Some properties of Intuitionistic fuzzy modules

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Abstract

In this paper, we discuss the structure of intuitionistic fuzzy projective modules and investigate some properties of them. Also we study about intuitionistic fuzzy homomorphisms between intuitionistic fuzzy modules.

We study about exact sequences, products and co-products, functors and relating topics in $IFR - Mod$ and investigate the relationship between them, where $IFR - Mod$ is category whose objects are intuitionidtic fuzzy modules and morphisms are intuitionistic fuzzy homomorphisms.

For a commutative ring R and two intuitionistic fuzzy R- modules

$A = (\mu_A, \nu_A) \leq_{IF} M$, $B = (\mu_B, \nu_B) \leq_{IF} N$ we show that

$\mathrm{Hom}_{IF-R}(A, B) = (\alpha, \beta)$ is an intuitionistic fuzzy R-module.

Also for a commutative ring R, if

$$0 \longrightarrow A \xrightarrow{f^\sim} B \xrightarrow{g^\sim} C$$

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is *an exact sequence in IFR-Mod, where* $\tilde{f}$ *is IF split homomorphism, then we show that* $Hom_{IF-R}(D,-)$ *preserves the sequence, for every* $D \in$ IFR − Mod.

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1 Preliminaries and introduction

After the definition of fuzzy sets by Zadeh [27], a number of applications of this fundamental concept have come up. Rosenfeld [24] was the first one to define the concept of fuzzy subgroups of a group. Negoita and Ralescu [20] applied this concept to modules and defined fuzzy submodules of a module.

As a generalization of fuzzy sets, the concept of intuitionistic fuzzy sets was introduced by K. T. Atanassov in [3]. Using this idea, B. Davvaz [10] established the intuitionistic fuzzification of the concept of submodules of a module. The intuitionistic fuzzy set theory is useful in various application areas, such as algebraic structures, robotics, control systems, agriculture areas, computer, irrigation, economy and various engineering fields. The knowledge and semantic representation of intuitionistic fuzzy set become more meaningful, resourceful and applicable since it includes the degree of membership, the degree of non-membership and the hesitation margin.

[1, 4, 5, 9, 23] are some other researches about intuitionidtic fuzzy groups, ideals and modules and [6, 7, 8, 11, 12, 13, 14, 18, 19, 25, 26] are some recent researches and applications of intuitionistic fuzzy sets.

Pan [21, 22] studied on functors $Hom(\mu_A, -) : FR - Mod \longrightarrow FR - Mod$ and $Hom(-, \nu_B) : FR - Mod \longrightarrow FR - Mod$ in category of fuzzy modules. Isaac [15] gave an alternate definition for projective L-modules and investigated these fuzzy modules. Chen [7] studied the relation between projective S-acts and Hom functors in the category of S-acts and Liu [17] studied the Hom functors and tensor product functors in the category of fuzzy S-acts. In this paper, we study the properties of Hom functor in intuitionistic fuzzy modules category. We obtain some properties about intuitionistic fuzzy modules category and Hom functors in this category. intuitionistic fuzzy projective modules and their relationship with exact sequences in this category will be investigated.

Various applications of intuitionistic fuzzy set have been carried out through distance measures approach. Many researchers have explored various applications of intuitionistic fuzzy set such as medical diagnosis, medical application, career determination, real life situations.

A category C is given by a collection C_0 of *objects* and a collection C_1 of *morphisms* which have the following structures:

(i) Each arrow has a *domain* and a *codomain* which are objects; one writes $f : X \longrightarrow Y$ if X is the domain of the morphism f, and Y its codomain.

(ii) Given two morphisms f and g such that $cod(f) = dom(g)$, the composition of f and g, written gf, is defined and has domain $dom(f)$ and codomain $cod(g)$: $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that $X \xrightarrow{gf} Z$.

(iii) Composition is *associative*, that is: given $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ and $h : Z \longrightarrow W$, $h(gf) = (hg)f$.

(iv) For every object X there is an identity morphism $id_X : X \longrightarrow X$, satisfying $id_X g = g$ for every $g : Y \longrightarrow X$ and $f id_X = f$ for every $f : X \longrightarrow Y$.

For example the category of (fuzzy) $R-$modules has objects $R-$ modules and morphisms (fuzzy) $R-$homomorphisms.

By a *fuzzy set (or fuzzy subset)* of a module M, we mean the μ from M to $[0, 1]$. By $[0, 1]^M$ we will denote the set of all fuzzy subsets of M.

For each fuzzy subset μ of M and any $\alpha \in [0, 1]$, we define two sets $U(\mu, \alpha) = \{x \in M \,|\, \mu(x) \geq \alpha\}$, $L(\mu, \alpha) = \{x \in M \,|\, \mu(x) \leq \alpha\}$, which are called an *upper level cut* and a *lower level cut* of μ, respectively. The complement of μ, denoted by μ^c, is the fuzzy set on M defined by $\mu^c(x) = 1 - \mu(x)$.

If $N \subseteq M$ and $\alpha \in [0, 1]$ then α_N is defined as,

$$\alpha_N(x) = \begin{cases} \alpha, & x \in N \\ 0,, & otherwise \end{cases}$$

If $N = \{x\}$ then α_x is often called a *fuzzy point* and is denoted by x_α. When $\alpha = 1$, then 1_N is known as the characteristic function of N. From nowonwards, we will denote the characteristic function of N as χ_N.

Now let $X \subseteq M$ and $\mu, \ \sigma \in [0, 1]^X$, then

$\mu \subseteq \sigma$ if and only if $\mu(x) \leq \sigma(x)$, for every $x \in X$;

$(\mu \cup \sigma)(x) = max\{\mu(x), \sigma(x)\} = \mu(x) \vee \sigma(x)$, for every $x \in X$;

$(\mu \cap \sigma)(x) = min\{\mu(x), \sigma(x)\} = \mu(x) \wedge \sigma(x)$, for every $x \in X$;

$(\mu \times \sigma)(x, y) = min\{\mu(x), \sigma(y)\} = \mu(x) \wedge \sigma(y)$, for every $x, y \in X$.

Also for any family $\{\mu_i | i \in I\}$ of fuzzy subsets of M, where I is any nonempty index set

$(\bigcup_{i\in I} \mu_i)(x) = sup_{i\in I}\mu_i(x) = \bigvee_{i\in I}\mu_i(x)$, for every $x \in M$;

$(\bigcap \mu_i)(x) = inf_{i\in I}\mu_i(x) =$

$= \bigvee \mu_i(x)$, for every

$x \in M$.

Definition 1.1 A fuzzy set μ of a ring R is called a *fuzzy ideal*, if it satisfies the following properties:

(1) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$, for every $x, y \in R$.

(2) $\mu(xy) \geq \mu(x) \vee \mu(y)$, for every $x, y \in R$.

Definition 1.2 ([20]) Let M be a left R-module. The $\mu \in [0, 1]^M$ is called a *fuzzy(left) R-module* (fuzzy R-submodule of M) if

1) $\mu(x + y) \geq \mu(x) \wedge \mu(y)$, for every $x, y \in M$;

2) $\mu(rx) \geq \mu(x)$ for every $x \in M,\ r \in R$;

3) $\mu(0) = 1$.

Similarly, we can define fuzzy right R-modules.

Example 1.3 Let μ and ν be two fuzzy subset of $\mathbb{Z}$-module $\mathbb{Z}$ defined by

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{2} & \text{if } x \in 2\mathbb{Z} \setminus \{0\} \\ \frac{1}{3} & \text{if } x \notin 2\mathbb{Z} \end{cases}$$

$$\nu(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{3} & \text{if } x \in 2\mathbb{Z} \setminus \{0\} \\ \frac{1}{2} & \text{if } x \notin 2\mathbb{Z} \end{cases}$$

then μ is a fuzzy submodule of $\mathbb{Z}$ but ν is not a fuzzy submodule of $\mathbb{Z}$ ($\frac{1}{3} = \nu(1 + 3) < \nu(1) \wedge \nu(3) = \frac{1}{2}$).

Let A, B be two R-modules. For two fuzzy R-modules μ_A of A and ν_B of B, a function $\tilde{f} : \mu_A \longrightarrow \nu_B$ is called a *fuzzy R-homomorphism*, if $f : A \longrightarrow B$ is an R-homomorphism and satisfies $\nu(f(a)) \geq \mu(a)$ for every $a \in A$.

For simplicity, denote by $Hom(\mu_A, \nu_B)$ the set of all fuzzy R-homomorphisms from μ_A to ν_B.

If μ and σ are two fuzzy submodules of a module M, then

$$(\mu + \sigma)(x) = \vee\{\mu(y) \wedge \sigma(z) \mid y, z \in M, y + z = x\}$$

for every $x \in M$. It is not difficult to see that if μ and σ are two fuzzy submodules of M, then $\mu + \sigma$ is a fuzzy submodule of M.

Example 1.4 Let μ and σ be two fuzzy submodule of $\mathbb{Z}$-module $\mathbb{Z}$ defined by

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{2} & \text{if } x \in 2\mathbb{Z} \setminus \{0\} \\ \frac{1}{3} & \text{if } x \notin 2\mathbb{Z} \end{cases}$$

$$\sigma(x) = \begin{cases} 1 & \text{if } x \in 2\mathbb{Z} \\ \frac{1}{4} & \text{if } x \notin 2\mathbb{Z} \end{cases}$$

Then

$$(\mu + \sigma)(x) = \begin{cases} 1 & \text{if } x \in 2\mathbb{Z} \\ \frac{1}{3} & \text{if } x \notin 2\mathbb{Z} \end{cases}.$$

Definition 1.5 Let X and Y be any two nonempty sets, and $f : X \longrightarrow Y$ be a mapping. Let $\mu \in [0, 1]^X$ and $\sigma \in [0, 1]^Y$. Then the image $f(\mu) \in [0, 1]^Y$ and the inverse image $f^{-1}(\sigma) \in [0, 1]^X$ are defined as follows: for every $y \in Y$

$$f(\mu)(y) = \begin{cases} \vee\{\mu(x) \mid x \in X, f(x) = y\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and $f^{-1}(\sigma) = \sigma(f(x))$ for every $x \in X$.

2 Intuitionistic fuzzy modules, basic notions and properties

Throughout article R means an associative ring with unity and M denotes a unitary left R-module while R-Mod denotes the category of all left R-modules.

In this section we discuss on *intuitionistic fuzzy submodules* of a module and present initial concepts related with them.

An *intuitionistic fuzzy set* (briefly an IFS) A of a non-void set X is an object having the form $A = \{(x, \mu_A(x), \nu_A(x)); x \in X\}$, where the maps $\mu_A : X \longrightarrow [0, 1]$ and $\nu_A : X \longrightarrow [0, 1]$, are fuzzy subsets of X, denote respectively the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for every $x \in X$.

For the sake of simplicity, we denote an IFS, $A = \{(x, \mu_A(x), \nu_A(x)); x \in X\}$ of the set X by $A = (\mu_A, \nu_A)$ or briefly A, and the set of all IFS of X by $IFS(X)$.

If X is a nonempty set and $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$ are two IFS of X, then

$A \subseteq B$, if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, for every $x \in X$;

$A = B$ if and only if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$, for every $x \in X$;

$A^c = (\nu_A, \mu_A)$;

$A \cap B = \{(x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)); x \in X\}$;

$A \cup B = \{(x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)); x \in X\}$.

Let $\{A_i = (\mu_{A_i}, \nu_{A_i})\}_{i\in I}$ be a family of IFS of X. Then ([3])

$$\bigcap_{i\in I} A_i = (\mu_{(\cap_{i\in I} A_i)}, \nu_{(\cap_{i\in I} A_i)}) = \{(x, \wedge_{i\in I}\, \mu_{A_i}(x), \vee_{i\in I}\, \nu_{A_i}(x)); x \in X\}$$ and

$$\bigcup_{i\in I} A_i = (\mu_{(\cup_{i\in I} A_i)}, \nu_{(\cup_{i\in I} A_i)}) = \{(x, \vee_{i\in I}\, \mu_{A_i}(x), \wedge_{i\in I}\, \nu_{A_i}(x)); x \in X\}$$

Definition 2.1 Let M be an $R-$module and $A = (\mu_A, \nu_A)$ an IFS of M. Then A is called an *intuitionistic fuzzy submodule of* M if A satisfies the following conditions:

(1) $\mu_A(0) = 1,\ \nu_A(0) = 0$

(2) $\mu_A(x + y) \geq \mu_A(x) \wedge \mu_A(y)$, for every $x, y \in M$
, $\nu_A(x + y) \leq \nu_A(x) \vee \nu_A(y)$, for every $x, y \in M$,

(3) $\mu_A(rx) \geq \mu_A(x)$, for every $x \in M$ and $r \in R$
, $\nu_A(rx) \leq \nu_A(x)$, for every $x \in M$ and $r \in R$.

Example 2.2 Let

$$\mu(x) = \begin{cases} 1, & x = 0,2,4 \\ \frac{1}{2}, & x = 1,3,5. \end{cases}$$

and

$$\nu(x) = \begin{cases} 0, & x = 0,2,4 \\ \frac{1}{3}, & x = 1,3,5. \end{cases}$$

Then $A = (\mu, \nu)$ is an intuitionistic fuzzy submodule of $\mathbb{Z}_6$ as $\mathbb{Z}$-module.

If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy submodule of an R-module M, we write A is an IFM of M and denote by $A \leq_{IF} M$. In this case we say A is an intuitionistic fuzzy module too.
We use by $IFS(M)$, the set of all IFM of M and $IFR - Mod$, the category of all IF R-modules.

Definition 2.3 ([4]) Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IFM's of M. Then the IFM, $A+B$ of M is $A+B = \{(x, \mu_{A+B}(x), \nu_{A+B}(x); x \in M\}$ defined as

$$\mu_{A+B}(x) = \vee\{\mu_A(y) \wedge \mu_B(z) \mid x = y + z;\ y, z \in M\}$$

$$\nu_{A+B}(x) = \bigwedge \{\nu_A(y) \vee \nu_B(z) \mid x = y + z;\ y, z \in M\}$$

.

For an IFM, $A = (\mu_A, \nu_A)$ of M and for any $r \in R$, define the IFS, $rA = (\mu_{rA}, \nu_{rA})$ such that for every $x \in M$,

$$\mu_{rA}(x) = \bigvee \{\mu_A(y) \mid x = ry\ ; y \in M\} \text{ and } \nu_{rA}(x) = \bigwedge \{\nu_A(y) \mid x = ry\ ; y \in M\}.$$

So the IFM, $-A = (\mu_{-A}, \nu_{-A})$ will be defined as $\mu_{-A}(x) = \mu_A(-x)$ and $\nu_{-A}(x) = \nu_A(-x)$ for every $x \in M$.

Proposition 2.4 *Let A, B be two IFM's of an R-module M. Then $A + B$ and rA for every $r \in R$, are IFM's of M.*

Proof. It is straightforward and follows from definitions. Q

Let M be an R-module, $N \subseteq M$ and $\alpha \in [0, 1]$. Then the IFS $\alpha_N = (\mu_{\alpha_N}, \nu_{\alpha_N})$ of M is defined by

$$\mu_{\alpha_N}(x) = \begin{cases} \alpha & x \in N \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \nu_{\alpha_N}(x) = \begin{cases} 1 - \alpha & x \in N \\ 1 & \text{otherwise} \end{cases}$$

for every $x \in M$.

If $\alpha = 1$, then $\mu_{\alpha_N} = \chi_N$ and $\nu_{\alpha_N} = \chi^c_N$, where χ_N denotes the characteristic function of N. In this case we write $\alpha_N = \chi^{IF}_N = (\chi_N, \chi^c_N)$.

We denote χ^{IF}_0 by $\bar{0}$ or 0^{IF}_N and χ^{IF}_N by 1^{IF}_N, too.

If $A \leq_{IF} M$, then $\chi^{IF}_0 \leq_{IF} A \leq_{IF} \chi^{IF}_M$.

Proposition 2.5 *Let M be an R-module and $N \subseteq M$. Then $N \leq M$ if and only if $\chi^{IF}_N \leq_{IF} M$.*

Proof. Suppose that N is a submodule of M. Then $0 \in N$ and hence $\chi_N(0) = 1$ and $\chi^c_N(0) = 0$.

Now let x, $y \in M$. If $x, y \in N$, then $x + y \in N$, so $1 = \chi_N(x + y) \geq \chi_N(x) \wedge \chi_N(y)$ and $0 = \chi^c_N(x + y) \leq \chi^c_N(x) \vee \chi^v_N(y)$.

If $x \notin N$, then

$$\chi_N(x + y) \geq \chi_N(x) \wedge \chi_N(y) = 0 \quad and \quad \chi^c_N(x + y) \leq \chi^c_N(x) \vee \chi^c_N(y) = 1.$$

Similar to this case we get if $y \notin N$.

Now let $x \in M$ and $r \in R$. If $x \in N$, then $rx \in N$ and so we have $1 = \chi_N(rx) \geq \chi_N(x)$ and $0 = \chi^c_N(x) \leq \chi^c_N(rx)$.

If $x \notin N$, then $0 = \chi_N(x) \leq \chi_N(rx)$ and also $1 = \chi^c_N(x) \geq \chi^c_N(rx)$.

Therefore χ^{IF}_N is an IFM of M.

Conversely suppose that χ^{IF}_N is an IFM of M. So $\chi_N(0) = 1$ and hence $0 \in N$. Now let $x, y \in N$ and $r \in R$, then $\chi_N(rx + y) \geq \chi_N(rx) \wedge \chi_N(y) \geq \chi_N(x) \wedge \chi_N(y) = 1$. So $rx + y \in N$. That is N is a submodule of M.

Example 2.6

(1) Since $n\mathbb{Z} \leq \mathbb{Z}$ so $\chi^{IF}_{n\mathbb{Z}} = (\chi_{n\mathbb{Z}}, \chi^c_{n\mathbb{Z}})$ is IFM of $\mathbb{Z}$ for every $n \in \mathbb{Z}$.

(2) $\mathbb{Z} \leq \mathbb{Q}$ and hence $\chi^{IF}_{\mathbb{Z}}$ is an intuitionistic fuzzy submodule of $\mathbb{Q}$.

(3) $\mathbb{Z}^\infty_p \leq \frac{\mathbb{Q}}{\mathbb{Z}}$ and hence $\chi^{IF}_{\mathbb{Z}_P}$ is an intuitionistic fuzzy submodule of $\frac{\mathbb{Q}}{\mathbb{Z}}$.

Let M, N be two R-modules and $f : M \longrightarrow N$ an R-homomorphism. Let $A = (\mu_A, \nu_A) \leq_{IF} M$ and $B = (\mu_B, \nu_B) \leq_{IF} N$, too. Then $f(A) = (\mu_{f(A)}, \nu_{f(A)}) \leq_{IF} N$ and $f^{-1}(B) = (\mu_{f^{-1}(B)}, \nu_{f^{-1}(B)}) \leq_{IF} M$ define by

$$(\mu_{f(A)})(y) = \begin{cases} \bigvee\{\mu_A(x) \mid y = f(x)\} & y \in Im(f) \\ 0 & y \notin Im(f) \end{cases}$$

,

$$(\nu_{f(A)})(y) = \begin{cases} \bigwedge\{\nu_A(x) \mid y = f(x)\} & y \in Im(f) \\ 1 & y \notin Im(f) \end{cases}$$

and

$$(\mu_{f^{-1}(B)})(x) = \mu_B(f(x)) \quad , \quad (\nu_{f^{-1}(B)})(x) = \nu_B(f(x)).$$

Proposition 2.7 *If $A \leq_{IF} M$, $B \leq_{IF} N$ and $f : M \longrightarrow N$ be an $R-$homomorphism. Then $f(A) \leq_{IF} N$ and $f^{-1}(B) \leq_{IF} M$.*

Proof. It is clear.

3 Intuitionistic fuzzy homomorphisms

In this section first we introduce intuitionistic fuzzy R-homomorphisms and then discuss about them.

Definition 3.1 Let R be a ring and *M, N* be R-modules such that $A = (\mu_A, \nu_A) \leq_{IF} M$ and $B = (\mu_B, \nu_B) \leq_{IF} N$. The function $\tilde{f} : A \to B$ is called an *intuitionistic fuzzy R-homomorphism*, if $f : M \longrightarrow N$ is an R-homomorphism and $\mu_B(f(x)) \geq \mu_A(x)$ and $\nu_B(f(x)) \leq \nu_A(x)$ for every $x \in M$.

For simplicity, we denote by $Hom_{IF-R}(A, B)$ the set of all intuitionistic fuzzy R-homomorphisms from A to B.

Example 3.2

(1) The identity map $id : M \longrightarrow M$ is an IF homomorphism ($\tilde{f} : A \longrightarrow A$) for every intuitionistic fuzzy submodule A of M.

(2) Let A,B be two intuitionistic fuzzy submodules of $\mathbb{Z}$ defined by $\mu_A(x) = \begin{cases} 1 & x \in 2\mathbb{Z} \\ \frac{1}{2} & x \notin 2\mathbb{Z} \end{cases}$, $\nu_A(x) = \begin{cases} 0 & x \in 2\mathbb{Z} \\ \frac{1}{3} & x \notin 2\mathbb{Z} \end{cases}$ and $\mu_B(x) = \begin{cases} 1 & x \in 3\mathbb{Z} \\ \frac{1}{5} & x \notin 3\mathbb{Z} \end{cases}$, $\nu_B(x) = \begin{cases} 0 & x \in 3\mathbb{Z} \\ \frac{1}{5} & x \notin 3\mathbb{Z} \end{cases}$.

Define $\tilde{f} : A \longrightarrow B$ such that ($f : \mathbb{Z} \longrightarrow \mathbb{Z}$) $f(x) = 2x$ for every $x \in \mathbb{Z}$. Then f is an R-homomorphism but $\tilde{f}$ is not an IF homomorphism. (see that $\frac{1}{5} = \mu_B(f(2)) < \mu_A(2) = 1$).

Definition 3.3 An intuitionistic fuzzy R-homomorphism $\tilde{f} \in Hom_{IF-R}(A, B)$ is called *fuzzy split*, if there is an intuitionistic fuzzy R-homomorphism $\tilde{t} \in Hom_{IF-R}(B, A)$ such that the composition $\tilde{t}\tilde{f} = id$.

Definition 3.4 An intuitionistic fuzzy R-homomorphism $\tilde{f} \in Hom_{IF-R}(A, B)$ is called *intuitionistic fuzzy quasi-isomorphism* if *f* is an isomorphism.

If $\tilde{f} : A \longrightarrow B$ is an IF R-homomorphism, define $Ker\tilde{f} = \left\{a \in A;\ \begin{matrix} \mu_B(\tilde{f}(a)) = 1; \\ \nu_B(\tilde{f}(a)) = 0 \end{matrix}\right\}$

and $Im\tilde{f} = \{\tilde{f}(a) | a \in A\}$.

Definition 3.5 An intuitionistic fuzzy R-homomorphism $\tilde{f} \in Hom_{IF-R}(A, B)$ is called *intuitionistic fuzzy isomorphism*, if *f* is an isomorphism and $\mu_B(\tilde{f}(a)) = \mu_A(a)$, $\nu_B(\tilde{f}(a)) = \nu_A(a)$ for every $a \in M$.

Note that $ker\tilde{f} = kerf$ is not true in general, but $kerf \subseteq ker\tilde{f}$.

If $ker\tilde{f} = \{0\}$ then $\tilde{f}$ is monomorphism because if

$$f(x) = f(y) \Rightarrow f(x-y) = 0 \Rightarrow \begin{cases} \mu_N(\tilde{f}(x-y)) = 1; \\ \nu_N(\tilde{f}(x-y)) = 0. \end{cases} \Rightarrow x-y \in ker\tilde{f} = \{0\} \Rightarrow x = y.$$

But the reverse is not true, it means if $\tilde{f}$ is a monomorphism then it need not that $ker\tilde{f} = \{0\}$.

Example 3.6 If $B = 1^{IF}_M$ then $Ker\tilde{f} = M$, for every $A \in IFR-Mod$ and $\tilde{f} \in Hom_{IF-R}(A, B)$. Especially let $M = N = \mathbb{Z}$, $A = B = 1^{IF}_M$ and $\tilde{f} : A \to B$ be the identity map. Then $kerf = \{0\}$ but $ker\tilde{f} = \mathbb{Z}$.

Proposition 3.7 *Let R be a ring. If $\tilde{f} \in Hom_{IF-R}(A, B)$, where A and B are two intuitionistic fuzzy R-modules, such that $A = (\mu_A, \nu_A) \leq_{IF} M$ and $B = (\mu_B, \nu_B) \leq_{IF} N$, then*

(1) $Ker\tilde{f}$ is a submodule of M,

(2) Define $\mu'|_{ker\tilde{f}} : Ker\tilde{f} \to [0, 1]$, $\nu'|_{ker\tilde{f}} : Ker\tilde{f} \to [0, 1]$ by $\begin{cases} \mu'(k) = \mu(k); \\ \nu'(k) = \nu(k). \end{cases}$

for every $k \in ker\tilde{f}$.

Then $A' = (\mu'|_{ker\tilde{f}}, \nu'|_{ker\tilde{f}})$ is an intuitionistic fuzzy submodule of A.

Proof.

(1) Let A be the zero element of M. Obviously, we have $A \in Ker\tilde{f}$. Given $a \in Ker\tilde{f}$ and $r \in R$, then

$$\mu_B(\tilde{f}(ra)) = \mu_B(r\tilde{f}(a)) \geq \mu_B(\tilde{f}(a)) = 1,$$

$$\nu_B(\tilde{f}(ra)) = \nu_B(r\tilde{f}(a)) \leq \nu_B(\tilde{f}(a)) = 0.$$

So , we get $ra \in Ker\tilde{f}$. Particularly, we have $-a \in Ker\tilde{f}$. If $a, b \in Ker\tilde{f}$, we can easily get $a + b \in Ker\tilde{f}$. This proves that $Ker\tilde{f}$ is a submodule of A.

(2) It is clear.

4 IF Exact Sequences and IF Hom functors in IFR-Mod

Here we will define a function from $Hom_{IF-R}(A, B)$ to [0, 1] and make $Hom_{IF-R}(A, B)$ into an intuitionistic fuzzy R-module.

In R-Mod, the sequence

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} K \longrightarrow 0$$

is called a *short exact sequence* if f is a monomorphism, g is an epimorphism and $Imf = Kerg$. We know that f is monomorphism *iff* $Kerf = \{0\}$. Suppose that $\bar{0}$ denote the intuitionistic fuzzy zero module. Next we define the exact sequence in IFR-Mod.

Definition 4.1 Let A, B and C be intuitionistic fuzzy R-modules of M, N and K respectively. A short exact sequence is a sequence of the form

$$0 \longrightarrow A \xrightarrow{\tilde{f}} B \xrightarrow{\tilde{g}} C \longrightarrow \bar{0}$$

where $\tilde{f}$ is a monomorphism, $\tilde{g}$ is an epimorphism and $Im\tilde{f} = ker\tilde{g}$.

Note that $Ker\tilde{f}$ is usually larger than {0} by Definition 3.5. Hence, the crisp case of the definition is different from the well-known notion of short exact sequence in the category IFR-mod.

If $C = 1_K$, we get that $Im\tilde{f} = Ker\tilde{g} = N$. As $\tilde{f}$ is monic, we can get that $\tilde{f}$ is quasi-isomorphism.

Example 4.2 Let $A = \chi^{IF}_{\mathbb{Z}}$, $B = \chi^{IF}_{n\mathbb{Z}}$ and $C = \chi^{IF}_{\frac{\mathbb{Z}}{n\mathbb{Z}}}$. Then

$$\bar{0} \longrightarrow A \xrightarrow{\tilde{i}} B \xrightarrow{\tilde{\pi}} C \longrightarrow \bar{0}$$

is an exact sequence where $\tilde{i}$ and $\tilde{\pi}$ are inclusion map and natural epimorphism, respectively.

Definition 4.3 An intuitionistic fuzzy R-module $P = (\mu_P, \nu_P)$ is called *projective* if for every surjective IF R-homomorphism $\tilde{f} : A \longrightarrow B$ and for every IF R-homomorphism $\tilde{g} : P \longrightarrow B$, there exists an IFR-homomorphism $\tilde{h} : P \longrightarrow A$ such that $\tilde{f}\tilde{h} = \tilde{g}$.

Remark 4.4 *If P is a projective intuitionistic fuzzy R-module, then $P = \bar{0}$.*

Proof. Let $P = (\mu_P, \nu_P) \leq_{IF} M$ be a projective intuitionistic fuzzy R-module and
$A = \bar{0} \leq_{IF} M$, $B = P \leq_{IF} M$. Let $\tilde{f} = \tilde{g} = id_M$. Then there exists $\tilde{h} = id : P \longrightarrow A$, since P is projective. We must have $\mu_A(h(x)) \geq \mu_P(x)$ and $\nu_A(h(x)) \leq \nu_P(x)$ for every $x \in P$. This implies $P \subseteq A = \bar{0}$.

Theorem 4.5 *Let R be a commutative ring and $A = (\mu_A, v_A) \leq_{IF} M$, $B = (\mu_B, v_B) \leq_{IF} N$ be two intuitionistic fuzzy R- modules. Then $Hom_{IF-R}(A, B) = (\alpha, \beta)$ is an intuitionistic fuzzy R-module with membership function*
$\alpha : Hom_{IF-R}(A, B) \longrightarrow [0, 1]$ and non-membership function
$\beta : Hom_{IF-R}(A, B) \longrightarrow [0, 1]$ defined by

$$\alpha(\tilde{f}) = \bigwedge \{\mu_B(\tilde{f}(x)) \mid x \in M\} \text{ and } \beta(\tilde{f}) = \bigvee \{v_B(\tilde{f}(x)) \mid x \in M\}.$$

Proof. Assume $r \in R$ and $\tilde{f} \in Hom_{IF-R}(A, B)$. Define a function $r{\cdot}\tilde{f} : A \longrightarrow B$ by $r{\cdot}\tilde{f}(x) = r\tilde{f}(x)$ for every $x \in M$.
Then we have

$$\mu_B(r{\cdot}\tilde{f}(x)) = \mu_B(r\tilde{f}(x)) \geq \mu_B(\tilde{f}(x)) \geq \mu_A(x)$$

$$v_B(r{\cdot}\tilde{f}(x)) = v_B(r\tilde{f}(x)) \leq v_B(\tilde{f}(x)) \leq v_A(x)$$

This concludes that $r{\cdot}\tilde{f} \in Hom_{IF-R}(A, B)$. Hence we show that $Hom_{IF-R}(A, B)$ is an R-module.
We now have to prove that $Hom_{IF-R}(A, B)$ is an intuitionistic fuzzy R-module.
Suppose that
$r \in R, \tilde{f} \in Hom_{IF-R}(A, B)$ and $x \in M$. By

$$\mu_B(r{\cdot}\tilde{f}(x)) = \mu_B(r\tilde{f}(x)) \geq \mu_B(\tilde{f}(x)),$$

$$v_B(r{\cdot}\tilde{f}(x)) = v_B(r\tilde{f}(x)) \leq v_B(\tilde{f}(x))$$

we have, $\bigvee\{\mu_B(r\tilde{f}(x)) \mid x \in M\} \geq \bigvee\{\mu_B(\tilde{f}(x)) \mid x \in M\}$ and $\bigvee\{v_B(r\tilde{f}(x)) \mid x \in M\} \leq \bigvee\{v_B(\tilde{f}(x)) \mid x \in M\}$. This implies that $\alpha(r{\cdot}\tilde{f}) \geq \alpha(\tilde{f})$ and $\beta(r{\cdot}\tilde{f})$

If $\tilde{f}, \tilde{g} \in Hom_{IF-R}(A, B)$ then

$$\alpha(\tilde{f}+\tilde{g}) = \bigwedge\{\mu_B(\tilde{f}(x)+\tilde{g}(x)) \mid x \in M\} \geq \bigwedge\{\bigwedge(\mu_B(\tilde{f}(x)), \mu_B(\tilde{g}(x))) \mid x \in M\}$$

$$\geq \bigwedge\{\{\bigwedge(\mu_B(\tilde{f}(x)) \mid x \in M\}, \{\bigwedge(\mu_B(\tilde{g}(x)) \mid x \in M\}\} = \bigwedge\{\alpha(\tilde{f}), \alpha(\tilde{g})\}$$

$$\beta(\tilde{f}+\tilde{g}) = \bigvee\{v_B(\tilde{f}(x)+\tilde{g}(x)) \mid x \in M\} \leq \bigvee\{\bigvee(v_B(\tilde{f}(x)), v_B(\tilde{g}(x))) \mid x \in M\}$$

$$\leq \bigvee\{\{\bigvee(v_B(\tilde{f}(x)) \mid x \in M\}, \{\bigvee(v_B(\tilde{g}(x)) \mid x \in M\}\} = \bigvee\{\beta(\tilde{f}), \beta(\tilde{g})\}$$

We obviously have that $\begin{cases} \alpha(0) = 1; \\ \beta(0) = 0. \end{cases}$ Therefore $Hom_{IF-R}(A, B)$ is an intuitionistic fuzzy R-module.

Example 4.6 Let $\mathbb{Z}$ be the set of integers and $M = 4\mathbb{Z}$. Define $B = (\mu_B, \nu_B) \leq_{IF} \mathbb{Z}$ such that $\mu_B : \mathbb{Z} \longrightarrow [0, 1]$, $\nu_B : \mathbb{Z} \longrightarrow [0, 1]$ by

$$\mu_B(n) = \begin{cases} \frac{1}{10}, & if\ n \neq 0; \\ 1, & if\ n = 0. \end{cases}$$

$$\nu_B(n) = \begin{cases} \frac{4}{10}, & if\ n \neq 0; \\ 0, & if\ n = 0. \end{cases}$$

and $C = (\mu_C, \nu_C) \leq_{IF} \mathbb{Z}$

$$\mu_C(\bar{k}) = \begin{cases} \frac{1}{2}, & if\ \bar{k} \neq \bar{0}; \\ 1, & if\ \bar{k} = \bar{0}. \end{cases}$$

$$\nu_C(\bar{k}) = \begin{cases} \frac{1}{3}, & if\ \bar{k} \neq \bar{0}; \\ 1, & if\ \bar{k} = \bar{0}. \end{cases}$$

Then both B and C are intuitionistic fuzzy $\mathbb{Z}$-modules and

$$\bar{0} \longrightarrow 0_M \xrightarrow{\tilde{f}} B \xrightarrow{\tilde{g}} C \longrightarrow \bar{0}$$

is a short exact sequence of intuitionistic fuzzy $\mathbb{Z}$-modules and IF $\mathbb{Z}$-homomorphisms, where $\tilde{f}$ is the inclusion homomorphism and $\tilde{g}$ is the natural epimorphism.

Example 4.7 Let $F = Hom_{IF-R}(B, -)$. Consider the short exact sequence

$$\bar{0} \longrightarrow 0_M \xrightarrow{\tilde{f}} B \xrightarrow{\tilde{g}} C \longrightarrow \bar{0}$$

in Example 4.6.

We claim that the sequence

$$\bar{0} \longrightarrow Hom_{IF-R}(B, 0_M) \xrightarrow{F\tilde{f}} Hom_{IF-R}(B, B) \xrightarrow{F\tilde{g}} Hom_{IF-R}(B, C)$$

is not exact. Define $\tilde{h}_1 : \mathbb{Z} \longrightarrow \mathbb{Z}$ by putting $\tilde{h}_1(n) = 6n$ and define $\tilde{h}_2 : \mathbb{Z} \longrightarrow \mathbb{Z}$ by putting $\tilde{h}_2(n) = 12n$. We can check that both $\tilde{h}_1$ and $\tilde{h}_2$ are in $KerF\tilde{g}$ (note that $\tilde{f}$ is the inclusion map, $\tilde{g}$ is the natural epimorphism and see the definition of $F = Hom_{IF-R}(B, -)$ from Theorem 4.5).

Hence $|\ kerF\tilde{g}\ | \geq 2$. Since $Hom(B, 0_M)$ contains only zero morphism, we have $ImF\tilde{f} \neq KerF\tilde{g}$. So $Hom_{IF-R}(B, -)$ is not exact.

Theorem 4.8 *Let R be a commutative ring and let*

$$0 \longrightarrow A \xrightarrow{\tilde{f}} B \xrightarrow{\tilde{g}} C$$

be an exact sequence in IFR-Mod, where $\tilde{f}$ is IF split homomorphism. Then $Hom_{IF-R}(D,-)$ preserves the sequence, for every $D \in$ IFR − Mod.

Proof. Let $F = Hom_{IF-R}(D, -)$ such that $A = (\mu_A, \nu_A) \leq_{IF} M$, $B = (\mu_B, \nu_B) \leq_{IF} N$, $C = (\mu_c, \nu_c) \leq_{IF} K$ and $D = (\mu_D, \nu_D) \leq_{IF} H$. We will show that the sequence

$$\bar{0} \longrightarrow Hom_{IF-R}(D, A) \xrightarrow{F\tilde{f}} Hom_{IF-R}(D, B) \xrightarrow{F\tilde{g}} Hom_{IF-R}(D, C)$$

is exact. Put $F\tilde{f} = f_*$ and $F\tilde{g} = g_*$.

$Hom_{IF-R}(D, -)$ is left exact, so it is clear that f_* is monic.

Let $Hom_{IF-R}(D, A) = (\alpha_1, \alpha_2)$ that $\alpha_1 : Hom_{IF-R}(D, A) \longrightarrow [0, 1]$, $\alpha_2 : Hom_{IF-R}(D, A) \longrightarrow [0, 1]$ are membership and non-membership functions, respectively, such that

$\alpha_1(\tilde{\psi}) = \bigvee\{\mu_A(\psi(x)) \mid x \in H\}$, $\alpha_2(\tilde{\psi}) = \bigwedge\{\nu_A(\psi(x)) \mid x \in H\}$. Also let $Hom_{IF-R}(D, C) = (\delta_1, \delta_2)$.

Now if $\tilde{\lambda} \in Imf_*$, then $\delta_2(\tilde{g}o\tilde{f}o\tilde{\psi}) = \bigwedge\{\nu_C((\tilde{g}o\tilde{f}o\tilde{\psi})(x)) \mid x \in H\} = \bigwedge\{\nu_C((\tilde{g}o\tilde{f}(\tilde{\psi}))(x)) \mid x \in H\} = \bigwedge\{0\} = 0$. So $ImF\tilde{f} \subseteq KerF\tilde{g}$.

Now we will show that $ImF\tilde{f} \supseteq KerF\tilde{g}$.

Assume $\tilde{\lambda} \in Kerg_*$. Hence we have

(1) $1 = \delta_1(g_*(\tilde{\lambda})) = \bigvee\{\mu_C(go\lambda(x)) \mid x \in H\}$, hence $\mu_C(go\lambda(x)) = 1$, for every $x \in H$,

(2) $0 = \delta_2(g_*(\tilde{\lambda})) = \bigwedge\{\nu_C(go\lambda(x)) \mid x \in H\}$, hence $\nu_C(go\lambda(x)) = 0$, for every $x \in H$.

From (1), (2) we conclude that $Im\tilde{\lambda} \in Ker\tilde{g} = Im\tilde{f}$.

Now define $\tilde{\rho} : D \longrightarrow A$ by $\tilde{\rho}(x) = m$ for every $x \in H$, where $\tilde{\lambda}(x) = \tilde{f}(m)$.

It is not difficult to see that $\tilde{f}o\tilde{\rho} = \tilde{\lambda}$(i.e $\tilde{\lambda} \in Imf_*$), that completes the proof.

Theorem 4.9 *Let R be a commutative ring and*

$$A \xrightarrow{\tilde{f}} B \xrightarrow{\tilde{g}} C \longrightarrow \bar{0}$$

be an exact sequence of intuitionistic fuzzy R-modules, where $\tilde{f}$ is IF split. Let $G = Hom_{IF-R}(-, D)$. Then the following sequence is exact

$$\bar{0} \longrightarrow Hom_{IF-R}(C, D) \xrightarrow{G\tilde{g}} Hom_{IF-R}(B, D) \xrightarrow{G\tilde{f}} Hom_{IF-R}(A, D)$$

Proof. It is similar to the proof of Theorem 4.8.

Let $E(R)$ be the set of all idempotent members of the ring R.

Now, we study the functor $Hom_{IF-R}(M, -)$, where $M = 0^{IF}_{Re}$ for $e \in E(R)$. First suppose that $A \leq_{IF-R} M$ and $e \in E(R)$. Then the intuitionistic fuzzy R-module $eA \leq_{IF} eM$ is defined by $eA = (\mu_{eA}, \nu_{eA})$ such that

$$\begin{cases} \mu_{eA}(em) = \mu_A(em) \\ \nu_{eA}(em) = \nu_A(em) \end{cases}.$$

Lemma 4.10 *Let R be a commutative ring and $A \in IFR - Mod$. Then $\Gamma_A : Hom(0^{IF}_{Re} A) \longrightarrow eA$ defined by $\tilde{f} \longrightarrow \tilde{f}(e)$, is an intuitionistic fuzzy R-module isomorphism.*

Proof. Assume $em \in eM$ where $A = (\mu_A, \nu_A) \leq_{IF} M$. Define a map $\tilde{f} : 0^{IF}_{Re} \longrightarrow eA$ by putting $\tilde{f}(re) = rem$. We can easily check that $\tilde{f} \in Hom_{IF-R}(0^{IF}_{Re}, A)$ and $\Gamma_A(f) = em$. It follows that Γ_A is a surjective map. If $\tilde{f} \in Hom_{IF-R}(0^{IF}_{Re}, A)$, then we can see that f is determined by $f(e)$. This shows that Γ_A is an injective map. Let $\tilde{f} \in Hom_{IF-R}(0^{IF}_{Re}, A)$. We have $\alpha_1(\tilde{f}) = \mu_A(\tilde{f}(e))$, $\alpha_2(\tilde{f}) = \nu_A(\tilde{f}(e))$ where $Hom_{IF-R}(0^{IF}_{Re}, A) = (\alpha_1, \alpha_2)$. This shows Γ_A is an intuitionistic fuzzy isomorphism.

Proposition 4.11 *Let R be a ring and the following diagram of intuitionistic fuzzy R-modules is commutative:*

$$\begin{array}{ccccccccc} \bar{0} & \longrightarrow & A & \xrightarrow{\tilde{f}} & B & \xrightarrow{\tilde{g}} & C & \longrightarrow & \bar{0} \\ & & \downarrow\tilde{\alpha} & & \downarrow\tilde{\beta} & & \downarrow\tilde{\gamma} & & \\ \bar{0} & \longrightarrow & D & \xrightarrow{\tilde{h}} & E & \xrightarrow{\tilde{p}} & F & \longrightarrow & \bar{0} \end{array}$$

where $\tilde{\alpha}$, $\tilde{\gamma}$ are IF isomorphisms and $\tilde{\beta}$ is an IF quasi-isomorphism. Then the bottom row is a short exact sequence if and only if so is the top row.

Proof. Let $A \leq_{IF} M$, $B \leq_{IF} N$, $C \leq_{IF} K$, $D \leq_{IF} H$, $E \leq_{IF} X$ and $F \leq_{IF} Y$.

Suppose that the bottom row is exact. First we prove that $\tilde{f}$ is monomorphism. Assume $f(m_1) = f(m_2)$; $(m_1, m_2 \in M)$. We have $\tilde{h}\tilde{\alpha}(m_1) = \tilde{\beta}\tilde{f}(m_1) = \tilde{\beta}\tilde{f}(m_2) = \tilde{h}\tilde{\alpha}(m_2)$. So $m_1 = m_2$, as $\tilde{h}\tilde{\alpha}$ is monomorphism. So $\tilde{f}$ is monic.

Let $k \in K$ and $\tilde{\gamma}(k) = y$. Since $\tilde{p}$ is epimorphism, so there exists $x \in X$ such that $\tilde{p}(x) = y$. Suppose that $x = \tilde{\beta}(n)$ for some $n \in N$. Now $y = \tilde{p}(x) = \tilde{p}\tilde{\beta}(n) = \tilde{\gamma}\tilde{g}(n)$ *i.e.* $\tilde{\gamma}(x) = \tilde{\gamma}\tilde{g}(x)$ and hence $x = \tilde{g}(x)$ as $\tilde{\gamma}$ is monic. This implies $\tilde{g}$ is an epimorphism.

Now let $m \in M$. We will show that $\tilde{f}(m) \in Ker\tilde{g}$. For this first we have $\tilde{h}\tilde{\alpha}(m) \in Im\tilde{h} = Ker\tilde{p}$. Therefore $\begin{cases} \mu_F(\tilde{p}(\tilde{h}\tilde{\alpha}(m))) = 1; \\ \nu_F(\tilde{p}(\tilde{h}\tilde{\alpha}(m))) = 0 \end{cases}$ and so $\begin{cases} \mu_F(\tilde{\gamma}(\tilde{g}\tilde{f}(m))) = 1; \\ \nu_F(\tilde{\gamma}(\tilde{g}\tilde{f}(m))) = 0 \end{cases}$.

Then since $\tilde{\gamma}$ is isomorphism, hence $\begin{cases} \mu_C(\tilde{g}(\tilde{f}(m))) = \mu_F(\tilde{\gamma}(\tilde{g}\tilde{f}(m))) = 1; \\ \nu_C(\tilde{g}(\tilde{f}(m))) = \nu_F(\tilde{\gamma}(\tilde{g}\tilde{f}(m))) = 0 \end{cases}$ and this implies $\tilde{f}(m) \in Ker\tilde{g}$. So $Im\tilde{f} \subseteq Ker\tilde{g}$.

Now suppose that $n \in Ker\tilde{g}$, so $\begin{cases} \mu_C(\tilde{g}(n)) = 1; \\ \nu_C(\tilde{g}(n)) = 0 \end{cases}$ and hence $\begin{cases} \mu_F(\tilde{\gamma}\tilde{g}(n)) = 1; \\ \nu_F(\tilde{\gamma}\tilde{g}(n)) = 0 \end{cases}$

and by commutativity we can obtain $\begin{cases} \mu_F(\tilde{p}\tilde{\beta}(n)) = 1; \\ \nu_F(\tilde{p}\tilde{\beta}(n)) = 0 \end{cases}$. This implies $\tilde{\beta}(n) \in Ker\tilde{p} = Im\tilde{h}$ and so there exists $t \in H$ such that $\tilde{h}(t) = \tilde{\beta}(n)$. Also there exists $m \in M$ such that $\tilde{\alpha}(m) = t$.

Now $\tilde{\beta}\tilde{f}(m) = \tilde{h}\tilde{\alpha}(m) = \tilde{h}(t) = \tilde{\beta}(n)$ and hence $n = \tilde{f}(m)$, as $\tilde{\beta}$ is isomorphism. Thus $Ker\tilde{g} \subseteq Im\tilde{f}$.
Similarly the converse can be shown.

Definition 4.12 Let M, N be two $R-$modules and $A \leq_{IF} M$, $B \leq_{IF} N$. If $\tilde{f} : A \longrightarrow B$ is an IF homomorphism and $e \in E(R)$, then we define $e\tilde{f} : eA \longrightarrow eB$ by $e\tilde{f}(em) = \tilde{f}(em) = e\tilde{f}(m)$, for every $m \in M$.

Proposition 4.13 *Let R be a commutative ring and $\bar{0} \longrightarrow A \xrightarrow{\tilde{f}} B \xrightarrow{\tilde{g}} C \longrightarrow \bar{0}$ be a short exact sequence of intuitionistic fuzzy R-module. Let $e \in E(R)$, $e\tilde{f} = \tilde{f}|_{eA}$ and $e\tilde{g} = \tilde{g}|_{eB}$. Then the sequence $\bar{0} \longrightarrow eA \xrightarrow{e\tilde{f}} eB \xrightarrow{e\tilde{g}} eC \longrightarrow \bar{0}$ is exact.*

Proof. Suppose that $A \leq_{IF} M$, $B \leq_{IF} N$, $C \leq_{IF} K$. Let $ek \in eK$. Since $\tilde{g}$ is an epimorphism, there exists $n \in N$ satisfying $\tilde{g}(n) = ek$. We have $en \in eN$ and $e\tilde{g}(en) = e\tilde{g}(n) = ek$, since e is an idempotent.

This proves that $e\tilde{g}$ is an epimorphism. Since $Im\tilde{f} \subseteq Ker\tilde{g}$, it is clear that $Im e\tilde{f} \subseteq Ker e\tilde{g}$.
Suppose that $en \in ker(e\tilde{g})$. We have an element $m \in M$ satisfying $\tilde{f}(m) = en$,

because $ker\tilde{eg} \subseteq ker\tilde{g}$. Hence $em \in eM$ and $\tilde{ef}(em) = \tilde{f}(em) = e\tilde{f}(m) = en$, i.e., $ker(e\tilde{g}) \subseteq Im(e\tilde{f})$, as desired.

Proposition 4.14 *Let R be a commutative ring and Let $e \in E(R)$. The functor $Hom(0'^F_{Re}, -)$ preserves the sequence $\bar{0} \longrightarrow A \xrightarrow{\tilde{f}} B \xrightarrow{\tilde{g}} C \longrightarrow \bar{0}$ of intuitionistic fuzzy R-modules.*

Proof. The sequence $\bar{0} \longrightarrow eA \xrightarrow{e\tilde{f}} eB \xrightarrow{e\tilde{g}} eC \longrightarrow \bar{0}$ is a short exact sequence by Proposition 4.13. Consider the following commutative diagram of intuitionistic fuzzy R-modules

$$\begin{array}{ccccccccc} \bar{0} & \longrightarrow & Hom(0'^F_{Re}, A) & \xrightarrow{\tilde{f}} & Hom(0'^F_{Re}, B) & \xrightarrow{\tilde{g}} & Hom(0'^F_{Re}, C) & \longrightarrow & \bar{0} \\ & & \downarrow \eta_A & & \downarrow \eta_B & & \downarrow \eta_C & & \\ \bar{0} & \longrightarrow & eA & \xrightarrow{\tilde{h}} & eB & \xrightarrow{\tilde{p}} & eC & \longrightarrow & \bar{0} \end{array}$$

Where η_A, η_B, η_C are IF isomorphisms by Lemma 4.10. Now by Proposition 4.11, the top row is exact.

5 Product and Coproduct in IFR-Mod

Definition 5.1 Let $\{A_i = (\mu_{A_i}, \nu_{A_i}) \leq_{IF} M_i \mid i \in I\}$ be a family of intuitionistic fuzzy R-modules. Then $\coprod_{i\in I} A_i = (\mu, \nu)$ is the *coproduct* of this family such that the maps $(\mu, \nu) : \coprod_{i\in I} M_i \to [0, 1]$ are defined by

$$\mu((a_i)_{i\in I}) = \wedge\{\mu_i(a_i) \mid i \in I\} \text{ and } \nu((a_i)_{i\in I}) = \vee\{\nu_i(a_i) \mid i \in I\}.$$

Similarly, the *product* of this family denoted by $\prod_{i\in I} A_i = (\mu, \nu)$, where $(\mu, \nu) : \prod M_i \to [0, 1]$ are defined by

$$\mu((a_i)_{i\in I}) = \wedge\{\mu_i(a_i) \mid i \in I\}, \ \nu((a_i)_{i\in I}) = \vee\{\nu_i(a_i) \mid i \in I\}.$$

If $\{A_i \leq_{IF} M_i \mid i \in I\}$ is a family of intuitionistic fuzzy R-modules, then it is not difficult to see that $\coprod_{i\in I} A_i$ and $\prod_{i\in I} A_i$ are intuitionistic fuzzy submodules of $\coprod_{i\in I} M_i$ and $\prod M_i$, respectively.

Proposition 5.2 *Let M be an R-module, $A = (\mu_A, \nu_A) \leq_{IF} M$ and $e_i \in E(R)$ for any $i \in I$. Then*

$$Hom(\textstyle\coprod_{i\in I} 0'^F_{Re_i}, A) \cong Hom(\textstyle\prod_{i\in I} 0'^F_{Re_i}, A).$$

Proof. Let $\kappa : Hom(\coprod_{i\in I} Re_i,\ M) \longrightarrow Hom(\prod_{i\in I} Re_i,\ M)$

given by

$$f \longrightarrow (f\lambda_i = f_i)_{i\in I}$$

be the isomorphism, where λ_j is the injection $Re_j \longrightarrow \coprod_{i\in I} Re_i$, for every $j \in I$.

By the proof of Lemma 4.10, for every $i \in I$, when $Hom(0^{IF}_{Re_i}, A) = (\alpha_i, \beta_i)$, $Hom(\coprod_{i\in I} 0^{IF}_{Re_i}, A) = (\alpha, \beta)$, we have $\alpha_i(f_i) = \mu_A(f_i(e_i))$, $\beta_i(f_i) = \nu_A(f_i(e_i))$.

Note that $\mu_A(\sum_{i\in I} f_i(r_ie_i)) = \bigvee_{i\in I} \mu_A(f_i(r_ie_i))$, $\nu_A(\sum_{i\in I} f_i(r_ie_i)) = \bigwedge_{i\in I} \nu_A(f_i(r_ie_i))$ and $\mu_A(f_i(r_ie_i)) = \mu_A(r_if_i(e_i)) \geq \mu_A(f_i(e_i))$, $\nu_A(f_i(r_ie_i)) = \nu_A(r_if_i(e_i)) \leq \nu_A(f_i(e_i))$, where $r_i \in R$ forevery $i \in I$. For every $f \in Hom(\coprod_{i\in I} 0^{IF}_{Re_i}, A)$ we have

$$\alpha(f) = \bigwedge\{\mu_A(f((r_ie_i)_{i\in I})) \mid (r_ie_i)_{i\in I} \in \coprod_{i\in I} Re_i\} = \bigwedge\{\mu_A(\sum_{i\in I} f_i(r_ie_i)) \mid r_ie_i \in Re_i\}$$

$$= \bigwedge\{\mu_A(f_i(e_i))_{i\in I} \mid i \in I\} = \bigwedge\{\alpha_i(f_i) \mid i \in I\} = (\alpha_i)_{i\in I}((f_i)_{i\in I}) = (\alpha_i)_{i\in I}(\kappa(f))$$

and

$$\beta(f) = \overline{\ }\{\mu_B(f((r_ie_i)_{i\in I})) \mid (r_ie_i)_{i\in I} \in \coprod_{i\in I} Re_i\} = \overline{\ }\{\mu_B(\sum_{i\in I} f_i(r_ie_i)) \mid r_ie_i \in Re_i\}$$

$$= \overline{\ }\{\mu_B(f_i(e_i))_{i\in I} \mid i \in I\} = \overline{\ }\{\beta_i(f_i) \mid i \in I\} = (\beta_i)_{i\in I}((f_i)_{i\in I}) = (\beta_i)_{i\in I}(\kappa(f)).$$

Similarly, (Note that $\{f_i(e_i) \mid e_i \in Re_i\} \subseteq \{\sum_{i\in I} f_i(r_ie_i) \mid r_ie_i \in Re_i\}$). So κ is an IF isomorphism, that is $Hom(\coprod_{i\in I} 0^{IF}_{Re_i},\ A) \cong Hom(\prod_{i\in I} 0^{IF}_{Re_i},\ A)$.

Proposition 5.3 Suppose that $e_iA = (\mu\_{e_iA}; \nu\_{e_iA}) \leq\_{IF} e_iM$, Where $e_i \in E(R)$ for $i \in I$. Then

$$Hom(\coprod_{i\in I} 0_{R_{e_i}}{}^{IF}, A) \cong \prod_{i\in I} e_iA.$$

Proof. It follows from Lemma 4.10 and Proposition 5.2.

Let R be a ring and I an ideal of R. Then we say *idempotents lift modulo I* if for every idempotent $e + I$ in R/I, there exists an idempotent e' of R such that $e + I = e' + I$.
A ring R is called *semiperfect* if $R/J(R)$ is semisimple and idempotents lift modulo $J(R)$, where $J(R)$ is the Jacobson radical of R.

Proposition 5.4 *Let R be a semiperfect ring and P a projective R-module. Then* $P \cong \bigoplus_{i\in I} Re_i$, *for some* $e_i \in E(R)$.

Proof. See [2, Theorem 27.11].

By Proposition 5.4, for any semiperfect ring R and projective intuitionistic fuzzy R-module P, we have $P \cong \coprod_{i\in I} 0^{IF}_{Re_i}$, for some $e_i \in E(R)$ (Since every projective intuitionistic fuzzy R-module is zero intuitionistic fuzzy R-module).

Proposition 5.5 *Let R be a commutative semiperfect ring. Then* $Hom(P, -)$ *preserves the short exact sequence* $0 \longrightarrow A \xrightarrow{\tilde{f}} B \xrightarrow{\tilde{g}} C \longrightarrow \bar{0}$ *of intuitionistic fuzzy R-modules if and only if P is an intuitionistic fuzzy projective R-module.*

Proof. Suppose that Hom(P,-) preserves that exact sequence. Let $B \leq_{IF} M$ and $C \leq_{IF} N$ and $\tilde{g} : B \longrightarrow C$ be an IF epimorphism. Let

$$K = ker\tilde{g} = \left\{ x \in M \;\middle|\; \begin{array}{l} \mu_c(g(x)) = 1; \\ \nu_c(g(x)) = 0 \end{array} \right\}$$

and $B' = (\mu_B, \nu_B)|_K$. Then B' is an intuitionistic fuzzy submodule of B. So we obtain the short exact sequence $\bar{0} \longrightarrow K \xrightarrow{\tilde{i}} B \xrightarrow{\tilde{g}} C \longrightarrow \bar{0}$ where $\tilde{i}$ is the inclusion map. Since $Hom(P, -)$ preserves the sequence, $Hom(P, -)$ preserves the epimorphism $\tilde{g}$. By this we can conclude that P is an intuitionistic fuzzy projective R-module.

Conversely since P is an intuitionistic fuzzy projective R-module, we have $P \cong \coprod_{i\in I} 0^{IF}_{Re_i}$ where $e_i \in E(R)$. Let $0 \longrightarrow A \xrightarrow[\tilde{h}]{} B \xrightarrow[\tilde{g}]{} C \longrightarrow \bar{0}$ be a short exact sequence of intuitionistic fuzzy R-modules. Then the sequence

$$\bar{0} \longrightarrow \prod_{i\in I} e_iA \longrightarrow \prod_{i\in I} e_iB \longrightarrow \prod_{i\in I} e_iC \longrightarrow \bar{0}$$

is also a short exact sequences by Proposition 4.13. Using Proposition 5.3 we have the following commutative diagram

$$\bar{0} \longrightarrow \quad Hom(\coprod_{i\in I} 0_{R_{e_i}}{}'^{F},\ A) \longrightarrow Hom(\coprod_{i\in I} 0_{R_{e_i}}{}'^{F},\ B) \longrightarrow Hom(\coprod_{i\in I} 0_{R_{e_i}}{}'^{F},\ C) \longrightarrow \ \bar{0}$$

$$\bar{0} \ \longrightarrow \ \prod_{i\in I} e_i A \ \longrightarrow \ \prod_{i\in I} e_i B \ \longrightarrow \ \prod_{i\in I} e_i C \ \longrightarrow \ \bar{0}$$

Now since the bottom row is a short exact sequence, so the top row is also a short exact sequence, by Proposition 4.11. This completes the proof.

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