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Fermatean Fuzzy Modified Composite Relation and its Application in Pattern Recognition

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Abstract

Fermatean Fuzzy Sets (FFSs) provide an effective way to handle uncertainty and vagueness by expanding the scope of membership and Non-Membership Degrees (NMDs) of Intuitionistic Fuzzy Set (IFS) and Pythagorean Fuzzy Set (PFS), respectively. FFS handles uncertain information more easily in the process of decision making. The concept of composite relation is an operational information measure for decision making. This study establishes Fermatean fuzzy composite relation based on max-average rule to enhance the viability of FFSs in machine learning via soft computing approach. Some numerical illustrations are provided to show the merit of the proposed max-average approach over existing the max-min-max computational process. To demonstrate the application of the approach, we discuss some pattern recognition problems of building materials and mineral fields with the aid of the Fermatean fuzzy modified composite relation and Fermatean fuzzy max-min-max approach to underscore comparative analyses. In recap, the objectives of the paper include: 1) discussion of FFS and its composite relations, 2) numerical demonstration of Fermatean fuzzy composite relations, 3) establishment of a decision application framework under FFS in pattern recognition cases, and 4) comparative analyses to showcase the merit of the new approach of Fermatean fuzzy composite relation. In future, this Fermatean fuzzy modified composite relation could be studied in different environments like picture fuzzy sets, spherical fuzzy sets, and so on.

Keywords: Intuitionistic fuzzy sets, Pythagorean fuzzy sets, Fermatean fuzzy sets, Fermatean fuzzy composite relation, Pattern recognition.

1 | Introduction

Zadeh [1] proposed the concept of fuzzy sets which deal with imprecise and vague information, and also serve as an effective tool to solve decision-making problems. However, fuzzy set considered the Membership Degree (MD) only. Sequel to this setback, Atanassov [2] proposed the Intuitionistic Fuzzy Sets (IFSs) to characterize uncertainty information by incorporating MD and Non-Membership Degree (NMD). The introduction of IFSs received a lot of attention in different fields, such as in medical diagnosis, pattern recognition [3]-[5] etc. But in a case where the sum of MD and NMD is greater than 1, IFS is no longer applicable.

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Yager [6] introduced Pythagorean Fuzzy Sets (PFSs) where the squared sum of its MD and NMD is less than or equal to 1. Since the concept was brought up it has been widely applied in different fields such as service control of domestic air lines [7], investment decision making [8], career placements based on academic performance [9], etc.

Although IFS and PFS facilitate the resolution of fuzzy decision problems, they still have obvious shortcomings, especially in extremely contradictory decision environments. PFS and IFS are unable to handle a situation where the sum of MD and NMD is greater than 1 and the sum of squares is still greater than 1, but the sum of cube is less than or equal to 1. For such cases, Senapati and Yager [10] developed the novel concept called Fermatean Fuzzy Set (FFS), which satisfies the criterion that the sum of the third power of MD and NMD must be less than or equal to 1. With comparison to IFS and PFS, FFS gains a stronger ability to describe uncertain information by expanding the spatial scope of MD and NMD. Based on FFS, Wang et al. [11] developed a hesitant Fermatean fuzzy multicriteria decision-making method using Archimedean Bonferroni mean operators, Senapati and Yager [10] proposed Fermatean fuzzy information weighted aggregation operators, and Liu et al. [12] developed a distance measure method for Fermatean fuzzy linguistic term sets. Furthermore, Liu et al. [13] defined a new concept of a Fermatean fuzzy linguistic set and Senapati and Yager [14] developed some new operations between Fermatean fuzzy Numbers (FFNs).

Sahoo [15] presented a similarity measure for FFSs with group decision-making application. Some score functions on FFSs have been studied and applied in transportation problem and bride selection [16] and [17]. Some uncertain approaches have been studied and applied [18]-[20]. Ejegwa et al. [21] studied composite relation on FFSs based on max-min-max approach with application to medical diagnosis. The Fermatean fuzzy composite relation presented in [21] used the extreme values of MD and NMD of the FFSs. This approach is not reliable because it cannot show the relation between two similar FFSs. Hence, this study seeks to explore Fermatean Fuzzy Max-Min-Max Composite Relation (FFMMMCR), highlight its setback and modify its applications in pattern recognition cases. The specific objectives of the work are to:

Discuss the FFS and its composite relations with some properties of the Fermatean fuzzy composite relation, numerically demonstrate the Fermatean fuzzy composite relations, establish a decision application framework under FFSs in pattern recognition cases, and present comparative analyses to showcase the merit of the new approach of Fermatean fuzzy composite relation.

This paper is presented as follows: Section two presents IFS, PFS, FFSs and their characterizations, Section three discusses Fermatean fuzzy composite relations with some examples, Section four presents the application of the studied Fermatean fuzzy composite relation and the approach in [21] to pattern recognition problems and discusses the results, and Section five summaries the paper with recommendations.

2 | Preliminaries

Definition 1. [2]. IFS A is defined on a non-empty set X as object having the form

$$A=\{\langle x,\alpha_A(x),\beta_A(x)\rangle : x \in X\}.$$

Where the functions

$$\alpha_A: X\rightarrow [0,1] \text{ and } \beta_A: X\rightarrow [0,1].$$

Denote MD and NMD of each element x ∈ X to the set A, respectively, and

$$0 \le \alpha_A(x) + \beta_A(x) \le 1 \text{ for all } x \in X.$$

Obviously, when $\beta_A(x)=1-\alpha_A(x)$ for all x ∈ X, the set A becomes a fuzzy set. Furthermore, we have

$$\pi_A(x)=1-\alpha_A(x)-\beta_A(x).$$

Called the IFS index or hesitation margin of x in A. The function $\pi_A(x)$ is the degree of indeterminacy of x ∈ X to the IFS A and $\pi_A(x)\in[0,1]$ i.e., π_A: $X\rightarrow[0,1]$ and $0\leq\pi_A(x)\leq1$ for every x ∈ X . $\pi_A(x)$ expresses the lack of knowledge of whether x belong to IFS A or not.

Definition 2. [7]. Let A and B be IFSs, then we have

I. $A\subseteq B \Leftrightarrow \alpha_A(x)\leq\alpha_B(x)$ and $\beta_A(x)\geq\beta_B(x)\forall$ x ∈ X.
II. $Ac=\{\langle x,\beta_A(x),\alpha_A(x)\rangle{:}x\in X\}$.
III. $A\cup B=\{\langle x,\max(\alpha_A(x),\alpha_B(x)),min(\beta_A(x),\beta_B(x))\rangle{:}x\in X\}$.
IV. $A\cap B=\{\langle x,\min(\alpha_A(x),\alpha_B(x)),max(\beta_A(x),\beta_B(x))\rangle{:}x\in X\}$.
V. $A\oplus B=\{\langle x,\alpha_A(x)+\alpha_B(x)-\alpha_A(x)\alpha_B(x),\beta_A(x),\beta_B(x)\rangle{:}\ x\in X\}$.
VI. $A\otimes B=\{\langle x\alpha_A(x)\alpha_B(x),\beta_A(x)+\ \beta_B(x)-\beta_A(x)\ \beta_B(x)\rangle{:}x\in X\}$.

Definition 3. [7]. PFS defined on a non-empty set X is an object having the form $P=\{\langle x,\alpha_P(x),\beta_P(x)\rangle{:}x\in X\}$, where the functions α_P: X⟶[0,1] and β_P: X⟶[0,1] denote the degree of membership and the degree of NMD of each element x ∈ X to the set P, respectively, and

$$0\leq\alpha_P^2(x)+\beta_P^2(x)\leq1.$$

For every x ∈ X.

For any PFS P, the function $\pi_P(x)=\sqrt{1-\alpha_P^2(x)-\beta_P^2(x)}$ is called the degree of indeterminacy of x to P. A Pythagorean fuzzy number of a PFS P is denoted by $P=(\alpha_P,\beta_P)$.

Definition 4. [7]. Given two PFNs $P=(\alpha_P,\beta_P)$ and $Q=(\alpha_Q,\beta_Q)$ where $\alpha_P,\beta_P\in[0,1]$, $\alpha_Q,\beta_Q\in[0,1]$, then some arithmetic operations can be described as follows:

I. $P\cup Q=(max\{\alpha_P,\alpha_Q\},min\{\beta_P,\beta_Q\})$.
II. $P\cap Q=(min\{\alpha_P,\alpha_Q\},max\{\beta_P,\beta_Q\})$.
III. $P^c=(\beta_P,\alpha_P)$.

Definition 5. [14]. Let X be a universe of discourse. A FFS F in X is an object having the form

$$F=\{\langle x,\boldsymbol{\alpha_F(x)},\boldsymbol{\beta_F(x)}\rangle{:}\ x\in X\}.$$

Where α_F : X⟶ [0, 1] and β_F : X⟶ [0, 1] and $0\leq\alpha_F^3(x)+\beta_F^3(x)\leq1$, for all x ∈ X. The numbers $\alpha_F(x)$ and $\beta_F(x)$ represent MD and NMD, respectively of F. For any FFS F, the function $\pi_F(x)=\sqrt[3]{1-\alpha_F^3(x)-\beta_F^3(x)}$ is identified as the degree of indeterminacy of x ∈ X in F.

For convenience, Senapati and Yager [14] called $(\alpha_F(x),\beta_F(x))$ a FFN denoted by $F=(\alpha_F,\beta_F)$.

Definition 6. [14]. Let $F=(\alpha_F,\beta_F)$, $F_1=(\alpha_{F_1},\beta_{F_1})$ and $F_2=(\alpha_{F_2},\beta_{F_2})$ be three FFNs, then their operations are defined as follows:

I. $F_1\cap F_2=(min\ \{\alpha_{F_1},\alpha_{F_2}\},max\ \{\beta_{F_1},\beta_{F_2}\})$.
II. $F_1\cup F_2=(max\ \{\alpha_{F_1},\alpha_{F_2}\},min\ \{\beta_{F_1},\beta_{F_2}\})$.
III. $F^c=(\beta_F,\alpha_F)$.

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Definition 7. [21]. Fermatean Fuzzy Values (FFVs) are describe by $\langle x, y\rangle$ for $x^3 + y^3 \leq 1$ where $x, y \in [0,1]$. FFVs assess the FFS where the constituents x *and* y are taken to mean MD and NMD, respectively.

FFS generalizes IFS/PFS such that $IFS \subset PFS \subset FFS$. For their differences, see *Table 1*.

Table 1. IFS, PFS and FFS.

IFS	PFS	FFS
$0 \leq \alpha + \beta \leq 1$	$0 \leq \alpha^2 + \beta^2 \leq 1$	$0 \leq \alpha^3 + \beta^3 \leq 1$
$\pi = 1 - \alpha - \beta$	$\pi = \sqrt{1 - \alpha^2 - \beta^2}$	$\pi = \sqrt[3]{1 - \alpha^3 - \beta^3}$
$\boldsymbol{\alpha + \beta + \pi = 1}$	$\alpha^2 + \beta^2 + \pi^2 = 1$	$\alpha^3 + \beta^3 + \pi^3 = 1$

3 | Fermatean Fuzzy Composite Relation

Here, we discuss the Fermatean fuzzy max-min-max introduced in [21], and present a new approach that do not use the extreme values adopted in [21].

3.1 | Fermatean Fuzzy Max-Min-Max Composite Relation (FFMMMCR)

The concept of max-min-max composite relation under IFSs and PFSs were presented in [22]-[24] with application to decision-making. To improve on it we explore information measure called FFMMMCR as presented in [21].

Assume X and Y are nonempty sets. Then the Fermatean Fuzzy Relation (FFR), ϕ from X to Y is a FFS of $X\ x\ Y$ consisting of MD, α_ϕ and NMD, β_ϕ and represented by $\boldsymbol{\phi}$ **(**$X \rightarrow Y$).

Definition 8. Suppose ϕ and ψ are FFRs of $X \times Y$ and $Y \times Z$ denoted by ϕ **(**$X \rightarrow Y$) and ψ ($Y \rightarrow Z$), respectively. Then FFMMMCR, $\sigma = \phi \circ \psi$ of $X \times Z$ is of the form

$$\sigma = \left\{\langle (x,z), \alpha_\sigma(x,z), \beta_\sigma(x,z)\rangle : (x,z) \in X \times Z\right\},$$

where

$$\alpha_\sigma(x,z) = \max\left\{\min\left(\alpha_\phi(x,y),\ \alpha_\psi(y,z)\right)\right\}.$$
$$\beta_\sigma(x,z) = \min\left\{\max\left(\alpha_\phi(x,y),\ \alpha_\psi(y,z)\right)\right\}.$$

For all $(x, y) \in X \times Y$ and $(y, z) \in Y \times Z$.

From Definition 8, FFMMCR, $\sigma = \phi \circ \psi$ can be computed by

$$\sigma = \alpha_\sigma(x,z) - \beta_\sigma(x,z)\pi_\sigma(x,z), \text{ for all } (x,z) \in X \times Z.$$

3.2 | Modified Fermatean Fuzzy Composite Relation

The Fermatean fuzzy composite relation presented in [21] uses the extreme values, i.e., the maximum of the minimum of MD and the minimum of the maximum of NMD. This approach is not reliable because it cannot show the relation between two similar FFSs. Hence, we modify the approach in [21] based on maximum average approach to resolve the limitation therein.

Definition 9. Suppose $\check{\phi}$ and $\check{\psi}$ are FFRs of $X \times Y$ and $Y \times Z$ denoted by $\check{\phi}$ **(**$X \rightarrow Y$) and $\check{\psi}$ ($Y \rightarrow Z$), respectively. Then the modified Fermatean fuzzy composite relation, $\check{\sigma} = \check{\phi} \circ \check{\psi}$ of $X \times Z$ is of the form

$$\check{\sigma} = \left\{\langle (x,z), \alpha_{\check{\sigma}}(x,z), \beta_{\check{\sigma}}(x,z)\rangle : (x,z) \in X \times Z\right\},$$

where

$$\alpha_{\breve{o}}(x,z) = \max\left\{\text{average}\left(\alpha_{\breve{\phi}}(x,y),\ \alpha_{\breve{\psi}}(y,z)\right)\right\}.$$
$$\beta_{\breve{o}}(x,z) = \min\left\{\text{average}\left(\alpha_{\breve{\phi}}(x,y),\ \alpha_{\breve{\psi}}(y,z)\right)\right\}.$$

For $0 \leq \alpha_{\breve{o}}^3(x,z) + \beta_{\breve{o}}^3(x,z) \leq 1$, and certainly, $\pi_{\breve{o}}(x,z) = \sqrt[3]{1-\alpha_{\breve{o}}^3(x,z)-\beta_{\breve{o}}^3(x,z)} \in [0,1]$ for all $(x,y) \in X \times Y$, $(y,z) \in Y \times Z$ and $(x,z) \in X \times Z$.

From Definition 9, modified Fermatean fuzzy composite relation, $\breve{o} = \breve{\phi} \circ \breve{\psi}$ can be computed by

$$\breve{o} = \alpha_{\breve{o}}(x,z) - \beta_{\breve{o}}(x,z)\pi_{\breve{o}}(x,z), \text{ for all } (x,z) \in X \times Z.$$

Definition 10. Given a binary FFR ϕ between X and Y, then ϕ^{-1} between Y and X is defined by

$$\alpha_{\phi^{-1}}(y,x) = \alpha_{\phi}(x,y),\ \beta_{\phi^{-1}}(y,x) = \beta_{\phi}(x,y),\ \forall\ (y,x) \in Y \times X \text{ and } \forall\ (x,y) \in X \times Y.$$

is called the inverse relation of ϕ.

Definition 11. If ϕ and ψ are two FFRs in $X \times Y$, then

I. $\phi \leq \psi$ iff $\alpha_{\phi}(x,y) \leq \alpha_{\psi}(x,y)$ and $\beta_{\phi}(x,y) \geq \beta_{\psi}(x,y)$ $\forall (x,y) \in X \times Y$.
II. $\phi \preccurlyeq \psi$ iff $\alpha_{\phi}(x,y) \leq \alpha_{\psi}(x,y)$ and $\beta_{\phi}(x,y) \leq \beta_{\psi}(x,y)$ $\forall (x,y) \in X \times Y$.
III. $\phi \vee \psi = \left\{\langle (x,y), \max(\alpha_{\phi}(x,y), \alpha_{\psi}(x,y)), \min(\beta_{\phi}(x,y), \beta_{\psi}(x,y)) \rangle : (x,y) \in X \times Y\right\}$.
IV. $\phi \wedge \psi = \left\{\langle (x,y), \min(\alpha_{\phi}(x,y), \alpha_{\psi}(x,y)), \max(\beta_{\phi}(x,y), \beta_{\psi}(x,y)) \rangle : (x,y) \in X \times Y\right\}$.
V. $\phi^c = \left\{\langle (x,y), \beta_{\phi}(x,y), \alpha_{\phi}(x,y) \rangle : \in X \times Y\right\}$, $\psi^c = \left\{\langle (x,y), \beta_{\psi}(x,y), \alpha_{\psi}(x,y) \rangle : X \times Y\right\}$.

3.3 | Some Properties of Modified Fermatean Fuzzy Composite Relation

Theorem 1. Suppose ϕ, ψ, φ be three FFRs in X x Y, then

I. $\phi \leq \psi \Rightarrow \phi^{-1} \leq \psi^{-1}$.
II. $(\phi \vee \psi)^{-1} = \phi^{-1} \vee \psi^{-1}$.
III. $(\phi \wedge \psi)^{-1} = \phi^{-1} \wedge \psi^{-1}$.
IV. $(\phi^{-1})^{-1} = \phi$.
V. $\phi \wedge (\psi \vee \varphi) = (\phi \wedge \psi) \vee (\phi \wedge \varphi)$.
VI. $\phi \vee (\psi \wedge \varphi) = (\phi \vee \psi) \wedge (\phi \vee \varphi)$.
VII. if $\phi \geq \psi$ and $\phi \geq \varphi$, then $\phi \geq \psi \wedge \varphi$.
VIII. if $\phi \leq \psi$ and $\phi \leq \varphi$, then $\phi \leq \psi \vee \varphi$.
IX. if $\phi \wedge \psi \leq \phi$ then $\phi \wedge \psi \leq \psi$.
X. if $\phi \vee \psi \leq \phi$ then $\phi \vee \psi \leq \psi$.

Proof.

First we prove (i). Assume $\phi \leq \psi$, then $\alpha_{\phi^{-1}}(y,x) = \alpha_{\phi}(x,y) \leq \alpha_{\psi}(x,y) = \alpha_{\psi^{-1}}(y,x)$ and

$$\beta_{\phi^{-1}}(y,x) = \beta_{\phi}(x,y) \geq \beta_{\psi}(x,y) = \beta_{\psi^{-1}}(y,x)\ \forall\ (x,y) \in X \times Y.$$

For the proof of (ii), we have

$$\alpha_{(\phi \vee \psi)^{-1}}(y,x) = \alpha_{(\phi \vee \psi)}(x,y) = \max\{\alpha_{\phi}(x,y), \alpha_{\psi}(x,y)\} =$$
$$\max\{\alpha_{\phi^{-1}}(y,x), \alpha_{\psi^{-1}}(y,x)\} = \alpha_{\phi^{-1} \vee \psi^{-1}}(y,x)\ \forall\ (x,y) \in X \times Y.$$

Similarly,

$$\beta_{(\phi \vee \psi)^{-1}}(y,x) = \beta_{(\phi \vee \psi)}(x,y) = \min\{\beta_{\phi}(x,y), \beta_{\psi}(x,y)\} =$$
$$\min\{\beta_{\phi^{-1}}(y,x), \beta_{\psi^{-1}}(y,x)\} = \beta_{\phi^{-1} \vee \psi^{-1}}(y,x)\ \forall\ (x,y) \in X \times Y.$$

The proof of (iii) is similar to (ii). Now, we proof (iv) is thus:

$$\alpha_{(\phi^{-1})^{-1}}(y,x) = \alpha_{\phi^{-1}}(x,y) = \alpha_{\phi}(y,x) = \alpha_{\phi}(x,y), \text{ alike } \beta_{(\phi^{-1})^{-1}}(y,x) =. \beta_{\phi}(x,y).$$

The proof of (v) is thus:

$$\alpha_{\phi \wedge (\psi \vee \varphi)}(x,y) = \min\{\alpha_{\phi}(x,y), \max\{\psi(x,y), \alpha_{\varphi}(x,y)\} =$$
$$\max\{\min\{\alpha_{\phi}(x,y), \psi(x,y)\}, \min\{\alpha_{\phi}(x,y), \alpha_{\varphi}(x,y)\}\} =$$
$$\max\{\alpha_{\phi\wedge\psi}(x,y), \alpha_{\phi\wedge\varphi}(x,y)\} = \alpha_{(\phi\wedge\psi)\vee(\phi\wedge\varphi)}(x,y)\ \forall\ (x,y) \in X \times Y.$$

In like manner, $\beta_{\phi \wedge (\psi \vee \varphi)}(x,y) = \beta_{(\phi \wedge \psi)\vee(\phi \wedge\varphi)}(x,y)\ \forall\ (x,y) \in X \times Y.$

The proofs of (vi) to (x) are forthright.

Theorem 2. We have $(\psi\, o\, \phi)^{-1} = \phi^{-1} o\, \psi^{-1}$ if ϕ is a FFR in $(X \times Y)$ and ψ is a FFR in $(Y \times Z)$, respectively.

Proof. For the proof, we have

$$\alpha_{(\psi\, o\, \phi)^{-1}}(z,x) = \alpha_{\psi\, o\, \phi}(x,z) = \max\{\frac{\alpha_{\phi}(x,y)+\alpha_{\psi}(y,z)}{2}\} = \max\left\{\frac{\alpha_{\phi^{-1}}(y,x)+\alpha_{\psi^{-1}}(z,y)}{2}\right\} = \max$$
$$\left\{\frac{\alpha_{\psi^{-1}}(z,y)+\alpha_{\phi^{-1}}(y,x)}{2}\right\} = \alpha_{\phi^{-1} o\, \psi^{-1}}(z,x)\ \forall\ (z,x) \in (Z \times X).$$

Similarly, we have

$$\beta_{(\psi\, o\, \phi)^{-1}}(z,x) = \beta_{\psi\, o\, \phi}(x,z) = \min\{\frac{\beta_{\phi}(x,y)+\beta_{\psi}(y,z)}{2}\} = \min\left\{\frac{\beta_{\phi^{-1}}(y,x)+\beta_{\psi^{-1}}(z,y)}{2}\right\} = \min$$
$$\left\{\frac{\beta_{\psi^{-1}}(z,y)+\beta_{\phi^{-1}}(y,x)}{2}\right\} = \beta_{\phi^{-1} o\, \psi^{-1}}(z,x)\ \forall\ (z,x) \in (Z \times X).$$

Theorem 3. Let ϕ and ψ be FFRs in $(Y \times Z)$ and φ be FFR in $(X \times Y)$, then

I. $(\phi \vee \psi)\ o\ \varphi \geq (\phi\ o\ \varphi)\ \vee\ (\psi\ o\ \varphi)$.
II. $(\phi\ \wedge\ \psi)\, o\, \varphi\ \leq\ (\phi\, o\, \varphi)\ \wedge\ (\psi\, o\, \varphi)$.

Proof. From Theorem 1, we have either $\phi \vee \psi\ \geq \phi$ or $\phi\ \vee \psi\ \geq \psi$. Thus, we have $(\phi \vee \psi) o\, \varphi \geq (\phi o\, \varphi)$ or $(\phi\ \vee \psi) o\, \varphi\ \geq (\psi o\, \varphi)$. Hence, it is clear that

$$(\phi \vee \psi)\ o\ \varphi \geq (\phi\ o\ \varphi)\ \vee\ (\psi\ o\ \varphi),$$

which proves (i).

Similarly, it is either $\phi \wedge \psi\ \geq \phi$ or $\phi\ \wedge \psi\ \geq \psi$, and so $(\phi \wedge \psi) o\, \varphi \leq (\phi o\, \varphi)$ or $(\phi \wedge \psi) o\, \varphi\ \leq (\psi o\, \varphi)$.

Therefore, $(\phi \wedge \psi)\ o\ \varphi \leq (\phi\ o\ \varphi)\ \wedge\ (\psi\ o\ \varphi)$, which proves (ii).

3.4 | Computation of Composite Relations under IFSs, PFSs and FFSs

We apply the proposed FFMMMCR, σ and the modified Fermatean fuzzy composite relation, $\breve{\sigma}$ to compute the composite relation between FFSs $F_1 = \{\langle x_1, 0.7, 0.2\rangle, \langle x_2, 0.5, 0.6\rangle, \langle x_3, 0.5, 0.4\rangle\}$ and $F_2 = \{\langle x_1, 0.8, 0.3\rangle, \langle x_2, 0.6, 0.1\rangle, \langle x_3, 0.5, 0.2\rangle\}$ in $X = \{x_1, x_2, x_3\}$.

The algorithm for the computation using the new approach of Fermatean fuzzy composite relation includes:

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Step 1. Establish a relation between F_1 and F_2 as FFVs.

Step 2. Identify MD and NMD of $\breve{\sigma} = \breve{\phi} \circ \breve{\psi}$ between F_1 and F_2 in X.

Step 3. Calculate $\breve{\sigma} = \breve{\phi} \circ \breve{\psi}$ between F_1 and F_2 in X using the information from Step 2 and the formula of $\breve{\sigma}$.

First, we apply the approach in [21] as follows:

$\alpha_\sigma(x_i, x_j) = \max\{0.7, 0.5, 0.5\} = 0.7$, and $\beta_\sigma(x_i, x_j) = \min\{0.3, 0.6, 0.4\} = 0.3$, where x_i is from F_1 and x_j is from F_2. Now, the composite relation between F_1 and F_2 using σ is:

$$\sigma = 0.7 - (0.3 \times 0.8573) = 0.4428.$$

The same example can also be captured in the frameworks of IFSs and PFSs, since 0.7 and 0.3 can define IFS and PFS. Using intuitionistic fuzzy information, we get:

$$\sigma = 0.7 - (0.3 \times 0) = 0.7.$$

Using Pythagorean fuzzy information, we get:

$$\sigma = 0.7 - (0.3 \times 0.6481) = 0.5056.$$

Next, we use the modified approach as follows:

$\alpha_{\breve{\sigma}}(x_i, x_j) = \max\{0.75, 0.55, 0.5\} = 0.75$, and $\beta_{\breve{\sigma}}(x_i, x_j) = \min\{0.25, 0.35, 0.3\} = 0.25$, where x_i is from F_1 and x_j is from F_2. Now, the composite relation between F_1 and F_2 using $\breve{\sigma}$ is:

$$\breve{\sigma} = 0.75 - (0.25 \times 0.8255) = 0.5436.$$

The same example can also be captured in the frameworks of IFSs and PFSs, since 0.75 and 0.25 can define IFS and PFS. Using intuitionistic fuzzy information, we get:

$$\breve{\sigma} = 0.75 - (0.25 \times 0) = 0.75.$$

Using Pythagorean fuzzy information, we get:

$$\breve{\sigma} = 0.75 - (0.25 \times 0.6124) = 0.5969.$$

The results are contained in *Table 2* for quick comparison.

Table 2. Results of composite relations.

FFCRs	IFS	PFS	FFS
FFMMMCR [21]	0.7	0.5056	0.4428
New Method	0.75	0.5969	0.5436

The information in *Table 2* is represented by *Fig. 1*. *Table 2* and *Fig. 1* clearly indicate that the composite relations in [21] and our approach under IFS gives the best relation between F_1 and F_2, follows by the results under PFS, and the results under FFS is the least; because of the inability of IFS and PFS to reliably curb uncertainties. However, in each of the environments, our new method gives the best measure of composite relation between F_1 and F_2.

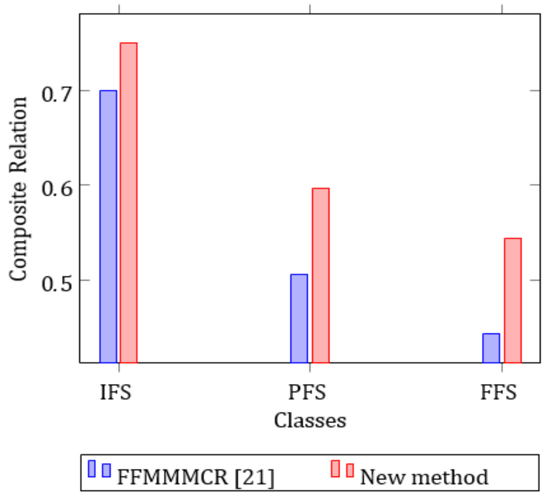

Fig. 1. Graphical representation of composite relations in terms of IFS, PFS, FFS.

4 | Application of Fermatean Fuzzy Composite Relation in Pattern Recognition Cases

Pattern recognition is the process of recognizing patterns by using a machine learning algorithm. Pattern recognition can be defined as the classification of data based on knowledge already gained or on statistical information extracted from patterns or their representation. Due to the presence of uncertainties in the process of pattern recognition and the ability of FFSs to curb uncertainties, it is expedient to carry out pattern recognition under Fermatean fuzzy information. In this section, we present the application of Fermatean fuzzy composite relation in pattern recognition problems based on max-average approach and the approach in [21]. The patterns are adopted from the work of Wang and Xin [25], but presented as Fermatean Fuzzy data. The pattern recognition problems are as follow:

4.1 | Problem 1

Given a pattern recognition problem about the classification of building materials. Four classes of building material, each is represented by FFS M_1, M_2, M_3, M_4 in the space $S = \{s_1, s_2, \ldots, s_{12}\}$ (see *Table 3*). Suppose we have another kind of unknown building material N, our aim is to show which class the unknown pattern N belong to.

Table 3. Pattern of building materials in FFVs.

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}
α_{M1}	0.173	0.102	0.530	0.965	0.420	0.008	0.331	1.000	0.215	0.432	0.750	0.432
β_{M1}	0.524	0.818	0.326	0.008	0.351	0.956	0.512	0.000	0.625	0.534	0.126	0.432
α_{M2}	0.510	0.627	1.000	0.125	0.026	0.732	0.556	0.650	1.000	0.145	0.047	0.760
β_{M2}	0.365	0.125	0.000	0.648	0.823	0.153	0.303	0.267	0.000	0.762	0.923	0.231
α_{M3}	0.495	0.603	0.987	0.073	0.037	0.690	0.147	0.213	0.501	1.000	0.324	0.045
β_{M3}	0.387	0.298	0.006	0.849	0.923	0.268	0.812	0.653	0.284	0.000	0.483	0.912
α_{M4}	1.000	1.000	0.857	0.734	0.021	0.076	0.152	0.113	0.489	1.000	0.386	0.028
β_{M4}	0.000	0.000	0.123	0.158	0.896	0.912	0.712	0.756	0.389	0.000	0.485	0.912
α_N	0.978	0.980	0.798	0.693	0.051	0.123	0.152	0.113	0.494	0.987	0.376	0.012
β_N	0.003	0.012	0.132	0.213	0.876	0.756	0.721	0.732	0.368	0.000	0.423	0.897

By deploying our method using the information in *Table 3*, we get the following results in *Table 4*.

Table 4. Result from problem 1.

FFCR	(M_1, N)	(M_2, N)	(M_3, N)	(M_4, N)
New method	0.781	0.881	0.994	0.994
Ranking	3rd	2nd	1st	1st

From the result in *Table 4*, we can see that the composite relation between M_3 and N, and between M_4 and N are the greatest. Hence the unknown building material N can be classified into either M_3 or M_4, respectively.

For comparison sake, we deploy the method in [21] and our method to the information in *Table 3*, and get results as seen in *Table 5* and *Fig. 2*.

Table 5. Comparison for the FFCRs for problem 1

FFCRs	(M_1, N)	(M_2, N)	(M_3, N)	(M_4, N)
FFMMMCR [21]	0.5078	0.6995	0.987	0.987
New method	0.781	0.881	0.994	0.994

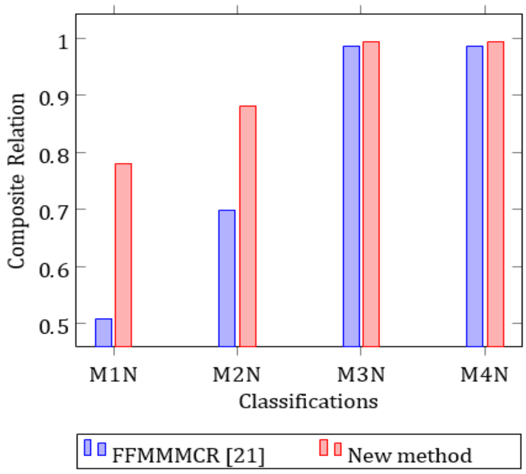

Fig. 2. Graphical representation of FFCRs for problem 1.

From the results, we observe that our new method of finding FFCR is better than the existing FFMMMCR method [21] because our method yields greater values of the composite relation.

Table 6. Patterns of mineral fields in FFVs

	s_1	s_2	s_3	s_4	s_5	s_6
$\alpha_{C_1}, \beta_{C_1}$	0.739, 0.125	0.033, 0.818	0.188, 0.626	0.492, 0.358	0.020, 0.628	0.739, 0.125
$\alpha_{C_2}, \beta_{C_2}$	0.124, 0.665	0.030, 0.825	0.048, 0.800	0.136, 0.648	0.823, 1.000	0.393, 0.653
$\alpha_{C_3}, \beta_{C_3}$	0.449, 0.387	0.662, 0.298	1.000, 0.000	1.000, 0.000	0.000, 0.188	1.000, 0.000
$\alpha_{C_4}, \beta_{C_4}$	0.280, 0.715	0.521, 0.368	0.470, 0.423	0.295, 0.658	0.806, 0.049	0.735, 0.118
$\alpha_{C_5}, \beta_{C_5}$	0.326, 0.452	1.000, 0.000	0.182, 0.725	0.156, 0.765	0.806, 0.049	0.675, 0.263
α_B, β_B	0.629, 0.303	0.524, 0.356	0.210,0. 689	0.218, 0.753	0.069, 0.876	0.658, 0.256

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4.2 | Problem 2

Given five kind of mineral fields, each is featured by the content of six minerals and has one kind of typical hybrid mineral. We can express the five kinds of typical hybrid minerals by five FFSs C_1, C_2, C_3, C_4 in the feature space $S = \{s_1, s_2, \ldots, s_6\}$ (see *Table 6*). Given another kind of hybrid mineral B in S, we find which field this kind of mineral B belongs to using our method and FFMMMCR approach [21].

From the data in *Table 6*, the results in *Table 7* are obtained based on our method.

Table 7. Results from problem 2.

FFCR	(C_1, B)	(C_2, B)	(C_3, B)	(C_4, B)	(C_5, B)
New Method	0.574	0.737	0.723	0.781	0.784
Ranking	5^{th}	3^{rd}	4^{th}	2^{nd}	1^{st}

From *Table 7*, it is clearly shown that the hybrid mineral B could likely be produced by the mineral field C_5 since $(C_5, B) \geq (C_4, B) \geq (C_2, B) \geq (C_3, B) \geq (C_1, B)$.

For the sake of comparison, we apply the method in [21] and our method to the information in *Table 6*, and get results as seen in *Table 8* and *Fig. 3*.

Table 8. Comparison for the FFCRs for problem 2.

FFCRs	(C_1, B)	(C_2, B)	(C_3, B)	(C_4, B)	(C_5, B)
FFMMMCR [21]	0.431	-0.175	0.431	0.431	0.425
New method	0.574	0.737	0.723	0.781	0.784

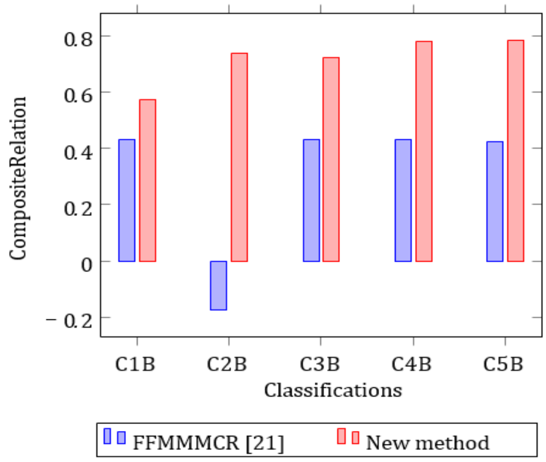

Fig. 3. Graphical representation of FFCRs for problem 2.

From *Table 8* and *Fig. 3*, we see that our new method yields better composite relations compare to the approach in [21], because the existing method is strictly based on extreme values whereas our method is based on max-mean values.

4.3 | Advantages of the New Approach of Fermatean Fuzzy Composite Relation

The method of Fermatean fuzzy composite relation in [21] incorporated the approach of extreme values. But from the knowledge of central tendency, such approach cannot yield a reliable result. This setback informed the present study. The following are some of the advantages of the new method of Fermatean fuzzy composite relation, which include:

- *The new method of Fermatean fuzzy composite relation uses maximum-average approach against the max-min-max approach adopted in [21], to avoid error due to the use of extreme values.*

- *The new method of Fermatean fuzzy composite relation can measure the composite relation between similar FFSs against the Fermatean fuzzy max-min-max approach adopted in [21].*
- *The new method of Fermatean fuzzy composite relation yields better result with regards to accuracy compare to the method in [21].*

5 | Conclusion

FFSs are generalizations of IFSs and PFSs which are capable of handling higher levels of uncertainties more efficient than IFSs and PFSs. In order to better appreciate the application of FFSs, this work studied a new composite relation under FFSs which modified the approach in [21] with better output. In this work, we succinctly differentiated FFSs from other generalized fuzzy sets with some characteristics. Numerical examples on pattern recognition were carried out under FFSs to demonstrate the application of our new method. The values gotten through this process clearly show where each pattern should belong. To demonstrate the merit our new approach, we presented certain comparative analyses between our method and the method in [21], which show that our method is more reliable, efficient and proficient than the approach in [21]. Due to the reliability of FFS as a competent soft computing tool, it can be used in multiple attributes decision making based on the new approach of Fermatean fuzzy composite relation. The limitation of this approach is that, it cannot be used to measure the composite relation between some generalized fuzzy sets like picture fuzzy sets, spherical fuzzy sets, etc. To achieve a reliable result, we recommend that the new approach of Fermatean fuzzy composite relation could be modified to be used in other environments like picture fuzzy sets, spherical fuzzy sets, etc.

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