $\mathcal{T}_A$  is the soft complement of  $\mathcal{S}_A$ . Thus we have the situation that  $e_{1_s} \tilde{\in} \tilde{U}$  but  $e_{1_s} \not\tilde{\in} \mathcal{S}_A$  and  $e_{1_s} \not\tilde{\in} \mathcal{S}_A^c$ .

Senel [13] modified the *Definition 10* and compared his definition with the *Definitions 10* and *Definitions 13*.

**Definition 11. [13].** A soft set  $\mathcal{S}_A$  is called a soft point if for the element  $e_i \in A$ ,  $\mathcal{S}_A(e_i) \neq \emptyset$ , and  $\mathcal{S}_A(e_j) = \emptyset$ , for all  $e_j \in A - \{e_i\}$ . A soft point is denoted by  $(e_{i_s})_j$  for all  $i, j \in \mathbb{N}^+$ .

A soft point  $(e_{i_s})_j$  is said to belong to a soft set  $\mathcal{T}_A$  if  $\mathcal{S}(e_i) \subseteq \mathcal{T}(e_i)$  for every  $e_i \in A$ , denoted by  $(e_{i_s})_j \tilde{\in} \mathcal{T}_B$ .

The restrictions on the *Definition 11* are the same as in the case of the *Definition 8*. Moreover it seems that the *Definition 11* works only if the set of parameters is a countable set.

Now we restate the definition given by Aygünoğlu and Aygün [7]. They also discussed its limitation on the soft union.

**Definition 12. [7].** A soft set  $\mathcal{S}_A$  is called a soft point if there exists a  $u_0 \in U$  and  $A \subseteq E$  such that  $\mathcal{S}(e) = \{u_0\}$ , for all  $e \in A$  and  $\mathcal{S}(e) = \emptyset$ , for all  $e \in E - A$ . A soft point is denoted by  $\mathcal{S}_A^{u_0}$ .

A soft point  $\mathcal{S}_A^{u_0}$  is said to belong to a soft set  $\mathcal{T}_B$  if  $u_0 \in \mathcal{T}(e)$  for each  $e \in A$ , denoted by  $\mathcal{S}_A^{u_0} \tilde{\in} \mathcal{T}_B$ .

If  $\mathcal{S}_A, \mathcal{T}_A \in \mathcal{S}(U)$  then  $\mathcal{S}_A^{u_0} \tilde{\in} \mathcal{S}_A \tilde{\cup} \mathcal{T}_A$  does not imply that  $\mathcal{S}_A^{u_0} \tilde{\in} \mathcal{S}_A$  or  $\mathcal{S}_A^{u_0} \tilde{\in} \mathcal{T}_A$ . Thus there may exists a soft point  $\mathcal{S}_A^{u_0} \tilde{\in} \tilde{U}$  and a soft set  $\mathcal{S}_A$  such that  $\mathcal{S}_A^{u_0} \not\tilde{\in} \mathcal{S}_A$  and  $\mathcal{S}_A^{u_0} \not\tilde{\in} \mathcal{S}_A^c$ .

This is shown by the following example.

**Example 6.** Let  $U = \{u_0, u_1, u_2, u_3\}$  and  $E = A = \{e_1, e_2, e_3\}$ . Consider the following soft sets

$$\mathcal{S}_A = \{(e_1, \{u_0, u_1\}), (e_2, \{u_0, u_1, u_2\}), (e_3, \{u_1, u_2, u_3\})\},$$

$$\mathcal{T}_A = \{(e_1, \{u_2, u_3\}), (e_2, \{u_3\}), (e_3, \{u_0\})\}.$$

Note that  $\mathcal{S}_A^{u_0} \tilde{\in} \mathcal{S}_A \tilde{\cup} \mathcal{T}_A = \tilde{U}$  but  $\mathcal{S}_A^{u_0} \not\tilde{\in} \mathcal{S}_A$  and  $\mathcal{S}_A^{u_0} \not\tilde{\in} \mathcal{T}_A$ . Also  $\mathcal{T}_A$  is the soft complement of  $\mathcal{S}_A$ . Hence we have the situation that  $\mathcal{S}_A^{u_0} \tilde{\in} \tilde{U}$  but  $\mathcal{S}_A^{u_0} \not\tilde{\in} \mathcal{S}_A$  and  $\mathcal{S}_A^{u_0} \not\tilde{\in} \mathcal{S}_A^c$ .

The following definition is given by Wardowski [17] and, Das and Samanta [8] independently.

**Definition 13. [8], [17].** A soft set  $\mathcal{S}_A$  over  $U$  is said to be a soft point if there is exactly one  $e \in A$  such that  $\mathcal{S}(e) = \{u\}$  for some  $u \in U$  and  $\mathcal{S}(e) = \emptyset$ , for all  $e \in A \setminus \{e\}$ . We denote such a soft point by  $(\mathcal{S}_e^u, A)$  or simply by  $\mathcal{S}_e^u$ .

A soft point  $\mathcal{S}_e^u$  is said to belong to  $\mathcal{S}_A$ , denoted by  $\mathcal{S}_e^u \tilde{\in} \mathcal{S}_A$ , if  $\mathcal{S}_e^u(e) = \{u\} \subset \mathcal{S}(e)$ .

Wardowski [17] formulated all basic operations of union and intersection for soft points. He proved that the *Definition 13* of soft point follows the same operations as in the crisp case in the following manner:

**Proposition 1.** Let  $\mathcal{S}_A, \mathcal{T}_A \in S(U)$ . Then the following holds

- I.  $\mathcal{S}_e^u \tilde{\in} \mathcal{S}_A \tilde{\cup} \mathcal{T}_A$  if and only if  $\mathcal{S}_e^u \tilde{\in} \mathcal{S}_A$  or  $\mathcal{S}_e^u \tilde{\in} \mathcal{T}_A$ .
- II.  $\mathcal{S}_e^u \tilde{\in} \mathcal{S}_A \tilde{\cap} \mathcal{T}_A$  if and only if  $\mathcal{S}_e^u \tilde{\in} \mathcal{S}_A$  and  $\mathcal{S}_e^u \tilde{\in} \mathcal{T}_A$ .
- III.  $\mathcal{S}_e^u \tilde{\in} \mathcal{S}_A \tilde{\setminus} \mathcal{T}_A$  if and only if  $\mathcal{S}_e^u \tilde{\in} \mathcal{S}_A$  and  $\mathcal{S}_e^u \not\tilde{\in} \mathcal{T}_A$ .
- IV. If every  $\mathcal{S}_e^u \tilde{\in} \mathcal{S}_A \Rightarrow \mathcal{S}_e^u \tilde{\in} \mathcal{T}_A$  if and only if  $\mathcal{S}_A \tilde{\subseteq} \mathcal{T}_A$ .

The following definition is given by Das and Samanta [9]. They defined soft element of a soft set as a mapping. They also used soft element to define soft real number.

**Definition 14. [9].** If  $g$  is a single valued mapping on  $A \subset E$  taking values in  $U$ , then the pair  $(g, A)$ , or simply  $g$  is called a soft element of  $U$ . A soft element  $g$  of  $U$  is said to belongs to a soft set  $\mathcal{S}_A$  over  $U$ , denoted by  $g \tilde{\in} \mathcal{S}_A$ , if  $g(e) \in \mathcal{S}(e)$ , for each  $e \in A$ .

Again there is a following restriction on the soft union:

Suppose that  $\mathcal{S}_A$  and  $\mathcal{T}_A$  are soft sets over  $U$  and  $g$  is a soft element such that  $g \tilde{\in} \mathcal{S}_A \tilde{\cup} \mathcal{T}_A$ . Then it does not imply that  $g \tilde{\in} \mathcal{S}_A$  or  $g \tilde{\in} \mathcal{T}_A$ . Thus there may exists a soft element  $g \tilde{\in} U$  such that  $g \not\tilde{\in} \mathcal{S}_A$  and  $g \not\tilde{\in} \mathcal{S}_A^c$ .

To observe these facts see the following example:

**Example 7.** Let  $U = \{0,1,2\}$  and  $E = A = \{e_1, e_2, e_3\}$ . Consider the following soft real sets

$$\mathcal{S}_A = \{(e_1, \{0,1\}), (e_2, \{0\}), (e_3, \{2\})\},$$

$$\mathcal{T}_A = \{(e_1, \{2\}), (e_2, \{1,2\}), (e_3, \{0,1\})\}.$$

Note that

$$\mathcal{S}_A \tilde{\cup} \mathcal{T}_A = \{(e_1, \{0,1,2\}), (e_2, \{0,1,2\}), (e_3, \{0,1,2\})\}.$$

Then the soft real number  $g$  is defined by

$$g(e_1) = 0; g(e_2) = 1; g(e_3) = 2;$$

belongs to  $\mathcal{S}_A \tilde{\cup} \mathcal{T}_A$ , that is,  $g \tilde{\in} \mathcal{S}_A \tilde{\cup} \mathcal{T}_A = \tilde{U}$  but  $g \not\tilde{\in} \mathcal{S}_A$  and  $g \not\tilde{\in} \mathcal{T}_A$ . Also  $\mathcal{T}_A$  is the soft complement of  $\mathcal{S}_A$ . Hence we have the situation that that  $g \tilde{\in} \tilde{U}$  but  $g \not\tilde{\in} \mathcal{S}_A$  and  $g \not\tilde{\in} \mathcal{S}_A^c$ .

**Remark 1.** In the above example,  $\mathcal{S}_A, \mathcal{T}_A$  are soft real sets and  $g$  is a soft element corresponding to the soft real number

$$\mathcal{V}_A = \{(e_1, \{0\}), (e_2, \{1\}), (e_3, \{2\})\}.$$

Then  $\mathcal{V}_A \tilde{\subseteq} \mathcal{S}_A \tilde{\cup} \mathcal{T}_A$  but  $\mathcal{V}_A \not\tilde{\subseteq} \mathcal{S}_A$  and  $\mathcal{V}_A \not\tilde{\subseteq} \mathcal{T}_A$ , that is, a soft real number belongs to a union of two soft real sets but not to the individual sets.

### 3 | Definition of Soft Point in Different Environments

In this section, we recall the definitions of different variants of soft points in different environments such as fuzzy soft point, interval valued soft point, hesitant soft point, and intuitionistic soft point.

There are two definitions of fuzzy soft point available in the literature. The first one is similar to the *Definition 13* of soft point and thus follows the basic rules of crisp set operations.

We will denote  $FS(U)$  by the collection of all fuzzy soft sets over  $U$ ,  $IS(U)$  by the collection of all interval valued fuzzy soft sets over  $U$  and  $IFS(U)$  by the collection of all intuitionistic fuzzy soft sets over  $U$ .

**Definition 15. [10].** A fuzzy soft point  $e_{u_\lambda}$  over  $U$  is a fuzzy soft set over  $U$  defined as follows:

$$e_{u_\lambda}(e') = \begin{cases} u_\lambda & \text{if } e' = e, \\ 0_U & \text{if } e' \neq e, \end{cases}$$

where  $u_\lambda$  is the fuzzy point in  $U$  with support  $u$  and value  $\lambda$ ,  $\lambda \in (0,1)$  and  $0_U$  is null fuzzy set.

Following is the second definition given by Neog et al. [11].

**Definition 16. [11].** Let  $\mathcal{S}_A \in FS(U)$ . Then  $\mathcal{S}_A$  is called a fuzzy soft point, denoted by  $e(\mathcal{S}_A)$ , if for  $e \in A$ ,  $\mathcal{S}(e) \neq \bar{0}$  and  $\mathcal{S}(e') = \bar{0}$  for  $e' \in A \setminus \{e\}$  (where  $\bar{0}$  denotes the null fuzzy set).

A fuzzy soft point  $e(\mathcal{S}_A)$  is said to belong to a fuzzy soft set  $\mathcal{T}_A$  if  $e(\mathcal{S}_A)$  is a fuzzy soft subset of  $\mathcal{T}_A$ .

In the above definition, one faces the same problem with an operation of union as discussed earlier in the case of the *Definition 10*.

The following example illustrates the situation:

**Example 8.** Suppose that  $U = \{u_0, u_1, u_2\}$  and  $E = \{e_1, e_2, e_3\}$ . Consider the following

$$\mathcal{S}_E = \left\{ \left( e_1, \left\{ \frac{u_0}{0.5}, \frac{u_1}{0.1} \right\} \right), \left( e_2, \left\{ \frac{u_0}{0.3} \right\} \right), \left( e_3, \left\{ \frac{u_2}{0.5} \right\} \right) \right\},$$

$$\mathcal{T}_E = \left\{ \left( e_1, \left\{ \frac{u_2}{0.4} \right\} \right), \left( e_2, \left\{ \frac{u_1}{0.2}, \frac{u_2}{0.3} \right\} \right), \left( e_3, \left\{ \frac{u_0}{0.5}, \frac{u_1}{0.7} \right\} \right) \right\}.$$

Note that

$$\mathcal{S}_E \tilde{\cup} \mathcal{T}_E = \left\{ \left( e_1, \left\{ \frac{u_0}{0.5}, \frac{u_1}{0.1}, \frac{u_2}{0.4} \right\} \right), \left( e_2, \left\{ \frac{u_0}{0.3}, \frac{u_1}{0.2}, \frac{u_2}{0.3} \right\} \right), \left( e_3, \left\{ \frac{u_0}{0.5}, \frac{u_1}{0.7}, \frac{u_2}{0.5} \right\} \right) \right\}.$$

If we define a fuzzy soft point  $e_1(\mathcal{S}_E)$  as follows:

$$\mathcal{V}_E = \left\{ \left( e_1, \left\{ \frac{u_0}{0.5}, \frac{u_1}{0.1}, \frac{u_2}{0.4} \right\} \right) \right\},$$

then  $e_1(\mathcal{V}_E) \tilde{\in} \mathcal{S}_E \tilde{\cup} \mathcal{T}_E$  but  $e_1(\mathcal{V}_E) \not\tilde{\in} \mathcal{S}_E$  and  $e_1(\mathcal{V}_E) \not\tilde{\in} \mathcal{T}_E$ .

Now we discuss the points of an interval valued fuzzy soft set. It is worth mentioning that an interval valued fuzzy soft point defined in any of the way stated in the previous section, it lacks some basic operations. Even in the case of the definition similar to the *Definition 13*, we can construct an example showing that there is an interval valued fuzzy soft point which belongs to the union but not to the individual sets. Thus until now, there is no definition of interval valued fuzzy soft point which follows basic rules of union and intersection.

Now we define interval valued fuzzy soft point similar to the *Definition 13* and give an example to support the above observation.

**Definition 17.** Let  $\mathcal{S}_A \in IS(U)$  and  $u \in U$ . An interval valued fuzzy soft point, denoted by  $IP^{(e,u)}$  is defined as follows: if for the element  $e \in A, \mathcal{S}(e)(v) = [0,0]$  for every  $v \neq u$  and  $\mathcal{S}(e') = \bar{0}$  for  $e' \in A \setminus \{e\}$  (where  $\bar{0}$  denotes the null interval valued fuzzy set).

An interval valued fuzzy soft point  $IP_u^e$  is said to belong to an interval valued fuzzy soft set  $\mathcal{S}_A$  if  $IP^{(e,u)}$  is an interval valued fuzzy soft subset of  $\mathcal{S}_A$ .

**Example 9.** Let  $U = \{u_1, u_2\}$  and  $E = \{e_1, e_2\}$  be the set of parameters. Consider the interval valued fuzzy soft sets  $\mathcal{S}_E$  and  $\mathcal{T}_E$  given by

$$\mathcal{S}_E = \left\{ \left( e_1, \left\{ \frac{u_1}{[0.5,0.8]}, \frac{u_2}{[1.0,1.0]} \right\} \right), \left( e_2, \left\{ \frac{u_1}{[0.2,0.5]}, \frac{u_2}{[0.3,0.6]} \right\} \right) \right\},$$

and

$$\mathcal{T}_E = \left\{ \left( e_1, \left\{ \frac{u_1}{[0.4,0.9]}, \frac{u_2}{[1.0,1.0]} \right\} \right), \left( e_2, \left\{ \frac{u_1}{[0.2,0.5]}, \frac{u_2}{[0.3,0.6]} \right\} \right) \right\}.$$

Their union is

$$\mathcal{V}_E = \left\{ \left( e_1, \left\{ \frac{u_1}{[0.5,0.9]}, \frac{u_2}{[1.0,1.0]} \right\} \right), \left( e_2, \left\{ \frac{u_1}{[0.2,0.5]}, \frac{u_2}{[0.3,0.6]} \right\} \right) \right\}.$$

Now consider the interval valued fuzzy soft point of  $\mathcal{V}_E$  defined as

$$IP^{(e_1,u_1)} = \left\{ \left( e_1, \left\{ \frac{u_1}{[0.5,0.9]} \right\} \right) \right\}$$

The point  $IP^{(e_1,u_1)}$  is an interval valued fuzzy soft point of  $\mathcal{V}_E$ . Note that  $IP^{(e_1,u_1)}$  is not the interval valued fuzzy soft point of  $\mathcal{S}_E$  or  $\mathcal{T}_E$ .

For other types of definitions of interval valued fuzzy soft point, we can construct examples in the similar fashion as discussed in the previous section.

**Remark 2.** It is easy to observe that the definition for hesitant fuzzy soft point faces the same problems.

Now we discuss the definition of intuitionistic fuzzy soft point.

The definition of intuitionistic fuzzy soft point given in [12] faces the same limitations as in the case of fuzzy soft point.

**Definition 18. [12].** Let  $\mathcal{S}_A \in IFS(U)$ . Then  $\mathcal{S}_A$  is called an intuitionistic fuzzy soft point, denoted by  $e_{\mathcal{S}}$ , if for the element  $e \in A, \mathcal{S}(e) \neq \bar{0}$  and  $\mathcal{S}(e') = \bar{0}$  for  $e' \in A \setminus \{e\}$  (where  $\bar{0}$  denotes the null intuitionistic fuzzy set).

An intuitionistic fuzzy soft point  $e_{\mathcal{S}}$  is said to belong to an intuitionistic fuzzy soft set  $\mathcal{T}_A$  if  $e_{\mathcal{S}}$  is an intuitionistic fuzzy soft subset of  $\mathcal{T}_A$ .

**Example 10.** Let  $U = \{u_0, u_1, u_2\}$  and  $E = \{e_1, e_2, e_3\}$ . Consider the following intuitionistic fuzzy soft sets as follows:

$$\mathcal{S}_E = \left\{ \left( e_1, \left\{ \frac{u_0}{(0.5,0.3)}, \frac{u_1}{(0.1,0.2)} \right\} \right), \left( e_2, \left\{ \frac{u_0}{(0.3,0.5)} \right\} \right), \left( e_3, \left\{ \frac{u_2}{(0.5,0.7)} \right\} \right) \right\};$$

$$\mathcal{T}_E = \left\{ \left( e_1, \left\{ \frac{u_2}{(0.4,0.3)} \right\} \right), \left( e_2, \left\{ \frac{u_1}{(0.2,0.4)}, \frac{u_2}{(0.3,0.6)} \right\} \right), \left( e_3, \left\{ \frac{u_0}{(0.5,0.8)}, \frac{u_1}{(0.7,0.3)} \right\} \right) \right\}.$$



Note that

$$\begin{aligned} & \mathcal{S}_E \tilde{\cup} \mathcal{T}_E \\ &= \{(e_1, \{\frac{u_0}{(0.5,0.3)}, \frac{u_1}{(0.1,0.2)}, \frac{u_2}{(0.4,0.3)}\}), (e_2, \{\frac{u_0}{(0.3,0.5)}, \frac{u_1}{(0.2,0.4)}, \frac{u_2}{(0.3,0.6)}\}), \\ & (e_3, \{\frac{u_0}{(0.5,0.8)}, \frac{u_1}{(0.7,0.3)}, \frac{u_2}{(0.5,0.7)}\})\}. \end{aligned}$$

If we define an intuitionistic fuzzy soft point  $e_{1\mathcal{V}}$  by

$$\mathcal{V}_E = \{(e_1, \{\frac{u_0}{(0.5,0.3)}, \frac{u_1}{(0.1,0.2)}, \frac{u_2}{(0.4,0.3)}\})\}.$$

Then  $e_{1\mathcal{V}} \tilde{\in} \mathcal{S}_E \tilde{\cup} \mathcal{T}_E$  but  $e_{1\mathcal{V}} \not\tilde{\in} \mathcal{S}_E$  and  $e_{1\mathcal{V}} \not\tilde{\in} \mathcal{T}_E$ .

Now we define the intuitionistic fuzzy soft point in the similar manner as the *Definition 13* of soft point.

**Definition 19.** Let  $\mathcal{S}_E \in IFS(U)$  and  $u \in U$ .  $\mathcal{S}_{\bar{E}}$  is called an intuitionistic fuzzy soft point, denoted by  $IFP^{(e,u)}$ , if for the element  $e \in A$ ,  $\mathcal{S}(e)(v) = (0,1)$  for every  $v \neq u$  and  $\mathcal{S}(e') = \bar{0}$  for  $e' \in A \setminus \{e\}$  (where  $\bar{0}$  denotes the null intuitionistic fuzzy set).

**Remark 3.** It is easy to observe that the above definition of intuitionistic fuzzy soft point follows the basic rules of union and intersection similar to the crisp case.

## 4 | Conclusion

The study of soft points of a soft set is of much importance due to its applications in many mathematical structures such as soft metric spaces, soft topological spaces, soft normed spaces etc. Several definitions of soft points are available in the literature which have their own limitations and merits, therefore it is natural to unify these definitions. In this paper, we listed out those limitations and presented some examples to support our claim. We also considered and emphasized the challenges to define soft points in the framework of fuzzy soft set, interval valued fuzzy soft set, hesitant fuzzy soft set and intuitionistic fuzzy soft set theories.

As a future work, one may carry out the study of soft point to investigate exponential operation laws and operators for interval-valued q-rung orthopair fuzzy sets sine trigonometric operational laws and Pythagorean fuzzy aggregation operators are based on them, and research on the connection number based q-rung orthopair fuzzy set and their applications in decision-making processes.

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## Authors Declaration

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