FUZZY METRIC TOPOLOGY SPACE AND MANIFOLD

MAHDI MOLLAEI ARANI

Department of Mathematics, University of Payame Noor, Tehran, Iran.
Email: m.mollaei@pnu.ac.ir

Abstract. This paper, considers the fuzzy topological subsets, fuzzy topological spaces and introduces a novel concept of fuzzy Hausdorff space and fuzzy manifold space in this regards. Based on these concepts, we present a concept of fuzzy metric Hausdorff spaces and fuzzy metric manifold spaces via the notations of KM-fuzzy metric spaces. This study, generalises the concept of fuzzy metric space to union and product of fuzzy metric spaces in classical logic and in this regard investigates the some product of fuzzy metric fuzzy manifold spaces. We apply the notation of valued-level subsets and make a relation between of topological space, Husdorff space, manifold space and fuzzy topological space, fuzzy Husdorff space and fuzzy manifold space. In final, we extended the fuzzy topological space, fuzzy Husdorff space and fuzzy manifold space to fuzzy metric topological space, fuzzy metric Husdorff space and fuzzy metric manifold space Indeed, this study analyses the notation of fuzzy metric manifold based on valued-level subset.

Keywords: Fuzzy metric space, t-norm, fuzzy (metric)Hausdorff space, fuzzy (metric)manifold space.

1. Introduction

As a generalization of the classical set theory, fuzzy set theory was introduced by Zadeh to deal with uncertainties[17]. Fuzzy set theory is playing an important role in modeling and controlling unsure systems in nature, society and industry. Fuzzy set theory also plays a vital role in complex phenomena which is not easily characterized by classical set theory. After the pioneering work of Zadeh, there has been a great effort to obtain fuzzy analogues of classical theories. Among other fields, a progressive developments is made in the field of fuzzy topology. Fuzzy topology is a fundamental branch of fuzzy theory which has become an area of active research in the last arse because of its wide range of applications. One of the most important problems in fuzzy topology is to obtain an proprieties concept of fuzzy metric space. This problem has been investigated by many authors from different points view. In particular, George and Veeramani have introduced and studied a notion of fuzzy metric space with respect to the concept of t-norms. Furthermore, the class of topological spaces that are fuzzy metrizable agrees with the class of metrizable- topological spaces. This result permits Gregori and Romaguera to restate some classical theorems on metric completeness and metric (pre) compactness in the realm of fuzzy metric [4]. Kaleva, generalized the notion of the metric space by setting the distance between two points to be a non-negative fuzzy number. They by defining an ordering and an addition

http://dx.doi.org/10.22105/jfea.2022.338931.1217

in the set of fuzzy numbers obtained a triangle inequality which is analogous to the ordinary triangle inequality [11]. Historically, the notion of a differentiable manifold, that is, a set that looks locally like Euclidean Space, has been an integral part of various fields of mathematics. One may note their applications in the fields of Differential Topology [7], Algebraic Geometry, Algebraic Topology, and Lie Groups and their associated algebras. They will base our work upon the already well-established fuzzy structures viz. fuzzy topological spaces, fuzzy topological vector spaces, fuzzy derivatives. However, the definition of a fuzzy derivative provided in Foster [7], is not easily generalized to a general k derivatives. Consequently, the existence of a fuzzy differentiable manifold of class greater than one has not yet been established. They shall give topological separation axioms that have not been given previously, for sake of completeness. Our principal approach is quite similar to the methods used in [7]. Namely, They shall take definitions in [3] of fuzzy continuity and fuzzy topological vector space, and use these notions to give a fuzzy topological vector space differential structure by constructing a fuzzy homemorphism and, naturally, a fuzzy diffeomorphism of class k. To do so, They provide a new definition of a fuzzy object, known as fuzzy vectors. They then define proper fuzzy directional derivatives along these abstract fuzzy vectors, and allude to their applications in manifold learning. For completeness, they shall define tangent vector spaces to these fuzzy manifolds. Further materials regarding fuzzy topological spaces are available in the literature too [1, 2, 8, 9, 10, 11, 12, 13, 15, 16].

Regarding these points, we apply the concept of fuzzy metric spaces and make a connection between of manifolds and fuzzy subsets. The fuzzy metric spaces are not necessarily finite space, so one of our motivation of this work is a construction of finite fuzzy metric space. This study presents a concept of fuzzy (metric) Hausdorff space and fuzzy (metric) manifold space. The main our motivation of this work is the concept of fuzzy (metric) Hausdorff space and fuzzy (metric) manifold space based on t-norm such as Domby t-norm, Godel t-norm and etc.

Motivation Fuzzy subsets are mail tools in construction of classic structures via valued cuts. We try to construct topology spaces, Hausdorff spaces and manifolds in this regard. Thus this probleem is a main motivation to extension of topology spaces, Hausdorff spaces and manifolds to fuzzy (metric) Hausdorff space and fuzzy (metric) manifold space.

2. Preliminaries

In this section, we recall some definitions and results, which we need in what follows.

Theorem 2.1. [6] (*Inverse Function Theorem*) *Let U be an open subset of $\mathbb{R}^n$, and let $p \in U$. Let $g : U \longrightarrow \mathbb{R}^n$ be a smooth map. If $dg_p \in Hom(\mathbb{R}^n, \mathbb{R}^n)$ is a linear isomorphism, then there exist open neighbourhoods U_0, V_0 of $p, g(p)$ such that $g|_{U_0} : U_0 \longrightarrow V_0$ is a diffeomorphism.*

Definition 2.2. [6] *Let M be a Hausdorff topological space. We say that M is an n-dimensional topological manifold if it satisfies the following condition: for any $p \in M$, there exists*
(1) an open subset U with $p \in U \subseteq M$,
(2) an open subset $E \subseteq \mathbb{R}^n$, and

(3) a homeomorphism $\psi : U \longrightarrow E$.
Such a U is called a (local) coordinate neighbourhood, and ψ is called a (local) coordinate function. We write $x = \psi(p)$ and regard $(x_1, \ldots, x_n)$ as local coordinates for the manifold M.

Definition 2.3. [6] *Let M be a topological manifold. Let A be a set. We say that S is a C^0-atlas (or coordinate neighbourhood system) for M if $S = \{(U_\alpha, \psi_\alpha) | \alpha \in A\}$ where*
(1) U_α is an open subset of M, for all $\alpha \in A$
(2) $\psi_\alpha : U_\alpha \to E_\alpha$ is a homeomorphism to an open subset E_α of $\mathbb{R}^n$, for all $\alpha \in A$
(3) $\bigcup_{\alpha \in A} U_\alpha = M$.

Definition 2.4. [6] *Let S be a C^0-atlas for M. If $\psi_\alpha \circ \psi_\beta^{-1}$ is a C^∞-map for all $\alpha, \beta \in A$, we say that M is a C^∞-atlas for M. We call $\psi_\alpha \circ \psi_\beta^{-1}$ a coordinate transformation or transition function. The domain of the map $\psi_\alpha \circ \psi_\beta^{-1}$ is assumed to be $\psi_\beta(U_\alpha \cap U_\beta)$ (which could be the empty set). Thus, $\psi_\alpha \circ \psi_\beta^{-1}$ is a homeomorphism from $\psi_\beta(U_\alpha \cap U_\beta)$ to $\psi_\alpha(U_\alpha \cap U_\beta)$.*

Definition 2.5. [6] *Let M be a topological manifold. Let S be a C^∞-atlas for M. We say that M is a C^∞-manifold (or smooth manifold, or differentiable manifold). The concept of C^r-manifold can be defined in a similar way. However, from now on, manifold will always mean C^∞-manifold. The concept of complex manifold can be defined in a similar way, using coordinate charts $\psi : U \to \mathbb{C}^n$. However, the term complex manifold will always mean complex manifold with holomorphic (complex analytic) transition functions.*

Definition 2.6. [6] *Let M, M' be smooth manifolds with $\dim M = n$, $\dim M' = n'$. Let $\phi : M \to M'$ be a map. Let $p \in M$.*
(1) We say that ϕ is smooth (or C^∞) at p if $\psi' \circ \phi \circ \psi^{-1}$ is smooth at $\psi(p)$ for some local coordinate functions $\psi : U \longrightarrow E \subseteq \mathbb{R}^n$, $\psi' : U' \to E' \subseteq \mathbb{R}^{n'}$ with $p \in U$, $\phi(p) \in U'$.
(2) We say that ϕ is a smooth map (or C^∞-map) if ϕ is smooth at all points of M.

If $\phi : M \to M'$ satisfies the conditions (1) ϕ is bijective, (2) ϕ is smooth and (3) ϕ^{-1} is smooth. Then we say that ϕ is a diffeomorphism and M, M' are diffeomorphic.

Definition 2.1. [17] *Let X be a non-empty set. A fuzzy subset A of X is defined as, $A = \{(x, \mu_A) \mid x \in A\} = \mu_A$ where $\mu_A : A \to [0, 1]$.*

Definition 2.2. [14] *A map $T : [01] \times [01] \to [01]$ ois called a t-norm, if for all $x, y, z \in X$*

(i) $T(x, y) = T(y, x)$;
(ii) $T(x, 1) = T(1, x) = x$;
(iii) $T(x, T(y, z)) = T(T(x, y), z)$;
(iv) *if* $x \leq y$, *then* $T(x, z) \leq T(y, z)$.

Definition 2.7. [14] *A triplet (X, ρ, T) is called a KM-fuzzy metric space, if X is an arbitrary non–empty set, T is a left-continuous t-norm and $\rho : X^2 \times \mathbb{R}^{\geq 0} \to [0, 1]$ is a fuzzy set, such that for each $x, y, z, \in X$ and $t, s \geq 0$, we have:*

(i) $\rho(x, y, 0) = 0$,
(ii) $\rho(x, x, t) = 1$ *for all* $t > 0$,
(iii) $\rho(x, y, t) = \rho(y, x, t)$(*commutative property*),
(iv) $T(\rho(x, y, t), \rho(y, z, s)) \le \rho(x, z, t + s)$(*triangular inequality*),
(vi) $\rho(x, y, -) : \mathbb{R}^{\ge 0} \to [0, 1]$ *is a left-continuous map,*
(vii) $\rho(x, y, t) \to 1$, *when* $t \to \infty$.
(viii) $\rho(x, y, t) = 1, \forall\ t > 0$ *implies that* $x = y$.

If (X, ρ, T) satisfies in conditions (i)–(vii), then it is called KM-fuzzy pseudometric space and ρ is called a KM-fuzzy pseudometric.

3. Fuzzy Topological Space

In this section, we recall some definitions and results, which we need in what follows.

Definition 3.1. *Let M be an arbitrary set and $F(M) = \{\mu : M \to [0, 1]\}$. A family $\mathcal{F}_\tau$ of subset of $F(M)$ is called a fuzzy topological if satisfied in the following conditions:*

(i) $\mu_0 \in \mathcal{F}_\tau$ *and* $\mu_1 \in \mathcal{F}_\tau$;
(ii) *every* $i \in I, \mu_i \in \mathcal{F}_\tau$, *implies that* $(\bigcup_{i\in I} \mu_i) \in \mathcal{F}_\tau$;
(iii) *each* $1 \le i \le n, \mu_i \in \mathcal{F}_\tau$, *implies that* $(\bigcap_{i=1}^{n} \mu_i) \in \mathcal{F}_\tau$.

So $(M, \mathcal{F}_\tau)$ is called a fuzzy topological space and the members of $\mathcal{F}_\tau$ are called open fuzzy subsets, where $\mu_0 \equiv 0$ and $\mu_1 \equiv 1$.

Example 3.2. *Consider $M = \mathbb{R}$ and $\mathcal{F}_\tau = \{\mu_1, \mu_i \mid$ where $\mu_i = \dfrac{i}{i + x^2}$ and $i \in \mathbb{N}^*\}$. One can see that $(M, \mathcal{F}_\tau)$ is a fuzzy topological space.*

Theorem 3.3. *Let $\alpha \in [0, 1]$. Then $(M, \mathcal{F}_\tau)$ is a fuzzy topological space if and only if $(M, \mathcal{F}_\tau^{+\alpha})$ is a topological space.*

Proof. Since $\mu_0 \in \mathcal{F}_\tau$,we get that $\mu_0^{\alpha^+} = \{x \mid \mu_0(x) > \alpha\} = \emptyset$ and so $\emptyset \in \mathcal{F}_\tau^{+\alpha}$. In additon, $\mu_1 \in \mathcal{F}_\tau$, implies that $\mu_0^{\alpha^+} = \{x \mid \mu_0(x) > \alpha\} = X$, then $X \in \mathcal{F}_\tau^{\alpha^+}$. Let $\{\mu_i^{\alpha^+}\}_{i\in I} \in \mathcal{F}_\tau^{\alpha^+}$. Since $\bigvee_{i\in I} \mu_i(x) = (\bigcup_{i\in I} \mu_i)(x)$ and for all $i \in I$, $\mu_i(x) > \alpha$, we have $\bigvee_{i\in I} \mu_i(x) > \alpha$ and so $\bigcup_{i\in I} \mu_i^{\alpha^+} \in \mathcal{F}_\tau^{\alpha^+}$. If $\{\mu_i^{\alpha^+}\}_{i=1}^n \in \mathcal{F}_\tau^{\alpha^+}$,then $(\bigcap_{i=1}^{n} \mu_i)(x) = (\bigwedge_{i=1}^{n} \mu_i)(x) = \bigwedge_{i=1}^{n} (\mu_i(x)) > \alpha$. It follows that $(X, \mathcal{F}_\tau^{\alpha^+})$ is a topological space. The converse is similar to. □

Example 3.4. *Consider the fuzzy topological space $(M, \mathcal{F}_\tau)$ in Example 3.2 and $\alpha = \frac{1}{2}$. Then $\mu_0^{\frac{1}{2}^+} = \{x \in M \mid 0 > \frac{1}{2}\} = \emptyset$, $\mu_1^{\frac{1}{2}^+} = \{x \in M \mid 1 > \frac{1}{2}\} = \mathbb{R}$, and for $i \ge 2$, $\mu_i^{\frac{1}{2}^+} = \{x \in M \mid \dfrac{i}{i + x^2} > \frac{1}{2}\} = \{x \in M \mid -\sqrt{i} < x < \sqrt{i}\}$. So $(\mathbb{R}, \mathcal{F}_\tau^{\frac{1}{2}^+})$ is a topological space, where $\mathcal{F}_\tau^{\frac{1}{2}^+} = \{\emptyset, \mathbb{R}, (-\sqrt{i}, \sqrt{i}) \mid i \ge 2\}$.*

Theorem 3.5. *Let M' be a set where $|M| = |M'|$ and $(M, \mathcal{F}_\tau)$ be a fuzzy topological space. Then there exists a fuzzy topology $\mathcal{F'}_\tau$ on M' in such a way that $(M', \mathcal{F'}_\tau)$ is a fuzzy topological space.*

Proof. Since $|M| = |M'|$, there is a bijection $\phi : M' \to M$. Consider $\mathcal{F'}_\tau = \{\mu_i \circ \phi \mid \mu_i \in \mathcal{F}_\tau\}$, clearly $(M', \mathcal{F'}_\tau)$ is a fuzzy topological space. □

Example 3.6. *Consider the fuzzy topological space $(\mathbb{R}, \mathcal{F}_\tau)$ in Example 3.2 and $M = (0, 1)$. Define a map $F : \mathbb{R} \to M$ by $f(x) = \dfrac{1}{1+e^x}$. Then $((0, 1), \mathcal{F'}_\tau)$ is a fuzzy topological space.*

Corollary 3.7. *Let M be a set where $|M| = |\mathbb{R}|$. Then there exists a topology $\mathcal{F}_\tau$ on M in such a way that $(M, \mathcal{F}_\tau)$ is a fuzzy topological space.*

Definition 3.1. *Let $(M, \mathcal{F}_\tau)$ be a fuzzy topological space. A subfamily $\mathcal{FB}_\tau$ of $\mathcal{F}_\tau$ is called a base if (i), for all $x \in M$, we have $\bigvee_{\mu \in \mathcal{FB}_\tau} \mu(x) = 1$ and (ii), $\mu_1, \mu_2 \in \mathcal{FB}_\tau$ implies that $\mu_1 \cap \mu_2 \in \mathcal{FB}_\tau$.*

Theorem 3.8. *Let $(M, \mathcal{F}_\tau)$ be a fuzzy topological space and $\mathcal{FB}_\tau$ be a base for $\mathcal{F}_\tau$. Then every element of $\mathcal{F}_\tau$ is inclused in union of elements of $\mathcal{FB}_\tau$.*

Proof. Let $\mu \in \mathcal{F}_\tau$. Then for all $x \in M$, we have $\mu(x) = \mu(x) \bigvee 1 = \mu(x) \vee (\bigvee_{\mu_i \in \mathcal{FB}_\tau} \mu_i(x)) \leq \bigvee_{\mu_i \in \mathcal{FB}_\tau} \mu_i(x)$. It follows that $\mu \subseteq \bigcup_{\mu_i \in \mathcal{FB}_\tau} \mu_i$. □

Definition 3.9. *Let $(M, \mathcal{F}_\tau)$ be a fuzzy topological space and $\mathcal{FB}_\tau$ be a base for $\mathcal{F}_\tau$. Define $\langle \mathcal{FB}_\tau \rangle = \{\nu \in \mathcal{F}_\tau | \ \exists \ \mu \in \mathcal{F}_\tau, \ \mu \subseteq \nu\}$ and it called by generated fuzzy topology by $\mathcal{FB}_\tau$.*

Theorem 3.10. *Let $(M, \mathcal{F}_\tau)$ be a fuzzy topological space. Then $\langle \mathcal{FB}_\tau \rangle$ is a fuzzy topology on M.*

Proof. Since for all $\mu \in \mathcal{F}_\tau, \mu \subseteq \mu_1$, we get that $\mu_1 \in \langle \mathcal{FB}_\tau \rangle$. If $\mu_0 \in \langle \mathcal{FB}_\tau \rangle$, then $\mu \subseteq \mu_0$ implies that $\mu = \mu_0$. Let $\{\nu_i\}_{i \in I}$ a family of elements $\langle \mathcal{FB}_\tau \rangle$ and $\bigcup_{i \in I} \nu_i = \nu$. Then there exist $\mu_i \in \mathcal{F}_\tau$ in such a way that $\mu_i \subseteq \nu_i$. So $\mu = \bigcup_{i \in I} \mu_i \subseteq \bigcup_{i \in I} \nu_i = \nu$. Because $\mu \in \mathcal{F}_\tau$ and $\mu \subseteq \nu$, we get that $\nu \in \langle \mathcal{FB}_\tau \rangle$. Now, if for $n \in \mathbb{N}$, $\{\nu_i\}_{i=1}^n$ is a family of elements of $\langle \mathcal{FB}_\tau \rangle$ in a similar way we have $\bigcap_{i=1}^n \nu_i \in \langle \mathcal{FB}_\tau \rangle$. □

Proposition 3.2. *Let $(M, \mathcal{F}_\tau)$ be a fuzzy topological space and $\mathcal{FB}_\tau$ be a base for $(M, \mathcal{F}_\tau)$. Then $\mathcal{FB}_\tau^{\alpha^+}$ is a base for topological space $(M, \mathcal{F}_\tau^{\alpha^+})$.*

Proof. Let $\mathcal{B} = \{\mu^{\alpha^+} | \ \mu \in \mathcal{FB}_\tau, \alpha \in [0, 1]\}, \ x \in M$ and $\mu(x) = \alpha$. Then $x \in \mu^{\alpha^+} \subseteq \bigcup_{\alpha \in \mathcal{FB}_\tau} \mu^{\alpha^+}$ and so $\bigcup_{\alpha \in \mathcal{FB}_\tau} \mu^{\alpha^+} = M$. Suppose $x \in \mu_1^{\alpha^+} \cap \mu_2^{\alpha^+}$. Since $\mathcal{FB}_\tau$ is a base for fuzzy topological space $(M, \mathcal{F}_\tau)$, we get $\mu_1 \cap \mu_2 \in \mathcal{FB}_\tau$. Now, $x \in (\mu_1 \cap \mu_2)^{\alpha^+} \subseteq \mu_1^{\alpha^+} \cap \mu_2^{\alpha^+}$ and so $\mathcal{B}$ is a base for $\mathcal{FB}_\tau$.In the following, we will construct fuzzy topological space via topological spaces.

□

Definition 3.11. *Let (M, τ_M) be a topological space. For all $M_i \in \tau_M$, define $\tau_\emptyset = \mu_0, \tau_M = \mu_1$ and for any $i \in I, \tau_{M_i} : M \to [0,1]$ by $\tau_{M_i}(x) = \mu_i(x) = \alpha_i \in [0,1]$.*

So we have the following lemma.

Theorem 3.12. *Let (M, τ_M) be a topological space. Then $(M, \mathcal{F}_\tau = \{\tau_{M_i}\}_{i\in I})$ is a fuzzy topological space.*

Proof. Since (M, τ_M) is a topological space, we have $\emptyset, M \in \tau_M$ so by definition $\mu_0 = \tau_\emptyset, \mu_0 = \tau_M \in \tau$. Let for any $i \in I$, we have $\mu_i \in \mathcal{F}_\tau$. Since for all $x \in M$, $(\bigcup_{i\in I} \mu_i)(x) = \bigvee_{i\in I} \mu_i(x) = \bigvee_{M_i \in \tau_M} \tau_{M_i}(x) = (\bigcup_{M_i\in\tau_M} \tau_{M_i})(x)$ and $(M.\tau_M)$ is a topological space, we get that $\bigcup_{M_i\in\tau_M} \tau_{M_i} \in \tau_M$ and so $\bigcup_{i\in I} \mu_i \in \mathcal{F}_\tau$. Let $n \in \mathbb{N}$ and $\{\mu_i\}_{i=1}^n$ be a set of elements $\mathcal{F}_\tau$. In a similar way, it is shoed that $\bigcap_{i=1}^{n} \mu_i \in \mathcal{F}_\tau$. therefore, $(M, \mathcal{F}_\tau$ is a fuzzy topological space. □

Corollary 3.13. *Let M be a non-empty set. Then there exists a fuzzy topology $\mathcal{F}_\tau$ on M such that $(M, \mathcal{F}_\tau)$ is a fuzzy topological space.*

Definition 3.14. *Let $(M, \mathcal{F}_\tau)$ and $(M', \mathcal{F}_{\tau'})$ be fuzzy topological spaces and $f : (M, \mathcal{F}_\tau) \to (M', \mathcal{F}_{\tau'})$ be a homeomorphism, define $f^{\alpha^+} : (M, \mathcal{F}_\tau^{\alpha^+}) \to (M', (\mathcal{F}_{\tau'})^{\alpha^+})$ by $f^{\alpha^+}(x) = f(x)$, where $x \in M$.*

Theorem 3.15. *Let $(M, \mathcal{F}_\tau)$ and $(M', \mathcal{F}_{\tau'})$ be fuzzy topological spaces. If $f : (M, \mathcal{F}_\tau) \to (M', \mathcal{F}_{\tau'})$ is a bijection and a fuzzy continuous map, then $f^{\alpha^+} : (M, \mathcal{F}_\tau^{\alpha^+}) \to (M', (\mathcal{F}_{\tau'})^{\alpha^+})$ is a continuous map.*

Proof. Let $(\mu')^{\alpha^+} \in (\mathcal{F}_{\tau'})^{\alpha^+}$. Then

$$\begin{aligned}
(f^{\alpha^+})^{-1}(\mu')^{\alpha^+} &= \{x \in M \mid f^{\alpha^+}(x) \in (\mu')^{\alpha^+}\} \\
&= \{x \in M \mid f(x) \in (\mu')^{\alpha^+}\} = \{x \in M \mid \mu'(f(x)) > \alpha\} \\
&= \{x \in M \mid \exists \mu \in \mathcal{F}_\tau \text{ s.t } \mu(x) > \alpha\} \\
&= \mu^{\alpha^+} \in \mathcal{F}_\tau.
\end{aligned}$$

Since f^{α^+} is a bijection, we get f^{α^+} is a homeomorphism.

□

3.1. **Fuzzy manifold space.** In this subsection, we introduce a concept of fuzzy Hausdorff space based on valued-cuts and in this regards, the concept of fuzzy manifold space is presented.

Definition 3.16. *Let $(M, \mathcal{F}_\tau)$ be a fuzzy topological space. Then $(M, \mathcal{F}_\tau)$ is called a fuzzy Hausdorff space if, for all $x, y \in M$ there exist $\mu_1, \mu_2 \in \mathcal{F}_\tau$ and $0 \leq \alpha < \beta \leq 1$ in such a way that $x \in \mu_1^{(\alpha,\beta)^+}, y \in \mu_2^{(\alpha,\beta)^+}$ and $\mu_1^{(\alpha,\beta)^+} \cap \mu_2^{(\alpha,\beta)^+} = \emptyset$, where for all $\mu \in \mathcal{F}_\tau$, we have $\mu^{(\alpha,\beta)^+} = \{x \in M \mid \alpha < \mu(x) < \beta\}$.*

Example 3.17. *Consider the fuzzy topological space, which is defined in Example 3.2. A simple computations show that for $i \neq i' \in \mathbb{N}^*$*

$$J_i = (-\sqrt{i(\frac{1}{\alpha}-1)}, -\sqrt{i(\frac{1}{\beta}-1)}) \cup (\sqrt{i(\frac{1}{\beta}-1)}, \sqrt{i(\frac{1}{\alpha}-1)})$$
$$J_{i'} = (-\sqrt{i'(\frac{1}{\alpha}-1)}, -\sqrt{i'(\frac{1}{\beta}-1)}) \cup (\sqrt{i'(\frac{1}{\beta}-1)}, \sqrt{i'(\frac{1}{\alpha}-1)})$$

, where $J_i = \mu_i^{(\alpha,\beta)^+}$ and $J_{i'} = \mu_{i'}^{(\alpha,\beta)^+}$. If $x, y \in \mathbb{R}$, since $\mathbb{R}$ is a Hausdorff space, there exists $i, i' \in \mathbb{N}^*$ such that $x \in J_i$, $y \in J_{i'}$, and $J_i \cap J_{i'} = \emptyset$. (For all $x \in \mathbb{R}$, consider $0 \leq \alpha < \beta \leq 1$, then $\frac{\alpha x^2}{1-\alpha} < i < \frac{\beta x^2}{1-\beta}$.)

Theorem 3.18. *Let $(M, \mathcal{F}_\tau)$ be a fuzzy Hausdorff space and $\alpha \in [0,1]$. Then $(M, \mathcal{F}_\tau^\alpha)$ is a Hausdorff space.*

Proof. Since $(M, \mathcal{F}_\tau)$ is a fuzzy topological space, $(M, \mathcal{F}_\tau^\alpha)$ is a topological space. Let $x, y \in M$ and $\alpha, \beta \in [0,1]$, because of $(M, \mathcal{F}_\tau)$ is a fuzzy Hausdorff space, there exist, $\mu_1, \mu_2 \in \mathcal{F}_\tau$ and $0 \leq \alpha' < \beta' < 1$ such that $x \in \mu_1^{(\alpha',\beta')^+}$ and $y \in \mu_2^{(\alpha',\beta')^+}$. Now consider $\alpha \leq T_{min}\{\alpha', \beta'\}$. It follows that, $x \in \mu^\alpha$ and $y \in \mu^\beta$ and $\mu^\alpha \cap \mu^\beta = \emptyset$. So $(M, \mathcal{F}_\tau^\alpha)$ is a Haussdorff space. □

Definition 3.19. *Let $(M, \mathcal{F}_\tau)$ be a fuzzy Hausdorff space. Then $(M, \mathcal{F}_\tau)$ is called a fuzzy manifold if, for all $x \in M$, there exists $\mu \in \mathcal{F}_\tau$ and homeomorphism $\phi : supp(\mu) \to \mathbb{R}^n$ such that $x \in Supp(\mu)$. Each (μ, ϕ) is called a fuzzy chart and $\mathcal{A} = \{(\mu, \phi) \mid \mu \in \mathcal{F}_\tau,\ \phi : supp(\mu) \to \mathbb{R}^n\}$ is called a fuzzy atlas. Let $(\mu, \phi), (\upsilon, \psi)$ be two fuzzy charts of fuzzy atlas $\mathcal{A}$. Then $(\mu, \phi), (\upsilon, \psi)$ are called C^∞-compatible charts if $\phi : supp(\mu) \to \mathbb{R}^n$, $\psi : supp(\upsilon) \to \mathbb{R}^n$ and $\phi \circ \psi^{-1} : \psi(Supp(\mu) \cap Supp(\upsilon)) \to \phi(Supp(\mu) \cap Supp(\upsilon))$ are a C^1-fuzzy diffeomorphism.*

Example 3.20. *Consider the fuzzy Hausdorff space, which is defined in Example 3.25. It is clear that for all $x \in \mathbb{R}$, and for all $i \in \mathbb{N}$, we have $x \in Supp(\mu_i) = \mathbb{R}$, we get that $x \in Supp(\mu_i)$. Define $(Ln)_i : Supp(\mu_i) \longrightarrow \mathbb{R}$, so $\mathcal{A} = \{(\mu_i, (Ln)_i) \mid i \in \mathbb{N}\}$ is a fuzzy atlas.*

Theorem 3.21. *Let $(M, \mathcal{F}_\tau)$ be a fuzzy manifold and $\alpha \in [0,1]$. Then $(M, \mathcal{F}_\tau^\alpha)$ is a manifold.*

Proof. Since $(M, \mathcal{F}_\tau)$ is a fuzzy manifold, by Theorem 3.18, $(M, \mathcal{F}_\tau^\alpha)$ is a Hausdorff topological space. Inaddition, for all $x \in M$, there exists $\mu \in \mathcal{F}_\tau$ and homeomorphism $\phi : supp(\mu) \to \mathbb{R}^n$ such that $x \in Supp(\mu)$. Now consider $\mathcal{A}^\alpha = \{(\mu^\alpha, \phi) \mid \mu \in \mathcal{F}_\tau,\ \phi : supp(\mu) \to \mathbb{R}^n\}$. One can see that $\mathcal{A}^\alpha$ is an atlas. Thus $(M, \mathcal{F}_\tau^\alpha)$ is a manifold.

□

In the following, we wnant to extend two fuzzy topological space to a larger class of fuzzy topological spaces.

Definition 3.22. *Let $\mathcal{F}_\tau$ and $\mathcal{F}_{\tau'}$ be fuzzy topology on M and M', respectively. Define $\mathcal{F}_\tau \times \mathcal{F}_{\tau'} = \{\mu \times \mu' \mid \mu \in \mathcal{F}_\tau, \mu' \in \mathcal{F}_{\tau'}\}$, where for all $x, y \in M \times M'$ and $(\mu \times \mu')(x, y) = T_{min}(\mu(x), \mu(y))$.*

Theorem 3.23. *Let $(M, \mathcal{F}_\tau)$ and $(M', \mathcal{F}_{\tau'})$ be fuzzy topological spaces. Then $(M \times M', \mathcal{F}_\tau \times \mathcal{F}_{\tau'})$ is a fuzzy topological space.*

Proof. Since $\mu_0 \in \mathcal{F}_\tau \cap \mathcal{F}_{\tau'}$ for all $(x,y) \in M \times M'$ we have $(\mu_0 \times \mu_0)(x,y) = T_{min}\{\mu_0(x), \mu_0(y)\} = 0$, and so $(\mu_0 \times \mu_0) \in \mathcal{F}_\tau \times \mathcal{F}_{\tau'}$. In addition, $\mu_1 \in \mathcal{F}_\tau \cap \mathcal{F}_{\tau'}$, implies that $(\mu_1 \times \mu_1)(x,y) = T_{pr}\{\mu_1(x), \mu_1(y)\} = 1$ and so $\mu_1 \times \mu_1 \in \mathcal{F}_\tau \times \mathcal{F}_{\tau'}$. Let $\{\mu_i\}_{i\in I}$ and$\{\mu'_j\}_{j\in J}$ be two families of fuzzy topologies on M and M', respectively. Then $(\bigcup\limits_{i\in I, j\in J} (\mu_i \times \mu'_j))(x,y) = (\bigvee\limits_{i\in I, j\in J} (\mu_i \times \mu'_j))(x,y) = \bigvee\limits_{i\in I, j\in J} (T_{min}(\mu_i \times \mu'_j))$ implies that $\bigcup\limits_{i\in I, j\in J} (\mu_i \times \mu'_j) \in \mathcal{F}_\tau \times \mathcal{F}_{\tau'}$. Let $\{\mu_i\}_{i=1}^n$ and $\{\mu'_j\}_{j=1}^m$ be two families of fuzzy topologies on M and M',respectively. Then $(\bigcap\limits_{i=1,j=1}^{n,m} (\mu_i \times \mu'_j))(x,y) = (\bigvee\limits_{i=1,j=1}^{n,m} (\mu_i \times \mu'_j))(x,y) = \bigvee\limits_{i=1,j=1}^{n,m} (T_{min}(\mu_i, \mu'_j))$ implies that $\bigcap\limits_{i=1,j=1}^{n,m} (\mu_i \times \mu'_j) \in \mathcal{F}_\tau \times \mathcal{F}_{\tau'}$. Thus $(M \times M', \mathcal{F}_\tau \times \mathcal{F}_{\tau'})$ is a fuzzy topological space.

□

3.2. **Fuzzy metric manifolds.** In this subsection, we introduce the concept of fuzzy metric topological space and investigate its properties.

Definition 3.24. *Let (M, ρ, T) be a fuzzy metric space and $F(M) = \{\mu : M \to [0,1]\}$. Then $(M, \rho, T, \mathcal{F}_\tau)$ is called a fuzzy metric topological space, if*

(i) $\mathcal{F}_\tau$ is a fuzzy topology on M;

(ii) for all $x, y \in M$, $t \in \mathbb{R}^+$ and $\mu_1 \neq \mu \in \mathcal{F}_\tau$, we have $T(\mu(x), \mu(y)) \leq \rho(x,y,t)$.

Example 3.25. *Let $M = \mathbb{R}$. Then (M, T_{pr}, ρ) is a fuzzy metric space, where for all $x, y \in \mathbb{R}$ and $t \in \mathbb{R}^+, \rho(x,y,t) = |\dfrac{\min\{x,y\}+t}{\max\{x,y\}+t}|$. It is easy to see that $(M, \rho, T, \mathcal{F}_\tau)$ is a fuzzy metric topological space, where $\mathcal{F}_\tau = \{\mu_1, \mu_i \mid$ where $\mu_i = \dfrac{i}{i+x^2}$ and $i \in \mathbb{N}^*\}$.*

Theorem 3.26. *Let (M, τ_M) be a topological space and T be a t-norm on M. Then there exists a fuzzy subset $\rho : M^2 \times \mathbb{R}^+ \to [0,1]$ such that $(M, \rho, T, \mathcal{F}_\tau)$ is a fuzzy metric topological space.*

Proof. Let $\mu \in \mathcal{F}_\tau, t \in \mathbb{R}^+$ and $x, y \in M$. Define

$$\rho(x,y,t) = \begin{cases} T(\mu(x), \mu(y)) & \text{if } x \neq y, \\ 1 & \text{otherwise.} \end{cases}$$

Now, we show that $H = (M, \rho, T, \mathcal{F}_\tau)$ is a fuzzy metric topological space. By definition, for all $x \in M$, $\rho(x,x,t) = 1$ and for all $y \in M$, $\rho(x,y,t) > 0$. Let $x, y, z \in M$. Then for all $t, s \in \mathbb{R}^+$

$$\begin{aligned} T(\rho(x,y,t), \rho(x,y,s)) &= T(T(\mu(x), \mu(y), T(\mu(y), \mu(z))) \\ &\leq T(\mu(x), \mu(z)) = \rho(x,y,t+s). \end{aligned}$$

So (M, ρ, T) is a fuzzy metric space. Let $x, y \in M, t \in \mathbb{R}^+$ and $\mu_1 \neq \mu_2 \in \mathcal{F}_\tau$. Hence $T(\mu(x), \mu(y)) = \rho(x,y,t) \leq \rho(x,y,t)$. Therefore, $(M, \rho, T, \mathcal{F}_\tau$ is a fuzzy metric topological space. □

Theorem 3.27. *Let $(M,\rho,T,\mathcal{F}_\tau)$ be a fuzzy metric topological space, $x,y\in M$ and $\alpha\in[0,1]$. Then*

(i) If $\mu=\mu_1$, then $T(\mu(x),\mu(y))\leq\rho(x,y,t)$ implies that $x=y$.

(ii) For any $i\in I,\mu_i\in\mathcal{F}_\tau$, we have $T((\bigcup_{i\in I}\mu_i)(x),(\bigcup_{i\in I}\mu_i)(y))\leq\rho(x,y,t)$,

(iii) For any $i\in\mathbb{N},\mu_i\in\mathcal{F}_\tau$, we get $T((\bigcap_{i=1}^{n}\mu_i)(x),(\bigcap_{i=1}^{n}\mu_i)(y))\leq\rho(x,y,t)$.

Proof. (i) since for all $x\in M$, $\mu(x)=1$, we get $T(\mu(x),\mu(x))=T(1,1)=1\leq\rho(x,y,t)$. Thus $\rho(x,y,t)=1$ and so $x=y$.

(ii) Let $x,y\in M$. Then $T((\bigcap_{i=1}^{n}\mu_i)(x),(\bigcap_{i=1}^{n}\mu_i)(y)) = T(\bigwedge_{i=1}^{n}\mu_i(x),\bigwedge_{i=1}^{n}\mu_i(y) \leq T(\mu_i(x),\mu_i(y))\leq \rho(x,y,t)$.

(iii) Since $(M,\mathcal{F}_\tau)$ is a fuzzy topological space, for all $i\in I$ and $\mu_i\in\mathcal{F}_\tau,\mu=\bigcup_{i\in I}\mu_i\in\mathcal{F}_\tau$. So

$$T((\bigcup_{i\in I}\mu_i)(x),(\bigcup_{i\in I}\mu_i)(y))=T(\mu(x),\mu(y))\leq\rho(x,y,t).$$

□

Theorem 3.28. *Let $(M,\rho,T,\mathcal{F}_\tau)$ be a fuzzy metric topological space and $\alpha\in[0,1]$. Then $(M,\rho,T,\mathcal{F}_\tau^{+\alpha})$ is a topological space.*

Proof. It is similar to Theorem 3.10. □

Theorem 3.29. *Let M be a non-empty set. Then there exists a t-norm T, a fuzzy metric subset $\rho:M^2\times\mathbb{R}^+\to[0,1]$, and fuzzy topology $\mathcal{F}_\tau$ on M such that $(M,\rho,T,\mathcal{F}_\tau)$ is a fuzzy metric topological space.*

Proof. Let M be a non-empty set. Then there exists a topology τ on M such that (M,τ) is a topological space. Using Theorem 3.13, there exists a fuzzy topology $\mathcal{F}_\tau$ on M such that $(M,\mathcal{F}_\tau)$ is a fuzzy topological space. Hence there exists a fuzzy subset $\rho:M^2\times\mathbb{R}^+\to[0,1]$, such that (M,ρ,T_{pr}) is a fuzzy metric space. Thus there exists a t-norm T, a fuzzy metric subset $\rho:M^2\times\mathbb{R}^+\to[0,1]$, and fuzzy topology $\mathcal{F}_\tau$ such that $(M,\rho,T,\mathcal{F}_\tau)$ is a fuzzy metric topological space. □

Theorem 3.30. *Let M' be a set where $|M|=|M'|$ and $(M,\rho,T,\mathcal{F}_\tau)$ be a fuzzy metric topological space. Then there exists a topology ${\mathcal{F}'}_\tau$ on M' and fuzzy subset $\rho:M^2\times\mathbb{R}^+\to[0,1]$ in such a way that $(M',\rho,T,{\mathcal{F}'}_\tau)$ is a fuzzy metric topological space.*

Proof. Since $|M|=|M'|$, there is a bijection $\phi:M'\to M$. Consider ${\mathcal{F}'}_\tau=\{\mu_i\circ\phi\mid\mu_i\in\mathcal{F}_\tau\}$, clearly $(M',{\mathcal{F}'}_\tau)$ is a fuzzy topological space. Based on Theorem 3.26, there exists a fuzzy subset $\rho:M^2\times\mathbb{R}^+\to[0,1]$ and a t-norm T such that $(M',\rho,T,{\mathcal{F}'}_\tau)$ is a fuzzy metric topological space. □

Corollary 3.31. *Let M be a set where $|M|=|\mathbb{R}|$. Then there exists a fuzzy topology $\mathcal{F}_\tau$ on M, a fuzzy subset $\rho:M^2\times\mathbb{R}^+\to[0,1]$ and a t-norm T such that $(M,\rho,T,\mathcal{F}_\tau)$ is a fuzzy metric topological space.*

In the following, results are similar to perevious section.

Definition 3.3. *Let $(M,\rho,T,\mathcal{F}_\tau)$ be a fuzzy metric topological space. A subfamily $\mathcal{FB}_\tau$ of $\mathcal{F}_\tau$ is called a base if (i), for all $x\in M$, we have $\bigvee_{\mu\in\mathcal{FB}_\tau}\mu(x)=1$ and $(ii), \mu_1,\mu_2\in\mathcal{FB}_\tau$ implies that $\mu_1\cap\mu_2\in\mathcal{FB}_\tau$.*

Theorem 3.32. *Let $(M,\rho,T,\mathcal{F}_\tau)$ be a fuzzy metric topology space and $\mathcal{FB}_\tau$ be a base for $\mathcal{F}_\tau$. Then every element of $\mathcal{F}_\tau$ is inclosed in union of elements of $\mathcal{FB}_\tau$.*

Proof. The proof is similar to Theorem 3.8. □

Definition 3.33. *Let $(M,\rho,T,\mathcal{F}_\tau)$ be a fuzzy metric topological space and $\mathcal{FB}_\tau$ be a base for $\mathcal{F}_\tau$. Define $\langle\mathcal{FB}_\tau\rangle=\{\nu\in\mathcal{F}_\tau |\ \exists\ \mu\in\mathcal{F}_\tau,\ \mu\subseteq\nu\}$ and it called by generated topology by $\mathcal{FB}_\tau$.*

Theorem 3.34. *Let $(M,\rho,T,\mathcal{F}_\tau)$ be a fuzzy metric topological space. Then $\langle\mathcal{FB}_\tau\rangle$ is a fuzzy topology on M.*

Proof. The proof is similar to Theorem 3.10. □

Proposition 3.4. *Let $(M,\rho,T,\mathcal{F}_\tau)$ be a fuzzy metric topological space and $\mathcal{FB}_\tau$ be a base for $(M,\rho,T,\mathcal{F}_\tau)$. Then $\mathcal{FB}_\tau^{\alpha^+}$ is a base for topological space $(M,\mathcal{F}_\tau^{\alpha^+})$.*

Definition 3.35. *Let $(M,\rho,T,\mathcal{F}_\tau)$ and $(M',\rho,T,\mathcal{F}_{\tau'})$ be fuzzy metric topological spaces and $f:(M,\mathcal{F}_\tau)\to(M',\mathcal{F}_{\tau'})$ be a homeomorphism, define $f^{\alpha^+}:(M,\mathcal{F}_\tau^{\alpha^+})\to(M',(\mathcal{F}_{\tau'})^{\alpha^+})$ by $f^{\alpha^+}(x)=f(x)$, where $x\in M$.*

Theorem 3.36. *Let $(M,\rho,T,\mathcal{F}_\tau)$ and $(M',\rho,T,\mathcal{F}_{\tau'})$ be fuzzy metric topological spaces. If $f:(M,\rho,T,\mathcal{F}_\tau)\to(M',\rho,T,\mathcal{F}_{\tau'})$ be a fuzzy continuous map, then $f^{\alpha^+}:(M,\mathcal{F}_\tau^{\alpha^+})\to(M',(\mathcal{F}_{\tau'})^{\alpha^+})$ is a continuous map.*

Proof. The proof is similar to Theorem 3.15. □

Definition 3.37. *Let $(M,\rho,T,\mathcal{F}_\tau)$ be a fuzzy metric topological space. Then $(M,\rho,T,\mathcal{F}_\tau)$ is called a fuzzy metric Hausdorff space if, for all $x,y\in M$ there exist $\mu_1,\mu_2\in\mathcal{F}_\tau$ in such a way that $x\in supp(\mu_1), y\in supp(\mu_2)$ and $\mu_1\cap\mu_2=\emptyset$ where $Supp(\mu)=\{x\mid\mu(x)\neq 0\}$ and $\mu_1\cap\mu_2=\emptyset$ means that $(\mu_1\cap\mu_2)(x)=0$.*

Definition 3.38. *Let $(M,\rho,T,\mathcal{F}_\tau)$ be a fuzzy metric Hausdorff space. Then $(M,\rho,T,\mathcal{F}_\tau)$ is called a fuzzy metric manifold if, for all $x\in M$, there exists $\mu\in\mathcal{F}_\tau$ and homeomorphism $\phi:supp(\mu)\to\mathbb{R}^n$ such that $x\in Supp(\mu)$.Each (μ,ϕ) is called a fuzzy chart and $\mathcal{A}=\{(\mu,\phi)\mid\mu\in\mathcal{F}_\tau,\ \phi:supp(\mu)\to\mathbb{R}^n\}$ is called a fuzzy atlas.Let $(\mu,\phi),(\upsilon,\psi)$ be two fuzzy chart of fuzzy atlas $\mathcal{A}$. Then $(\mu,\phi),(\upsilon,\psi)$ are called C^∞-compatible charts if phi $:supp(\mu)\to\mathbb{R}^n$, $\psi:supp(\upsilon)\to\mathbb{R}^n$ and $\phi\circ\psi^{-1}:\psi(Supp(\mu)\cap Supp(\upsilon))\to\phi(Supp(\mu)\cap Supp(\upsilon))$ are a C^1-fuzzy diffeomorphism.*

Example 3.39. *Consider the fuzzy Hausdorff space, which is defined in Example 3.25. It is clear that for all $x\in\mathbb{R}$, and for all $i\in\mathbb{N}$, we have $x\in Supp(\mu_i)=\mathbb{R}$, we get that $x\in Supp(\mu_i)$. Define $(Ln)_i:Supp(\mu_i)\longrightarrow\mathbb{R}$, so $\mathcal{A}=\{(\mu_i,(Ln)_i)\mid i\in\mathbb{N}\}$ is a fuzzy atlas. Now, define for all $x,y\in\mathbb{R}$ and*

$t \in \mathbb{R}^+, \rho(x,y,t) = |\frac{\min\{x,y\}+t}{\max\{x,y\}+t}|$. *It is easy to see that* $(M, \rho, T, \mathcal{F}_\tau)$ *is a fuzzy metric topological space, where* $\mathcal{F}_\tau = \{\mu_1, \mu_i \mid$ *where* $\mu_i = \frac{i}{i+x^2}$ *and* $i \in \mathbb{N}^*\}$ *and consequently* $(M, \rho, T, \mathcal{F}_\tau)$ *is a fuzzy metric manifold.*

4. **Conclusion**

The current paper has introduced a novel concept of fuzzy Hausdorff space, fuzzy manifold space. Also:

(i) Based on fuzzy toplology, every non empty set converted to a fuzzy Hausdorff space.

(ii) It is showed that the product and union of fuzzy Hausdorff spaces is a fuzzy Hausdorff space.

(iii) The extended fuzzy metric spaces are constructed using the some algebraic operations on fuzzy metric spaces.

(iv) The concept of fuzzy Hausdorff space and fuzzy manifold space is defined and investigated some its properties.

We hope that these results are helpful for further studies in theory of fuzzy metric Hausdorff space fuzzy metric manifold space. In our future studies, we hope to obtain more results regarding instuitic metric Hausdorff spaces, neutrosophic metric manifold spaces and their applications.

References

[1] C. L. Change, Fuzzy Topological Spaces, J. Math. Anal. Appl., **24** (1987), 182-190.

[2] M. Ferraro and D. Foster, C^1 fuzzy manifolds, Fuzzy Sets and System, **54**(1993), 99-106.

[3] M. Ferraro, David Foster, Differentiation of Fuzzy Continuous Mappings on Fuzzy Topological Vector Spaces Journal of Mathematical Analysis and Applications, **121(2)** (1987), 589-601.

[4] V. Gregori, S. Morillas and A. Sapena, Examples of fuzzy metrics and applications, Fuzzy Sets and Systems, **70** (2011), 95–111.

[5] A. George and P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets and Systems **64** (1994), 395–399.

[6] M. Lee, Introduction to Smooth Manifolds, Springer Science, New York, 2003.

[7] M. Hirsch, Differential Topology, Springer, New York, 6th edition, 1997.

[8] M. Hamidi, S. Jahanpanah, A. Radfar, Extended graphs based on KM-fuzzy metric spaces, Iranian Journal of Fuzzy Systems, **17(5)** (2020), 81-95.

[9] M. Hamidi, S. Jahanpanah, A. Radfar, On KM-Fuzzy Metric Hypergraphs, Fuzzy Information and Engineering, **12(3)** (2021), 300–321.

[10] K. Katsara and D. B. Liu, Fuzzy vector space and fuzzy topological vector spaces, J. Math. Anal. Appl., **58** (1977), 135-146.

[11] O. Kaleva and S. Seikkala, On Fuzzy Metric Spaces, Fuzzy Sets and Systems, **12** (1984), 215–229.

[12] R. Lowen, Fuzzy topological space and fuzzy compactness, J. Math. Anal. Appl., **56** (1976), 621-633.

[13] B. M. Pu and Y. M. Liu, Fuzzy topology I: Neighborhood structure of a fuzzy point and Moore smith convergence, J. Math. Anal. Appl., **76** (1980), 571-599

[14] I. Mardones-Perez and M. A. de Prada Vicente, Fuzzy pseudometric spaces vs fuzzifying structures, Fuzzy Sets and Systems **267** (2015), 117–132.

[15] G. Rano and T. Bag, Fuzzy Normed Linear Spaces, International Journal Of Mathematics and Scientific Computing, **2** (2012), 16-19.

[16] R. Saadati and M. Vaezpour, Some Results On Fuzzy Banach Spaces, J. Appl. Math. Comput., **17** (2005), 475-484.

[17] L. A. Zadeh, Fuzzy sets, Information and Control, **8(3)** (1965), 338–353.