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Some Picture Fuzzy Mean Operators and Their Applications in Decision-Making

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Abstract

Picture fuzzy set is the generalization of fuzzy set and intuitionistic fuzzy set. It is useful for handling uncertainty by considering the positive membership, neutral membership and negative membership degrees independently for each element of a universal set. The main objective of this article is to develop some picture fuzzy mean operators, including Picture Fuzzy Harmonic Mean (PFHM), Picture Fuzzy Weighted Harmonic Mean (PFWHM), Picture Fuzzy Arithmetic Mean (PFAM), Picture Fuzzy Weighted Arithmetic Mean (PFWAM), Picture Fuzzy Geometric Mean (PFGM) and Picture Fuzzy Weighted Geometric Mean (PFWGM), to aggregate the picture fuzzy sets. Moreover, we discuss some relevant properties of these operators. Furthermore, we apply these mean operators to make decisions with practical examples. Finally, to show the efficiency and the validity of the proposed operators, we compare our results with the results of existing methods and concluded from the comparison that our proposed methods are more effective and reliable.

Keywords: Picture fuzzy set, Harmonic mean operator, Arithmetic mean operator, Geometric mean operator.

1 | Introduction

Many branches of science and engineering and also in the field of medical science, management science, economics, environmental science and so on, we face various problems where data are more ambiguous than precise. To describe such ambiguous data Zadeh [31] introduced the concept of fuzzy set in 1965. Then this was successfully applied in different branches of science and engineering by a host of researchers. There are some generalizations of fuzzy set according to the application in different fields of our real life problems. One of the generalization of fuzzy set is intuitionistic fuzzy fuzzy [1] which is capable to describe uncertainty more precisely than fuzzy set by taking positive membership and negative membership of an element of a universal set. But in many cases of our real life, we face some problems, where the term neutrality becomes essential to describe uncertainty. Voting is an example of such situation, where the human voters may be divided into four groups of those who: vote for, abstain, vote against, the refusal of the voting.

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To describe such situation Cuong and Kreinovich [3] developed the latest generalization of fuzzy set as picture fuzzy set which is the direct extension of intuitionistic fuzzy set. Later a huge amount of works have been emerged on diverse aspects of picture fuzzy sets and their applications [see ([2], [4]-[6], [7], [10], [11], [13]-[15], [17], [19]-[24], [27], [30], [32])]. The averaging operators on picture fuzzy sets along their applications are also becoming a great attention by numerous researchers. In 2017, Wei [28] discussed the arithmetic and geometric operations for picture fuzzy sets and applied them in multiple attribute decision-making problems. Khan et al. [11], [12] investigated the information aggregation operators' method under the picture fuzzy environment with the help of Einstein norms operations and applied a group decision-making problem in 2019 [12]. In 2018, Wei [30] discussed the multiple attribute decision-making problem based on the arithmetic, geometric aggregation operators and Hamacher operations of picture fuzzy [30]. Luo and Long [16] studied picture fuzzy geometric aggregation operators based on a trapezoidal fuzzy number and applied it to Multi-Attribute Decision-Making (MADM) and pattern recognition in 2021. Picture fuzzy aggregation operators are also discussed some researchers (see, [8], [9], [12], [18], [25], [26], [28]). In the above aggregation operators related to picture fuzzy set, the authors described the score function in such a way, where the properties of neutrality coincided with negative membership degree. But the properties neutrality should coincide of the term of positive membership degree. So in this article, we redefine the score function, where the the properties of neutrality coincide with the positive membership degree. On the other hand the existing aggregation operators are more complicated, because the aggregated value cannot find from the direct definition of the aggregation. In this article, to overcome these difficulties, we have defined some mean operators, where the aggregated value can be found from direct definition. Some related properties of the operators are also explored. The practical application of these methods is also described with comparison in existing methods.

The article is organized as follows: In Section 2, some basic definitions are given which are essential to rest of the paper. In Section 3, Picture Fuzzy Harmonic Mean (PFHM) operator and Picture Fuzzy Weighted Harmonic Mean (PFWHM) operator are discussed. In Section 4, picture fuzzy arithmetic operator and Picture Fuzzy Weighted Arithmetic Mean (PFWAM) operator are discussed. In Section 5, Picture Fuzzy Geometric Mean (PFGM) operator and picture Fuzzy Weighted Geometric Mean (PFWGM) operator are deliberated. In Section 6, the application of the proposed methods is illustrated. In Section 7, the comparison studies are showed.

2 | Preliminaries

In this section, we recall some basic definitions which are used in later sections.

Definition 1 ([31]). Let X be non-empty set. A fuzzy set A in X is given by

$$A = \{(x, \mu_A(x)) : x \in X\}, \text{ where } \mu_A : X \to [0,1].$$

Definition 2 ([1]). Let X be non-empty set. An intuitionistic fuzzy set A in X is given by

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}, \text{ where } \mu_A : X \to [0,1] \text{ and } \nu_A : X \to [0,1].$$

The values $\mu_A(x)$ and $\nu_A(x)$ represent the membership degree and non-membership degree of the element x to the set A respectively. The pair $(\mu_A(x),\ \nu_A(x))$ is called intuitionistic fuzzy value satisfying the condition,

$$0 \le \mu_A(x) + \nu_A(x) \le 1\ \forall x \in X.$$

For any intuitionistic fuzzy set A on the universal set X, for $x \in X$,

$$\pi_A(x) = 1 - (\mu_A(x) + \nu_A(x)).$$

is called the hesitancy degree (or intuitionistic fuzzy index) of an element x in A. It is the degree of indeterminacy membership of the element x whether belonging to A or not.

Obviously, $0 \leq \pi_A(x) \leq 1$ for any $x \in X$.

Definition 3 ([3]). A picture fuzzy set A on a universal set $X \neq \emptyset$ is of the form

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) : x \in X\},$$

where $\mu_A(x) \in [0,1]$ is the degree of positive membership, $\eta_A(x) \in [0,1]$ is the degree of neutral membership and $\nu_A(x) \in [0,1]$ is the degree of negative membership of x in A, where $\mu_A(x), \eta_A(x)$ and $\nu_A(x)$ satisfy the following condition,

$$0 \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1 \ \forall x \in X.$$

Here $1 - (\mu_A(x) + \eta_A(x) + \nu_A(x)); \forall x \in X$ is called the degree of refusal membership of x in A. The pair (μ_A, η_A, ν_A) is called picture fuzzy value.

Definition 4 ([3]). Let $A = (\mu_A, \eta_A, \nu_A)$ and $B = (\mu_B, \eta_B, \nu_B)$ be two picture fuzzy values of X. Then

I. $A \leq B$ iff $\mu_A \leq \mu_B, \eta_A \leq \eta_B$ and $\nu_A \geq \nu_B$.
II. $A = B$ iff $\mu_A = \mu_B, \eta_A = \eta_B$ and $\nu_A = \nu_B$.

Definition 5. Let $A = (\mu_A, \eta_A, \nu_A)$ be a picture fuzzy value. Then the score function $S(A)$ and the accuracy function $H(A)$ are defined as

$$S(A) = \mu_A + \eta_A - \nu_A.$$

and

$$H(A) = \mu_A + \eta_A + \nu_A,$$

where $S(A) \in [-1,1]$ and $H(A) \in [0,1]$.

Definition 6. Let $A = (\mu_A, \eta_A, \nu_A)$ and $A = (\mu_A, \eta_A, \nu_A)$ be two picture fuzzy values. Then the following comparison rules can be used:

I. If $S(A) > S(B)$, then A is greater than B, denoted by $A \succ B$.
II. If $S(A) = S(B)$, then
III. $H(A) = H(B)$, implies that A is equivalent to B, denoted by $A \sim B$.
IV. $H(A) > H(B)$, implies that A is greater than B, denoted by $A \succ B$.

3 | PFHM Operators

Definition 7. Let $A_i = (\mu_{A_i}, \eta_{A_i}, \nu_{A_i})$ $(i = 1, 2, \cdots, n)$ be collection of picture fuzzy values. Then the PFHM operator is mapping $PFHM: A^n \to A$ such that

$$PFHM(A_1, A_2, \cdots, A_n) = \begin{pmatrix} n\left(\sum_{i=1}^{n}(\mu_{A_i})^{-1}\right)^{-1}, n\left(\sum_{i=1}^{n}(\eta_{A_i})^{-1}\right)^{-1}, \\ n\left(\sum_{i=1}^{n}(\nu_{A_i})^{-1}\right)^{-1} \end{pmatrix}.$$

Definition 8. Let $A_i = \left(\mu_{A_i}, \eta_{A_i}, \nu_{A_i}\right)$ $(i = 1, 2, \cdots, n)$ be collection of picture fuzzy values and $w = (w_1, w_2, \cdots, w_n)^T$ be the weighting vector of A_i $(i = 1, 2, \cdots, n)$ such that $w_i \in [0,1]$, $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} w_i = 1$. Then the PFWHM operator is a mapping $PFWHM: A^n \to A$ such that

$$\mathrm{PFWHM}(A_1, A_2, \cdots, A_n) = \left(\left(\sum_{i=1}^{n} \frac{w_i}{\mu_{A_i}}\right)^{-1}, \left(\sum_{i=1}^{n} \frac{w_i}{\eta_{A_i}}\right)^{-1}, \left(\sum_{i=1}^{n} \frac{w_i}{\nu_{A_i}}\right)^{-1}\right),$$

where $\mu_{A_i}, \eta_{A_i}, \nu_{A_i} \neq 0$.

The following axioms are satisfied for *PFHM* and *PFWHM*:

Theorem 1 (Idempotency). Let $A_i = \left(\mu_{A_i}, \eta_{A_i}, \nu_{A_i}\right)$ $(i = 1, 2, \cdots, n)$ be collection of picture fuzzy values.

If $A_i = A$, $(i = 1, 2, \cdots, n)$, then

$$\mathrm{PFHM}(A_1, A_2, \cdots, A_n) = A,$$

And

$$\mathrm{PFWHM}(A_1, A_2, \cdots, A_n) = A.$$

Proof. For $A_i = A$ and $\sum_{i=1}^{n} w_i = 1$, we have

$$\mathrm{PFHM}(A_1, A_2, \cdots, A_n) = \begin{pmatrix} n\left(\sum_{i=1}^{n}\left(\mu_{A_i}\right)^{-1}\right)^{-1}, n\left(\sum_{i=1}^{n}\left(\eta_{A_i}\right)^{-1}\right)^{-1}, \\ n\left(\sum_{i=1}^{n}\left(\nu_{A_i}\right)^{-1}\right)^{-1} \end{pmatrix} =$$

$$\begin{pmatrix} n\left(\sum_{i=1}^{n}(\mu_A)^{-1}\right)^{-1}, n\left(\sum_{i=1}^{n}(\eta_A)^{-1}\right)^{-1}, \\ n\left(\sum_{i=1}^{n}(\nu_A)^{-1}\right)^{-1} \end{pmatrix} = \left(\frac{n}{n(\mu_A)^{-1}}, \frac{n}{n(\eta_A)^{-1}}, \frac{n}{n(\nu_A)^{-1}}\right) = (\mu_A, \eta_A, \nu_A) = A.$$

And where

$$\mathrm{PFWHM}(A_1, A_2, \cdots, A_n) = \left(\left(\sum_{i=1}^{n} \frac{w_i}{\mu_{A_i}}\right)^{-1}, \left(\sum_{i=1}^{n} \frac{w_i}{\eta_{A_i}}\right)^{-1}, \left(\sum_{i=1}^{n} \frac{w_i}{\nu_{A_i}}\right)^{-1}\right) \mu_{A_i}, \eta_{A_i}, \nu_{A_i} \neq 0$$

$$= \left(\left(\sum_{i=1}^{n} \frac{w_i}{\mu_A}\right)^{-1}, \left(\sum_{i=1}^{n} \frac{w_i}{\eta_A}\right)^{-1}, \left(\sum_{i=1}^{n} \frac{w_i}{\nu_A}\right)^{-1}\right) =$$

$$\begin{pmatrix} \left(\sum_{i=1}^{n} w_i\right)^{-1}\left((\mu_A)^{-1}\right)^{-1}, \left(\sum_{i=1}^{n} w_i\right)^{-1}\left((\eta_A)^{-1}\right)^{-1}, \\ \left(\sum_{i=1}^{n} w_i\right)^{-1}\left((\nu_A)^{-1}\right)^{-1} \end{pmatrix}; \ \sum_{i=1}^{n} w_i = 1 = (\mu_A, \eta_A, \nu_A) = A.$$

Theorem 2 (Monotonicity). If $A_i \leq A_i^*$, then

$$\mathrm{PFHM}(A_1, A_2, \cdots, A_n) \leq \mathrm{PFHM}(A^*_1, A^*_2, \cdots, A^*_n).$$

And

$$\mathrm{PFWHM}(A_1, A_2, \cdots, A_n) \leq \mathrm{PFWHM}(A^*_1, A^*_2, \cdots, A^*_n).$$

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Proof.

$$\mathrm{PFHM}(A_1, A_2, \cdots, A_n) - \mathrm{PFHM}(A^*_1, A^*_2, \cdots, A^*_n)$$
$$= \begin{pmatrix} \dfrac{n}{\frac{1}{\mu_{A_1}} + \frac{1}{\mu_{A_2}} + \cdots + \frac{1}{\mu_{A_n}}} - \dfrac{n}{\frac{1}{\mu_{A_1^*}} + \frac{1}{\mu_{A_2^*}} + \cdots + \frac{1}{\mu_{A_n^*}}}, \\ \dfrac{n}{\frac{1}{\eta_{A_1}} + \frac{1}{\eta_{A_2}} + \cdots + \frac{1}{\eta_{A_n}}} - \dfrac{n}{\frac{1}{\eta_{A_1^*}} + \frac{1}{\eta_{A_2^*}} + \cdots + \frac{1}{\eta_{A_n^*}}}, \\ \dfrac{n}{\frac{1}{\nu_{A_1}} + \frac{1}{\nu_{A_2}} + \cdots + \frac{1}{\nu_{A_n}}} - \dfrac{n}{\frac{1}{\nu_{A_1^*}} + \frac{1}{\nu_{A_2^*}} + \cdots\cdots + \frac{1}{\nu_{A_n^*}}} \end{pmatrix} \leq 0.$$

Since $A_i \leq A_i^*$ or $\frac{1}{A_i} \geq \frac{1}{A_i^*}$, for $i = 1, 2, \cdots, n$.

Similarly, we can prove that

$$\mathrm{PFWHM}(A_1, A_2, \cdots, A_n) - \mathrm{PFWHM}(A_1^*, A_2^*, \cdots, A_n^*) \leq 0.$$

Theorem 4 (Boundedness). Let $A_{min} = min(A_1, A_2, \cdots, A_n)$ and $A_{max} = max(A_1, A_2, \cdots A_n)$, for $i = 1, 2, \cdots, n$. Then $A_{min} \leq PFHM(A_1, A_2, \cdots, A_n) \leq A_{max}$ and $A_{min} \leq PFWHM(A_1, A_2, \cdots A_n) \leq A_{max}$.

Proof. Boundedness is the consequence of monotonicity and idempotency.

Theorem 5 (Commutatively). *If* $\left(A_1^0, A_2^0, \cdots, A_n^0\right)$ be any permutation of $(A_1, A_2, \cdots, A_n)$, then

$$\mathrm{PFHM}(A_1, A_2, \cdots, A_n) = \mathrm{PFHMO}\left(A^0_1, A^0_2, \cdots, A^0_n\right).$$

And

$$\mathrm{PFWHM}(A_1, A_2, \cdots, A_n) = \mathrm{PFWHM}\left(A^0_1, A^0_2, \cdots, A^0_n\right).$$

Proof.

$$\mathrm{PFHM}(A_1, A_2, \cdots, A_n) - \mathrm{PFHMO}\left(A^0_1, A^0_2, \cdots, A^0_n\right)$$
$$= \begin{pmatrix} n\left(\sum_{i=1}^{n}\left(\mu_{A_i}\right)^{-1}\right)^{-1} - n\left(\sum_{i=1}^{n}\left(\mu_{A_i^0}\right)^{-1}\right)^{-1}, \\ n\left(\sum_{i=1}^{n}\left(\eta_{A_i}\right)^{-1}\right)^{-1} - n\left(\sum_{i=1}^{n}\left(\eta_{A_i^0}\right)^{-1}\right)^{-1}, \\ n\left(\sum_{i=1}^{n}\left(\nu_{A_i}\right)^{-1}\right)^{-1} - n\left(\sum_{i=1}^{n}\left(\nu_{A_i^0}\right)^{-1}\right)^{-1} \end{pmatrix} = 0,$$

because $\left(A_1^0, A_2^0, \cdots\cdots, A_n^0\right)$ be any permutation of $(A_1, A_2, \cdots\cdots, A_n)$.

Hence, we have

$$\mathrm{PFHM}(A_1, A_2, \cdots, A_n) = \mathrm{PFHM}\left(A^0_1, A^0_2, \cdots, A^0_n\right)$$

Again,

$$\mathrm{PFWHM}(A_1, A_2, \cdots\cdots, A_n) - \mathrm{PFWHM}(A^0_1, A^0_2, \cdots\cdots, A^0_n)$$
$$= \begin{pmatrix} \left(\sum_{i=1}^{n} w_i(\mu_{A_i})^{-1}\right)^{-1} - \left(\sum_{i=1}^{n} w_i(\mu_{A_i^0})^{-1}\right)^{-1}, \\ \left(\sum_{i=1}^{n} w_i(\eta_{A_i})^{-1}\right)^{-1} - \left(\sum_{i=1}^{n} w_i(\eta_{A_i^0})^{-1}\right)^{-1}, \\ \left(\sum_{i=1}^{n} w_i(\nu_{A_i})^{-1}\right)^{-1} - \left(\sum_{i=1}^{n} w_i(\nu_{A_i^0})^{-1}\right)^{-1} \end{pmatrix}.$$

Because $(A_1^0, A_2^0, \cdots, A_n^0)$ be any permutation of $(A_1, A_2, \cdots, A_n)$.

Hence, we have

$$\mathrm{WHM}(A_1, A_2, \cdots, A_n) = \mathrm{PFWHM}(A^0_1, A^0_2, \cdots, A^0_n).$$

4 | Picture Fuzzy Arithmetic Mean (PFAM) Operators

Definition 9. Let $A_i = (\mu_{A_i}, \eta_{A_i}, \nu_{A_i})$ $(i = 1, 2, \cdots, n)$ be collection of picture fuzzy values. Then the PFAM operator is mapping $PFAM: A^n \to A$ such that

$$\mathrm{PFAM}(A_1, A_2, \cdots, A_n) = \left(\frac{1}{n}\sum_{i=1}^{n} \mu_{A_i}, \frac{1}{n}\sum_{i=1}^{n} \eta_{A_i}, \frac{1}{n}\sum_{i=1}^{n} \nu_{A_i}\right).$$

Definition 10. Let $A_i = (\mu_{A_i}, \eta_{A_i}, \nu_{A_i})$ $(i = 1, 2, \cdots, n)$ be collection of picture fuzzy values and $w = (w_1, w_2, \cdots, w_n)^T$ be the weighting vector of A_i $(i = 1, 2, \cdots, n)$ such that $w_i \in [0,1]$, $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} w_i = 1$. Then the PFWAM operator is a mapping $PFWAM: A^n \to A$ such that

$$\mathrm{PFWAM}(A_1, A_2, \cdots, A_n) = \left(\frac{1}{n}\sum_{i=1}^{n} w_i\mu_{A_i}, \frac{1}{n}\sum_{i=1}^{n} w_i\eta_{A_i}, \frac{1}{n}\sum_{i=1}^{n} w_i\nu_{A_i}\right).$$

The following axioms are satisfied for $PFAM$ and $PFWAM$.

Theorem 6 (Idempotency). Let $A_i = (\mu_{A_i}, \eta_{A_i}, \nu_{A_i})$ $(i = 1, 2, \cdots, n)$ be collection of picture fuzzy values.

If $A_i = A$, $(i = 1, 2, \cdots, n)$, then $PFAM(A_1, A_2, \cdots, A_n) = A$ and $PFWAM(A_1, A_2, \cdots, A_n) = A$.

Proof. For $A_i = A$ and $\sum_{i=1}^{n} w_i = 1$, we have

$$\mathrm{PFAM}(A_1, A_2, \cdots, A_n) = \left(\frac{1}{n}\sum_{i=1}^{n} \mu_{A_i}, \frac{1}{n}\sum_{i=1}^{n} \eta_{A_i}, \frac{1}{n}\sum_{i=1}^{n} \nu_{A_i}\right)$$
$$= \left(\frac{1}{n}\sum_{i=1}^{n} \mu_{A}, \frac{1}{n}\sum_{i=1}^{n} \eta_{A}, \frac{1}{n}\sum_{i=1}^{n} \nu_{A}\right)$$
$$= \left(\frac{1}{n} n\mu_A, \frac{1}{n}.n\eta_A, \frac{1}{n}.n\nu_A\right) = (\mu_A, \eta_A, \nu_A) = A.$$

Theorem 7 (Monotonicity). If $A_i \leq A_i^*$, then

$$\mathrm{PFAM}(A_1, A_2, \cdots, A_n) \leq \mathrm{PFAM}(A^*_1, A^*_2, \cdots, A^*_n).$$

And

$$\mathrm{PFWAM}(A_1, A_2, \cdots, A_n) \leq \mathrm{PFWAM}(A^*_1, A^*_2, \cdots, A^*_n).$$

Proof.

$$\mathrm{PFAM}(A_1, A_2, \cdots, A_n) - \mathrm{PFAM}(A^*_1, A^*_2, \cdots, A^*_n) = \begin{pmatrix} \frac{\mu_{A_1}+\mu_{A_2}+\cdots\cdots+\mu_{A_n}}{n} - \frac{\mu_{A_1^*}+\mu_{A_2^*}+\cdots\cdots+\mu_{A_n^*}}{n}, \\ \frac{\eta_{A_1}+\eta_{A_2}+\cdots\cdots+\eta_{A_n}}{n} - \frac{\eta_{A_1^*}+\eta_{A_2^*}+\cdots\cdots+\eta_{A_n^*}}{n}, \\ \frac{\nu_{A_1}+\nu_{A_2}+\cdots\cdots+\nu_{A_n}}{n} - \frac{\nu_{A_1^*}+\nu_{A_2^*}+\cdots\cdots+\nu_{A_n^*}}{n} \end{pmatrix} \leq 0.$$

Since $A_i \leq A_i^*$ or $\frac{1}{A_i} \geq \frac{1}{A_i^*}$, for $i = 1, 2, \cdots \cdots \cdots, n$.

Similarly, we can prove that

$$\mathrm{PFWAM}(A_1, A_2, \cdots, A_n) - \mathrm{PFWAM}(A^*_1, A^*_2, \cdots, A^*_n) \leq 0.$$

This proves the monotonicity of *PFAM* and *PFWAM*.

Theorem 8 (Boundedness). Let $A_{min} = min(A_1, A_2, \cdots, A_n)$ and $A_{max} = max(A_1, A_2, \cdots A_n)$, for $i = 1, 2, \cdots, n$, then $A_{min} \leq PFAM(A_1, A_2, \cdots, A_n) \leq A_{max}$ and $A_{min} \leq PFWAM(A_1, A_2, \cdots A_n) \leq A_{max}$.

Proof. Boundedness is the consequence of monotonicity and idempotency.

Theorem 9 (Commutatively). If $\left(A_1^0, A_2^0, \cdots, A_n^0\right)$ be any permutation of $(A_1, A_2, \cdots, A_n)$, then

$$\mathrm{PFAM}(A_1, A_2, \cdots, A_n) = \mathrm{PFAM}\left(A^0_1, A^0_2, \cdots, A^0_n\right),$$

and

$$\mathrm{PFWAM}(A_1, A_2, \cdots, A_n) = \mathrm{PFWAM}\left(A^0_1, A^0_2, \cdots, A^0_n\right).$$

Proof.

$$\mathrm{PFAM}(A_1, A_2, \cdots, A_n) - \mathrm{PFAM}\left(A^0_1, A^0_2, \cdots, A^0_n\right) = \begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n}\mu_{A_i} - \frac{1}{n}\sum_{i=1}^{n}\mu_{A_i^0}, \\ \frac{1}{n}\sum_{i=1}^{n}\eta_{A_i} - \frac{1}{n}\sum_{i=1}^{n}\eta_{A_i^0}, \\ \frac{1}{n}\sum_{i=1}^{n}\nu_{A_i} - \frac{1}{n}\sum_{i=1}^{n}\nu_{A_i^0} \end{pmatrix} = 0$$

Hence, we have

$$\mathrm{PFAM}(A_1, A_2, \cdots, A_n) = \mathrm{PFAM}\left(A^0_1, A^0_2, \cdots, A^0_n\right).$$

Again,

$$\mathrm{PFWAM}(A_1, A_2, \cdots\cdots, A_n) - \mathrm{PFWAM}\left(A^0_1, A^0_2, \cdots\cdots, A^0_n\right) = \begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n} w_i\mu_{A_i} - \frac{1}{n}\sum_{i=1}^{n} w_i\mu_{A_i^0}, \\ \frac{1}{n}\sum_{i=1}^{n} w_i\eta_{A_i} - \frac{1}{n}\sum_{i=1}^{n} w_i\eta_{A_i^0}, \\ \frac{1}{n}\sum_{i=1}^{n} w_i\nu_{A_i} - \frac{1}{n}\sum_{i=1}^{n} w_i\nu_{A_i^0} \end{pmatrix} = 0$$

because $\left(A_1^0, A_2^0, \cdots, A_n^0\right)$ be any permutation of $(A_1, A_2, \cdots, A_n)$.

Hence, we have

$$\mathrm{PFWAM}(A_1, A_2, \cdots, A_n) = \mathrm{PFWAM}\left(A^0_1, A^0_2, \cdots, A^0_n\right).$$

5 | PFGM Operators

Definition 11. Let $A_i = \left(\mu_{A_i}, \eta_{A_i}, \nu_{A_i}\right)$ $(i = 1,2,\cdots,n)$ be collection of picture fuzzy values. Then the PFGM operator is mapping $PFGM: A^n \to A$ such that

$$\mathrm{PFGM}(A_1, A_2, \cdots, A_n) = \left(\sqrt[n]{\prod_{i=1}^{n} \mu_{A_i}}, \sqrt[n]{\prod_{i=1}^{n} \eta_{A_i}}, \sqrt[n]{\prod_{i=1}^{n} \nu_{A_i}}\right).$$

Definition 12. Let $A_i = \left(\mu_{A_i}, \eta_{A_i}, \nu_{A_i}\right)$ $(i = 1,2,\cdots,n)$ be collection of picture fuzzy values and $w = (w_1, w_2, \cdots, w_n)^T$ be the weighting vector of A_i $(i = 1,2,\cdots,n)$ such that $w_i \in [0,1]$, $(i = 1,2,\cdots,n)$ and $\sum_{i=1}^{n} w_i = 1$. Then the picture PFWGM operator is a mapping $PFWGM: A^n \to A$ such that

$$\mathrm{PFWGM}(A_1, A_2, \cdots, A_n) = \left(\sqrt[n]{\prod_{i=1}^{n} w_i\mu_{A_i}}, \sqrt[n]{\prod_{i=1}^{n} w_i\eta_{A_i}}, \sqrt[n]{\prod_{i=1}^{n} w_i\nu_{A_i}}\right).$$

The following axioms are satisfied for *PFGM* and *PFWGM*.

Theorem 10 (Idempotency). Let $A_i = \left(\mu_{A_i}, \eta_{A_i}, \nu_{A_i}\right)$ $(i = 1,2,\cdots,n)$ be collection of picture fuzzy values. If $A_i = A$, $(i = 1,2,\cdots,n)$, then

$$PFGM(A_1, A_2, \cdots, A_n) = A.$$

And

$$\mathrm{PFWGM}(A_1, A_2, \cdots, A_n) = A.$$

Proof. For $A_i = A$ and $\sum_{i=1}^{n} w_i = 1$, we have

$$\mathrm{PFGM}(A_1, A_2, \cdots, A_n) = \left(\sqrt[n]{\prod_{i=1}^{n} \mu_{A_i}}, \sqrt[n]{\prod_{i=1}^{n} \eta_{A_i}}, \sqrt[n]{\prod_{i=1}^{n} \nu_{A_i}}\right)$$

$$= \left(\begin{array}{c}\sqrt[n]{\prod_{i=1}^{n} \mu_{A}}, \sqrt[n]{\prod_{i=1}^{n} \eta_{A}}, \\ \sqrt[n]{\prod_{i=1}^{n} \nu_{A}}\end{array}\right) = \left(\sqrt[n]{(\mu_A)^n}, \sqrt[n]{(\eta_A)^n}, \sqrt[n]{(\nu_A)^n}\right)$$

$$= (\mu_A, \eta_A, \nu_A) = A.$$

Theorem 11 (Monotonicity). If $A_i \leq A_i^*$, then

$$\mathrm{PFGM}(A_1, A_2, \cdots, A_n) \leq \mathrm{PFGM}(A^*_1, A^*_2, \cdots, A^*_n).$$

$$\mathrm{PFWGM}(A_1, A_2, \cdots, A_n) - \mathrm{PFWGM}(A^*_1, A^*_2, \cdots, A^*_n) \leq 0.$$

And

$$\mathrm{PFWGM}(A_1, A_2, \cdots, A_n) \leq \mathrm{PFWGM}(A^*_1, A^*_2, \cdots, A^*_n).$$

Proof.

$$\mathrm{PFGM}(A_1, A_2, \cdots, A_n) - \mathrm{PFGM}(A^*_1, A^*_2, \cdots, A^*_n) =$$

$$\left(\begin{array}{c}\left(\mu_{A_1}\cdot\mu_{A_2}\cdot\cdots\cdots\cdot\mu_{A_n}\right)^{\frac{1}{n}} - \left(\mu_{A_1^*}\cdot\mu_{A_2^*}\cdot\cdots\cdots\cdot\mu_{A_n^*}\right)^{\frac{1}{n}}, \\ \left(\eta_{A_1}\cdot\eta_{A_2}\cdot\cdots\cdots\cdot\eta_{A_n}\right)^{\frac{1}{n}} - \left(\eta_{A_1^*}\cdot\eta_{A_2^*}\cdot\cdots\cdots\cdot\eta_{A_n^*}\right)^{\frac{1}{n}}, \\ \left(\nu_{A_1}\cdot\nu_{A_2}\cdot\cdots\cdots\cdot\nu_{A_n}\right)^{\frac{1}{n}} - \left(\nu_{A_1^*}\cdot\nu_{A_2^*}\cdot\cdots\cdots\cdot\nu_{A_n^*}\right)^{\frac{1}{n}}\end{array}\right) \leq 0.$$

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Similarly, we can prove that

This proves the monotonicity of *PFGM* and *PFWGM*.

Theorem 12 (**Boundedness**). Let $A_{min} = min(A_1, A_2, \cdots, A_n)$ and $A_{max} = max(A_1, A_2, \cdots A_n)$, for $i = 1, 2, \cdots, n$, Then $A_{min} \leq PFGM(A_1, A_2, \cdots, A_n) \leq A_{max}$ and $A_{min} \leq PFWGM(A_1, A_2, \cdots A_n) \leq A_{max}$.

Proof. Boundedness is the consequence of monotonicity and idempotency.

Theorem 13 (Commutatively). If $\left(A_1^0, A_2^0, \cdots, A_n^0\right)$ be any permutation of $\left(A_1, A_2, \cdots, A_n\right)$, the

$$PFGM(A_1, A_2, \cdots, A_n) = PFGM\left(A^0_1, A^0_2, \cdots, A^0_n\right).$$

And

$$PFWGM(A_1, A_2, \cdots, A_n) = PFWGM\left(A^0_1, A^0_2, \cdots, A^0_n\right).$$

Proof.

$$PFGM(A_1, A_2, \cdots, A_n) - PFGM\left(A^0_1, A^0_2, \cdots, A^0_n\right) = \begin{pmatrix} \sqrt[n]{\prod_{i=1}^n \mu_{A_i}} - \sqrt[n]{\prod_{i=1}^n \mu_{A_i^0}}, \\ \sqrt[n]{\prod_{i=1}^n \eta_{A_i}} - \sqrt[n]{\prod_{i=1}^n \eta_{A_i^0}}, \\ \sqrt[n]{\prod_{i=1}^n \nu_{A_i}} - \sqrt[n]{\prod_{i=1}^n \nu_{A_i^0}} \end{pmatrix} = 0$$

because $\left(A_1^0, A_2^0, \cdots\cdots, A_n^0\right)$ be any permutation of $\left(A_1, A_2, \cdots\cdots, A_n\right)$.

Hence, we have

$$PFGM(A_1, A_2, \cdots, A_n) = PFGM\left(A^0_1, A^0_2, \cdots, A^0_n\right).$$

Again,

$$PFWGM(A_1, A_2, \cdots\cdots, A_n) - PFWGM\left(A^0_1, A^0_2, \cdots\cdots, A^0_n\right) = \begin{pmatrix} \sqrt[n]{\prod_{i=1}^n w_i\mu_{A_i}} - \sqrt[n]{\prod_{i=1}^n w_i\mu_{A_i^0}}, \\ \sqrt[n]{\prod_{i=1}^n w_i\eta_{A_i}} - \sqrt[n]{\prod_{i=1}^n w_i\eta_{A_i^0}}, \\ \sqrt[n]{\prod_{i=1}^n w_i\nu_{A_i}} - \sqrt[n]{\prod_{i=1}^n w_i\nu_{A_i^0}} \end{pmatrix} = 0.,$$

Because $\left(A_1^0, A_2^0, \cdots, A_n^0\right)$ be any permutation of $(A_1, A_2, \cdots, A_n)$.

Hence, we have

$$PFWGM(A_1, A_2, \cdots, A_n) = PFWGM\left(A^0_1, A^0_2, \cdots, A^0_n\right).$$

6 | Application of the Picture Fuzzy Weighted Mean Operators to Multiple Attribute Decision-Making

MADM problems are common in everyday decision environments. An MADM problem is to find a great concession solution from all possible alternatives measured on multiple attributes.

Let the discrete set of alternatives and attributes are $A = \{A_1, A_2, \cdots A_n\}$ and $C = \{C_1, C_2, \cdots C_m\}$ respectively. Let $w = (w_1, w_2, \cdots, w_m)^T$ be the weighting vector of attributes C_j $(j = 1, 2, \cdots, m)$ such that $w_j \in [0,1]$, $(j = 1, 2, \cdots, m)$ and $\sum_{j=1}^{m} w_j = 1$. Suppose decision maker gives the picture fuzzy values for the alternatives A_i $(i = 1, 2, \cdots, n)$ on attributes C_j $(j = 1, 2, \cdots, m)$ are $k_{ij} = \left(\mu_{k_{ij}}, \eta_{k_{ij}}, \nu_{k_{ij}}\right)$, where $\mu_{k_{ij}}, \eta_{k_{ij}}$ and $\nu_{k_{ij}}$are positive, neutral and negative membership values of A_i under C_j respectively. Here $\mu_{k_{ij}}, \eta_{k_{ij}}, \nu_{k_{ij}} \in [0,1]$ and $0 \le \mu_{k_{ij}} + \eta_{k_{ij}} + \nu_{k_{ij}} \le 1$. Hence, an MADM problem can be briefly stated in a picture fuzzy decision matrix

$$\mathrm{K} = \left(\mathrm{k}_{\mathrm{ij}}\right)_{\mathrm{n}\times\mathrm{m}}.$$

Step 1. Utilize the decision information given in matrix K, and the $PFWHM, PFWAM$ and $PFWGM$ operators to derive the overall preference values d_i $(i = 1, 2, \cdots, n)$of the alternative A_i $(i = 1, 2, \cdots, n)$.

Step 2. Calculate the scores $S(d_i)$ $(i = 1, 2, \cdots, n)$ of the overall picture fuzzy values d_i $(i = 1, 2, \cdots, n)$.

Step 3. Rank all the alternatives A_i $(i = 1, 2, \cdots, n)$ in accordance with the values of $S(d_i)$ $(i = 1, 2, \cdots, n)$ and select the best one(s). If there is no difference between two scores $S(d_i)$ and $S(d_j)$, then we need to calculate the accuracy degrees $H(d_i)$ and $H(d_j)$ of the overall picture fuzzy values d_i and d_j, respectively, and then rank the alternatives A_i and A_j in accordance with the accuracy degrees $H(d_i)$ and $H(d_j)$.

Step 4. End.

6.1 | Numerical Example

A ceramic factory is looking for a general manager. There are five applicants $A = \{A_1, A_2, A_3, A_4, A_5\}$ for this position. The company is also looking for four attributes $C = \{C_1, C_2, C_3, C_4\}$ from these applicants. These attributes are leadership, problem-solving skill, communication skill, and experimentation. An expert will be graded for the four attributes. The decision matrix $K = \left(k_{ij}\right)_{5\times4}$ is presented in *Table 1*, where k_{ij} $(i = 1, 2, \cdots, 5,\ j = 1, 2, \cdots, 4)$ are in the form of picture fuzzy values.

Table 1. Picture fuzzy decision matrix.

	C_1	C_2	C_3	C_4
A_1	(0.5,0.1,0.3)	(0.4,0.2,0.4)	(0.7,0.1,0.1)	(0.2,0.4,0.1)
A_2	(0.4,0.3,0.3)	(0.2,0.5,0.2)	(0.4,0.2,0.4)	(0.5,0.1,0.3)
A_3	(0.2,0.3,0.4)	(0.5,0.2,0.3)	(0.5,0.2,0.1)	(0.5,0.4,0.1)
A_4	(0.8,0.1,0.1)	(0.7,0.2,0.1)	(0.4,0.2,0.4)	(0.3,0.2,0.4)
A_5	(0.3,0.2,0.4)	(0.6,0.1,0.1)	(0.4,0.2,0.2)	(0.5,0.2,0.3)

The information about the attribute weights is known as: $w = (0.30, 0.35, 0.15, 0.20)$.

Step 1. Utilize the decision information given in matrix K and $PFWHM, PFWAM$ and $PFWGM$ operators, we have overall preference values d_i as following *Table 2*.

Table 2. Preference values $d_i (i = 1, 2, \cdots, 5)$ for the operators PFWHM, PFWAM and PFWGM.

	d_1	d_2	d_3	d_4	d_5
PFWHM	(0.37, 0.15, 0.19)	(0.31, 0.22, 0.26)	(0.34, 0.25, 0.18)	(0.52, 0.15, 0.14)	(0.42, 0.15, 0.18)
PFWAM	(0.11, 0.05, 0.07)	(0.09, 0.08, 0.07)	(0.10, 0.07, 0.07)	(0.15, 0.04, 0.05)	(0.12, 0.04, 0.06)
PFWGM	(0.10, 0.04, 0.04)	(0.08, 0.06, 0.07)	(0.09, 0.06, 0.04)	(0.12, 0.04, 0.05)	(0.10, 0.04, 0.05)

Step 2. The scores $S(d_i)$ $(i = 1, 2, \cdots, 5)$ of the overall picture fuzzy values d_i $(i = 1, 2, \cdots, 5)$ are as following *Table 3*.

Table 4. Ranking all the alternatives A_i $(i = 1, 2, \cdots, 5)$ in accordance with the values of $S(d_i)$ $(i = 1, 2, \cdots, 5)$.

Operators	Ranking	Best Alternatives
PFWHM	$A_4 \succ A_3 \succ A_5 \succ A_1 \succ A_2$	A_4
PFWAM	$A_4 \succ A_3 \succ A_2 \succ A_5 \succ A_1$	A_4
PFWGM	$A_4 \succ A_3 \succ A_1 \succ A_5 \succ A_2$	A_4

7 | Comparison Studies

Comparing our results with the method using Picture fuzzy aggregation operator Wei [29] we get following score values of weighted picture fuzzy aggregation operator

$S(d_1) = 0.23.$

$S(d_2) = 0.09.$

$S(d_3) = 0.20.$

$S(d_4) = 0.49.$

$S(d_5) = 0.26.$

Rank all the alternatives A_i $(i = 1, 2, \cdots ,5)$ in accordance with the values of $S(d_i)$ $(i = 1, 2, \cdots ,5)$,

$$A_4 \succ A_5 \succ A_1 \succ A_3 \succ A_2.$$

Hence the best alternative is A_4, which is same as our result.

We compare our result with method of some geometric aggregation operators given by Wang et al. [26] we have following score values of weighted geometric aggregation operator

$S(d_1) = 0.18.$

$S(d_2) = 0.11.$

$S(d_3) = 0.14.$

$S(d_4) = 0.37.$

$S(d_5) = 0.20.$

Rank all the alternatives A_i $(i = 1, 2, \cdots ,5)$ in accordance with the values of $S(d_i)$ $(i = 1, 2, \cdots ,5)$,

$$A_4 \succ A_5 \succ A_1 \succ A_3 \succ A_2.$$

Hence the best alternative is A_4, which is same as our result.

8 | Conclusions

Mean operators are very useful tools to aggregate some picture fuzzy sets. It also helps us to make a decision in many problems of our real life. In literature, a host of researchers studied on different kind of aggregation operators of picture fuzzy sets and applied them to solve many problems in practical life. In this article, we have introduced some picture fuzzy mean operators and explored some related properties of them. A practical example is illustrated by using our proposed operators. Comparison studied are also discussed to show the effectiveness of our proposed operators.

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