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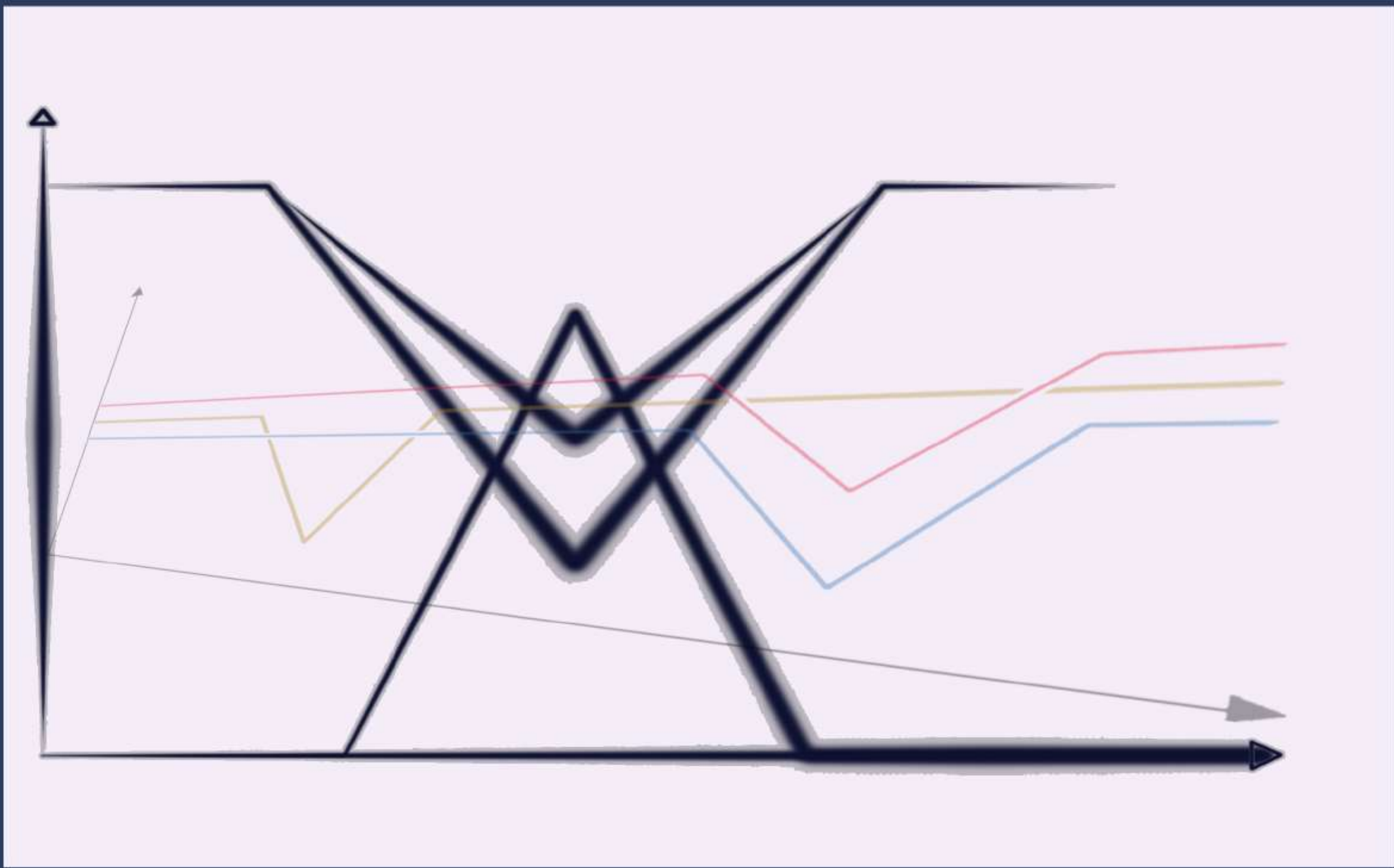

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A Novel Mathematical Approach for Fuzzy Multi-Period Multi-Objective Portfolio Optimization Problem under Uncertain Environment and Practical Constraints

Pejman Peykani[1,*], Mojtaba Nouri[1], Farzad Eshghi[2], Mohammad Khamechian[3], Hamed Farrokhi-Asl[4]

[1] School of Industrial Engineering, Iran University of Science and Technology, Tehran, Iran; pejman.peykani@yahoo.com; mojtabanuri93@gmail.com.

[2] Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran; eshghise@gmail.com

[3] Department of Industrial and Manufacturing Engineering, University of Wisconsin-Milwaukee, Milwaukee, USA; khamech@uwm.edu.

[4] Sheldon B. Lubar School of Business, University of Wisconsin-Milwaukee, Milwaukee, Wisconsin, USA; farrokh5@uwm.edu.

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Abstract

Investment Portfolio Optimization (IPO) is one of the most important problems in the financial area. Also, one of the most important features of financial markets is their embedded uncertainty. Thus, it is essential to propose a novel IPO model that can be employed in the presence of uncertain data. Accordingly, the main goal of this paper is to propose a novel Fuzzy Multi-Period Multi-Objective Portfolio Optimization (FMPMOPO) model that is capable to be used under data ambiguity and practical constraints including budget constraint, cardinality constraint, and bound constraint. It should be noted that three objectives including terminal wealth, risk, and liquidity as well as practical constraints are considered in proposed FMPMOPO model. Also, the alpha-cut method is employed to deal with fuzzy data. Finally, the proposed Fuzzy Multi-Period Wealth-Risk-Liquidity (FMPWRL) model is implemented in real-world case study from Tehran Stock Exchange (TSE). The experimental results show the applicability and efficacy of the proposed FMPWRL model for fuzzy multi-period multi-objective investment portfolio optimization problem under fuzzy environment.

Keywords: Portfolio optimization, Multi-Period problem, Fuzzy optimization, Alpha-Cut method, Data ambiguity, Investment problem, Stock market.

1 | Introduction

Portfolio optimization problem is selection of the best combination of financial assets to achieve the maximum return and minimum risk for portfolio as much as possible. So far, many models, approaches, and algorithms have been proposed by different researchers to achieve the optimal portfolio [1]-[6]. The most important and influential researches in this field have been done by Markowitz [7] and Sharp [8]. Please note that one of the most important points that should be considered in the proposed approach for portfolio optimization problem is the uncertainty of data in financial markets [9]-[13].

Corresponding Author: pejman.peykani@yahoo.com

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Because, in real cases, financial data such as rate of return and liquidity can be tainted by uncertainty. Thus, it is essential to propose a new portfolio optimization model that can be applied under ambiguity and uncertainty. Fuzzy Mathematical Programming (FMP) is one of the popular and effective approaches to deal with data uncertainty and linguistic variables [14]-[20]. Sadjadi et al. [21] proposed Fuzzy Multi-Period Portfolio Selection (FMPPS) model with different rates for borrowing and lending.

Liu et al. [22] introduced FMPPS approach by considering return, transaction cost, risk and skewness of portfolio. Zhang et al. [23] presented possibilistic mean-semi variance-entropy model for FMPPS problem, and designed hybrid intelligent algorithm for solving the presented model. Zhang et al. [24] applied Particle Swarm Optimization (PSO) algorithm for solving fuzzy multi-period portfolio optimization model under possibility measures. Zhang and Zhang [25] proposed fuzzy multi-period mean-absolute deviation (MAD) for portfolio selection problem by considering risk control and cardinality constraints. Zhang et al. [26] presented a new FMP approach for multi-period portfolio optimization with return demand and risk control. Mehlawat [27] proposed credibilistic mean-entropy models for FMPPS with multi-choice aspiration levels by considering wealth, risk, transaction cost, liquidity, and cardinality constraint. Wang et al. [28] introduced a new uncertain multi-period portfolio selection model with dynamic risk/expected-return level that is capable to be used in the presence of fuzzy random uncertainty.

Liu and Zhang [29] proposed fuzzy multi-period portfolio selection optimization model based on possibility theory and applied Genetic Algorithm (GA) to solve the proposed model. Liagkouras and Metaxiotis [30] discussed the multi-period mean–variance portfolio optimization problem with transaction costs and fuzzy variables to count for the uncertainty of future returns and liquidities on assets. Cao [31] employed PSO algorithm for solving multi-objective portfolio optimization problem under fuzzy environment, in which the return rates and the turnover rates are characterized by fuzzy variables. Liu & Zhang [32] examined possibilistic moment models for FMPPS with fuzzy returns by taking into account some realistic constraints including budget constraint, higher moments, cardinality constraint, round-lot constraint, and bound constraint. Gupta et al. [33] proposed intuitionistic fuzzy optimistic and pessimistic multi-period portfolio optimization models. Last but not the least, Gupta et al. [34] used coherent fuzzy numbers to model the returns and the investor attitude in credibilistic multi-period multi-objective portfolio optimization problem.

The goal of this paper is to propose a new Fuzzy Multi-Period Multi-Objective Portfolio Optimization (FMPMOPO) model by considering three objectives including wealth, risk, and liquidity. Notably, to reach this goal, the alpha-cut technique and goal programming approach are applied. Additionally, to show the applicability and efficacy of proposed FMPMOPO model, a real-world case study from Tehran stock market is utilized. The rest of this paper is organized as follows. The research background of the paper will be explained in Section 2. The mathematical modeling of fuzzy multi-period multi-objective portfolio optimization approach will be proposed in Section 3. Then, the implementation of the proposed FMPMOPO model in Tehran Stock Exchange (TSE) will be discussed in Section 4. Finally, conclusions as well as some directions for future research will be introduced in Section 5.

2 | Research Background

In this section, the research background of the paper to propose FMPMOPO model including alpha-cut method and goal programming technique as well as required equations, formulations, and explanations will be introduced.

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2.1 | Alpha-Cut Method

An alpha-cut operation is one of the important solution methods that widely applied in literature to solve Fuzzy Linear Programming (FLP) problem [35]. For more details, let $\tilde{\beta}$ be a triangular fuzzy number that is determined by $\tilde{\beta}=(\beta_1,\beta_2,\beta_3), \beta_1 \le \beta_2 \le \beta_3$. The membership function of $\tilde{\beta}$ is defined as *Eq. (1)*:

$$\mu_{\tilde{\beta}}(x)=\begin{cases} 0, & \text{if } x \le \beta_1; \\ \dfrac{x-\beta_1}{\beta_2-\beta_1}, & \text{if } \beta_1 \le x \le \beta_2; \\ \dfrac{\beta_3-x}{\beta_3-\beta_2}, & \text{if } \beta_2 \le x \le \beta_3; \\ 0, & \text{if } x \ge \beta_3. \end{cases} \tag{1}$$

The alpha-cut of the fuzzy number $\tilde{\beta}$ is defined as $\beta_\alpha=\{x \in R / \mu_{\tilde{\beta}}(x) \ge \alpha\}$ where α is the confidence level [36]. Accordingly, the alpha-cut of $\tilde{\beta}$ is actually a close interval of the real number field as follows:

$$\beta_\alpha=[\beta_L(\alpha),\beta_R(\alpha)]=[(\beta_2-\beta_1)\alpha+\beta_1\ ,\ \beta_3-(\beta_3-\beta_2)\alpha]. \quad \alpha \in [0,1] \tag{2}$$

The graphical presentation of alpha-cut method for triangular fuzzy number is shown in *Fig. 1*:

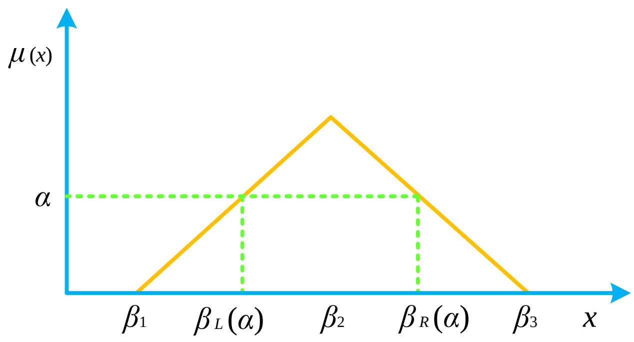

Fig. 1. Alpha-Cut of triangular fuzzy number.

Notably, by applying the alpha-cut method, FLP can be transformed to Interval-Parameter Linear Programming (IPLP). Finally, the resulting IPLP problem can be solved as a crisp linear programming (CLP) model for a given α with some variable substitutions.

2.2 | Goal Programming Technique

So far, several approaches and algorithms have been proposed to solve Multiple-Objectives Decision Making (MODM) problems in which some objectives are conflicting and non-commensurable. Goal Programming (GP) is one of the popular, powerful, and effective solution methods for MODM problems. The major variants of GP in terms of underlying distance metric are lexicographic, weighted, and Chebyshev goal programming. Also, GP in terms of the mathematical nature of the decision variables and/or goals can be categorized into fuzzy, integer, binary, and fractional goal programming [37]-[39]. Please consider following multiple objective linear programming (MOLP) problem that c, a, and b, are the objective function coefficient, the technological coefficient, and the right-hand-side, respectively:

$$\text{Max} \quad \sum_{j=1}^{J} c_{j1} x_j \tag{3}$$

$$\vdots$$

$$\text{Max} \quad \sum_{j=1}^{J} c_{jk} x_j$$

$$\vdots$$

$$\text{Max} \quad \sum_{j=1}^{J} c_{jK} x_j$$

$$\text{S.t.} \quad \sum_{j=1}^{J} a_{ij} x_j \le b_i, \quad \forall i$$

$$x_j \ge 0, \quad \forall j.$$

Assume that a set of K goals $\{\Theta_1, \dots, \Theta_k, \dots, \Theta_K\}$ is specified by the Decision Maker (DM) for objective functions. Goal programming try to achieve an optimal solution "as close as possible" to the set of specified goals which may not be simultaneously attainable. The equivalent weighted GP mathematical formulation to the above MOLP is written as follows:

$$\text{Min} \quad \sum_{k=1}^{K} \left(\lambda_k^- \delta_k^- + \lambda_k^+ \delta_k^+ \right) \tag{4}$$

$$\text{S.t.} \quad \sum_{j=1}^{J} c_{jk} x_j + \delta_k^- - \delta_k^+ = \Theta_k, \quad \forall k$$

$$\sum_{j=1}^{J} a_{ij} x_j \le b_i, \quad \forall i$$

$$x_j, \delta_k^-, \delta_k^+ \ge 0. \quad \forall j, k$$

It should be explained that non-negative variables δ_k^- and δ_k^+ are deviational variables of goal k. Also, λ_k^- and λ_k^+ are weights assigned to the deviational variables of goal k that determined by the DM. Notably, the weighted GP mathematical formulation can be extended to handle the objectives (goals) at different priority levels and classes.

3 | The Proposed FMPMOPO Model

In this section, the fuzzy multi-period multi-objective portfolio optimization model will be introduced. It should be noted that three objectives including wealth, risk, and liquidity as well as practical constraints are considered in FMPMOPO model. The indices, parameters, and decision variables that will be used in this study are described as follows:

i : the indices of risky assets $i = 1,...,I$.

t : the indices of investment periods $t = 1,...,T$.

$\tilde{\gamma}_{it}$: the return of i th risky asset in t th investment period (triangular fuzzy number).

$\tilde{\eta}_{it}$: The liquidity of i th risky asset in t th investment period (triangular fuzzy number).

l_{it} : The lower bound of budget allocation for i th risky asset in t th investment period.

u_{it} : The upper bound of budget allocation for i th risky asset in t th investment period.

φ_{it} : The transaction cost rate of i th risky asset in t th investment period.

Φ_t : The expected return of portfolio x_t in t th investment period.

Υ_t : The total transaction cost of portfolio x_t in t th investment period.

Γ_t : The return of portfolio x_t in t th investment period

Ω_t : The maximum number of risky assets of portfolio x_t in t th investment period.

Ψ_t : The expected value of wealth at the beginning of investment period t .

Δ_t : the absolute deviation of portfolio in t th investment period.

ω_{it} : The weight of i th risky asset in portfolio in t th investment period.

ξ_{it} : A binary variable which will be one if i th risky asset is selected in t th investment period and zero otherwise.

Assume that the investor acquires his initial wealth at the beginning of the first period and the terminal wealth at the end of period T. The investor can reinvest the wealth among these risky assets at the beginning of each $T-1$ sub-periods. The investor does not invest any additional wealth in the entire investment horizon. Also, the return and the liquidity of risky assets have a triangular fuzzy distribution $\tilde{\gamma}(\gamma^1, \gamma^2, \gamma^3)$ and $\tilde{\eta}(\eta^1, \eta^2, \eta^3)$ in which $\gamma^1 \le \gamma^2 \le \gamma^3$ and $\eta^1 \le \eta^2 \le \eta^3$.

In the following, the objective functions and constraints of FMPMOPO model will be described. Notably, the terminal wealth is the investor's wealth in the last period of his investment. To find the terminal wealth, it is necessary to obtain the general relation of the wealth gained in each period. The wealth earned in each period consists of two components: the expected return of portfolio, and the transaction cost. The expected return of portfolio is calculated as follows:

$$\Phi_t = \sum_{i=1}^{I} \omega_{it} \tilde{\gamma}_{it} . \quad \forall t \tag{5}$$

In order to calculate the transaction cost, the V-shape function is used, which is the difference between two consecutive portfolios:

$$\Upsilon_t = \sum_{i=1}^{I} \varphi_{it} \left| \omega_{it} - \omega_{it-1} \right| . \quad \forall t \tag{6}$$

As a result, the return of portfolio in t th investment period is defined as follows:

$$\Gamma_t = \Phi_t - \Upsilon_t. \quad \forall t \tag{7}$$

Then, the wealth of investor in t^{th} investment period is calculated as follows:

$$\Psi_t = \Psi_{t-1}\left(1+\Gamma_t\right). \quad \forall t \tag{8}$$

By replacing *Eqs. (5) - (7)* in *Eq. (8)*, the wealth of investor in each investment period is described as follows:

$$\Psi_t = \Psi_{t-1}\left(1+\sum_{i=1}^{I}\omega_{it}\tilde{\gamma}_{it} - \sum_{i=1}^{I}\varphi_{it}\left|\omega_{it}-\omega_{it-1}\right|\right). \quad \forall t \tag{9}$$

Accordingly, the fuzzy multi-objective wealth-risk-liquidity model for multi-period portfolio optimization problem under fuzzy environment is proposed as *Model (10)*:

$$\text{Max} \quad \Psi_T, \tag{10}$$

$$\text{Min} \quad \frac{1}{T}\sum_{t=1}^{T}\Delta_t,$$

$$\text{Max} \quad \frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{I}\omega_{it}\tilde{\eta}_{it},$$

$$\text{S.t.} \quad \Delta_t \geq \left(\frac{1}{T}\sum_{t=1}^{T}\Psi_t\right) - \Psi_t, \quad \forall t$$

$$\Delta_t \geq \Psi_t - \left(\frac{1}{T}\sum_{t=1}^{T}\Psi_t\right), \quad \forall t$$

$$\Psi_t = \Psi_{t-1}\left(1+\sum_{i=1}^{I}\omega_{it}\tilde{\gamma}_{it} - \sum_{i=1}^{I}\varphi_{it}\left|\omega_{it}-\omega_{it-1}\right|\right), \quad \forall t$$

$$\sum_{i=1}^{I}\omega_{it} = 1, \quad \forall t$$

$$\sum_{i=1}^{I}\xi_{it} = \Omega_t, \quad \forall t$$

$$\omega_{it} \leq \xi_{it}u_{it}, \quad \forall i,t$$

$$\omega_{it} \geq \xi_{it}l_{it}, \quad \forall i,t$$

$$\xi_{it} \in \{0,1\}, \quad \forall i,t$$

$$\omega_{it} \geq 0. \quad \forall i,t$$

It should be explained that some practical constraints including budget constraint, cardinality constraint, and bound constraint are taking into account in proposed FMPMOPO model. Finally, the FMPMOPO model is transformed to single objective model by applying goal programming technique. Also, alpha-cut method is employed to deal with fuzzy data of return and liquidity of risky assets.

$$\text{Min}\ \left(\lambda_1^- \delta_1^- + \lambda_2^- \delta_2^- + \lambda_3^- \delta_3^-\right) \tag{11}$$

$$\text{s.t.}\ \ \Psi_T + \delta_1^- = \Theta_1$$

$$\frac{1}{T}\sum_{t=1}^{T} \Delta_t + \delta_2^- = \Theta_2,$$

$$\frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{I} \omega_{it}\left[(\eta_{it}^2 - \eta_{it}^1)\alpha + \eta_{it}^1,\ \eta_{it}^3 - (\eta_{it}^3 - \eta_{it}^2)\alpha \right] + \delta_3^- = \Theta_3,$$

$$\Delta_t \ge \left(\frac{1}{T}\sum_{t=1}^{T} \Psi_t \right) - \Psi_t,\ \ \forall t$$

$$\Delta_t \ge \Psi_t - \left(\frac{1}{T}\sum_{t=1}^{T} \Psi_t \right),\ \ \forall t$$

$$\Psi_t = \Psi_{t-1}\left(1 + \sum_{i=1}^{I} \omega_{it}\left[(\gamma_{it}^2 - \gamma_{it}^1)\alpha + \gamma_{it}^1,\ \gamma_{it}^3 - (\gamma_{it}^3 - \gamma_{it}^2)\alpha \right] - \sum_{i=1}^{I} \varphi_{it}\left(P_{it} + Q_{it}\right) \right),\ \ \forall t$$

$$\sum_{i=1}^{I} \omega_{it} = 1,\ \ \forall t$$

$$\sum_{i=1}^{I} \xi_{it} = \Omega_t,\ \ \forall t$$

$$\omega_{it} \le \xi_{it} u_{it},\ \ \forall i, t$$

$$\omega_{it} \ge \xi_{it} l_{it},\ \ \forall i, t$$

$$\omega_{it} - \omega_{it-1} = P_{it} - Q_{it},\ \ \forall i, t$$

$$\xi_{it} \in \{0, 1\},\ \ \forall i, t$$

$$\omega_{it}, \delta_k^-, \delta_k^+, P_{it}, Q_{it} \ge 0.\ \ \forall i, t, k$$

To solve the proposed FMPMOPO model, *Model (11)* is run once for the lower bound of the alpha-cut interval and again for the upper bound of the alpha-cut interval.

4 | Case Study and Experimental Results

In this section, the proposed fuzzy multi-period multi-objective portfolio optimization model will be implemented for a real-world case study from the Tehran Stock Exchange (TSE). TSE, with a history of nearly half a century, is one of the most attractive financial markets in the Middle East region. Accordingly, the data set of 30 stocks for 5 periods are extracted from TSE. *Tables (1)* and *(2)* show the data set for return and liquidity of 30 stocks for 5 periods under triangular fuzzy distribution:

Table 1. Fuzzy data set for return.

Stocks	First Period	Second Period	Third Period	Fourth Period	Fifth Period
Stock 01	(-0.042, 0.000, 0.049)	(-0.036, -0.004, 0.050)	(-0.049, 0.002, 0.040)	(-0.036, 0.001, 0.040)	(-0.049, -0.004, 0.050)
Stock 02	(-0.022, 0.001, 0.001)	(-0.002, 0.000, 0.002)	(-0.049, -0.002, 0.050)	(-0.002, 0.004, 0.012)	(-0.026, 0.045, 0.048)
Stock 03	(-0.046, 0.001, 0.040)	(-0.049, -0.004, 0.050)	(-0.034, -0.030, -0.004)	(-0.049, 0.001, 0.054)	(-0.048, 0.032, 0.049)
Stock 04	(-0.025, -0.006, 0.047)	(-0.026, -0.003, 0.015)	(-0.010, -0.003, 0.007)	(-0.026, 0.001, 0.048)	(-0.046, 0.040, 0.047)
Stock 05	(-0.005, 0.004, 0.008)	(-0.003, 0.000, 0.003)	(-0.026, -0.003, 0.048)	(-0.003, -0.001, 0.040)	(-0.211, 0.000, 0.049)
Stock 06	(-0.183, 0.004, 0.012)	(-0.026, 0.045, 0.048)	(-0.002, 0.002, 0.045)	(-0.026, -0.002, 0.049)	(-0.046, 0.043, 0.060)
Stock 07	(-0.047, 0.008, 0.049)	(-0.041, -0.002, 0.028)	(-0.046, 0.014, 0.032)	(-0.041, 0.011, 0.048)	(-0.030, 0.013, 0.049)
Stock 08	(-0.021, 0.001, 0.031)	(-0.013, -0.006, 0.001)	(-0.048, 0.002, 0.049)	(-0.013, -0.006, 0.064)	(-0.050, 0.000, 0.066)
Stock 09	(-0.045, 0.001, 0.054)	(-0.048, 0.032, 0.049)	(-0.009, 0.032, 0.045)	(-0.048, -0.003, 0.050)	(-0.025, 0.002, 0.101)
Stock 10	(-0.046, 0.000, 0.038)	(-0.017, 0.005, 0.015)	(-0.018, -0.003, 0.021)	(-0.017, -0.002, 0.028)	(-0.043, 0.006, 0.019)
Stock 11	(-0.016, 0.001, 0.048)	(-0.007, -0.002, 0.000)	(-0.046, 0.004, 0.047)	(-0.007, -0.004, 0.020)	(-0.033, 0.000, 0.034)
Stock 12	(-0.022, 0.001, 0.048)	(-0.046, 0.040, 0.047)	(-0.013, 0.037, 0.040)	(-0.046, 0.000, 0.023)	(-0.050, 0.000, 0.049)
Stock 13	(-0.211, -0.003, 0.032)	(-0.022, -0.005, 0.031)	(-0.029, 0.006, 0.029)	(-0.022, 0.000, 0.050)	(-0.040, 0.012, 0.047)
Stock 14	(0.003, -0.001, 0.040)	(-0.001, -0.003, 0.004)	(-0.211, -0.001, 0.049)	(-0.001, 0.000, 0.044)	(-0.048, 0.039, 0.069)
Stock 15	(-0.019, -0.001, 0.040)	(-0.211, 0.000, 0.049)	(-0.003, -0.002, 0.000)	(-0.211, -0.017, 0.044)	(-0.049, 0.018, 0.058)
Stock 16	(-0.046, 0.003, 0.041)	(-0.028, 0.000, 0.021)	(-0.028, -0.010, 0.041)	(-0.028, -0.046, 0.048)	(-0.047, 0.004, 0.065)
Stock 17	(-0.019, -0.002, 0.018)	(-0.014, -0.008, 0.003)	(-0.046, 0.012, 0.060)	(-0.014, -0.003, 0.046)	(-0.049, 0.008, 0.049)
Stock 18	(-0.022, -0.002, 0.049)	(-0.046, 0.043, 0.060)	(0.024, 0.043, 0.049)	(-0.046, 0.039, 0.048)	(-0.049, -0.012, 0.095)
Stock 19	(-0.022, -0.003, 0.035)	(-0.026, 0.011, 0.028)	(-0.027, 0.002, 0.034)	(-0.026, -0.032, 0.042)	(-0.138, 0.007, 0.049)
Stock 20	(-0.015, 0.011, 0.022)	(-0.017, -0.005, -0.001)	(-0.030, 0.005, 0.049)	(-0.017, -0.004, 0.050)	(-0.075, 0.001, 0.047)
Stock 21	(-0.033, 0.011, 0.048)	(-0.030, 0.013, 0.049)	(-0.018, 0.009, 0.013)	(-0.030, 0.045, 0.048)	(-0.183, 0.004, 0.014)
Stock 22	(-0.024, 0.001, 0.035)	(-0.040, -0.001, 0.048)	(-0.050, -0.014, 0.050)	(-0.040, 0.032, 0.049)	(-0.048, 0.001, 0.054)
Stock 23	(-0.020, -0.006, 0.022)	(-0.001, -0.001, 0.006)	(-0.050, -0.002, 0.066)	(-0.001, 0.040, 0.047)	(-0.126, 0.001, 0.056)
Stock 24	(-0.020, -0.006, 0.064)	(-0.050, 0.000, 0.066)	(-0.003, -0.002, 0.000)	(-0.050, 0.000, 0.049)	(-0.173, -0.001, 0.067)
Stock 25	(-0.022, -0.005, 0.050)	(-0.025, -0.008, 0.101)	(-0.008, 0.014, 0.050)	(-0.025, 0.043, 0.060)	(-0.082, -0.002, 0.050)
Stock 26	(-0.004, -0.003, 0.018)	(-0.015, 0.036, 0.049)	(-0.025, -0.001, 0.101)	(-0.015, 0.013, 0.049)	(-0.050, 0.011, 0.106)
Stock 27	(-0.042, -0.003, 0.050)	(-0.025, 0.002, 0.101)	(-0.001, 0.002, 0.002)	(-0.025, 0.000, 0.066)	(-0.053, -0.006, 0.064)
Stock 28	(-0.043, 0.002, 0.012)	(-0.010, -0.009, 0.017)	(-0.015, -0.008, 0.019)	(-0.010, 0.002, 0.101)	(-0.130, -0.003, 0.050)
Stock 29	(-0.005, -0.002, 0.023)	(0.011, -0.010, 0.000)	(-0.043, 0.000, 0.019)	(0.011, 0.006, 0.019)	(-0.034, -0.002, 0.028)
Stock 30	(-0.034, -0.002, 0.028)	(-0.043, 0.006, 0.019)	(-0.020, 0.000, 0.006)	(-0.043, 0.000, 0.034)	(-0.014, -0.004, 0.020)

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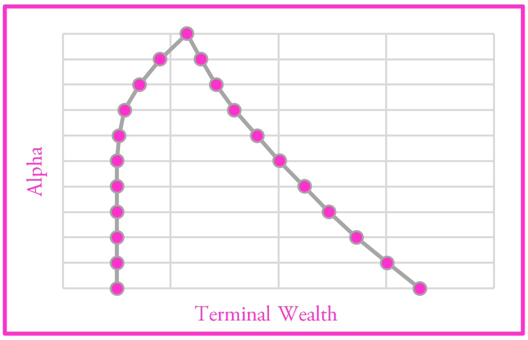
Table 2. Fuzzy data set for liquidity.

Stocks	First Period	Second Period	Third Period	Fourth Period	Fifth Period
Stock 01	(0.068, 0.024, 0.068)	(0.014, 0.021, 0.078)	(0.008, 0.038, 0.049)	(0.008, 0.038, 0.049)	(0.014, 0.021, 0.078)
Stock 02	(0.007, 0.024, 0.070)	(0.000, 0.023, 0.078)	(0.003, 0.022, 0.064)	(0.008, 0.021, 0.029)	(0.013, 0.015, 0.023)
Stock 03	(0.000, 0.025, 0.089)	(0.022, 0.023, 0.052)	(0.000, 0.015, 0.078)	(0.006, 0.024, 0.044)	(0.014, 0.018, 0.050)
Stock 04	(0.039, 0.032, 0.039)	(0.013, 0.015, 0.023)	(0.008, 0.021, 0.029)	(0.009, 0.034, 0.053)	(0.007, 0.025, 0.041)
Stock 05	(0.012, 0.032, 0.039)	(0.008, 0.018, 0.047)	(0.011, 0.030, 0.062)	(0.010, 0.016, 0.063)	(0.000, 0.023, 0.042)
Stock 06	(0.000, 0.022, 0.056)	(0.021, 0.025, 0.030)	(0.008, 0.043, 0.062)	(0.009, 0.025, 0.036)	(0.008, 0.012, 0.023)
Stock 07	(0.056, 0.030, 0.056)	(0.014, 0.018, 0.050)	(0.006, 0.024, 0.044)	(0.000, 0.024, 0.033)	(0.012, 0.014, 0.035)
Stock 08	(0.015, 0.030, 0.056)	(0.000, 0.014, 0.047)	(0.013, 0.034, 0.060)	(0.004, 0.012, 0.049)	(0.006, 0.030, 0.043)
Stock 09	(0.017, 0.033, 0.066)	(0.025, 0.034, 0.035)	(0.000, 0.028, 0.060)	(0.012, 0.033, 0.084)	(0.019, 0.040, 0.066)
Stock 10	(0.038, 0.013, 0.038)	(0.007, 0.025, 0.041)	(0.009, 0.034, 0.053)	(0.000,0.0000, 0.019)	(0.000, 0.000, 0.013)
Stock 11	(0.007, 0.013, 0.040)	(0.000, 0.029, 0.053)	(0.013, 0.025, 0.069)	(0.000, 0.000, 0.053)	(0.000, 0.005, 0.016)
Stock 12	(0.008, 0.007, 0.054)	(0.019, 0.025, 0.040)	(0.000, 0.039, 0.069)	(0.000, 0.000, 0.008)	(0.000, 0.000, 0.020)
Stock 13	(0.053, 0.021, 0.053)	(0.000, 0.023, 0.042)	(0.010, 0.016, 0.063)	(0.007, 0.020, 0.032)	(0.010, 0.016, 0.040)
Stock 14	(0.015, 0.021, 0.053)	(0.000, 0.023, 0.054)	(0.016, 0.051, 0.070)	(0.020, 0.053, 0.053)	(0.016, 0.026, 0.033)
Stock 15	(0.005, 0.019, 0.062)	(0.017, 0.030, 0.051)	(0.000, 0.023, 0.070)	(0.031, 0.064, 0.063)	(0.026, 0.048, 0.075)
Stock 16	(0.048, 0.028, 0.048)	(0.008, 0.012, 0.023)	(0.009, 0.025, 0.036)	(0.013, 0.030, 0.084)	(0.033, 0.059, 0.099)
Stock 17	(0.009, 0.028, 0.048)	(0.000, 0.022, 0.046)	(0.008, 0.016, 0.072)	(0.000, 0.034, 0.040)	(0.012, 0.027, 0.074)
Stock 18	(0.013, 0.010, 0.054)	(0.016, 0.017, 0.020)	(0.000, 0.037, 0.072)	(0.000, 0.020, 0.054)	(0.034, 0.082, 0.092)
Stock 19	(0.066, 0.026, 0.066)	(0.012, 0.014, 0.035)	(0.000, 0.024, 0.033)	(0.025, 0.037, 0.075)	(0.000, 0.034, 0.052)
Stock 20	(0.012, 0.026, 0.066)	(0.000, 0.014, 0.047)	(0.000, 0.024, 0.062)	(0.007, 0.024, 0.068)	(0.000, 0.000, 0.019)
Stock 21	(0.011, 0.021, 0.069)	(0.017, 0.024, 0.036)	(0.000, 0.032, 0.066)	(0.012, 0.032, 0.039)	(0.018, 0.038, 0.076)
Stock 22	(0.078, 0.041, 0.078)	(0.006, 0.030, 0.043)	(0.004, 0.012, 0.049)	(0.020, 0.030, 0.056)	(0.015, 0.042, 0.063)
Stock 23	(0.008, 0.041, 0.078)	(0.000, 0.030, 0.059)	(0.000, 0.011, 0.052)	(0.009, 0.013, 0.038)	(0.000, 0.029, 0.067)
Stock 24	(0.000, 0.041, 0.072)	(0.009, 0.011, 0.030)	(0.000, 0.024, 0.089)	(0.000, 0.021, 0.053)	(0.009, 0.037, 0.085)
Stock 25	(0.084, 0.055, 0.084)	(0.019, 0.040, 0.066)	(0.012, 0.033, 0.084)	(0.009, 0.028, 0.048)	(0.016, 0.031, 0.073)
Stock 26	(0.022, 0.055, 0.084)	(0.000, 0.025, 0.060)	(0.001, 0.032, 0.066)	(0.011, 0.026, 0.066)	(0.028, 0.060, 0.085)
Stock 27	(0.000, 0.024, 0.063)	(0.032, 0.035, 0.063)	(0.000, 0.021, 0.084)	(0.008, 0.041, 0.078)	(0.000, 0.046, 0.080)
Stock 28	(0.019, 0.000, 0.019)	(0.000, 0.000, 0.013)	(0.000, 0.000, 0.019)	(0.000, 0.055, 0.084)	(0.068, 0.024, 0.068)
Stock 29	(0.000, 0.000, 0.049)	(0.000, 0.005, 0.017)	(0.000, 0.000, 0.036)	(0.000, 0.000, 0.019)	(0.007, 0.024, 0.070)
Stock 30	(0.000, 0.000, 0.025)	(0.000, 0.001, 0.049)	(0.000, 0.011, 0.049)	(0.000, 0.000, 0.039)	(0.000, 0.025, 0.089)

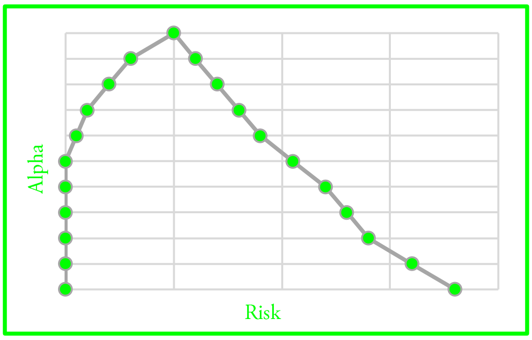
Finally, after collecting required data, the results of proposed FMPMOPO model are presented in *Table (3)* and *Fig. (2)* as follows:

Table 3. The results of FMPMOPO model under different Alpha-Cuts.

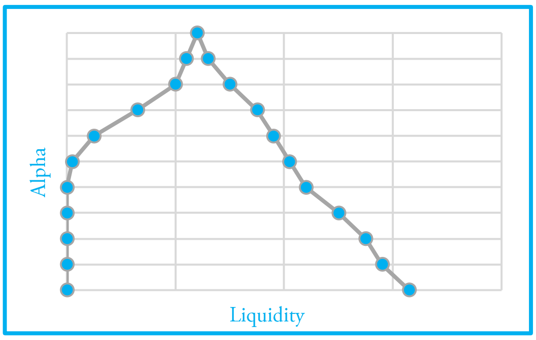
Alpha (α)	Terminal Wealth	Risk	Liquidity
0.00	(1.000, 1.281)	(0.000, 0.036)	(0.000, 0.063)
0.10	(1.000, 1.251)	(0.000, 0.032)	(0.000, 0.058)
0.20	(1.000, 1.222)	(0.000, 0.028)	(0.000, 0.055)
0.30	(1.000, 1.197)	(0.000, 0.026)	(0.000, 0.050)
0.40	(1.000, 1.174)	(0.000, 0.024)	(0.000, 0.044)
0.50	(1.000, 1.151)	(0.000, 0.021)	(0.001, 0.041)
0.60	(1.002, 1.130)	(0.001, 0.018)	(0.005, 0.038)
0.70	(1.007, 1.109)	(0.002, 0.016)	(0.013, 0.035)
0.80	(1.021, 1.092)	(0.004, 0.014)	(0.020, 0.030)
0.90	(1.040, 1.078)	(0.006, 0.012)	(0.022, 0.026)
1.00	(1.065, 1.065)	(0.010, 0.010)	(0.024, 0.024)



(a) First objective



(b) Second Objective



(C) Third objective

Fig. 2. The presentation of objective functions under different alpha-cuts.

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It should be explained that the parameters of the proposed FMPMOPO model including u_{it}, l_{it}, φ_{it}, Ω_t, and Ψ_t, are set equal to 0.1, 0, 0.1%., 10, and 1, respectively. Also, ideal goal of three objectives including terminal wealth, risk, and liquidity, are set equal to 2, 0, and 0.5, respectively. Notably, the results indicate on applicability and efficacy of the FMPMOPO model for multi-period portfolio optimization problem under ambiguity.

5 | Conclusions and Future Research Directions

In this paper, a new fuzzy multi-period multi-objective portfolio optimization model in the context of fuzzy uncertainty is proposed. Notably, three objectives including wealth, risk, and liquidity as well as practical investment constraints are considered to propose FMPMOPO model. Also, the proposed fuzzy multi-period wealth-risk-liquidity model is implemented in real-world case study from Tehran stock market. The experimental results show the applicability of the proposed FMPMOPO model. For future studies, Robust Convex Programming (RCP) and Scenario-Based Robust Optimization (SBRO) approach can be employed in order to deal with uncertainty of financial data [40]-[50]. Moreover, Data Envelopment Analysis (DEA) approach can be applied for stock selection [51]-[62].

Conflicts of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper

References

[1] Roman, D., & Mitra, G. (2009). Portfolio selection models: a review and new directions. *Wilmott journal: the international journal of innovative quantitative finance research*, *1*(2), 69-85.

[2] Kolm, P. N., Tütüncü, R., & Fabozzi, F. J. (2014). 60 years of portfolio optimization: practical challenges and current trends. *European journal of operational research*, *234*(2), 356-371.

[3] Mansini, R., Ogryczak, W., & Speranza, M. G. (2014). Twenty years of linear programming based portfolio optimization. *European journal of operational research*, *234*(2), 518-535.

[4] Mokhtar, M., Shuib, A., & Mohamad, D. (2014). Mathematical programming models for portfolio optimization problem: a review. *International journal of social, management, economics and business engineering*, *8*(2), 443-450.

[5] Zhang, Y., Li, X., & Guo, S. (2018). Portfolio selection problems with Markowitz's mean–variance framework: a review of literature. *Fuzzy optimization and decision making*, *17*(2), 125-158.

[6] Milhomem, D. A., & Dantas, M. J. P. (2020). Analysis of new approaches used in portfolio optimization: a systematic literature review. *Production*, *30*, e20190144. https://doi.org/10.1590/0103-6513.20190144

[7] Markowitz, H. (1952). Portfolio selection. *The journal of finance*, *7*(1), 77-91.

[8] Sharpe, W. F. (1963). A simplified model for portfolio analysis. *Management science*, *9*(2), 277-293.

[9] Kim, J. H., Kim, W. C., & Fabozzi, F. J. (2018). Recent advancements in robust optimization for investment management. *Annals of operations research*, *266*(1-2), 183-198.

[10] Peykani, P., Mohammadi, E., Jabbarzadeh, A., Rostamy-Malkhalifeh, M., & Pishvaee, M. S. (2020). A novel two-phase robust portfolio selection and optimization approach under uncertainty: a case study of Tehran stock exchange. *Plos One*, *15*(10), e0239810. https://doi.org/10.1371/journal.pone.0239810

[11] Peykani, P., Namakshenas, M., Arabjazi, N., Shirazi, F., & Kavand, N. (2021). Optimistic and pessimistic fuzzy data envelopment analysis: empirical evidence from Tehran stock market. *Fuzzy optimization and modeling journal*, *3*(2), 12-21.

[12] Yin, C., Perchet, R., & Soupé, F. (2021). A practical guide to robust portfolio optimization. *Quantitative finance*, *21*(6), 911-928.

[13] Zhu, B., & Zhang, T. (2021). Long-Term wealth growth portfolio allocation under parameter uncertainty: a non-conservative robust approach. *The North American journal of economics and finance*, *57*, 101432. https://doi.org/10.1016/j.najef.2021.101432

[14] Zadeh, L. A. (1978). Fuzzy sets as a basis for a theory of possibility. *Fuzzy sets and systems*, *1*(1), 3-28.

[15] Peykani, P., Mohammadi, E., Pishvaee, M. S., Rostamy-Malkhalifeh, M., & Jabbarzadeh, A. (2018). A novel fuzzy data envelopment analysis based on robust possibilistic programming: possibility, necessity and credibility-based approaches. *RAIRO-Operations research*, *52*(4-5), 1445-1463.

[16] Peykani, P., Mohammadi, E., Emrouznejad, A., Pishvaee, M. S., & Rostamy-Malkhalifeh, M. (2019). Fuzzy data envelopment analysis: an adjustable approach. *Expert systems with applications*, *136*, 439-452.

[17] Goli, A., Zare, H. K., Tavakkoli-Moghaddam, R., & Sadegheih, A. (2020). Multiobjective fuzzy mathematical model for a financially constrained closed-loop supply chain with labor employment. *Computational intelligence*, *36*(1), 4-34.

[18] Tirkolaee, E. B., Goli, A., & Weber, G. W. (2020). Fuzzy mathematical programming and self-adaptive artificial fish swarm algorithm for just-in-time energy-aware flow shop scheduling problem with outsourcing option. *IEEE transactions on fuzzy systems*, *28*(11), 2772-2783.

[19] Peykani, P., Hosseinzadeh Lotfi, F., Sadjadi, S. J., Ebrahimnejad, A., & Mohammadi, E. (2021). Fuzzy chance-constrained data envelopment analysis: a structured literature review, current trends, and future directions. *Fuzzy optimization and decision making*, 1-65.

[20] Peykani, P., Mohammadi, E., & Emrouznejad, A. (2021). An adjustable fuzzy chance-constrained network dea approach with application to ranking investment firms. *Expert systems with applications*, *166*, 113938. https://doi.org/10.1016/j.eswa.2020.113938

[21] Sadjadi, S. J., Seyedhosseini, S. M., & Hassanlou, K. (2011). Fuzzy multi period portfolio selection with different rates for borrowing and lending. *Applied soft computing*, *11*(4), 3821-3826.

[22] Liu, Y. J., Zhang, W. G., & Xu, W. J. (2012). Fuzzy multi-period portfolio selection optimization models using multiple criteria. *Automatica*, *48*(12), 3042-3053.

[23] Zhang, W. G., Liu, Y. J., & Xu, W. J. (2012). A possibilistic mean-semi variance-entropy model for multi-period portfolio selection with transaction costs. *European journal of operational research*, *222*(2), 341-349.

[24] Zhang, X., Zhang, W., & Xiao, W. (2013). Multi-Period portfolio optimization under possibility measures. *Economic modelling*, *35*, 401-408.

[25] Zhang, P., & Zhang, W. G. (2014). Multiperiod mean absolute deviation fuzzy portfolio selection model with risk control and cardinality constraints. *Fuzzy sets and systems*, *255*, 74-91.

[26] Zhang, W. G., Liu, Y. J., & Xu, W. J. (2014). A new fuzzy programming approach for multi-period portfolio optimization with return demand and risk control. *Fuzzy sets and systems*, *246*, 107-126.

[27] Mehlawat, M. K. (2016). Credibilistic mean-entropy models for multi-period portfolio selection with multi-choice aspiration levels. *Information sciences*, *345*, 9-26.

[28] Wang, B., Li, Y., & Watada, J. (2017). Multi-Period portfolio selection with dynamic risk/expected-return level under fuzzy random uncertainty. *Information sciences*, *385*, 1-18.

[29] Liu, Y. J., & Zhang, W. G. (2018). Multiperiod fuzzy portfolio selection optimization model based on possibility theory. *International journal of information technology & decision making*, *17*(03), 941-968.

[30] Liagkouras, K., & Metaxiotis, K. (2018). Multi-Period mean–variance fuzzy portfolio optimization model with transaction costs. *Engineering applications of artificial intelligence*, *67*, 260-269.

[31] Cao, J. L. (2019). Algorithm research based on multi period fuzzy portfolio optimization model. *Cluster computing*, *22*(2), 3445-3452.

[32] Liu, Y. J., & Zhang, W. G. (2019). Possibilistic moment models for multi-period portfolio selection with fuzzy returns. *Computational economics*, *53*(4), 1657-1686.

[33] Gupta, P., Mehlawat, M. K., Yadav, S., & Kumar, A. (2020). Intuitionistic fuzzy optimistic and pessimistic multi-period portfolio optimization models. *Soft computing*, *24*, 11931–11956.

[34] Gupta, P., Mehlawat, M. K., & Khan, A. Z. (2020). Multi-Period portfolio optimization using coherent fuzzy numbers in a credibilistic environment. *Expert systems with applications*, *167*, 114135. https://doi.org/10.1016/j.eswa.2020.114135

[35] Zadeh, L. A. (1996). Fuzzy sets. *Fuzzy sets, fuzzy logic, and fuzzy systems*, 394-432.

[36] Xu, J., & Zhou, X. (2011). *Fuzzy-Like multiple objective decision making*. Berlin: Springer.

[37] Charnes, A., & Cooper, W. W. (1977). Goal programming and multiple objective optimizations. *European journal of operational research*, *1*(1), 39-54.

Peykani et al.|J. Fuzzy. Ext. Appl. 2(3) (2021) 191-203

[38] Zanakis, S. H., & Gupta, S. K. (1985). A categorized bibliographic survey of goal programming. *Omega*, *13*(3), 211-222.

[39] Tamiz, M., Jones, D., & Romero, C. (1998). Goal programming for decision making: an overview of the current state-of-the-art. *European journal of operational research*, *111*(3), 569-581.

[40] Mulvey, J. M., Vanderbei, R. J., & Zenios, S. A. (1995). Robust optimization of large-scale systems. *Operations research*, *43*(2), 264-281.

[41] Yu, C. S., & Li, H. L. (2000). A robust optimization model for stochastic logistic problems. *International journal of production economics*, *64*(1-3), 385-397.

[42] Bertsimas, D., & Sim, M. (2004). The price of robustness. *Operations research*, *52*(1), 35-53.

[43] Ben-Tal, A., El Ghaoui, L., & Nemirovski, A. (2009). *Robust optimization*. Princeton University Press.

[44] Bertsimas, D., Brown, D. B., & Caramanis, C. (2011). Theory and applications of robust optimization. *SIAM review*, *53*(3), 464-501.

[45] Li, Z., Ding, R., & Floudas, C. A. (2011). A comparative theoretical and computational study on robust counterpart optimization: I. robust linear optimization and robust mixed integer linear optimization. *Industrial & engineering chemistry research*, *50*(18), 10567-10603.

[46] Li, Z., Tang, Q., & Floudas, C. A. (2012). A comparative theoretical and computational study on robust counterpart optimization: II. Probabilistic guarantees on constraint satisfaction. *Industrial & engineering chemistry research*, *51*(19), 6769-6788.

[47] Li, Z., & Floudas, C. A. (2014). A comparative theoretical and computational study on robust counterpart optimization: III. Improving the quality of robust solutions. *Industrial & engineering chemistry research*, *53*(33), 13112-13124.

[48] Gabrel, V., Murat, C., & Thiele, A. (2014). Recent advances in robust optimization: an overview. *European journal of operational research*, *235*(3), 471-483.

[49] Gorissen, B. L., Yanıkoğlu, İ., & Den Hertog, D. (2015). A practical guide to robust optimization. *Omega*, *53*, 124-137.

[50] Peykani, P., Mohammadi, E., Farzipoor Saen, R., Sadjadi, S. J., & Rostamy-Malkhalifeh, M. (2020). Data envelopment analysis and robust optimization: a review. *Expert systems*, *37*(4), e12534. https://doi.org/10.1111/exsy.12534

[51] Charnes, A., Cooper, W. W., & Rhodes, E. (1978). Measuring the efficiency of decision making units. *European journal of operational research*, *2*(6), 429-444.

[52] Banker, R. D., Charnes, A., & Cooper, W. W. (1984). Some models for estimating technical and scale inefficiencies in data envelopment analysis. *Management science*, *30*(9), 1078-1092.

[53] Liu, J. S., Lu, L. Y., Lu, W. M., & Lin, B. J. (2013). A survey of dea applications. *Omega*, *41*(5), 893-902.

[54] Liu, J. S., Lu, L. Y., & Lu, W. M. (2016). Research fronts in data envelopment analysis. *Omega*, *58*, 33-45.

[55] Emrouznejad, A., & Yang, G. L. (2018). A survey and analysis of the first 40 years of scholarly literature in DEA: 1978–2016. *Socio-Economic planning sciences*, *61*, 4-8.

[56] Adam, L., & Branda, M. (2021). Risk-Aversion in data envelopment analysis models with diversification. *Omega*, *102*, 102338. https://doi.org/10.1016/j.omega.2020.102338

[57] Amin, G. R., & Hajjami, M. (2021). Improving DEA cross-efficiency optimization in portfolio selection. *Expert systems with applications*, *168*, 114280. https://doi.org/10.1016/j.eswa.2020.114280

[58] Gong, X., Yu, C., Min, L., & Ge, Z. (2021). Regret theory-based fuzzy multi-objective portfolio selection model involving DEA cross-efficiency and higher moments. *Applied soft computing*, *100*, 106958. https://doi.org/10.1016/j.asoc.2020.106958

[59] Peykani, P., Farzipoor Saen, R., Seyed Esmaeili, F. S., & Gheidar-Kheljani, J. (2021). Window data envelopment analysis approach: a review and bibliometric analysis. *Expert systems*, e12721. https://doi.org/10.1111/exsy.12721

[60] Ren, T., Zhou, Z., & Xiao, H. (2021). Estimation of portfolio efficiency considering social responsibility: evidence from the multi-horizon diversification DEA. *RAIRO-Operations research*, *55*(2), 611-637.

[61] Xiao, H., Ren, T., Zhou, Z., & Liu, W. (2021). Parameter uncertainty in estimation of portfolio efficiency: evidence from an interval diversification-consistent dea approach. *Omega*, *103*, 102357. https://doi.org/10.1016/j.omega.2020.102357

[62] Zhou, Z., Gao, M., Xiao, H., Wang, R., & Liu, W. (2021). Big Data and portfolio optimization: a novel approach integrating DEA with multiple data sources. *Omega*, *104*, 102479. https://doi.org/10.1016/j.omega.2021.102479

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The New Algorithm for Fully Fuzzy Transportation Problem by Trapezoidal Fuzzy Number (A Generalization of Triangular Fuzzy Number)

Ladji Kané[1,*], Moctar Diakité[1], Souleymane Kané[1], Hawa Bado[1], Moussa Konaté[1], Koura Traoré[1]

[1] Faculty of Economics and Management, University of Social Sciences and Management of Bamako (USSGB), Quartier du Fleuve Rue 310, Porte 238, Mali; fsegmath@gmail.com; moctardia@yahoo.fr; garalo2018@gmail.com; badohawa@yahoo.fr; konatefseg@gmail.com.

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Abstract

Transportation problem is an important network structured linear programming problem that arises in several contexts and has deservedly received a great deal of attention in the literature. The central concept in this problem is to find the least total transportation cost of a commodity in order to satisfy demands at destinations using available supplies at origins in a crisp environment. In real life situations, the decision maker may not be sure about the precise values of the coefficients belonging to the transportation problem. The aim of this paper is to introduce a formulation of fully fuzzy transportation problem involving trapezoidal fuzzy numbers for the transportation costs and values of supplies and demands. We propose a two-step method for solving fuzzy transportation problem where all of the parameters are represented by triangular fuzzy numbers i.e. two interval transportation problems. Since the proposed approach is based on classical approach it is very easy to understand and to apply on real life transportation problems for the decision makers. To illustrate the proposed approach four application examples are solved. The results show that the proposed method is simpler and computationally more efficient than existing methods in the literature.

Keywords: Fully fuzzy linear programming, Transportation problem, Trapezoidal fuzzy numbers, Triangular fuzzy numbers.

1 | Introduction

Transportation problem is an important network structured linear programming problem that arises in several contexts and has deservedly received a great deal of attention in the literature. The central concept in this problem is to find the least total transportation cost of a commodity in order to satisfy demands at destinations using available supplies at origins. Transportation problem can be used for a wide variety of situations such as scheduling, production, investment, plant location, inventory control, employment scheduling and many others.

Corresponding Author: fsegmath@gmail.com

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In general, transportation problems are solved with the assumptions that the transportation costs and values of supplies and demands are specified in a precise way i.e., in crisp environment. However, in many cases the decision maker has no crisp information about the coefficients belonging to the transportation problem. If the nature of the information is vague, that is, if it has some lack of precision, the corresponding coefficients or elements defining the problem can be formulated by means of fuzzy sets, and thus fuzzy transportation problems arise. Several researchers have carried out investigations on fuzzy transportation problem. Zimmermann [4] developed Zimmermann's fuzzy linear programming into several fuzzy optimization methods for solving the transportation problems. ÓhÉigeartaigh [5] proposed an algorithm for solving transportation problems where the supplies and demands are fuzzy sets with linear or triangular membership functions. Chanas et al. [6] investigated the transportation problem with fuzzy supplies and demands and solved them via the parametric programming technique. Their method provided solution which simultaneously satisfies the constraints and the goal to a maximal degree.

In addition, Chanas et al. [7] formulated the classical, interval and fuzzy transportation problem and discussed the methods for solution for the fuzzy transportation problem. Kuchta [8] discussed the type of transportation problems with fuzzy cost coefficients and converted the problem into a bicriterial transportation problem with crisp objective function. Their method only gives crisp solutions based on efficient solutions of the converted problems. Jiménez and Verdegay [9], [10] investigated the fuzzy solid transportation problem in which supplies, demands and conveyance capacities are represented by trapezoidal fuzzy numbers and applied a parametric approach for finding the fuzzy solution. Liu and Kao [11] developed a procedure, based on extension principle to derive the fuzzy objective value of fuzzy transportation problem, in that the cost coefficients and the supply and demand quantities are fuzzy numbers. Gani and Razak [12] presented a two-stage cost minimizing fuzzy transportation problem in which supplies and demands are as trapezoidal fuzzy numbers and used a parametric approach for finding a fuzzy solution with the aim of minimizing the sum of the transportation costs in the two stages. Li et al. [13] proposed a new method based on goal programming for solving fuzzy transportation problem with fuzzy costs. Lin [14] used genetic algorithm for solving transportation problems with fuzzy coefficients. Dinagar and Palanivel [15] investigated fuzzy transportation problem, with the help of trapezoidal fuzzy numbers and applied fuzzy modified distribution method to obtain the optimal solution in terms of fuzzy numbers. Pandian and Natarajan [16] introduced a new algorithm namely, fuzzy zero-point method for finding fuzzy optimal solution for such fuzzy transportation problem in which the transportation cost, supply and demand are represented by trapezoidal fuzzy numbers. Kumar and Kaur [17] proposed a new method based on fuzzy linear programming problem for finding the optimal solution of fuzzy transportation problem. Gupta et al. [18] proposed a new method named as Mehar's method, to find the exact fuzzy optimal solution of fully fuzzy multi-objective transportation problems. Ebrahimnejad [19] applied a fuzzy bounded dual algorithm for solving bounded transportation problems with fuzzy supplies and demands. Shanmugasundari and Ganesan [20] developed the fuzzy version of Vogel's and MODI methods for obtaining the fuzzy initial basic feasible solution and fuzzy optimal feasible solution, respectively, without converting them into classical transportation problem. Also, [21] Chandran and Kandaswamy [21] proposed an algorithm to find an optimal solution of a fuzzy transportation problem, where supply, demand and cost coefficients all are fuzzy numbers. Their algorithm provides decision maker with an effective solution in comparison to existing methods. Ebrahimnejad [22] using an example showed that their method will not always lead to a fuzzy optimal solution.

Moreover, Kumar and Kaur [23] pointed out the limitations and shortcomings of the existing methods for solving fuzzy solid transportation problem and to overcome these limitations and shortcomings proposed a new method to find the fuzzy optimal solution of unbalanced fuzzy solid transportation problems. In addition, Ebrahimnejad [24] proposed a two-step method for solving fuzzy transportation problem where all of the parameters are represented by non-negative triangular fuzzy numbers. Some researchers applied generalized fuzzy numbers for solving transportation problems. Kumar and Kaur [25] proposed a new method based on ranking function for solving fuzzy transportation problem by assuming that transportation cost, supply and demand of the commodity are represented by generalized trapezoidal fuzzy numbers. After that, Kaur and Kumar [26] introduced a similar algorithm for solving a special type of fuzzy transportation problem by assuming that a decision maker is uncertain about the precise values of transportation cost only but there is no uncertainty about the supply and demand of the product. Ebrahimnejad [27] demonstrated that once the ranking function is chosen, the fuzzy transportation problem introduced by Kaur and Kumar [26] is converted into crisp one, which is easily solved by the standard transportation algorithms.

The contributions of the present study are summarized as follows: (a) in the *Eq. (15)* under consideration, all of the parameters, such as the transportation costs, supplies and demands are considered as fuzzy numbers. (b) According to the proposed approach, the *Eq. (15)* is converted into two interval transportation problems *Eq. (16)* and *Eq. (17)*. The integration of the optimal solution of the two sub-problems provides the optimal solution of the *Eq. (15)*. (c) In contrast to most existing approaches, which provide a precise solution, the proposed method provides a fuzzy optimal solution. (d) In contrast to existing methods that include negative parts in the obtained fuzzy optimal solution and fuzzy optimal cost, the proposed method provides a fuzzy optimal solution and optimal cost. (e) Similarly, to the competing methods in the literature, the proposed method is applicable for all types of trapezoidal fuzzy numbers. (f) The complexity of computation is greatly reduced compared with commonly used existing methods in the literature.

The rest of this paper is organized as follows: In Section 2, we recall the definitions of interval number linear programming, interval numbers and fully fuzzy transportation problem. In Section 3, a new method is proposed for obtaining the fuzzy optimal solution of the *Eq. (15)*. The advantages of the proposed method are discussed in Section 4. Two application examples are provided to illustrate the effectiveness of the proposed method in Section 5, and a comparative study in Section 6. Finally, concluding remarks are presented in Section 7.

2 | Materials and Methods

In this section, some basic definitions, arithmetic operations for closed Intervals numbers and of linear programming problems involving interval numbers are presented [28].

2.1 | A New Interval Arithmetic

In this section, some arithmetic operations for two intervals are presented [28].

Let $\mathfrak{R}=\left\{\bar{a}=\left[a^1,a^4\right]:a^1\le a^4,\ \ a^1,a^4\in\mathbb{R}\right\}$ be the set of all proper intervals. We shall use the terms "interval" and "interval number" interchangeably. The mid-point and width (or half-width) of an interval number $\bar{a}=\left[a^1,a^4\right]$ are defined as $m(\bar{a})=\frac{a^4+a^1}{2}$ and $w(\bar{a})=\frac{a^4-a^1}{2}$. The interval number $\bar{a}$ can also be expressed in terms of its midpoint and width as

$$\bar{a}=\left[a^1,a^4\right]=\langle m(\bar{a}),w(\bar{a})\rangle=\langle\frac{a^4+a^1}{2},\frac{a^4-a^1}{2}\rangle. \quad (1)$$

For any two intervals $\bar{a}=\left[a^1,a^4\right]=\langle m(\bar{a}),w(\bar{a})\rangle$ and $\bar{b}=\left[b^1,b^4\right]=\langle m(\bar{b}),w(\bar{b})\rangle$, the arithmetic operations on $\bar{a}$ and $\bar{b}$ are defined as:

Addition: $\bar{a}+\bar{b}=\langle m(\bar{a})+m(\bar{b}),w(\bar{a})+w(\bar{b})\rangle$. (2)

Subtraction: $\bar{a}-\bar{b}=\langle m(\bar{a})-m(\bar{b}),w(\bar{a})+w(\bar{b})\rangle$. (3)

Multiplication: $$\alpha\bar{a} = \begin{cases} \langle \alpha m(\bar{a}), \alpha w(\bar{a}) \rangle \text{ if } \alpha \geq 0, \\ \langle \alpha m(\bar{a}), -\alpha w(\bar{a}) \rangle \text{ if } \alpha < 0. \end{cases} \quad (4)$$

$$\bar{a} \times \bar{b} = \begin{cases} \langle m(\bar{a})m(\bar{b}) + w(\bar{a})w(\bar{b}), m(\bar{a})w(\bar{b}) + m(\bar{b})w(\bar{a}) \rangle \text{ if } a^1 \geq 0, b^1 \geq 0, \\ \langle m(\bar{a})m(\bar{b}) + m(\bar{a})w(\bar{b}), m(\bar{b})w(\bar{a}) + w(\bar{b})w(\bar{a}) \rangle \text{ if } a^1 < 0, b^1 \geq 0, \\ \langle m(\bar{a})m(\bar{b}) - w(\bar{a})w(\bar{b}), m(\bar{b})w(\bar{a}) - m(\bar{a})w(\bar{b}) \rangle \text{ if } a^4 < 0, b^1 \geq 0. \end{cases} \quad (5)$$

2.2| A New Interval Arithmetic for Trapezoidal Fuzzy Numbers via Intervals Numbers

The aim of this section is to present some notations, notions and results which are of useful in our further consideration.

A number $\tilde{a} = (a^1, a^2, a^3, a^4)$ (where $a^1 \leq a^2 \leq a^3 \leq a^4$) is said to be a trapezoidal fuzzy number if its membership function is given by [1]-[3]:

$$\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{x - a^1}{a^2 - a^1}, \; a^1 \leq x \leq a^2, \\ \dfrac{x - a^4}{a^3 - a^4}, \; a^3 \leq x \leq a^4. \end{cases} \quad (6)$$

Assume that $\tilde{a} = (a^1, a^2, a^3, a^4) = (\bar{a}^{14} / \bar{a}^{23}) = ([a^1, a^4] / [a^2, a^3])$,

and $\tilde{b} = (b^1, b^2, b^3, b^4) = (\bar{b}^{14} / \bar{b}^{23}) = ([b^1, b^4] / [b^2, b^3])$ are two trapezoidal fuzzy numbers. For any two trapezoidal fuzzy numbers $\tilde{a} = (\bar{a}^{14} / \bar{a}^{23})$ and $\tilde{b} = (\bar{b}^{14} / \bar{b}^{23})$, the arithmetic operations on $\tilde{a}$ and $\tilde{b}$ are defined as:

Addition: $$\tilde{a} + \tilde{b} = (\bar{a}^{14} | \bar{a}^{23}) + (\bar{b}^{14} | \bar{b}^{23}) = ([a^1, a^4] + [b^1, b^4] | [a^2, a^3] + [b^2, b^3]). \quad (7)$$

Multiplication: $$\tilde{a}\tilde{b} = (\bar{a}^{14} | \bar{a}^{23}) \times (\bar{b}^{14} | \bar{b}^{23}) = ([a^1, a^4] \times [b^1, b^4] | [a^2, a^3] \times [b^2, b^3]). \quad (8)$$

2.3| Formulation of a Transportation Problems Involving Interval Numbers

We consider the Transportation Problem involving Interval numbers as follows [28]:

$$\begin{cases} \operatorname{Min} \bar{Z}(\bar{x}) \approx \sum_{j=1}^{n}\sum_{i=1}^{m} \bar{c}_{ij}\bar{x}_{ij} \\ \text{Subject to the constraints} \\ \sum_{j=1}^{n} \bar{x}_{ij} \approx \bar{a}_i, \text{for } i = 1,2,..,m , \\ \sum_{i=1}^{m} \bar{x}_{ij} \approx \bar{b}_j, \text{ for } j = 1,2,..,n. \end{cases} \tag{9}$$

where $\bar{c}_{ij} = \left[c^1_{ij}, c^4_{ij} \right]$, $\bar{a}_i = \left[a^1_i, a^4_i \right]$, $\bar{b}_j = \left[b^1_j, b^4_j \right]$ are non-negative interval numbers and

$\bar{x}_{ij} = \left[x^1_{ij}, x^4_{ij} \right]$ are unrestricted interval numbers.

Objective function transformation.

$$\bar{Z}(\bar{x}) \approx \sum_{j=1}^{n}\sum_{i=1}^{m} \bar{c}_{ij}\bar{x}_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{m} \left[c^1_{ij}, c^4_{ij} \right]\left[x^1_{ij}, x^4_{ij} \right] = \sum_{j=1}^{n}\sum_{i=1}^{m} \langle m\left(\bar{c}_{ij}\bar{x}_{ij}\right), w\left(\bar{c}_{ij}\bar{x}_{ij}\right)\rangle ,$$

Where

$$m\left(\bar{c}_{ij}\bar{x}_{ij}\right) = \begin{cases} m\left(\bar{c}_{ij}\right)m\left(\bar{x}_{ij}\right) + w\left(\bar{c}_{ij}\right)w\left(\bar{x}_{ij}\right) \text{if } x^1_{ij} \geq 0, \\ m\left(\bar{c}_{ij}\right)m\left(\bar{x}_{ij}\right) + w\left(\bar{c}_{ij}\right)m\left(\bar{x}_{ij}\right) \text{if } x^1_{ij} \langle 0 \text{ and } x^4_{ij} \rangle 0, \\ m\left(\bar{c}_{ij}\right)m\left(\bar{x}_{ij}\right) - w\left(\bar{c}_{ij}\right)w\left(\bar{x}_{ij}\right) \text{if } x^4_{ij} < 0. \end{cases} \tag{10}$$

And

$$w\left(\bar{c}_{ij}\bar{x}_{ij}\right) = \begin{cases} m\left(\bar{c}_{ij}\right)w\left(\bar{x}_{ij}\right) + w\left(\bar{c}_{ij}\right)m\left(\bar{x}_{ij}\right) \text{if } x^1_{ij} \geq 0, \\ m\left(\bar{c}_{ij}\right)w\left(\bar{x}_{ij}\right) + w\left(\bar{c}_{ij}\right)w\left(\bar{x}_{ij}\right) \text{if } x^1_{ij} \langle 0 \text{ and } x^4_{ij} \rangle 0, \\ m\left(\bar{c}_{ij}\right)w\left(\bar{x}_{ij}\right) - w\left(\bar{c}_{ij}\right)m\left(\bar{x}_{ij}\right) \text{if } x^4_{ij} < 0. \end{cases} \tag{11}$$

Transformation of constraints.

$$\sum_{j=1}^{n} \bar{x}_{ij} \approx \bar{a}_i \Leftrightarrow \sum_{j=1}^{n} \langle m\left(\bar{x}_{ij}\right), w\left(\bar{x}_{ij}\right)\rangle \approx \langle m\left(\bar{a}_i\right), w\left(\bar{a}_i\right)\rangle \text{ for } i = 1,2,..,m \text{ and}$$

$$\sum_{i=1}^{m} \bar{x}_{ij} \approx \bar{b}_j \Leftrightarrow \sum_{i=1}^{m} \langle m\left(\bar{x}_{ij}\right), w\left(\bar{x}_{ij}\right)\rangle \approx \langle m\left(\bar{b}_j\right), w\left(\bar{b}_j\right)\rangle \text{ for } j = 1,2,..,n.$$

Now we can say that

$$\begin{cases} \text{Min}\,\bar{Z}(\bar{x}) \approx \sum_{j=1}^{n}\sum_{i=1}^{m}\langle m(\bar{c}_{ij}\bar{x}_{ij}), w(\bar{c}_{ij}\bar{x}_{ij})\rangle \\ \text{Subject to the constraints} \\ \sum_{j=1}^{n}\langle m(\bar{x}_{ij}), w(\bar{x}_{ij})\rangle \approx \langle m(\bar{a}_i), w(\bar{a}_i)\rangle, i=1,2,..,m, \\ \sum_{i=1}^{m}\langle m(\bar{x}_{ij}), w(\bar{x}_{ij})\rangle \approx \langle m(\bar{b}_j), w(\bar{b}_j)\rangle, j=1,2,..,n. \end{cases} \tag{12}$$

is equivalent to

$$\begin{cases} m(\text{Min}\,\bar{Z}(\bar{x})) = \sum_{j=1}^{n}\sum_{i=1}^{m} m(\bar{c}_{ij}\bar{x}_{ij}) \\ \text{Subject to the constraints} \\ \sum_{j=1}^{n} m(\bar{x}_{ij}) = m(\bar{a}_i), \text{for } i=1,2,..,m, \\ \sum_{i=1}^{m} m(\bar{x}_{ij}) = m(\bar{b}_j), \text{ for } j=1,2,..,n. \end{cases} \tag{13}$$

Optimal Solution (9) according to the choice of the decision maker:

$$\bar{x}_{ij}^{*} = \left[x_{ij}^{1}, x_{ij}^{4}\right] = \left[m(\bar{x}_{ij}) - w(\bar{x}_{ij}), m(\bar{x}_{ij}) + w(\bar{x}_{ij})\right] \text{ where } \sum_{j=1}^{n} w(\bar{x}_{ij}) \geq w(\bar{a}_i),\ i=1,2,..,m.$$

Remark.

I. $\sum_{j=1}^{n}\bar{x}_{ij} = \bar{a}_i$ if and only if $\sum_{j=1}^{n} m(\bar{x}_{ij}) = m(\bar{a}_i)$ and $\sum_{j=1}^{n} w(\bar{x}_{ij}) = w(\bar{a}_i)$ for $i=1,2,..,m$.

II. $\sum_{j=1}^{n}\bar{x}_{ij} \neq \bar{a}_i$ if and only if $\sum_{j=1}^{n} m(\bar{x}_{ij}) = m(\bar{a}_i)$ and $\sum_{j=1}^{n} w(\bar{x}_{ij}) > w(\bar{a}_i)$ for $i=1,2,..,m$.

III. $\sum_{i=1}^{m}\bar{x}_{ij} = \bar{b}_j$ if and only if $\sum_{i=1}^{m} m(\bar{x}_{ij}) = m(\bar{b}_j)$ and $\sum_{i=1}^{m} w(\bar{x}_{ij}) = w(\bar{b}_j)$ for $j=1,2,..,n$.

IV. $\sum_{i=1}^{m}\bar{x}_{ij} \neq \bar{b}_j$ if and only if $\sum_{i=1}^{m} m(\bar{x}_{ij}) = m(\bar{b}_j)$ and $\sum_{i=1}^{m} w(\bar{x}_{ij}) > w(\bar{b}_j)$ for $j=1,2,..,n$.

2.4 | Formulation of a Fully Fuzzy Transportation Problem

The fuzzy linear programming formulation of a fully fuzzy transportation problem can be written as follows as follows [2], [3]:

$$
\begin{cases}
\operatorname{Min} \tilde{Z}(\tilde{x}) \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{c}_{ij} \tilde{x}_{ij} \\
\text{Subject to the constraints} \\
\sum_{j=1}^{n} \tilde{x}_{ij} \approx \tilde{a}_i, \text{ for } i = 1,2,..,m \\
\sum_{i=1}^{m} \tilde{x}_{ij} \approx \tilde{b}_j, \text{ for } j = 1,2,..,n
\end{cases} \tag{14}
$$

with $\sum_{i=1}^{m} \tilde{a}_i = \sum_{j=1}^{n} \tilde{b}_j$ where $\tilde{x}_{ij}$ are unrestricted fuzzy numbers and $\tilde{c}_{ij}$, $\tilde{a}_i$ and $\tilde{b}_j$ are non-negatives fuzzy numbers.

3| Main Results

In this section, we will describe our method of solving.

3.1| Transportation Problem with Trapezoidal Fuzzy Numbers

For all the rest of this paper, we will consider the following transportation problem with trapezoidal fuzzy numbers as follows:

$$
\begin{cases}
\operatorname{Min} \tilde{Z}(\tilde{x}) \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{c}_{ij} \tilde{x}_{ij} \\
\text{Subject to the constraints} \\
\sum_{j=1}^{n} \tilde{x}_{ij} \approx \tilde{a}_i, \text{for } i = 1,2,..,m, \\
\sum_{i=1}^{m} \tilde{x}_{ij} \approx \tilde{b}_j, \text{ for } j = 1,2,..,n.
\end{cases} \tag{15}
$$

where $\tilde{c}_{ij} = \left(c_{ij}^1, c_{ij}^2, c_{ij}^3, c_{ij}^4\right)$, $\tilde{x}_j = \left(x_{ij}^1, x_{ij}^2, x_{ij}^3, x_{ij}^4\right)$, $\tilde{b}_j = \left(b_j^1, b_j^2, b_j^3, b_j^4\right)$ and $\tilde{a}_i = \left(a_i^1, a_i^2, a_i^3, a_i^4\right)$ are trapezoidal fuzzy numbers with

$\tilde{x}_{ij} = \left(\bar{x}_{ij}^{14} / \bar{x}_{ij}^{23}\right) = \left(\left[x_{ij}^1, x_{ij}^4\right] / \left[x_{ij}^2, x_{ij}^3\right]\right)$ where $\bar{x}_{ij}^{14} = \left[x_{ij}^1, x_{ij}^4\right]$ and $\bar{x}_{ij}^{23} = \left[x_{ij}^2, x_{ij}^3\right]$,

$\tilde{c}_{ij} = \left(\bar{c}_{ij}^{14} / \bar{c}_{ij}^{23}\right) = \left(\left[c_{ij}^1, c_{ij}^4\right] / \left[c_{ij}^2, c_{ij}^3\right]\right)$ where $\bar{c}_{ij}^{14} = \left[c_{ij}^1, c_{ij}^4\right]$ and $\bar{c}_{ij}^{23} = \left[c_{ij}^2, c_{ij}^3\right]$,

$\tilde{b}_j = \left(\bar{b}_j^{14} / \bar{b}_j^{23}\right) = \left(\left[b_j^1, b_j^4\right] / \left[b_j^2, b_j^3\right]\right)$ where $\bar{b}_j^{14} = \left[b_j^1, b_j^4\right]$ and $\bar{b}_j^{23} = \left[b_j^2, b_j^3\right]$ and

$\tilde{a}_i = \left(\bar{a}_i^{14} / \bar{a}_i^{23}\right) = \left(\left[a_i^1, a_i^4\right] / \left[a_i^2, a_i^3\right]\right)$ where $\bar{a}_i^{14} = \left[a_i^1, a_i^4\right]$ and $\bar{a}_i^{23} = \left[a_i^2, a_i^3\right]$.

For all the rest of this paper, we will consider the following transportation problems involving interval numbers $\bar{x}_{ij}^{14}$, $\bar{c}_{ij}^{14}$, $\bar{a}_i^{14}$ and $\bar{b}_j^{14}$ as follows:

$$\begin{cases} \operatorname{Min} \bar{Z}^{14}\left(\bar{x}^{14}\right) \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \bar{c}_{ij}^{14} \bar{x}_{ij}^{14} \\ \text{Subject to the constraints} \\ \sum_{j=1}^{n} \bar{x}_{ij}^{14} \approx \bar{a}_{i}^{14}, \text{for } i=1,2,..,m, \\ \sum_{i=1}^{m} \bar{x}_{ij}^{14} \approx \bar{b}_{j}^{14}, \text{for } j=1,2,..,n. \end{cases} \tag{16}$$

And the transportation problems involving interval numbers $\bar{x}_{ij}^{23}$, $\bar{c}_{ij}^{23}$, $\bar{a}_{i}^{23}$ and $\bar{b}_{j}^{23}$ as follows:

$$\begin{cases} \operatorname{Min} \bar{Z}^{23}\left(\bar{x}^{23}\right) \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \bar{c}_{ij}^{23} \bar{x}_{ij}^{23} \\ \text{Subject to the constraints} \\ \sum_{j=1}^{n} \bar{x}_{ij}^{23} \approx \bar{a}_{i}^{23}, \text{for } i=1,2,..,m, \\ \sum_{i=1}^{m} \bar{x}_{ij}^{23} \approx \bar{b}_{j}^{23}, \text{for } j=1,2,..,n. \end{cases} \tag{17}$$

3.2| Formulation of a Transportation Problem involving Midpoint

Thanks to the new interval arithmetic and *Eq. (16)*, we can write the following transportation problem involving midpoint ($\bar{x}_{ij}^{14}$, $\bar{c}_{ij}^{14}$, $\bar{a}_{i}^{14}$ and $\bar{b}_{j}^{14}$) [28] as follows:

$$\begin{cases} \operatorname{Min} Z^{14}\left(x^{14}\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} m\left(\bar{c}_{ij}^{14}\right) x_{ij}^{14} \\ \text{Subject to the constraints} \\ \sum_{j=1}^{n} x_{ij}^{14} = m\left(\bar{a}_{i}^{14}\right), \text{for } i=1,2,..,m, \\ \sum_{i=1}^{m} x_{ij}^{14} = m\left(\bar{b}_{j}^{14}\right), \text{for } j=1,2,..,n. \end{cases} \tag{18}$$

where $m\left(\bar{c}_{ij}^{14}\right) = \dfrac{c_{ij}^{4} + c_{ij}^{1}}{2}$, $m\left(\bar{a}_{i}^{14}\right) = \dfrac{a_{i}^{4} + a_{i}^{1}}{2}$, $m\left(\bar{b}_{j}^{14}\right) = \dfrac{b_{j}^{4} + b_{j}^{1}}{2}$ and $w\left(\bar{a}_{i}^{14}\right) = \dfrac{a_{i}^{4} - a_{i}^{1}}{2}$.

Thanks to the new interval arithmetic and *Eq. (17)*, we can write the following transportation problem involving midpoint ($\bar{x}_{ij}^{23}$, $\bar{c}_{ij}^{23}$, $\bar{a}_{i}^{23}$ and $\bar{b}_{j}^{23}$) [28] as follows:

$$\begin{cases} \text{Min} Z^{23}\left(x^{23}\right)=\sum_{i=1}^{m}\sum_{j=1}^{n} m\left(\bar{c}_{ij}^{23}\right)x_{ij}^{23} \\ \text{Subject to the constraints} \\ \sum_{j=1}^{n} x_{ij}^{23}=m\left(\bar{a}_i^{23}\right), \text{for } i=1,2,..,m, \\ \sum_{i=1}^{m} x_{ij}^{23}=m\left(\bar{b}_j^{23}\right), \text{for } j=1,2,..,n. \end{cases} \tag{19}$$

where $m\left(\bar{c}_{ij}^{23}\right)=\dfrac{c_{ij}^{3}+c_{ij}^{2}}{2}$, $m\left(\bar{a}_i^{23}\right)=\dfrac{a_i^{3}+a_i^{2}}{2}$, $m\left(\bar{b}_j^{23}\right)=\dfrac{b_j^{3}+b_j^{2}}{2}$ and $w\left(\bar{a}_i^{23}\right)=\dfrac{a_i^{3}-a_i^{2}}{2}$.

Thanks to the new interval arithmetic, we can write the following proposition [28]:

Proposition 1. If $\bar{x}_{ij}^{14}=\left[x_{ij}^{*1}, x_{ij}^{*4}\right]$ is an optimal solution to the *Eq. (16)* and $\bar{x}_{ij}^{23}=\left[x_{ij}^{*2}, x_{ij}^{*3}\right]$ is an optimal solution to the *Eq. (17)*, then $\tilde{x}^{*}=\left(\tilde{x}_{ij}^{*}\right)_{m\times n}$ is an optimal solution to the *Eq. (15)* with

$$\tilde{x}_{ij}^{*}=\left(\bar{x}_{ij}^{14}/\bar{x}_{ij}^{23}\right)=\left(\left[x_{ij}^{*1}, x_{ij}^{*4}\right]/\left[x_{ij}^{*2}, x_{ij}^{*3}\right]\right)=\left(x_{ij}^{*1}, x_{ij}^{*2}, x_{ij}^{*3}, x_{ij}^{*4}\right).$$

3.3 | The Steps of Our Computational Method

The steps of our method for solving the fully fuzzy transportation problem involving trapezoidal fuzzy numbers as follows.

3.3.1 | Solution procedure for transportation problem with trapezoidal fuzzy numbers

Step 1. Construct the fuzzy transportation *Problem (15)*, and then convert it into a balanced one if it is not. Else, go to Step 2.

Step 2. Solving *Eq. (17)* via *Eq. (19)*. Determine $\bar{x}_{ij}^{23}=\left[x_{ij}^{*2}, x_{ij}^{*3}\right]=\left[x_{ij}^{23}-w\left(\bar{x}_{ij}^{23}\right), x_{ij}^{23}+w\left(\bar{x}_{ij}^{23}\right)\right]$ by applying the following conditions:

I. $w\left(\bar{x}_{ij}^{23}\right)=0$ if and only if $x_{ij}^{23}=0$ and

II. $\sum_{x_{ij}^{23}\neq 0} w\left(\bar{x}_{ij}^{23}\right)=w\left(\bar{a}_i^{23}\right)$ with $w\left(\bar{x}_{ip}^{23}\right)\geq w\left(\bar{x}_{ik}^{23}\right)$ if $c_{ip}^{4}\leq c_{ik}^{4}$ for $i=1,..,m$.

III. Go to Step 3.

Step 3. Solving *Eq. (16)* via *Eq. (18)*. Determine $\bar{x}_{ij}^{14}=\left[x_{ij}^{*1}, x_{ij}^{*4}\right]=\left[x_{ij}^{14}-w\left(\bar{x}_{ij}^{14}\right), x_{ij}^{14}+w\left(\bar{x}_{ij}^{14}\right)\right]$ for $i=1,..,m$. Considering the following cases:

Case 1. If $E_i=\sum_{j=1}^{n}\left|x_{ij}^{14}-x_{ij}^{23}\right|+w\left(\bar{a}_i^{23}\right)\leq w\left(\bar{a}_i^{14}\right)$, then

$\bar{x}_{ij}^{14}=\left[x_{ij}^{*1}, x_{ij}^{*4}\right]=\left[x_{ij}^{14}-w\left(\bar{x}_{ij}^{14}\right), x_{ij}^{14}+w\left(\bar{x}_{ij}^{14}\right)\right]$ with $\sum_{x_{ij}^{14}\neq 0} w\left(\bar{x}_{ij}^{14}\right)=w\left(\bar{a}_i^{14}\right)$. Else, go to Case 2.

Case 2. If $E_i=\sum_{j=1}^{n}\left|x_{ij}^{14}-x_{ij}^{23}\right|+w\left(\bar{a}_i^{23}\right)>w\left(\bar{a}_i^{14}\right)$, then

$\bar{x}_{ij}^{14}=\left[x_{ij}^{*1},x_{ij}^{*4}\right]=\left[x_{ij}^{14}-w\left(\bar{x}_{ij}^{14}\right),x_{ij}^{14}+w\left(\bar{x}_{ij}^{14}\right)\right]$ with $w\left(\bar{x}_{ij}^{14}\right)=\left|x_{ij}^{14}-x_{ij}^{23}\right|+w\left(\bar{x}_{ij}^{23}\right)$ and go to Step 4.

Step 4. The *Optimal Solution (15)* according to the choice of the decision maker is:

$$Min\ \tilde{Z}(\tilde{x})\approx\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij}^{1},c_{ij}^{2},c_{ij}^{3},c_{ij}^{4}\right)\left(x_{ij}^{*1},x_{ij}^{*2},x_{ij}^{*3},x_{ij}^{*4}\right)=\left(Z^{*1},Z^{*2},Z^{*3},Z^{*4}\right) \text{ where}$$

$$\tilde{x}_{ij}^{*}=\left(\bar{x}_{ij}^{14}/\bar{x}_{ij}^{23}\right)=\left(\left[x_{ij}^{*1},x_{ij}^{*4}\right]/\left[x_{ij}^{*2},x_{ij}^{*3}\right]\right)=\left(x_{ij}^{*1},x_{ij}^{*2},x_{ij}^{*3},x_{ij}^{*4}\right).$$

3.3.2 | Solution procedure for transportation problem with triangular fuzzy numbers

The steps of our method for solving the fully fuzzy transportation problem involving triangular fuzzy numbers as follows:

The fuzzy number $\tilde{x}_{ij}^{*}=\left(x_{ij}^{*1},x_{ij}^{*2},x_{ij}^{*3},x_{ij}^{*4}\right)$ is said to be triangular if and only if $x_{ij}^{*2}=x_{ij}^{*3}$. Then $w\left(\bar{x}_{ij}^{23}\right)=\dfrac{x_{ij}^{*3}-x_{ij}^{*2}}{2}=0$. Therefore $w\left(\bar{x}_{ij}^{14}\right)=\left|x_{ij}^{14}-x_{ij}^{23}\right|$ and

$$\tilde{x}_{ij}^{*}=\left(x_{ij}^{*1},x_{ij}^{*2},x_{ij}^{*3},x_{ij}^{*4}\right)=\left(x_{ij}^{*1},x_{ij}^{*2},x_{ij}^{*2},x_{ij}^{*4}\right)=\left(x_{ij}^{*1},x_{ij}^{*2},x_{ij}^{*4}\right).$$

In this article, consider fuzzy triangular numbers by: $\tilde{x}_{ij}^{*}=\left(x_{ij}^{2}/\bar{x}_{ij}^{13}\right)=\left(x_{ij}^{2}/\left[x_{ij}^{*1},x_{ij}^{*3}\right]\right)=\left(x_{ij}^{*1},x_{ij}^{*2},x_{ij}^{*3}\right)$, where $\left[x_{ij}^{*1},x_{ij}^{*3}\right]=\left[x_{ij}^{13}-w\left(\bar{x}_{ij}^{13}\right),x_{ij}^{13}+w\left(\bar{x}_{ij}^{13}\right)\right]$.

Step 1. Construct the fuzzy transportation *Problem (15)*, and then convert it into a balanced one if it is not. Else, go to Step 2.

Step 2. Solving *Eq. (17)* via *Eq. (19)*. Determine $\bar{x}_{ij}^{23}=\left[x_{ij}^{*2},x_{ij}^{*3}\right]=\left[x_{ij}^{*2},x_{ij}^{*2}\right]=x_{ij}^{*2}$, for $i=1,..,m$ and go to Step 3.

Step 3. Solving *Eq. (16)* via *Eq. (18)*. Determine $\bar{x}_{ij}^{13}=\left[x_{ij}^{*1},x_{ij}^{*3}\right]=\left[x_{ij}^{13}-w\left(\bar{x}_{ij}^{13}\right),x_{ij}^{13}+w\left(\bar{x}_{ij}^{13}\right)\right]$ for $i=1,..,m$. Considering the following cases:

Case 1. If $E_i=\sum_{j=1}^{n}\left|x_{ij}^{13}-x_{ij}^{2}\right|\le w\left(\bar{a}_i^{13}\right)$, then $\bar{x}_{ij}^{13}=\left[x_{ij}^{*1},x_{ij}^{*3}\right]=\left[x_{ij}^{13}-w\left(\bar{x}_{ij}^{13}\right),x_{ij}^{13}+w\left(\bar{x}_{ij}^{13}\right)\right]$ with $\sum_{x_{ij}^{13}\neq 0}w\left(\bar{x}_{ij}^{13}\right)=w\left(\bar{a}_i^{13}\right)$. Else, go to Case 2.

Case 2. If $E_i=\sum_{j=1}^{n}\left|x_{ij}^{13}-x_{ij}^{2}\right|>w\left(\bar{a}_i^{13}\right)$, then $\bar{x}_{ij}^{13}=\left[x_{ij}^{*1},x_{ij}^{*3}\right]=\left[x_{ij}^{13}-w\left(\bar{x}_{ij}^{13}\right),x_{ij}^{13}+w\left(\bar{x}_{ij}^{13}\right)\right]$ with $w\left(\bar{x}_{ij}^{13}\right)=\left|x_{ij}^{13}-x_{ij}^{2}\right|$ and go to Step 4.

Step 4. The *Optimal Solution (15)* according to the choice of the decision maker is:

$$Min\ \tilde{Z}(\tilde{x}) \approx \sum_{i=1}^{m}\sum_{j=1}^{n} \left(c^1_{ij}, c^2_{ij}, c^3_{ij}\right)\left(x^{*1}_{ij}, x^{*2}_{ij}, x^{*3}_{ij}\right) = \left(Z^{*1}, Z^{*2}, Z^{*3}\right) \text{with}$$

$$\tilde{x}^*_{ij} = \left(x^2_{ij} / \tilde{x}^{13}_{ij}\right) = \left(x^2_{ij} / \left[x^{*1}_{ij}, x^{*3}_{ij}\right]\right) = \left(x^{*1}_{ij}, x^{*2}_{ij}, x^{*3}_{ij}\right).$$

4| Advantages of the Proposed Method

Let us explore the main advantages of the proposed method:

- *The new proposed method is applicable to all Eq. (15) where $\tilde{x}_{ij}$ are unrestricted triangular or trapezoidal fuzzy numbers and $\tilde{c}_{ij}$, $\tilde{a}_i$ and $\tilde{b}_j$ are non-negatives triangular or trapezoidal fuzzy numbers.*
- *The proposed technique does not use the goal and parametric approaches which are difficult to apply in real life situations.*
- *By applying the proposed approach for finding the fuzzy optimal solution, there is no need of much knowledge of fuzzy linear programming technique, Zimmerman approach and crisp linear programming which are difficult to learn for a new decision maker.*
- *The proposed method to solve Eq. (15) is based on traditional transportation algorithms. Thus, the existing and easily available software can be used for the same. However, the existing method [1], [2], [3], [11], [29] to solve Eq. (15) should be implemented into a programming language.*
- *To solve the Eq. (15) by using the existing method [1], [2], [3], [11], [29], there is need to use arithmetic operations of generalized fuzzy numbers. While, if the proposed technique is used for the same then there is need to use arithmetic operations of real numbers. This proves that it is much easy to apply the proposed method as compared to the existing method [1], [2], [3], [11], [29].*
- *Moreover, it is possible to assume a generic ranking index for comparing the fuzzy numbers involved in the Eq. (15), in such a way that each time in which the decision maker wants to solve the FFTP under consideration (s)he can choose (or propose) the ranking index that best suits the Eq. (15).*

5| Numerical Illustration

In this section, to illustrate the new method proposed, and the existing fully fuzzy transportation problem due to [1], [2], [3], [11], [29], [30], presented in *Examples 1-4*, are solved by the proposed method.

Example 1. [3].

Step 1. Construct the fuzzy transportation *Problem (15)*, and then convert it into a balanced one if it is not.

Table 1. Eq. (15) in trapezoidal unbalanced form.

	R_1	R_2	R_3	R_3	Supply $(\tilde{a}_i)$
A	$(10,10,10,10)$	$(50,50,50,50)$	$(80,80,80,80)$	$(0,0,0,0)$	$(70,90,90,100)$
B	$(60,70,80,90)$	$(60,60,60,60)$	$(20,20,20,20)$	$(0,0,0,0)$	$(40,60,70,80)$
C	$(0,0,0,0)$	$(0,0,0,0)$	$(0,0,0,0)$	$(0,0,0,0)$	$(0,0,10,50)$
Demand $(\tilde{b}_j)$	$(30,40,50,70)$	$(20,30,40,50)$	$(40,50,50,80)$	$(20,30,30,30)$	$\sum_{i=1}^{m}\tilde{a}_i = \sum_{j=1}^{n}\tilde{b}_j$

Step 2. Solving *Eq. (17)* via *Eq. (19)*. $MinZ^{23}\left(x^{23}\right)=25x^{23}_{11}+65x^{23}_{12}+100x^{23}_{13}+75x^{23}_{21}+90x^{23}_{22}+40x^{23}_{23}$ subject to the constraints $x^{23}_{11}+x^{23}_{12}+x^{23}_{13}+x^{23}_{14}=90$, $x^{23}_{21}+x^{23}_{22}+x^{23}_{23}+x^{23}_{24}=65$, $x^{23}_{31}+x^{23}_{32}+x^{23}_{33}+x^{23}_{34}=5$, $x^{23}_{11}+x^{23}_{21}+x^{23}_{31}=45$, $x^{23}_{12}+x^{23}_{22}+x^{23}_{32}=35$, $x^{23}_{13}+x^{23}_{23}+x^{23}_{33}=50$ and $x^{23}_{14}+x^{23}_{24}+x^{23}_{34}=30$.

Optimal solution is: $x^{23}_{11}=45$, $x^{23}_{12}=30$, $x^{23}_{13}=0$, $x^{23}_{14}=15$, $x^{23}_{21}=0$, $x^{23}_{22}=0$, $x^{23}_{23}=50$, $x^{23}_{24}=15$, $x^{23}_{31}=0$, $x^{23}_{32}=5$, $x^{23}_{33}=0$ and $x^{23}_{34}=0$. Furthermore $w\left(\bar{a}^{23}_{1}\right)=0$, $w\left(\bar{a}^{23}_{2}\right)=5$ and $w\left(\bar{a}^{23}_{3}\right)=5$.

For $i=1$, then $\sum_{x^{23}_{1j}\neq 0} w\left(\bar{x}^{23}_{1j}\right)=w\left(\bar{a}^{23}_{1}\right)=0$ with $w\left(\bar{x}^{23}_{11}\right)=w\left(\bar{x}^{23}_{12}\right)=w\left(\bar{x}^{23}_{13}\right)=w\left(\bar{x}^{23}_{14}\right)=0$. We get $\bar{x}^{23}_{11}=\lfloor 45,45 \rfloor$, $\bar{x}^{23}_{12}=\lfloor 30,30 \rfloor$, $\bar{x}^{23}_{13}=\bar{0}$ and $\bar{x}^{23}_{14}=\lfloor 15,15 \rfloor$.

For $i=2$, then $\sum_{x^{23}_{2j}\neq 0} w\left(\bar{x}^{23}_{2j}\right)=w\left(\bar{a}^{23}_{2}\right)=5$ with $w\left(\bar{x}^{23}_{21}\right)=w\left(\bar{x}^{23}_{22}\right)=0$, $w\left(\bar{x}^{23}_{23}\right)=1$ and

$w\left(\bar{x}^{23}_{24}\right)=4$. We get $\bar{x}^{23}_{21}=\lfloor 0,0 \rfloor$, $\bar{x}^{23}_{22}=\lfloor 0,0 \rfloor$, $\bar{x}^{23}_{23}=\lfloor 49,51 \rfloor$ and $\bar{x}^{23}_{24}=\lfloor 11,19 \rfloor$.

For $i=3$, then $\sum_{x^{23}_{3j}\neq 0} w\left(\bar{x}^{23}_{3j}\right)=w\left(\bar{a}^{23}_{3}\right)=5$ with $w\left(\bar{x}^{23}_{31}\right)=0$, $w\left(\bar{x}^{23}_{32}\right)=5$, $w\left(\bar{x}^{23}_{33}\right)=0$ and

$w\left(\bar{x}^{23}_{34}\right)=0$. We get $\bar{x}^{23}_{31}=\bar{0}$, $\bar{x}^{23}_{32}=\lfloor 0,10 \rfloor$, $\bar{x}^{23}_{33}=\bar{0}$ and $\bar{x}^{23}_{34}=\bar{0}$.

Step 3. Solving *Eq. (16)* via *Eq. (18)*. We get

$MinZ^{14}\left(x^{14}\right)=25x^{14}_{11}+70x^{14}_{12}+100x^{14}_{13}+75x^{14}_{21}+95x^{14}_{22}+40x^{14}_{23}$

subject to the constraints $x^{14}_{11}+x^{14}_{12}+x^{14}_{13}+x^{14}_{14}=85$, $x^{14}_{21}+x^{14}_{22}+x^{14}_{23}+x^{14}_{24}=60$,

$x^{14}_{31}+x^{14}_{32}+x^{14}_{33}+x^{14}_{34}=25$, $x^{14}_{11}+x^{14}_{21}+x^{14}_{31}=50$, $x^{14}_{12}+x^{14}_{22}+x^{14}_{32}=35$, $x^{14}_{13}+x^{14}_{23}+x^{14}_{33}=60$ and

$x^{14}_{14}+x^{14}_{24}+x^{14}_{34}=25$.

Optimal solution is: $x^{14}_{11}=50$, $x^{14}_{12}=10$, $x^{14}_{13}=0$, $x^{14}_{14}=25$, $x^{14}_{21}=0$, $x^{14}_{22}=0$, $x^{14}_{23}=60$, $x^{14}_{24}=0$,

$x^{14}_{31}=0$, $x^{14}_{32}=25$, $x^{14}_{33}=0$ and $x^{14}_{34}=0$. Furthermore $w\left(\bar{a}^{14}_{1}\right)=15$, $w\left(\bar{a}^{14}_{2}\right)=20$ and $w\left(\bar{a}^{14}_{3}\right)=25$

For $i=1$, we have

$$E_{1}=\sum_{j=1}^{4}\left|x^{14}_{1j}-x^{23}_{1j}\right|+w\left(\bar{a}^{23}_{1}\right)=\left|50-45\right|+\left|10-30\right|+\left|0-0\right|+\left|25-15\right|+0=35>15=w\left(\bar{a}^{14}_{1}\right),$$

then $w\left(\bar{x}^{14}_{1j}\right)=\left|x^{32}_{1j}-x^{14}_{1j}\right|+w\left(\bar{x}^{23}_{1j}\right)$. We have $w\left(\bar{x}^{14}_{11}\right)=5,$ $w\left(\bar{x}^{14}_{12}\right)=20$, $w\left(\bar{x}^{14}_{13}\right)=0$ and

$w\left(\bar{x}^{14}_{14}\right)=10$. We get $\bar{x}^{14}_{11}=\lfloor 45,55 \rfloor$, $\bar{x}^{14}_{12}=\lfloor -10,30 \rfloor$, $\bar{x}^{14}_{13}=\bar{0}$ and $\bar{x}^{14}_{14}=\lfloor 15,35 \rfloor$.

For $i=2$, we have

$$E_2=\sum_{j=1}^{4}\left|x_{2j}^{14}-x_{2j}^{23}\right|+w\left(\bar{a}_2^{23}\right)=\left|0-0\right|+\left|0-0\right|+\left|60-50\right|+\left|0-15\right|+5=30>20=w\left(\bar{a}_2^{14}\right),$$

then $w\left(\bar{x}_{2j}^{14}\right)=\left|x_{2j}^{14}-x_{2j}^{23}\right|+w\left(\bar{x}_{2j}^{23}\right)$. We have $w\left(\bar{x}_{21}^{14}\right)=w\left(\bar{x}_{22}^{14}\right)=0$,

$w\left(\bar{x}_{23}^{23}\right)=11$ and $w\left(\bar{x}_{24}^{14}\right)=19$. We get $\bar{x}_{21}^{14}=\left\lfloor 0,0\right\rfloor$, $\bar{x}_{22}^{14}=\left\lfloor 0,0\right\rfloor$, $\bar{x}_{23}^{14}=\left\lfloor 49,71\right\rfloor$ and

$\bar{x}_{24}^{14}=\left\lfloor -19,19\right\rfloor$.

For $i=3$, we have

$$E_3=\sum_{j=1}^{4}\left|x_{3j}^{14}-x_{3j}^{23}\right|+w\left(\bar{a}_3^{23}\right)=\left|0-0\right|+\left|25-5\right|+\left|0-0\right|+\left|0-0\right|+5=25\le 25=w\left(\bar{a}_3^{14}\right),$$

then $\sum_{x_{3j}^{14}\neq 0}w\left(\bar{x}_{3j}^{14}\right)=w\left(\bar{a}_3^{14}\right)=25$. We have $w\left(\bar{x}_{31}^{14}\right)=0$, $w\left(\bar{x}_{32}^{14}\right)=25$, $w\left(\bar{x}_{33}^{14}\right)=0$ and $w\left(\bar{x}_{34}^{14}\right)=0$

We get $\bar{x}_{31}^{14}=\bar{0}$, $\bar{x}_{32}^{14}=\left\lfloor 0,50\right\rfloor$, $\bar{x}_{33}^{14}=\bar{0}$ and $\bar{x}_{34}^{14}=\bar{0}$.

Step 4. The optimal solution according to the choice of the decision maker is:

$$Min\ \tilde{Z}\left(\tilde{x}\right)\approx\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij}^{1},c_{ij}^{2},c_{ij}^{3},c_{ij}^{4}\right)\left(x_{ij}^{*1},x_{ij}^{*2},x_{ij}^{*3},x_{ij}^{*4}\right)=\left(Z^{*1},Z^{*2},Z^{*3},Z^{*4}\right)=\left(930,2930,2970,3470\right)$$ and

$\tilde{x}_{ij}^{*}=\left(\bar{x}_{ij}^{14}/\bar{x}_{ij}^{23}\right)=\left(\left[x_{ij}^{*1},x_{ij}^{*4}\right]/\left[x_{ij}^{*2},x_{ij}^{*3}\right]\right)=\left(x_{ij}^{*1},x_{ij}^{*2},x_{ij}^{*3},x_{ij}^{*4}\right)$ with

$\tilde{x}_{11}^{*}=\left(45,\ 45,\ 45,55\right)$, $\tilde{x}_{12}^{*}=\left(-10,\ 30,\ 30,\ 30\right)$, $\tilde{x}_{13}^{*}=\tilde{0}$, $\tilde{x}_{14}^{*}=\left(15,\ 15,\ 15,\ 35\right)$,

$\tilde{x}_{21}^{*}=\tilde{0}$, $\tilde{x}_{22}^{*}=\tilde{0}$, $\tilde{x}_{23}^{*}=\left(49,\ 49,\ 51,\ 71\right)$, $\tilde{x}_{24}^{*}=\left(-19,\ 11,\ 19,\ 19\right)$,

$\tilde{x}_{31}^{*}=\tilde{0}$, $\tilde{x}_{32}^{*}=\left(0,0,10,50\right)$, $\tilde{x}_{33}^{*}=\tilde{0}$ and $\tilde{x}_{34}^{*}=\tilde{0}$.

Example 2. [29].

Step 1. Construct the fuzzy transportation *Problem (15)*, and then convert it into a balanced one if it is not.

Table 2. Eq. (15) in trapezoidal balanced form.

	R_1	R_2	R_3	Supply $\tilde{a}_i$
A	$(1,4,9,19)$	$(1,2,5,9)$	$(2,5,8,18)$	$(1,5,7,9)$
B	$(8,9,12,26)$	$(3,5,8,12)$	$(7,9,13,28)$	$(4,7,8,10)$
C	$(11,12,20,27)$	$(0,5,10,15)$	$(4,5,8,11)$	$(4,5,8,11)$
Demand $\tilde{b}_j$	$(3,5,8,12)$	$(4,8,9,10)$	$(2,4,6,8)$	$\sum_{i=1}^{m}\tilde{a}_i=\sum_{j=1}^{n}\tilde{b}_j$

Step 2. Solving *Eq. (17)* via *Eq. (19)*. Optimal solution is: $x_{11}^{23}=6$, $x_{12}^{23}=0$, $x_{13}^{23}=0$, $x_{21}^{23}=\frac{1}{2}$, $x_{22}^{23}=7$, $x_{23}^{23}=0$, $x_{31}^{23}=0$, $x_{32}^{23}=\frac{3}{2}$ and $x_{33}^{14}=5$. Furthermore $w\left(\bar{a}_1^{23}\right)=\frac{2}{2}=1$, $w\left(\bar{a}_2^{23}\right)=\frac{1}{2}$, $w\left(\bar{a}_3^{23}\right)=\frac{3}{2}$.

For $i=1$, then $\sum_{x_{1j}^{23}\neq 0} w\left(\bar{x}_{1j}^{23}\right)=w\left(\bar{a}_1^{23}\right)=1$ with $w\left(\bar{x}_{11}^{23}\right)=1$. We get $\bar{x}_{11}^{23}=\lfloor 5,7\rfloor$, $\bar{x}_{12}^{23}=\lfloor 0,0\rfloor$ and $\bar{x}_{13}^{23}=\lfloor 0,0\rfloor$.

For $i=2$, then $\sum_{x_{2j}^{23}\neq 0} w\left(\bar{x}_{2j}^{23}\right)=w\left(\bar{a}_2^{23}\right)=\frac{1}{2}$ with $w\left(\bar{x}_{21}^{23}\right)=\frac{1}{4}$ and $w\left(\bar{x}_{22}^{23}\right)=\frac{1}{4}$. We get $\bar{x}_{21}^{23}=\left[\frac{1}{4},\frac{3}{4}\right]$, $\bar{x}_{22}^{23}=\left[\frac{27}{4},\frac{29}{4}\right]$ and $\bar{x}_{23}^{23}=\lfloor 0,0\rfloor$.

For $i=3$, then $\sum_{x_{3j}^{23}\neq 0} w\left(\bar{x}_{3j}^{23}\right)=w\left(\bar{a}_3^{23}\right)=\frac{3}{2}$ with $w\left(\bar{x}_{32}^{23}\right)=\frac{1}{2}$ and $w\left(\bar{x}_{33}^{23}\right)=1$. We get $\bar{x}_{31}^{23}=\lfloor 0,0\rfloor$, $\bar{x}_{32}^{23}=\lfloor 1,2\rfloor$ and $\bar{x}_{33}^{23}=\lfloor 4,6\rfloor$.

Step 3. Solving *Eq. (16)* via *Eq. (18)*. Optimal solution is: $x_{11}^{14}=5$, $x_{12}^{14}=0$ and $x_{13}^{14}=0$, $x_{21}^{14}=\frac{5}{2}$, $x_{22}^{14}=\frac{9}{2}$, $x_{23}^{14}=0$, $x_{31}^{14}=0$, $x_{32}^{14}=\frac{5}{2}$ and $x_{33}^{14}=5$. Furthermore $w\left(\bar{a}_1^{14}\right)=4$, $w\left(\bar{a}_2^{14}\right)=3$, $w\left(\bar{a}_3^{14}\right)=\frac{7}{2}$.

For $i=1$, we have $E_1=\sum_{j=1}^{n}\left|x_{1j}^{14}-x_{1j}^{23}\right|+w\left(\bar{a}_1^{23}\right)=|5-6|+|0-0|+|0-0|+1=2\leq 4=w\left(\bar{a}_1^{14}\right)$, then $\sum_{x_{1j}^{14}\neq 0} w\left(\bar{x}_{1j}^{14}\right)=w\left(\bar{a}_1^{14}\right)=4$ with $w\left(\bar{x}_{11}^{14}\right)=4$. We get $\bar{x}_{11}^{14}=\lfloor 1,9\rfloor$, $\bar{x}_{12}^{14}=\lfloor 0,0\rfloor$ and $\bar{x}_{13}^{14}=\lfloor 0,0\rfloor$.

For $i=2$, we have $E_2=\sum_{j=1}^{n}\left|x_{2j}^{14}-x_{2j}^{23}\right|+w\left(\bar{a}_2^{23}\right)=\left|\frac{5}{2}-\frac{1}{2}\right|+\left|\frac{9}{2}-7\right|+|0-0|+\frac{1}{2}=5>w\left(\bar{a}_2^{14}\right)=3$, then $w\left(\bar{x}_{2j}^{14}\right)=\left|x_{2j}^{23}-x_{2j}^{14}\right|+w\left(\bar{x}_{2j}^{23}\right)$ with $w\left(\bar{x}_{21}^{14}\right)=\frac{9}{4}$ and $w\left(\bar{x}_{22}^{14}\right)=\frac{11}{4}$.

We get $\bar{x}_{21}^{14}=\left[\frac{1}{4},\frac{19}{4}\right]$, $\bar{x}_{22}^{14}=\left[\frac{7}{4},\frac{29}{4}\right]$ and $\bar{x}_{23}^{14}=\lfloor 0,0\rfloor$.

For $i=3$, we have $E_3=\sum_{j=1}^{n}\left|x_{3j}^{14}-x_{3j}^{23}\right|+w\left(\bar{a}_3^{23}\right)=\left|\frac{5}{2}-\frac{3}{2}\right|+|5-5|+|0-0|+\frac{3}{2}=\frac{5}{2}\leq w\left(\bar{a}_3^{14}\right)=\frac{7}{2}$, then $\sum_{x_{3j}^{14}\neq 0} w\left(\bar{x}_{3j}^{14}\right)=w\left(\bar{a}_3^{14}\right)=\frac{7}{2}$ with $w\left(\bar{x}_{32}^{14}\right)=\frac{3}{2}$ and $w\left(\bar{x}_{33}^{14}\right)=2$.

We get $\bar{x}_{31}^{14}=\lfloor 0,0\rfloor$, $\bar{x}_{32}^{14}=\lfloor 1,4\rfloor$ and $\bar{x}_{33}^{14}=\lfloor 3,7\rfloor$.

Step 4. The optimal solution according to the choice of the decision maker is:

$\tilde{x}^*_{ij} = \left(\bar{x}^{14}_{ij} / \bar{x}^{23}_{ij}\right) = \left(\left[x^{*1}_{ij}, x^{*4}_{ij}\right] / \left[x^{*2}_{ij}, x^{*3}_{ij}\right]\right) = \left(x^{*1}_{ij}, x^{*2}_{ij}, x^{*3}_{ij}, x^{*4}_{ij}\right)$. $Min\ \tilde{Z}(\tilde{x}) \approx \left(\frac{81}{4}, 81, 173, \frac{1037}{2}\right)$ with $\tilde{x}^*_{11} = (1, 5, 7, 9)$, $\tilde{x}^*_{12} = \tilde{0}$, $\tilde{x}^*_{13} = \tilde{0}$,

$\tilde{x}^*_{21} = \left(\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{19}{4}\right)$, $\tilde{x}^*_{22} = \left(\frac{7}{4}, \frac{27}{4}, \frac{29}{4}, \frac{29}{4}\right)$, $\tilde{x}^*_{23} = \tilde{0}$, $\tilde{x}^*_{31} = \tilde{0}$, $\tilde{x}^*_{32} = (1, 1, 2, 4)$ and $\tilde{x}^*_{33} = (3, 4, 6, 7)$.

Example 3. [29].

Step 1. Construct the fuzzy transportation *Problem (15)*, and then convert it into a balanced one if it is not.

Table 3. Eq. (15) in triangular balanced form.

	R_1	R_2	Supply ($\tilde{a}_i$)
A	$(22,31,34)$	$(15,19,29)$	$(150,201,246)$
B	$(30,39,54)$	$(8,10,12)$	$(50,99,154)$
Demand ($\tilde{b}_j$)	$(100,150,200)$	$(100,150,200)$	$\sum_{i=1}^{m} \tilde{a}_i = \sum_{j=1}^{n} \tilde{b}_j$

Step 2. Solving *Eq. (17)* via *Eq. (19)*. We get $Min Z^2(x^2) = 31x^2_{11} + 19x^2_{12} + 39x^2_{21} + 10x^2_{22}$ subject to the constraints $x^2_{11} + x^2_{12} = 201$, $x^2_{21} + x^2_{22} = 99$, $x^2_{11} + x^2_{21} = 150$ and $x^2_{12} + x^2_{22} = 150$.

Optimal solution is: $x^2_{11} = 150$, $x^2_{12} = 51$, $x^2_{21} = 0$ and $x^2_{22} = 99$.

Step 3. Solving *Eq. (16)* via *Eq. (18)*. We have $Min Z^{13}(x^{13}) = 28x^{13}_{11} + 22x^{13}_{12} + 42x^{13}_{21} + 10x^{13}_{22}$ subject to the

constraints $x^{13}_{11} + x^{13}_{12} = 198$, $x^{13}_{21} + x^{13}_{22} = 102$, $x^{13}_{11} + x^{13}_{21} = 150$ and $x^{13}_{12} + x^{13}_{22} = 150$.

Optimal solution is: $x^{13}_{11} = 150$, $x^{13}_{12} = 48$, $x^{13}_{21} = 0$ and $x^{13}_{22} = 102$. Furthermore $w\left(\bar{a}^{13}_1\right) = 48$,

$w\left(\bar{a}^{13}_2\right) = 52$.

For $i = 1$, we have $E_1 = \sum_{j=1}^{2} \left|x^{13}_{1j} - x^2_{1j}\right| = |150 - 150| + |48 - 51| = 3 \leq 48 = w\left(\bar{a}^{13}_1\right)$,

then $\sum_{x^{13}_{1j} \neq 0} w\left(\bar{x}^{13}_{1j}\right) = w\left(\bar{a}^{13}_1\right) = 48$ with $w\left(\bar{x}^{13}_{11}\right) = 10$ and $w\left(\bar{x}^{13}_{12}\right) = 38$.

We get $\bar{x}^{13}_{11} = \left[140, 160\right]$ and $\bar{x}^{13}_{12} = \left[10, 86\right]$.

For $i=2$, we have $E_2=\sum_{j=1}^{2}\left|x_{2j}^{13}-x_{2j}^{2}\right|=\left|0-0\right|+\left|102-99\right|=3\leq 52=w\left(\bar{a}_2^{13}\right)$,

then $\sum_{x_{2j}^{13}\neq 0} w\left(\bar{x}_{2j}^{13}\right)=w\left(\bar{a}_2^{13}\right)=52$ with $w\left(\bar{x}_{21}^{13}\right)=0$ and $w\left(\bar{x}_{22}^{13}\right)=52$.

We get $\bar{x}_{21}^{13}=\left\lfloor 0,0\right\rfloor$ and $\bar{x}_{22}^{13}=\left\lfloor 50,154\right\rfloor$.

Step 4. The optimal solution according to the choice of the decision maker is

$Min\tilde{Z}\left(\tilde{x}\right)\approx\sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij}^{1},c_{ij}^{2},c_{ij}^{3}\right)\tilde{x}_{ij}$ with $\tilde{x}_{ij}^{*}=\left(x_{ij}^{2}/\bar{x}_{ij}^{13}\right)=\left(x_{ij}^{2}/\left[x_{ij}^{*1},x_{ij}^{*3}\right]\right)=\left(x_{ij}^{*1},x_{ij}^{*2},x_{ij}^{*3}\right)$. We have

Min $\tilde{Z}^{*}\approx\left(6609/\left[3630,\ 9782\right]\right)=\left(3630,6609,\ 9782\right)$ where

$\tilde{x}_{11}^{*}=\left(140,150,\ 160\right)$, $\tilde{x}_{12}^{*}=\left(10,51,86\right)$,

$\tilde{x}_{21}^{*}=\tilde{0}$ and $\tilde{x}_{22}^{*}=\left(50,99,154\right)$.

5.1| Interpretation of Results

We will now interpret the minimum total fuzzy transportation cost obtained in *Example 3*. by using the proposed methods presented in Section 3. Similarly, the obtained fuzzy optimal solution will also be interpreted. By using the methods proposed the minimum total fuzzy transportation cost is $\left(3630,6609,\ 9782\right)$, which can be physically interpreted as follows:

- *The least amount of the minimum total transportation cost is* 3630.
- *The most possible amount of minimum total transportation cost is* 6609.
- *The greatest amount of the minimum total transportation cost is* 9782 *i.e., the minimum total transportation cost will always be greater than* 3630 *and less than* 6609, *and the highest chances are that the minimum total transportation cost will be* 9782.

Example 4. [1]-[3].

Step 1. Construct the fuzzy transportation *Problem (15)*, and then convert it into a balanced one if it is not.

Step 2. Solving *Eq. (17)* via *Eq. (19)*. We get

$$MinZ^{2}\left(x^{2}\right)=10x_{11}^{2}+22x_{12}^{2}+10x_{13}^{2}+20x_{14}^{2}15x_{21}^{2}+20x_{22}^{2}+12x_{23}^{2}+8x_{24}^{2}+20x_{31}^{2}+12x_{32}^{2}+10x_{33}^{2}+15x_{34}^{2}$$

subject to the constraints $x_{11}^{2}+x_{12}^{2}+x_{13}^{2}+x_{14}^{2}=8$,

$x_{21}^{2}+x_{22}^{2}+x_{23}^{2}+x_{24}^{2}=14$, $x_{31}^{2}+x_{32}^{2}+x_{33}^{2}+x_{34}^{2}=12$, $x_{11}^{2}+x_{21}^{2}+x_{31}^{2}=7$, $x_{12}^{2}+x_{22}^{2}+x_{32}^{2}=10$,

$x_{13}^{2}+x_{23}^{2}+x_{33}^{2}=8$ and $x_{14}^{2}+x_{24}^{2}+x_{34}^{2}=9$.

Optimal solution is: $x_{11}^2 = 7$, $x_{12}^2 = 0$, $x_{13}^2 = 1$, $x_{14}^2 = 0$, $x_{21}^2 = 0$, $x_{22}^2 = 0$, $x_{23}^2 = 5$, $x_{24}^2 = 9$,

$x_{31}^2 = 0$, $x_{32}^2 = 10$, $x_{33}^2 = 2$ and $x_{34}^2 = 0$.

Table 4. Eq. (15) in triangular balanced form (in U.S. dollar).

Source	Destination Taichung	Chiayi	Kaohsiung	Taipei	Supply ($\tilde{a}_i$) (000 dozen bottles)
Changhua	(\$8, \$10, \$10.8)	(\$20.4, \$22, \$24)	(\$8, \$10, \$10.6)	(\$18.8, \$20, \$22)	(7.2,8,8.8)
Touliu	(\$14, \$15, \$16)	(\$18.2, \$20, \$22)	(\$10, \$12, \$13)	(\$6, \$8, \$8.8)	(12,14,16)
Hsinchu	(\$18.4, \$20, \$21)	(\$9.6, \$12, \$13)	(\$7.8, \$10, \$10.8)	(\$14, \$15, \$16)	(10.2,12,13.8)
Demand ($\tilde{b}_j$) (000 dozen bottles)	(6.2,7,7.8)	(8.9,10,11.1)	(6.5,8,9.5)	(7.8,9,10.2)	$\sum_{i=1}^{m} \tilde{a}_i = \sum_{j=1}^{n} \tilde{b}_j$

Step 3. Solving *Eq. (16)* via *Eq. (18)*. We get

$$Min\, Z^{13}\left(x^{13}\right) = 9.4x_{11}^{13} + 22.2x_{12}^{13} + 9.3x_{13}^{13} + 20.4x_{14}^{13} + 15x_{21}^{13} + 20.1x_{22}^{13}$$

$+11.5x_{23}^{13} + 7.4x_{24}^{13} + 19.7x_{31}^{13} + 11.3x_{32}^{13} + 9.3x_{33}^{13} + 15x_{34}^{13}$ subject to the constraints

$x_{11}^{13} + x_{12}^{13} + x_{13}^{13} + x_{14}^{13} = 8$, $x_{21}^{13} + x_{22}^{13} + x_{23}^{13} + x_{24}^{13} = 14$, $x_{31}^{13} + x_{32}^{13} + x_{33}^{13} + x_{34}^{13} = 12$,

$x_{11}^{13} + x_{21}^{13} + x_{31}^{13} = 7$, $x_{12}^{13} + x_{22}^{13} + x_{32}^{13} = 10$, $x_{13}^{13} + x_{23}^{13} + x_{33}^{13} = 8$ and $x_{14}^{13} + x_{24}^{13} + x_{34}^{13} = 9$.

Optimal solution is: $x_{11}^{13} = 7$, $x_{12}^{13} = 0$, $x_{13}^{13} = 1$, $x_{14}^{13} = 0$, $x_{21}^{13} = 0$, $x_{22}^{13} = 0$, $x_{23}^{13} = 5$, $x_{24}^{13} = 9$,

$x_{31}^{13} = 0$, $x_{32}^{13} = 10$, $x_{33}^{13} = 2$ and $x_{34}^{13} = 0$. We have $w\left(\bar{a}_1^{13}\right) = \frac{4}{5}$, $w\left(\bar{a}_2^{13}\right) = 2$ and $w\left(\bar{a}_3^{13}\right) = \frac{9}{5}$.

Furthermore:

- $\bar{x}_{11}^{13} = \left[\frac{34}{5}, \frac{36}{5}\right]$, $\bar{x}_{12}^{13} = \left[0,0\right]$, $\bar{x}_{13}^{13} = \left[\frac{2}{5}, \frac{8}{5}\right]$ *and* $\bar{x}_{14}^{13} = \left[0,0\right]$ *where* $w\left(\bar{x}_{11}^{13}\right) = \frac{1}{5}$, $w\left(\bar{x}_{12}^{13}\right) = 0$, $w\left(\bar{x}_{13}^{13}\right) = \frac{3}{5}$ *and* $w\left(\bar{x}_{14}^{13}\right) = 0$.
- $\bar{x}_{21}^{13} = \left[0,0\right]$, $\bar{x}_{22}^{13} = \left[0,0\right]$, $\bar{x}_{23}^{13} = \left[\frac{9}{2}, \frac{11}{2}\right]$ *and* $\bar{x}_{24}^{13} = \left[\frac{15}{2}, \frac{21}{2}\right]$ *where* $w\left(\bar{x}_{21}^{13}\right) = 0$, $w\left(\bar{x}_{22}^{13}\right) = 0$, $w\left(\bar{x}_{23}^{13}\right) = \frac{1}{2}$ *and* $w\left(\bar{x}_{24}^{13}\right) = \frac{3}{2}$.

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– $\bar{x}_{31}^{13} = \left[0, 0\right]$, $\bar{x}_{32}^{13} = \left[\frac{47}{5}, \frac{53}{5}\right]$, $\bar{x}_{33}^{13} = \left[\frac{4}{5}, \frac{16}{5}\right]$ *and* $\bar{x}_{34}^{13} = \left[0, 0\right]$ *with* $w\left(\bar{x}_{31}^{13}\right) = 0$, $w\left(\bar{x}_{32}^{13}\right) = \frac{3}{5}$, $w\left(\bar{x}_{33}^{13}\right) = \frac{6}{5}$ *and* $w\left(\bar{x}_{34}^{13}\right) = 0$.

Step 4. The optimal solution according to the choice of the decision maker is

The value of the objective function: $Min\ \tilde{Z}\left(\tilde{x}\right) \approx \sum_{i=1}^{m}\sum_{j=1}^{n}\left(c_{ij}^{1}, c_{ij}^{2}, c_{ij}^{3}\right)\left(x_{ij}^{*1}, x_{ij}^{*2}, x_{ij}^{*3}\right) = \left(Z^{*1}, Z^{*2}, Z^{*3}\right)$

with $\tilde{x}_{ij}^{*} = \left(x_{ij}^{2} / \bar{x}_{ij}^{13}\right) = \left(x_{ij}^{2} / \left[x_{ij}^{*1}, x_{ij}^{*3}\right]\right) = \left(x_{ij}^{*1}, x_{ij}^{*2}, x_{ij}^{*3}\right)$.

We have Min $\tilde{Z}^{*} \approx \left(352\$ / \left[\frac{12204}{50}\$, \frac{21549}{50}\$\right]\right) = \left(\frac{12204}{50}\$, 352\$, \frac{21549}{50}\$\right)$ where

$\tilde{x}_{11}^{*} = \left(\frac{34}{5}, 7, \frac{36}{5}\right)$, $\tilde{x}_{12}^{*} = \tilde{0}$, $\tilde{x}_{13}^{*} = \left(\frac{2}{5}, 1, \frac{8}{5}\right)$, $\tilde{x}_{14}^{*} = \tilde{0}$,

$\tilde{x}_{21}^{*} = \tilde{0}$, $\tilde{x}_{22}^{*} = \tilde{0}$, $\tilde{x}_{23}^{*} = \left(\frac{9}{2}, 5, \frac{11}{2}\right)$, $\tilde{x}_{24}^{*} = \left(\frac{15}{2}, 9, \frac{21}{2}\right)$,

$\tilde{x}_{31}^{*} = \tilde{0}$, $\tilde{x}_{32}^{*} = \left(\frac{47}{5}, 10, \frac{53}{5}\right)$, $\tilde{x}_{33}^{*} = \left(\frac{4}{5}, 2, \frac{16}{5}\right)$ and $\tilde{x}_{34}^{*} = \tilde{0}$.

5.2| Interpretation of Results

We will now interpret the minimum total fuzzy transportation cost obtained in *Example 4* by using the proposed methods presented in *Section 3*. Similarly, the obtained fuzzy optimal solution will also be interpreted. By using the methods proposed the minimum total fuzzy transportation cost is

$\left(\frac{12204}{50}\$, 352\$, \frac{21549}{50}\$\right)$, which can be physically interpreted as follows:

- *The least amount of the minimum total transportation cost is* $\frac{12204}{50}\$$.
- *The most possible amount of minimum total transportation cost is* $352\$$.
- *The greatest amount of the minimum total transportation cost is* $\frac{21549}{50}\$$ *i.e., the minimum total transportation cost will always be greater than* $\frac{12204}{50}\$$ *and less than* $352\$$, *and the highest chances are that the minimum total transportation cost will be* $\frac{21549}{50}\$$.

6| A Comparative Study

Author's of [3], [11], [29] have proposed a method to find the crisp optimal solution of such fuzzy transportation problems in which all the parameters are represented by triangular or trapezoidal fuzzy numbers. Then, they have used their new method proposed to find the crisp optimal solution of a real-life fuzzy transportation problem.

However, it is often better to find a fuzzy optimal solution than a crisp optimal solution. In this section we will therefore show how in the problem considered by [3], [11], [29], we can obtain a fuzzy optimal solution of the same real-life problem using the new method proposed.

Example	Minimum Total Fuzzy Transportation Cost		
	[3]	[29]	Method Proposed in This Paper
1	$(2100,2900,3500,4200)$		$(930,2930,2970,3470)$
2		$(31,80,199,460)$	$\left(\frac{81}{4},81,173,\frac{1037}{2}\right)$
3			$(3630,6609,9782)$
4	$(238.44,347.8,428.9)$		$\left(\frac{12204}{50},352,\frac{21549}{50}\right)$

7| Concluding Remarks and Future Research Directions

7.1| Concluding Remarks

These days a number of researchers have shown interest in the area of fuzzy transportation problems and various attempts have been made to study the solution of these problems. In this paper, to overcome the shortcomings of the existing methods we introduced a new formulation of transportation problem involving trapezoidal (or triangular) fuzzy numbers for the transportation costs and values of supplies and demands. We propose a fuzzy linear programming approach for solving trapezoidal (or triangular) fuzzy numbers transportation problem based on the converting into two interval transportation problems *Eq. (16)* and *Eq. (17)*. To show the advantages of the proposed methods over existing methods, some fuzzy transportation problems, may or may not be solved by the existing methods, are solved by using the proposed methods and it is shown that it is better to use the proposed methods as compared to the existing methods for solving the transportation problems. From both theoretical and algorithmic considerations, and examples solved in this paper, it can be noticed that some shortcomings of the methods for solving the fuzzy transportation problems known from the literature can be resolved by using the new methods proposed in Section 5.

7.2| Future Research Directions

Finally, we feel that, there are many other points of research and should be studied later on interval numbers or fuzzy numbers. Some of these points are below:

We will consider the following transportation *Problems (14)* with fuzzy numbers as follows:

$Min\tilde{Z}(\tilde{x}) \approx \sum_{j=1}^{n}\sum_{i=1}^{m}\tilde{c}_{ij}\tilde{x}_{ij}$ subject to the constraints $\sum_{j=1}^{n}\tilde{x}_{ij} \approx \tilde{a}_i$ and $\sum_{i=1}^{m}\tilde{x}_{ij} \approx \tilde{b}_j$ where

$\tilde{c}_{ij} = (c^1_{ij}, c^2_{ij}, \dots, c^t_{ij})$, $\tilde{x}_{ij} = (x^1_{ij}, x^2_{ij}, \dots, x^t_{ij})$, $\tilde{a}_i = (a^1_i, a^2_i, \dots, a^t_i)$ and $\tilde{b}_j = (b^1_j, b^2_j, \dots, b^t_j)$ with

$t \in \mathbb{N}_{\geq 1}$. Let $\bar{x}^{pq}_{ij} = \lfloor x^p_{ij}, x^q_{ij} \rfloor$ where $p \leq q$ and $p, q \in \mathbb{N}_{\geq 1}$. The same applies to $\bar{c}^{pq}_{ij}$, $\bar{a}^{pq}_i$ and $\bar{b}^{pq}_j$.

I. Solution procedure for classical Transportation Problem $(t=1)$: $x_{ij} = (x^1_{ij})$.

II. Solution procedure for Transportation Problem with Interval numbers $(t=2)$: $\bar{x}^{12}_{ij} = \lfloor x^1_{ij}, x^2_{ij} \rfloor$.

III. Solution procedure for Transportation Problem with Triangular fuzzy numbers $(t=3)$: $\tilde{x}_{ij} = (x^1_{ij}, x^2_{ij}, x^3_{ij}) = (x^2_{ij}, \bar{x}^{13}_{ij})$.

IV. Solution procedure for Transportation Problem with Trapezoidal fuzzy numbers $(t=4)$: $\tilde{x}_{ij} = (x^1_{ij}, x^2_{ij}, x^3_{ij}, x^4_{ij}) = (\bar{x}^{23}_{ij}, \bar{x}^{14}_{ij})$.

V. Solution procedure for Transportation Problem with Pentagonal fuzzy numbers $(t=5)$: $\tilde{x}_{ij} = (x^1_{ij}, x^2_{ij}, x^3_{ij}, x^4_{ij}, x^5_{ij}) = (x^3_{ij}, \bar{x}^{24}_{ij}, \bar{x}^{15}_{ij})$

VI. Solution procedure for Transportation Problem with Hexagonal fuzzy numbers $(t=6)$: $\tilde{x}_{ij} = (x^1_{ij}, x^2_{ij}, x^3_{ij}, x^4_{ij}, x^5_{ij}, x^6_{ij}) = (\bar{x}^{34}_{ij}, \bar{x}^{25}_{ij}, \bar{x}^{16}_{ij})$.

VII. Solution procedure for Transportation Problem with Heptagonal fuzzy numbers $(t=7)$ $: \tilde{x}_{ij} = (x^1_{ij}, x^2_{ij}, x^3_{ij}, x^4_{ij}, x^5_{ij}, x^6_{ij}, x^7_{ij}) = (x^4_{ij}, \bar{x}^{35}_{ij}, \bar{x}^{26}_{ij}, \bar{x}^{17}_{ij})$.

VIII. Solution procedure for Transportation Problem with Octagonal fuzzy numbers $(t=8)$ $: \tilde{x}_{ij} = (x^1_{ij}, x^2_{ij}, x^3_{ij}, x^4_{ij}, x^5_{ij}, x^6_{ij}, x^7_{ij}, x^8_{ij}) = (\bar{x}^{45}_{ij}, \bar{x}^{36}_{ij}, \bar{x}^{27}_{ij}, \bar{x}^{18}_{ij})$.

IX. Solution procedure for Transportation Problem with Nonagonal fuzzy numbers $(t=9)$: $\tilde{x}_{ij} = (x^1_{ij}, x^2_{ij}, x^3_{ij}, x^4_{ij}, x^5_{ij}, x^6_{ij}, x^7_{ij}, x^8_{ij}, x^9_{ij}) = (x^5_{ij}, \bar{x}^{46}_{ij}, \bar{x}^{37}_{ij}, \bar{x}^{28}_{ij}, \bar{x}^{19}_{ij})$.

X. Solution procedure for Transportation Problem with Decagonal fuzzy numbers $(t=10)$: $\tilde{x}_{ij} = (x^1_{ij}, x^2_{ij}, x^3_{ij}, x^4_{ij}, x^5_{ij}, x^6_{ij}, x^7_{ij}, x^8_{ij}, x^9_{ij}, x^{10}_{ij}) = (\bar{x}^{56}_{ij}, \bar{x}^{47}_{ij}, \bar{x}^{38}_{ij}, \bar{x}^{29}_{ij}, \bar{x}^{110}_{ij})$.

XI. Solution procedure for Transportation Problem with Hendecagonal fuzzy numbers $(t=11)$: $\tilde{x}_{ij} = (x^1_{ij}, x^2_{ij}, x^3_{ij}, x^4_{ij}, x^5_{ij}, x^6_{ij}, x^7_{ij}, x^8_{ij}, x^9_{ij}, x^{10}_{ij}, x^{11}_{ij}) = (x^6_{ij}, \bar{x}^{57}_{ij}, \bar{x}^{48}_{ij}, \bar{x}^{39}_{ij}, \bar{x}^{210}_{ij}, \bar{x}^{111}_{ij})$.

XII. Solution procedure for Transportation Problem with Dodecagonal fuzzy numbers $(t=12)$ $: \tilde{x}_{ij} = (x^1_{ij}, x^2_{ij}, x^3_{ij}, x^4_{ij}, x^5_{ij}, x^6_{ij}, x^7_{ij}, x^8_{ij}, x^9_{ij}, x^{10}_{ij}, x^{11}_{ij}, x^{12}_{ij}) = (\bar{x}^{67}_{ij}, \bar{x}^{58}_{ij}, \bar{x}^{49}_{ij}, \bar{x}^{310}_{ij}, \bar{x}^{211}_{ij}, \bar{x}^{112}_{ij})$.

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Conflicts of interest

The authors declare no conflicts of interest.

References

[1] Kaur, J., & Kumar, A. (2016). *An introduction to fuzzy linear programming problems theory, methods and applications*. Springer.

[2] Nasseri, S. H., Ebrahimnejad, A., & Cao, B. Y. (2019). Fuzzy linear programming. In *Fuzzy linear programming: solution techniques and applications* (pp. 39-61). Springer, Cham.

[3] Kaur, A., Kacprzyk, J., Kumar, A., & Analyst, C. F. (2020). *Fuzzy transportation and transshipment problems*. Springer International Publishing.

[4] Zimmermann, H. J. (1978). Fuzzy programming and linear programming with several objective functions. *Fuzzy sets and systems*, *1*(1), 45-55.

[5] ÓhÉigeartaigh, M. (1982). A fuzzy transportation algorithm. *Fuzzy sets and systems*, *8*(3), 235-243.

[6] Chanas, S., Kołodziejczyk, W., & Machaj, A. (1984). A fuzzy approach to the transportation problem. *Fuzzy sets and systems*, *13*(3), 211-221.

[7] Chanas, S., Delgado, M., Verdegay, J. L., & Vila, M. A. (1993). Interval and fuzzy extensions of classical transportation problems. *Transportation planning and technology*, *17*(2), 203-218.

[8] Kuchta, D. (1996). A concept of the optimal solution of the transportation problem with fuzzy cost coefficients. *Fuzzy sets and systems*, *82*(299), 299-305.

[9] Jiménez, F., & Verdegay, J. L. (1998). Uncertain solid transportation problems. *Fuzzy sets and systems*, *100*(1-3), 45-57.

[10] Jiménez, F., & Verdegay, J. L. (1999). Solving fuzzy solid transportation problems by an evolutionary algorithm based parametric approach. *European journal of operational research*, *117*(3), 485-510.

[11] Liu, S. T., & Kao, C. (2004). Solving fuzzy transportation problems based on extension principle. *European journal of operational research*, *153*(3), 661-674.

[12] Gani, A. N., & Razak, K. A. (2006). Two stage fuzzy transportation problem. *J. Phys. Sci. 10*, 63–69. http://inet.vidyasagar.ac.in:8080/jspui/handle/123456789/720

[13] Li, L., Huang, Z., Da, Q., & Hu, J. (2008, May). A new method based on goal programming for solving transportation problem with fuzzy cost. *2008 international symposiums on information processing* (pp. 3-8). IEEE.

[14] Lin, F. T. (2009, August). Solving the transportation problem with fuzzy coefficients using genetic algorithms. *2009 IEEE international conference on fuzzy systems* (pp. 1468-1473). IEEE.

[15] Dinagar, D. S., & Palanivel, K. (2009). The transportation problem in fuzzy environment. *International journal of algorithms, computing and mathematics*, *2*(3), 65-71.

[16] Pandian, P., & Natarajan, G. (2010). A new algorithm for finding a fuzzy optimal solution for fuzzy transportation problems. *Applied mathematical sciences*, *4*(2), 79-90.

[17] Kumar, A., & Kaur, A. (2011). Application of linear programming for solving fuzzy transportation problems. *Journal of applied mathematics & informatics*, *29*(3_4), 831-846.

[18] Gupta, A., Kumar, A., & Kaur, A. (2012). Mehar's method to find exact fuzzy optimal solution of unbalanced fully fuzzy multi-objective transportation problems. *Optimization letters*, *6*(8), 1737-1751.

[19] Ebrahimnejad, A. (2015). A duality approach for solving bounded linear programming problems with fuzzy variables based on ranking functions and its application in bounded transportation problems. *International journal of systems science*, *46*(11), 2048-2060.

[20] Shanmugasundari, M., & Ganesan, K. (2013). A novel approach for the fuzzy optimal solution of fuzzy transportation problem. *Transportation*, *3*(1), 1416-1424.

[21] Chandran, S., & Kandaswamy, G. (2016). A fuzzy approach to transport optimization problem. *Optimization and engineering*, *17*(4), 965-980.

[22] Ebrahimnejad, A. (2016). Note on "A fuzzy approach to transport optimization problem". *Optimization and engineering*, *17*(4), 981-985.

[23] Kumar, A., & Kaur, A. (2014). Optimal way of selecting cities and conveyances for supplying coal in uncertain environment. *Sadhana*, *39*(1), 165-187.

[24] Ebrahimnejad, A. (2015). An improved approach for solving fuzzy transportation problem with triangular fuzzy numbers. *Journal of intelligent & fuzzy systems*, *29*(2), 963-974.

[25] Kaur, A., & Kumar, A. (2011). A new method for solving fuzzy transportation problems using ranking function. *Applied mathematical modelling*, *35*(12), 5652-5661.

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[26] Kaur, A., & Kumar, A. (2012). A new approach for solving fuzzy transportation problems using generalized trapezoidal fuzzy numbers. *Applied soft computing*, *12*(3), 1201-1213.

[27] Ebrahimnejad, A. (2014). A simplified new approach for solving fuzzy transportation problems with generalized trapezoidal fuzzy numbers. *Applied soft computing*, *19*, 171-176.

[28] Ramesh, G., & Ganesan, K. (2012). Duality theory for interval linear programming problems. *IOSR journal of mathematics*, *4*(4), 39-47.

[29] Bisht, D. C., & Srivastava, P. K. (2019). One point conventional Model to optimize Trapezoidal Fuzzy Transportation problem. *International journal of mathematical, engineering and management sciences*, *4*(5), 1251-1263.

[30] Kane, L., Sidibe, H., Kane, S., Bado, H., Konate, M., Diawara, D., & Diabate, L. (2021). A simplified new approach for solving fully fuzzy transportation problems with involving triangular fuzzy numbers. *Journal of fuzzy extension and applications*, *2*(1), 89-105.

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Ranking of Different of Investment Risk in High-Tech Projects Using TOPSIS Method in Fuzzy Environment Based on Linguistic Variables

Mohammad Ebrahim Sadeghi[1], Hamed Nozari[2,*], Hadi Khajezadeh Dezfoli[1], Mehdi Khajezadeh Dezfoli[3]

[1] Department of Industrial Management, Faculty of Management, University of Tehran, Tehran, Iran; sadeqi.m.e@gmail.com; mehkhajezadehd1@gmail.com.

[2] Department of Industrial Engineering, Islamic Azad University, Central Tehran Branch, Tehran, Iran; ham.nozari.eng@iauctb.ac.ir.

[3] Allameh Tabataba'i University, Tehran, Iran; hkhdez@gmail.com.

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Abstract

Examining the trend of the global economy shows that global trade is moving towards high-tech products. Given that these products generate very high added value, countries that can produce and export these products will have high growth in the industrial sector. The importance of investing in advanced technologies for economic and social growth and development is so great that it is mentioned as one of the strong levers to achieve development. It should be noted that the policy of developing advanced technologies requires consideration of various performance aspects, risks and future risks in the investment phase. Risk related to high-tech investment projects has a meaning other than financial concepts only. In recent years, researchers have focused on identifying, analyzing, and prioritizing risk. There are two important components in measuring investment risk in high-tech industries, which include identifying the characteristics and criteria for measuring system risk and how to measure them. This study tries to evaluate and rank the investment risks in advanced industries using fuzzy TOPSIS technique based on verbal variables.

Keywords: Investment risk, Advanced technology, Linguistc variables, Fuzzy TOPSIS.

1 | Introduction

Technology is one of the most important sources of changes in societies. Technology creates new solutions to man problem of daily life [1], [2]. So economic development of countries is intertwined with high tech development [3]. Technology has been a key factor in economic progress over the past few centuries and has played an undeniable role in improving and growing production around the world. New technological developments and constantly changing demands of customers have obliged companies to introduce their new or modified products faster [4].

Corresponding Author: ham.nozari.eng@iauctb.ac.ir

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Technological changes improve the production process of goods and services and increase the efficiency of the production process. In fact, technology change is an integral part of economic growth and development, and in other words, the engine of economic growth; As it is considered the most important factor of economic growth in developed countries in the twentieth century. By emergence of a knowledge-based economy, research productivity, knowledge production, technological innovation, and highly skilled manpower have become key determinants of economic growth [5]. A study of the economic condition of the countries of the world shows that 78% of the economic growth of Germany and 76% of the economic growth of France was due to their technological growth. That figure was 50 percent for the United States. That was 50 percent for the United States. One of the most important features of technology is its increasing return to scale and the fact that its transfer is low cost and after its creation and discovery by one firm, other firms use its knowledge spillover [6].

High-techs considered as the source of technological developments and future industrial revolution, therefore, the economic development and growth of a country in the near future depends on it. That is why many countries have empowered themselves in these technologies and try to utilize its advantages by innovation and technology policy [7]. These technologies are in the early stages of development and it is not possible to make accurate predictions about their development process and dimensions [8]. The importance of investing in high-techs for economic and social development is so great that it is mentioned as one of the strong elements for achieving development [9]. But it should be noted that just as paying attention to this can cause economic growth by shaping a virtues cycle, not paying attention to it may lead to economic decline and falling into a vicious cycle. Therefore, economic growth and increasing public welfare in the long term will not be possible without investing in these industries and paying attention to the risks associated with it.

Investment has two components of risk and return, and the relationship between them offers different combinations of investment. On the one hand, investors seek to maximize their return on investment, and on the other hand, they face conditions of uncertainty in the market and industry environment, which makes investing uncertain, therefore, success is not guaranteed. Of every 7 to 10 innovative product concepts, only one will gain commercial success. Also, forty percent of innovative products fail at launch, despite their successful development and passing performance tests [10]. In this regard many researchers expressed that uncertainty affects not only on real economic activity but also on the investment decisions of economic agents [11]. In other words, all investment decisions are based on the relationship between risk and return. However, the high volume of global trade in high-tech products motivates investors to enter these industries. However, it should be noted that the policy of high techs requires consideration of various dimensions and aspects of performance and risks in the investment process. These products require advanced technologies that are changing rapidly. It also requires adequate infrastructure, highly skilled human resources, and strong links between firms and relation between firms and research centers and universities.

In order to invest in these industries, two important factors must be paid special attention: firstly, these industries need high investments and secondly, the investment processes in these industries face complex risks. Risks in high-tech investment projects have dimensions other than financial ones only. There are risks in technological, competitive, managerial aspects and some other risk arising from the presence of asymmetric information [12]. In the process of investing and implementing high-tech projects, events may occur that jeopardize the occurrence, implementation and profitability of the project. Therefore, identifying, analyzing, prioritizing and having a plan to deal with these events, can play an important role in the success of the investment project in high techs. In recent years, researchers have focused on identifying, analyzing, and prioritizing risk in high tech. There are two important components in measuring investment risks in high-tech industries, which include determining the criteria for measuring system risk and how to measure them. In this research, we tried to evaluate and rank the investment risks in high tech industries with fuzzy TOPSIS technique based on linguistic variables.

2 | Literature Review

Studies on the risk of investing in high-tech projects date back to the 1960s. Numerous experimental studies have been performed by Myers and Marquitz. Their studies focused more on financial metrics, but market uncertainty and project technology were not considered [13]. Until in 1970, the criteria of technology, market and management were examined in related research. researchers divided the evaluation criteria into four categories: production, company capacity, environmental factors, and alternative competition. Other scholars Stated that venture capitalists should consider the five areas of skill, technology, production, market, and investment to evaluate a new investment project [14]. Based on the qualitative evaluation criteria, Tyebjee and Bruno [15], for the first time, used a questionnaire method to identify evaluation factors to structure the investment risk assessment model in US projects. They selected the 12 evaluation criteria that investors often referred to in their questionnaires. Fried and Hisrich [16] set 15 initial evaluation criteria and divided them into three areas: strategic thinking, management capacity, and research revenue. Manigart et al. [17] first interviewed and researched a number of venture capital project investors from the United Kingdom, Ireland, Belgium and France. They chose factors that affect investment income and investment risk. The team found that the risk of the firm and the target market management team had the greatest impact on investment risk, production innovation, and expected return on investment. They believed that the general economic situation had the least effect on the rate of return [17]. Chotigeat et al. [18] studied the risk assessment criteria for investment in Sri Lanka and Thailand and found that those countries have their own investment assessment criteria. Sri Lankan empirical studies have shown that venture capitalists first emphasize the future return on investment of the firm and then examine market demand, management team, market growth potential and investment liquidity. Venture capitalists in Thailand, on the other hand, first looked at the capacity of the management team and then at the return on investment. Kaplan and Stromberg's study highlighted the criteria for investment attractiveness (market size, strategy, technical, customer), competitiveness, and the subject matter of investment, which is often of interest to investors [19]. The study of the investment risk of Zutshi et al. [20] in Sri Lanka shows that entrepreneurial personality is the most important factor in evaluating investment and financial factors are the least important. To create an investment analysis system, researchers examined investment risks, which resulted in the formation of a 12-factor valuation system that falls into five areas: product differentiation, market attractiveness, capacity, and management capacity. Economic effectiveness, and environmental impact. Given the real situation in China where investing companies could not withdraw their capital, the investment risk decision model proposed by previous researchers modified and a new factor called the exit strategy, which included two exit strategies added to it. These two factors included the degree of difficulty or ease of withdrawing capital and the method of withdrawing capital. Some scholars proposed a combined assessment system that includes three subsystems: environmental assessment, risk assessment, and economic income assessment, and proposed 50 criteria and factors. Han [21] proposed six indicators of technical risk, production risk, market risk, operational risk, financial risk, and environmental risk, which include 26 two-tier criteria. Qiu-bai [22] divide the investment risks in the project into systematic and non-systematic risks. Systematic risks include political and social risks, economic risks and legal risks. Unsystematic risks include technical risk, production risk, management risk, financial risk and market risk.

Um and Kim [23] express task uncertainty as one of the most important uncertainties in innovative projects. They identified three major causes of innovation project task uncertainty which include: product complexity, technological novelty, and task interdependence. Fanousse et al. [24] by integrating the previous studies, indentify 11 main uncertainties in innovation projects. Three of them are the main uncertainties, emphasized by most scholars that are technological, market and task. Other 8 identified uncertainties are perceived environmental, regulatory/institutional, perceived social, collaboration, organisational, decision, financial, operational and ontological. Also, it is obvious that there are many differences between the environments and systems of different regions in different countries. Therefore, investment evaluation criteria are not the same in different countries [13].

Scholars used fuzzy mathematical methods and the Analytic Hierarchy Process to create a quantitative model for evaluating investment risks in high-tech projects [25]. Scholars used fuzzy mathematical methods and the Analytic Hierarchy Process to create a quantitative model for evaluating investment risks in high-tech projects [13]. Tang et al. [26] by considering mutual influence between factors use fuzzy network analysis to evaluate risks in urban rail transit projects. Zhao and Li [27] in their study of key risk factors of Ultra-High Voltage projects suggest a risk index structure. Their research is based on a cloud model and Fuzzy Comprehensive Evaluation (FCE) method. They combine the superiority of the cloud model for reflecting randomness and discreteness with the advantages of the fuzzy comprehensive evaluation method in handling uncertain and vague issues. Wang et al. [28] in their study of risk ranking of energy performance contracting project, develop a multi-criteria decision-making framework under the picture fuzzy environment model. In order to create a risk assessment framework, Wu et al. [29] use linguistic hesitant fuzzy sets based cloud model, for seawater pumped hydro storage project under three typical public-private partnership management modes. Baylan [30] combined AHP and TOPSIS methods to develop a multi-criteria based decision method that prioritizes project risks at the activity level.

In Iran, in the last two decades, many high techs projects have been launched. Both public and private sectors made a lot of investments in advanced technologies. However, many studies investigate risks in projects (for example see [31]), few studies appear to identify and rank investment risks in high-tech industries (for example see [32]). Of course, many positive moves have been made in the country to support innovative projects, such as the establishment of technology parks ([33], [34]) and venture capital funds, but there is still a gap in studies related to the identification and ranking of investment risks in the field of high technologies.

3 | Theoretical Framework of Research Method

There are various methods for ranking factors in different studies. The most famous method is the family of Multi-Criteria Decision Making (MCDM). The MCDM methods are among the best methods in dealing with decision-making problems [35]. Multi-criteria decision-making techniques are divided into two categories: MODM like TOPSIS, AHP [36] and MADM) like SAW [37]. MCDM techniques and group decision making have a wide range of applications in the literature and allow managers and decision makers to evaluate options in different dimensions. MCDM includes various techniques such as TOPSIS, AHP, etc. These methods are widely used due to their practicality, so that today their use has spread to all fields and disciplines [38].

The TOPSIS technique is one of the most popular classic MCDM techniques first introduced by Hwang and Yoon [39]. The basic logic of TOPSIS is the definition of ideal and anti-ideal solution. The ideal solution is a solution that maximizes positive criteria and minimizes negative criteria. the ideal solution contains all the best values of the available criteria, while the anti-ideal solution is a combination of the worst values of the available criteria. The optimal option is the option that has the shortest distance from the ideal solution and the longest distance from the anti-ideal solution [40]. Because TOPSIS is a popular method for classical MCDM problems, many researchers use it to solve ranking and prioritization problems. In fact, TOPSIS is a practical method that compares alternatives according to their values in each criterion and the weight of the criterion [41].

However, in many cases it is not possible to measure values with any particular degree of accuracy. Hence, inaccuracies occur in the information obtained. The sources of imprecision are unquantifiable information, incomplete information, non-obtainable information, and partial ignorance [42]. It should be noted that the problem is not a lack of information, but there is uncertainty in the information. Such uncertainty can be formulated with non-random intervals. In fact, these uncertainties can be easily modeled with fuzzy sets. [43]

The fuzzy logic method was first proposed by Zadeh [44]. There are many inaccurate concepts around us that are expressed on a daily basis in the form of various phrases. Fuzzy logic is a new process that replaces the methods that require advanced and sophisticated mathematics to design and model a system with linguistic quantities and expert knowledge [44]. Zadeh argues that humans do not have a lot of accurate information inputs, but are able to perform adaptive control extensively [45]. In fact, fuzzy logic provides an easy way to reach a definite result based on incomplete, erroneous, ambiguous, vague input information. In this regard, in the present study, TOPSIS technique in fuzzy environment is used to prioritize the factors affecting the investment risk in high-tech industries.

In the classical TOPSIS method, accurate and definite numerical values are used to rank the alternatives and determine the weight of each criterion. But it is not always possible for decision makers to express their thoughts and decisions accurately and quantitatively, so they use linguistic variables such as good, bad, poor, etc. to reflect their opinions. In such cases, it is possible to use the theory of fuzzy sets to express the views and evaluate the opinions of decision makers. The fuzzy TOPSIS algorithm is one of the efficient algorithms in the category of multi-criteria decision making problems in which the elements of the decision matrix or the weight of the criteria or both are expressed by linguistic variables. The important point in the ranking process is that the metrics of this model are expressed in terms of subjective, qualitative and linguistic variables [46].

Fuzzy set theory and fuzzy logic as mathematical theories are a very efficient and useful tool for modeling and formulating mathematical ambiguity and inaccuracy in human cognitive processes. Fuzzy set theory provides tools that mathematical formulate human reasoning and decision making mathematically, so these mathematical models can be used in a variety of fields of science and technology [47]. Using fuzzy concepts, evaluators can use verbal expressions in colloquial natural language to evaluate effective factors, and by linking these expressions to appropriate membership functions, analysis of scores and components will be more appropriate and accurate [48].

3.1 | Overview of TOPSIS Method and Fuzzy Calculations

In a multi-criteria decision making problem with m options and n criteria and using triangular fuzzy numbers, the following steps are used to rank the options[35], [47], [49]- [51]:

Step 1. creating decision matrix.

In the first step, we create the decision matrix according to the criteria and options:

$$\widetilde{D}=\begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \cdots & \tilde{x}_{1n} \\ \tilde{x}_{21} & \tilde{x}_{22} & \cdots & \tilde{x}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{x}_{m1} & \tilde{x}_{m2} & \cdots & \tilde{x}_{mn} \end{bmatrix} \quad (1)$$

Where $\tilde{x}_{ij}$ is a triangular fuzzy number corresponding to the ith option according to the j_{th} criterion and $i = 1,2,\dots, m$ and $j = 1,2,\dots, n$.

If there are K decision makers and the fuzzy ranking of the kth decision maker for $i = 1,2,\dots, m$ and $j = 1,2,\dots, n$ as a triangular fuzzy number is $\tilde{x}_{ijk} = a_{ijk}, b_{ijk}, c_{ijk}$ then the combined fuzzy ranking of decision makers' opinions about options is $\tilde{x}_{ij} = a_{ij}, b_{ij}, c_{ij}$ and can be obtained based on the following relationships:

$$a_{ij} = \mathrm{Min}_K\{a_{ijk}\},$$
$$b_{ij} = \frac{\sum_{k=1}^{K} b_{ijk}}{K}, \quad (2)$$
$$c_{ij} = \mathrm{Max}_K\{c_{ijk}\}.$$

Table 1. Relation of triangular numbers with linguistic variables.

Linguistic Variables	Fuzzy Triangular Numbers According to Linguistic Variables
Very low	(1, 1, 3)
weak	(1, 3, 5)
medium	(3, 5, 7)
high	(5, 7, 9)
Very high	(7, 9, 9)

Step 2. Determining the weighted criteria matrix.

If we consider the fuzzy matrix $\widetilde{W}=[\widetilde{w}_1, \widetilde{w}_2, \ldots, \widetilde{w}_n]$ as the weighted criteria matrix and $\widetilde{w}_j$ be a triangular fuzzy number as $\widetilde{w}_j = w_{j1}, w_{j2}, w_{j3}$ .and If the minimum number of decision makers is equal to K and the Kth decision maker's significance coefficient be a triangular fuzzy number as $\widetilde{w}_{jk} = w_{jk1}, w_{jk2}, w_{jk3}$ for $j = 1,2,\ldots, n$, then the combined fuzzy ranking $\widetilde{w}_j = w_{j1}, w_{j2}, w_{j3}$can be obtained by using the following equations:

$$w_{j1}= \mathrm{Min}_K\{w_{jk1}\},$$

$$w_{j2}=\frac{\sum_{k=1}^{K} w_{jk2}}{K}, \tag{3}$$

$$w_{j3}= \mathrm{Max}_K\{w_{jk3}\}.$$

Step 3. make the decision matrix dimensionless.

In this step, linear scale transformation is used to make the fuzzy decision matrix dimensionless so the comparison of different options is comparable. The components of the dimensionless decision matrix for positive and negative criteria are calculated from the following equations, respectively:

$$\tilde{r}_{ij}=\left(\frac{a_{ij}}{c_j^*}, \frac{b_{ij}}{c_j^*}, \frac{c_{ij}}{c_j^*}\right),$$

$$\tilde{r}_{ij}=\left(\frac{a_j^-}{c_{ij}}, \frac{a_j^-}{b_{ij}}, \frac{a_j^-}{a_{ij}}\right), \tag{4}$$

$$c_j^*=\mathrm{Max}_i c_{ij},$$

$$a_j^-=\mathrm{Min}_i a_{ij}.$$

According to the above steps, the dimensionless fuzzy decision matrix is obtained as follows:

$$\tilde{R}=[\tilde{r}_{ij}]_{m\times n} \ ,i=1,2,\ldots,m \ ; j=1,2,\ldots,n$$

$$\tilde{R}=\begin{bmatrix} \tilde{r}_{11} & \tilde{r}_{12} & \cdots & \tilde{r}_{1n} \\ \tilde{r}_{21} & \tilde{r}_{22} & \cdots & \tilde{r}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{r}_{m1} & \tilde{r}_{m2} & \cdots & \tilde{r}_{mn} \end{bmatrix}. \tag{5}$$

Where m and n represent the number of options and the number of criteria, respectively.

Step 4. Obtain the weighted decision matrix.

The weighted decision matrix is calculated by multiplying the coefficient of significance related to each of the criteria in the fuzzy scaleless matrix and the calculation method for the positive and negative criteria is as follows:

$$\tilde{v}_{ij}=\tilde{r}_{ij}\,.\,\widetilde{w}_j=\left(\frac{a_{ij}}{c_j^*},\frac{b_{ij}}{c_j^*},\frac{c_{ij}}{c_j^*}\right).\,(w_{j1},w_{j2},w_{j3})=\left(\frac{a_{ij}}{c_j^*}.\,w_{j1},\frac{b_{ij}}{c_j^*}.\,w_{j2},\frac{c_{ij}}{c_j^*}.\,w_{j3}\right).$$

$$\tilde{v}_{ij}=\tilde{r}_{ij}\,.\,\widetilde{w}_j=\left(\frac{a_j^-}{c_{ij}},\frac{a_j^-}{b_{ij}},\frac{a_j^-}{a_{ij}}\right).\,(w_{j1},w_{j2},w_{j3})=\left(\frac{a_j^-}{c_{ij}}.\,w_{j1},\frac{a_j^-}{b_{ij}}.\,w_{j2},\frac{a_j^-}{a_{ij}}.\,w_{j3}\right).\qquad(6)$$

Where $\widetilde{w}_j$ is the significance factor of the criterion *j*.

According to the above, we will have:

$$\widetilde{V}=[\tilde{v}_{ij}]_{m\times n}\ ,i=1,2,\dots,m\ ;j=1,2,\dots,n$$

$$\widetilde{V}=\begin{bmatrix}\tilde{v}_{11} & \tilde{v}_{12} & \cdots & \tilde{v}_{1n}\\ \tilde{v}_{21} & \tilde{v}_{22} & \cdots & \tilde{v}_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ \tilde{v}_{m1} & \tilde{v}_{m2} & \cdots & \tilde{v}_{mn}\end{bmatrix}.\qquad(7)$$

Step 5. Determine the positive and negative ideal solution.

Positive ideal and negative ideal solutions are defined as follows:

$$A^*=\{\tilde{v}_1^*,\tilde{v}_2^*,\dots,\tilde{v}_n^*\},$$

$$A^-=\{\tilde{v}_1^-,\tilde{v}_2^-,\dots,\tilde{v}_n^-\}.\qquad(8)$$

Where $\tilde{v}_i^*$ is the best value of criterion i among the options and $\tilde{v}_i^-$ is the worst value of criterion *i* among all available options. In fact, in this step we want to find the best and worst possible option.

Step 6. Distance from the ideal positive and negative fuzzy solution.

These distances can be calculated according to the following equations:

$$S_i^*=\sum_{j=1}^{n}d(\tilde{v}_{ij},\tilde{v}_j^*)\quad,\quad i=1,2,\dots,m$$

$$S_i^-=\sum_{j=1}^{n}d(\tilde{v}_{ij},\tilde{v}_j^-)\quad,\quad i=1,2,\dots,m\qquad(9)$$

Thus, the distance between two triangular fuzzy numbers (a_1,b_1,c_1) and (a_2,b_2,c_2) is calculated as follows:

$$d_v(\widetilde{M}_1,\widetilde{M}_2)=\sqrt{\frac{1}{3}[(a_1-a_2)^2+(b_1-b_2)^2+(c_1-c_2)^2]}.\qquad(10)$$

Step 7. Calculate the similarity index.

Similarity index can be calculated according to the following equation:

$$CC_i=\frac{S_i^-}{S_i^-+S_i^*}\qquad(11)$$

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Step 8. Ranking the options.

At this point the options are ranked according to the CC_i values. So, the options that have a higher similarity index will have better rankings.

4 | Methods and Results of Research

The present study seeks to identify and prioritize significant risks in investing in high-tech industries. Therefore, in line with the objectives of the research, first by reviewing the literature and measures done in other countries, as well as using the opinions of experts, thirty factors affecting investment risk in high-tech industries are identified and classified into six general categories including financial risk and technology risk, production risk, market risk, management risk, and environment risk.

Finally, these factors and components through a questionnaire and obtaining the opinions of experts (30 people) were measured by verbal variables. The experts have a master or PhD degree in financial management, metallurgical engineering, industrial engineering, chemical engineering (nanotechnology) and financial engineering from Tehran University, Iran University of Science and Technology, Amirkabir University, Tarbiat Modarres University, Allameh Tabatabai University and Economic Sciences University.

Also, some Experts of the capital and financial market activists of the country were included to the survey. It is necessary to explain that the selection of the number of people from each specialty as well as the type of specialization is based on the nature of high-tech projects and the type of relevant questions.

Table 2. Identified risks in investing in high-tech projects.

Type of Risk	
financial risk	Financial capability Ability to raise production capital Change in interest rates Change the exchange rate Capital market volume
Technology risk	Technological advantage Technological maturity Reliability of technology Alternative technology Professional work experience
Production risk	How difficult or easy it is to work with technology how standard the equipment and production process are Employee decisions Raw material supply capacity Raw material prices
Market risk	Product life cycle Capacity and time of admission Product competitiveness Potential rival effect Marketing capability Network readiness
Management risk	New technology acceptance network Quality and experience of managers The ease of obtaining information The rate of use of collective wisdom Project management mechanism
Environment risk	The desirability of legal environment policies Macroeconomic environment desirability Favorable social environment the environment condition

In the next step, the information obtained from the questionnaires was extracted and after entering them in EXCEL software and performing the fuzzy TOPSIS algorithm, the following results were extracted:

Table 3. The values of the ideals.

Positive Ideal	(1,1,1)
Negative Ideal	(0/111,0/111,0/111)

Table 4. Results extracted from questionnaires.

Types of Risks	Fuzzy Average of Expert Opinions	The Distance from The Positive Ideal	The Distance from The Negative Ideal	Similarity Index	Ranking
Financial capability	(3,7/133,9)	0.403	0.659	0.62	1
Ability to raise production capital	(3,6/866,9)	0.408	0.648	0.613	2
Change in interest rates	(1,6/266,9)	0.542	0.614	0.531	8
Change the exchange rate	(1,5/6,9)	0.557	0.592	0.515	11
Capital market volume	(1,4/466,9)	0.59	0.559	0.486	21
Technological advantage	(1,5/466,9)	0.561	0.587	0.511	12
Technological maturity	(1,5/066,9)	0.572	0.575	0.501	16
Reliability of technology	(1,5/066,9)	0.572	0.575	0.501	16
Alternative technology	(3,6/866,9)	0.408	0.648	0.613	2
Professional work experience	(1,6/4,9)	0.539	0.619	0.534	6
How difficult or easy it is to work with technology	(1,5/066,9)	0.572	0.575	0.501	17
how standard the equipment and production process are	(1,5/4,9)	0.562	0.585	0.51	13
Employee decisions	(1,4/666,9)	0.583	0.564	0.491	19
Raw material supply capacity	(1,5/133,9)	0.57	0.577	0.503	16
Raw material prices	(1,5/066,9)	0.572	0.575	0.501	16
Product life cycle	(1,6/2,9)	0.543	0.611	0.529	9
Capacity and time of admission	(1,5/866,9)	0.551	0.6	0.521	10
Product competitiveness	(1,6/533,9)	0.537	0.623	0.537	5

Table 4. (Continued).

Types of Risks	Fuzzy Average of Expert Opinions	The Distance from The Positive Ideal	The Distance from The Negative Ideal	Similarity Index	Ranking
Potential rival effect	(1,6/266,9)	0.542	0.614	0.531	8
Marketing capability	(1,6/6,9)	0.535	0.626	0.539	4
Network readiness	(1,5/266,9)	0.566	0.581	0.506	14
New technology acceptance network	(1,5,9)	0.573	0.573	0.5	18
Quality and experience of managers	(1,6/333,9)	0.541	0.616	0.532	7
The ease of obtaining information	(1,5/2,9)	0.568	0.579	0.505	15
The rate of use of collective wisdom	(1,5,9)	0.573	0.575	0.501	17
Project management mechanism	(1,5/2,9)	0.568	0.579	0.504	15
The desirability of legal environment policies	(1,5/2,9)	0.568	0.579	0.504	15
Macroeconomic environment desirability	(1,6/733,9)	0.533	0.631	0.542	3
Favorable social environment	(1,4/533,9)	0.588	0.56	0.487	20
the environment condition	(1,3/333,9)	0.629	0.534	0.459	22

In the final stage, using the fuzzy average method and the above information, six risk categories were ranked and the following results were obtained:

Table 5. Ranking of six risks.

Types of Risks Studied	Fuzzy Average of Expert Opinions	The Distance from the Positive Ideal	The Distance from the Negative Ideal	Similarity Index	Ranking
financial risk	(1,6/066,9)	0/546	0/607	0/526	1
Technology risk	(1,5/773,9)	0/553	0/597	0/519	3
Production risk	(1,5/066,9)	0/571	0/575	0/502	5
Market risk	(1,5/85,9)	0/551	0/6	0/521	2
Management risk	(1,5/433,9)	0/562	0/586	0/510	4
Environment risk	(1,4/95,9)	0/575	0/572	0/499	6

5 | Conclusion

Today, the trend of the global economy reflects the trend of trade towards products with advanced technologies. Naturally, countries that can produce and export these products will have high growth in the industrial sector, and on the contrary, neglecting it can cause economic decline in the future. Given the growing importance of these products in world trade, it is necessary to move to expand investment in these technologies in the country. The findings of this study show that investing in high-tech industries faces many risks. If we divide these risks into six categories: financial risk, market risk, environmental risk, technology risk, production risk and management risk, based on the opinion of elites and experts and using the scientific method of TOPSIS evaluation model in fuzzy environment based on verbal variables. The degree of importance of these risks in the country is in the form of financial risk, market risk, technology risk, management risk, production risk and finally environmental risk. This ranking and information reflect the opinion of the elites about the state of the country to invest in high-tech projects. Naturally, paying attention to it is necessary for principled planning and accurate policy-making in the country, as well as the awareness of domestic and foreign investors about the investment climate in the country.

Conflicts of Interest

All co-authors have seen and agree with the contents of the manuscript and there is no financial interest to report. We certify that the submission is original work and is not under review at any other publication.

References

[1] Mousakhani, M., Saghafi, F., Hasanzadeh, M., & Sadeghi, M. E. (2020). Proposing dynamic model of functional interactions of iot technological innovation system by using system dynamics and fuzzy DEMATEL. *Journal of operational research and its applications*, *17*(4), 1-21.

[2] Fallah, M., Sadeghi, M. E., & Nozari, H. (2021). Quantitative analysis of the applied parts of Internet of Things technology in Iran: an opportunity for economic leapfrogging through technological development. *Science and technology policy Letters*. (**In Persian**). http://stpl.ristip.sharif.ir/?_action=article&keywords=%D9%81%D9%84%D8%A7%D8%AD

[3] Mousakhani, M., Saghafi, F., Hasanzade, M., & Sadeghi, M. E. (2020). Presenting a policy framework for high technologies, using Identification of factors affecting the development of a technological innovation system with meta-synthesis. *Journal of decisions and operations research*, *5*(1), 13-27. (**In Persian**). DOI: 10.22105/dmor.2020.221888.1138

[4] Elyasi, M., Mohammadi, M., Baharloo, M., Khosropour, H., & Taheri, Z. (2021). Developing a knowledge management`s framework for identification of success factors in the product acquisition cycles - case of aviation industries organization. *International journal of innovation in engineering*, 1(1) 76-100.

[5] Shin, J. C., Li, X., Byun, B.-K., & Nam, I. (2020). Building a coordination system of HRD ,research and industry for knowledge and technology-driven economic development in South Asia. *International journal of educational development*, *74*, 102161. DOI: 10.1016/j.ijedudev.2020.102161

[6] Brown, M. M., Bryan, R., & Conley, J. (1999). Database protection in a digital world. *Richmond journal of law and technology*, *6*(1). https://scholarship.richmond.edu/jolt/vol6/iss1/4

[7] Ganda, F. (2019). The impact of innovation and technology investments on carbon emissions in selected organisation for economic Co-operation and development countries. *Journal of cleaner production*, *217*, 469-483. https://doi.org/10.1016/j.jclepro.2019.01.235

[8] Choudry, S. A., Sandmann, S., & Landgrebe, D. (2018). A methodical approach for an economic assessment of joining technologies under risk – optimized decision-making in automobile body development. *Procedia CIRP*, *69*, 31-36. https://doi.org/10.1016/j.procir.2017.11.045

[9] Chen, Y., Wang, Y., Hu, D., & Zhou, Z. (2020). Government R&D subsidies, information asymmetry, and the role of foreign investors: Evidence from a quasi-natural experiment on the shanghai-hong kong stock connect. *Technological forecasting and social change*, *158*, 120162. https://doi.org/10.1016/j.techfore.2020.120162

[10] Cooper, R. G. (2019). The drivers of success in new-product development. *Industrial marketing management*, *76*, 36-47. https://doi.org/10.1016/j.indmarman.2018.07.005

[11] Nartea, G. V., Bai, H., & Wu, J. (2020). Investor sentiment and the economic policy uncertainty premium. *Pacific-Basin finance journal*, *64*, 101438. https://doi.org/10.1016/j.pacfin.2020.101438

[12] Joshi, K. (2018). Managing the risks from high-tech Investments in India: differential strategies of foreign and domestic venture capital firms. *Journal of global entrepreneurship research, 8*(1), 21. DOI: 10.1186/s40497-018-0106-6

[13] Liu, P., Zhang, X., & Liu, W. (2011). A risk evaluation method for the high-tech project investment based on uncertain linguistic variables. *Technological forecasting and social change, 78*(1), 40-50. https://doi.org/10.1016/j.techfore.2010.04.011

[14] Park, W., & Maillie, J. (1982). *Strategic analysis for venture evaluations: the save approach to business decisions*. New York Van Nostrand Reinhold.

[15] Tyebjee, T. T., & Bruno, A. V. (1984). A model of venture capitalist investment activity. *Management science, 30*(9), 1051-1066. DOI: 10.1287/mnsc.30.9.1.51

[16] Fried, V. H., & Hisrich, R. D. (1994). Toward a model of venture capital investment decision making. *Financial management, 23*(3), 28-37. DOI: 10.2307/3665619

[17] Manigart, S., Wright, M., Robbie, K., Desbrières, P., & De Waele, K. (1997). Venture capitalists' appraisal of investment projects: an empirical european study. *Entrepreneurship theory and practice, 21*(4), 29-43. DOI: 10.1177/104225879702100403

[18] Chotigeat, T., Pandey, I. M., & Kim, D. (1997). Venture capital evaluation in emerging markets. *Multinational business review, 5*, 54-62 .

[19] Kaplan, S. N., & Strömberg, P. (2003). Financial contracting theory meets the real world: an empirical analysis of venture capital contracts. *The review of economic studies, 70*(2), 281-315.

[20] Zutshi, R. K., Tan, W. L., Allampalli, D. G., & Gibbons, P .G. (1999). Singapore Venture Capitalists (VCs) investment evaluation criteria: a re-examination. *Small business economics, 13*(1), 9-26.

[21] Han, J. X., & Ma, L. (2001). Analysis and measurement of venture of investment into high technology projects. *J. Harbin Inst. Technol*, *3*(7), 300-303.

[22] Shan, X. L., & Sun, Q. B. (2008). Risk analysis and evaluation on investment in high-new-tech project. *Journal of university of science and technology liaoning*, *5*, 014. https://en.cnki.com.cn/Article_en/CJFDTotal-ASGT200805014.htm

[23] Um, K.-H., & Kim, S.-M. (2018). Collaboration and opportunism as mediators of the relationship between NPD project uncertainty and NPD project performance. *International journal of project management, 36*(4), 659-672.

[24] Fanousse, R. I., Nakandala, D., & Lan, Y.-C. (2021). Reducing uncertainties in innovation projects through intra-organisational collaboration: a systematic literature review. *International journal of managing projects in business, ahead-of-print*(ahead-of-print). DOI: 10.1108/IJMPB-11-2020-0347

[25] Wengao, L. (2005). Comprehensive evaluation model for the investing risk of using artificial neural network in the hi-tech projects. *Technology and innovation management*, *26*(3), 8-11.

[26] Tang, W., Zhang, F., Ye, Z., & Li, J. (2011). Research on risk evaluation in urban rail transit project. *Procedia engineering, 15*, 5094-5099. https://doi.org/10.1016/j.proeng.2011.08.945

[27] Zhao, H., & Li, N. (2015). Risk evaluation of a UHV power transmission construction project based on a cloud model and FCE method for sustainability. *Sustainability, 7*(3), 2885-2914.

[28] Wang, L., Peng, J.-j., & Wang, J.-q. (2018). A multi-criteria decision-making framework for risk ranking of energy performance contracting project under picture fuzzy environment. *Journal of cleaner production, 191*, 105-118.

[29] Wu, Y., Zhang, T., Chen, K., & Yi, L. (2020). A risk assessment framework of seawater pumped hydro storage project in China under three typical public-private partnership management modes. *Journal of energy storage, 32*, 101753. https://doi.org/10.1016/j.est.2020.101753

[30] Baylan, E. B. (2020). A novel project risk assessment method development via AHP-TOPSIS hybrid algorithm .*Emerging science journal, 4*(5), 390-410.

[31] Dehgan Tezerjani, H., Ramazani Khorshid Doost, R., & Khajezadeh Dezfoli, M. (2021). Investigating Iran Housing market cycles and possible scenarios in short-term and midterm horizon using the system dynamics. *International journal of innovation in engineering, 1*(1), 120-133.

[32] Shirazi, F. (2021). Managing portfolio's risk for improving quality in a project oriented manufacture. *International journal of innovation in engineering, 1*(1), 55-63.

[33] Sadeghi, M. E., & Sadabadi, A. A. (2015). Evaluating science parks capacity to create competitive advantages: comparison of pardis technology park and sheikh bahaei science and technology park in Iran. *International journal of innovation and technology management*, *12*(06), 1550031. https://doi.org/10.1142/S0219877015500315

[34] Kaviani, M., Fakhrehosseini, S. (2020). Application of fuzzy logic in portfolio management: evidence from Iranian researches. *Journal of fuzzy extension and applications*, *1*(2), 115-118.

[35] Sagnak, M., Berberoglu, Y., Memis, İ., & Yazgan, O. (2021). Sustainable collection center location selection in emerging economy for electronic waste with fuzzy Best-Worst and fuzzy TOPSIS. *Waste management*, *127*, 37-47.

[36] Nozari, H., Sadeghi, M. E., Eskandari, J., & Ghorbani, E .(2012) .Using integrated fuzzy AHP and fuzzy TOPSIS methods to explore the impact of knowledge management tools in staff empowerment (Case study in knowledge-based companies located on science and technology parks in Iran). *International journal of information, business and management*, *4*(2), 75-92 .

[37] Sadeghi, M. E., Khodabakhsh, M., Ganjipoor, M. R., Kazemipoor, H., & Nozari, H. (2021). A new multi objective mathematical model for relief distribution location at natural disaster response phase. *International journal of innovation in engineering*, *1*(1), 29-54.

[38] Solangi, Y. A., Longsheng, C., & Shah, S. A. A. (2021). Assessing and overcoming the renewable energy barriers for sustainable development in Pakistan: An integrated AHP and fuzzy TOPSIS approach. *Renewable energy*, *173*, 209-222.

[39] Hwang, C. L., & Yoon, K. (1981). Methods for multiple attribute decision making. In *Multiple attribute decision making* (pp. 58-191). Springer, Berlin, Heidelberg.

[40] Varatharajulu, M., Duraiselvam, M., Kumar, M. B., Jayaprakash, G., & Baskar, N. (In Press). Multi criteria decision making through TOPSIS and COPRAS on drilling parameters of magnesium AZ91. *Journal of magnesium and alloys*. https://doi.org/10.1016/j.jma.2021.05.006

[41] Yu, D., & Pan, T. (2021). Tracing knowledge diffusion of TOPSIS: A historical perspective from citation network. *Expert systems with applications*, *168*, 114238. https://doi.org/10.1016/j.eswa.2020.114238

[42] Aliahmadi, A., Sadeghi, M. E., Nozari, H., Jafari-Eskandari, M., & Najafi, S. E. (2015). Studying key factors to creating competitive advantage in science park. *Proceedings of the ninth international conference on management science and engineering management* (pp. 977-987). Springer, Berlin, Heidelberg. https://doi.org/10.1007/978-3-662-47241-5_82

[43] Mathew, M., Chakrabortty, R. K., & Ryan, M. J. (2020). A novel approach integrating AHP and TOPSIS under spherical fuzzy sets for advanced manufacturing system selection. *Engineering applications of artificial intelligence*, *96*, 103988. https://doi.org/10.1016/j.engappai.2020.103988

[44] Zadeh, L. A. (1965). Fuzzy sets. *Information and control*, *8*(3), 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X

[45] Zadeh, L. A. (1975). The concept of a linguistic variable and its application to approximate reasoning—I. *Information sciences*, *8*(3), 199-249. https://doi.org/10.1016/0020-0255(75)90036-5

[46] Alkan, N., & Kahraman, C. (2021). Evaluation of government strategies against COVID-19 pandemic using q-rung orthopair fuzzy TOPSIS method. *Applied soft computing*, *110*, 107653. https://doi.org/10.1016/j.asoc.2021.107653

[47] Salih, M. M., Zaidan, B. B., Zaidan, A. A., & Ahmed, M. A .(2019) .Survey on fuzzy TOPSIS state-of-the-art between 2007 and 2017. *Computers & operations research*, *104*, 207-227.

[48] Rajak, M., & Shaw, K. (2019). Evaluation and selection of mobile health (mHealth) applications using AHP and fuzzy TOPSIS. *Technology in society*, *59*, 101186. https://doi.org/10.1016/j.techsoc.2019.101186

[49] Marzouk, M., & Sabbah, M. (2021). AHP-TOPSIS social sustainability approach for selecting supplier in construction supply chain. *Cleaner environmental systems*, *2*, 100034. https://doi.org/10.1016/j.cesys.2021.100034

[50] Jafar, M., Khan, F., Naveed, A. (2020). Prediction of Pakistan super league-2020 using TOPSIS and fuzzy TOPSIS methods. *Journal of fuzzy extension and applications*, *1*(2), 104-114.

[51] Khodadadi-Karimvand, M., & Shirouyehzad, H. (2021). Well drilling fuzzy risk assessment using fuzzy FMEA and fuzzy TOPSIS. *Journal of fuzzy extension and application*, *2*(2), 144-155.

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An Interval-Valued Atanassov's Intuitionistic Fuzzy Multi-attribute Group Decision Making Method Based on the Best Representation of the WA and OWA Operators

Ivanosca Andrade Da Silva[1], Berta Bedregal[2], Benjamın Bedregal[3,*], Regivan Hugo Nunes Santiago[3]

[1] Directorate of Material and Heritage, Federal University of Rio Grande do Norte, Natal, RN, Brazil; ivanosca@yahoo.com.br.
[2] Production Engineering Course state University of Rio de Janeiro; leticia.berta@gmail.com.
[3] Department of Informatics and Applied Mathematics, Federal University of Rio Grande do Norte, Natal, RN, Brazil; bedregal@dimap.ufrn.br; regivan@dimap.ufrn.br.

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Abstract

In this paper we extend the notion of interval representation for interval-valued Atanassov's intuitionistic representations, in short Lx-representations, and use this notion to obtain the best possible one, of the Weighted Average (WA) and Ordered Weighted Average (OWA) operators. A main characteristic of this extension is that when applied to diagonal elements, i.e. fuzzy degrees, they provide the same results as the WA and OWA operators, respectively. Moreover, they preserve the main algebraic properties of the WA and OWA operators. A new total order for interval-valued Atanassov's intuitionistic fuzzy degrees is also introduced in this paper which is used jointly with the best Lx-representation of the WA and OWA, in a method for multi-attribute group decision making where the assesses of the experts, in order to take in consideration uncertainty and hesitation, are interval-valued Atanassov's intuitionistic fuzzy degrees. A characteristic of this method is that it works with interval-valued Atanassov's intuitionistic fuzzy values in every moments, and therefore considers the uncertainty on the membership and non-membership in all steps of the decision making. We apply this method in two illustrative examples and compare our result with other methods.

Keywords: Interval-Valued Atanassov's intuitionistic fuzzy sets, WA and OWA operators, Lx-representations, Total orders, Multi-Attribute group decision making.

1 | Introduction

From the seminal paper [71] on fuzzy set theory, several extensions for this theory have been proposed [18]. Among them, we stress "Interval-valued Fuzzy Sets Theory" [10], [19], [72] and "Atanassov's Intuitionistic Fuzzy Sets Theory" [2], [5], [25], [26]. Although they are mathematically equivalents, they capture dif- ferent kinds of uncertainty in the membership degrees, i.e. they have different semantics [61]. The first one takes in account the intrinsic difficulty to determine the exact membership degree of an object to some linguistic term; in this case, an expert provides an interval which expresses his uncertainty on such degree. The second one adds an extra degree to the usual fuzzy sets in order to model the hesitation and uncertainty about the membership degree. In fuzzy set theory, the non-membership degree is by default the complement of the membership degree, i.e.

Corresponding Author: bedregal@dimap.ufrn.br
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1- $\mu_A(x)$, meaning that there is no doubt or hesitation in the membership degree. In [3], both extensions are mixed by considering that we can also have an uncertainty or imprecision in the membership and non-membership degrees if we model them with intervals. This results in other extension of fuzzy set theory, known as Interval-Valued Atanassov's Intuitionistic Fuzzy Sets (IVAIFS). Several applications of IVAIFS, and extensions of usual fuzzy notions to the IVAIFS framework have been made, see for example [4], [7], [21], [32], [51], [64].

Besides, Group Decision Making (GDM) and Multi-attribute Decision Making (MADM) are the most well know branches of decision making. GDM consists in choosing of one or more alternatives among several ones by a group of decision makers (experts), probably with a weight of confidence [24]. MADM choosing one or more alternatives among several ones based in the assesses of an expert his opinion of how much the alternative fulfills a criteria or satisfies an attribute. Usually, a weighting vector for the attributes is associated, in order to represent the importance of an attribute in the overall decision problem. Nevertheless, complex decision making problems usually need to consider a group of experts as well as a set of criteria or attributes, i.e. a Multi-attribute Group Decision Making (MAGDM) [28], [43], [55], [59].

Fuzzy logic, by their nature, has played an important role in the field of decision making, since decision makers can be subject to uncertainty expressed in terms of fuzzy degrees [46], [47], [55], [57]. An important mathematical tool for fuzzy decision-making are Weighted Average (WA) and the Ordered Weighted Average (OWA) operators introduced in [69], which have triggered their "extension" for Interval-Valued Atanassov's Intuitionistic Fuzzy Values (IVAIFV) – see for example [65], [67]. However, in the cited cases, the proposed interval-valued Atanassov's intuitionistic OWA, although of preserve some algebraic properties of the OWA (monotonicity, idempotency, symmetry and boundedness [16]), have not the same behaviour as the OWA when applied to diagonals elements.

In [11], [54], in order to formalize the principle of correctness of interval computation [37], it was introduce the notion of interval representation of real functions. In addition, the best of the interval representations of a real function models the notion of optimality in interval computing. This notion has been used in the context of interval-valued fuzzy functions, to obtain interval-valued t-norms (t-conorms, overlap functions, fuzzy negations and fuzzy implications) from t- norms (t-conorms, overlap functions, fuzzy negations and fuzzy implications) in [1], [8], [14], [34]. In this paper we extend the notion of interval-valued representation and the best interval-valued representation of fuzzy functions for the interval-valued Atanassov's intuitionistic representations of fuzzy and interval- valued fuzzy functions. In particular, we provide a novel extension of the WA and OWA operator for IVAIFS, based on the best interval-valued Atanassov's intu- itionistic fuzzy representation, which preserve the main properties of the OWA operators and when restrict to the diagonals elements it is an OWA in $0, 1$. This new IVAIFAF OWA together with some total orders for IVAIFV are used to develop a method to rank alternatives from the individual interval-valued Atanassov's intuitionistic decision matrices of a group of experts reflecting how much each alternative satisfy each attribute. Two illustrative examples are considered in order to show the use of the method and to show that the final ranking of alternatives obtained by the method is adequate.

This paper is organized as follows: Section 2 introduces Atanassov intuitionisc and interval-valued fuzzy sets, the score and accuracy index and the notion of representation in particular in the interval-valued and Atanassov intuitionisc best representation of the WA and OWA operators. In Section 3 it is consider the notion of interval-valued intuitionistic fuzzy set and some orders for interval-valued. Atanassov's intuitionistic fuzzy values are presented. In particular, based in a novel notion of membership and subsets, interval-valued intuitionistic fuzzy degrees are seen as an interval of interval-valued fuzzy degrees and based in this

point of view a new total order for IVAIFV is provided. In Section 4 it is introduced the notion of IVAIFV representation and it is provide a canonical way of obtain the best representation of an interval-valued fuzzy function and of a fuzzy func- tion, which is used to obtain the best IVAIFV representation of the WA and OWA operators. In Section 5 the total orders on IVAIFV and the best IVAIFV representation of the WA and OWA are used to develop a method to solve MAGDMP and this method is used in two illustrative examples. Finally in Section 6 some final remarks on the paper are provided.

2| Preliminaries

Atanassov in [2] extended the notion of fuzzy sets, by adding an extra degree to model the hesitation or uncertainty in the membership degree. This second degree is called non-membership degree. In fuzzy set theory, by default, this non- membership degree is given by the complement of the membership degree, i.e. one minus the membership degree, and therefore is fixed whereas in Atanassov intuitionistic fuzzy sets the non-membership degree may take any value between zero and one minus the membership degree.

Definition 1. [2]. Let X be a non-empty set and two functions $\mu_A, \nu_A : X \to [0,1]$. Then

$$A = \{(x, \mu_A(x), \nu_A(x)) / x \in X\},$$

is an Atanassov Intuitionistic Fuzzy Set (AIFS) over X if $\mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$.

The functions μ_A and ν_A provide the membership and non-membership degrees of elements in X to the AIFS A. Let $L^* = \{(x,y) \in [0,1]^2 / x + y \leq 1\}$. Elements of L^* are called L^* -values. We define the projections $l, r : L^* \to [0,1]$ by $l(x,y)=x$ and $r(x,y)=y$, but by notational simplicity, we will denote $\underset{\sim}{x}$ and $\tilde{x}$ instead of $l(x)$ and $r(x)$, respectively.

The usual partial order on L^* is the following:

$$x \leq_{L^*} y \text{ if } \underset{\sim}{x} \leq \underset{\sim}{y} \text{ and } \tilde{y} \leq \tilde{x}.$$

Deschrijver and Kerre [33] proved that $\langle L^*, \leq_{L^*} \rangle$ is a complete lattice and therefore that AIFS are a particular kind of L-fuzzy set, in the sense of Goguen [35].

Let A be an AIFS over X. The intuitionistic fuzzy index[1] of an element $x \in X$ to A is given by $\pi^*_A(x) = 1 - \mu_A(x) - \nu_A(x)$. In particular, the intuitionistic fuzzy index of $x \in L^*$ is defined in a similar way, i.e. $\pi^*_A(x) = 1 - l(x) - r(x)$. This index measures the hesitation degree in each $x \in L^*$.

In [27], Chen and Tan, introduce the notion of score of a L^* -value as the function $S^* : L^* \to [-1,1]$ defined by

$$S^*(x) = \underset{\sim}{x} - \tilde{x} \tag{1}$$

In [38], Hong and Choi, introduce the notion of accuracy function for an L^* -value as the function $h^* : L^* \to [0,1]$ defined by

$$h^*(x) = \underset{\sim}{x} + \tilde{x} \tag{2}$$

[1] In the seminal paper on AIFS, i.e. in [2], this index was called degree of indeterminacy of an element $x \in X$ to A.

Xu and Yager in [68], based on the score and accuracy index on L^* and with the goal of rank L^* -values, introduce the total order on L^* defined by

$$x \leq_{YY} y \text{ if } s^*(x) < s^*(y) \text{ or } (s^*(x) = s^*(y) \text{ and } h^*(x) \leq h^*(y)). \tag{3}$$

In [36], [40], [52], [72] and in an independent way, fuzzy set theory was extended by considering subintervals of the unit interval [0,1] instead of a single value in [0,1]. The main goal was to represent the uncertainty in the process of assigning the membership degrees.

Definition 2. Let X be a non-empty set and $L = \{[a,b] / 0 \leq a \leq b \leq 1\}$ be the set of closed subintervals of [0,1]. An Interval-Valued Fuzzy Set (IVFS) A over X is an expression

$$A = \{(x, \mu_A(x)) / x \in X\}.$$

Where $\mu_A : X \to L$.

Define the projections[1] $\nabla, \Delta : L \to [0,1]$ by $\nabla([a,b]) = a$ and $\Delta([a,b]) = b$.

For notational simplicity, for an arbitrary $X \in L$, we will denote $\nabla(x)$ and $\Delta(x)$ by $\underline{X}$ and $\overline{X}$, respectively. An interval $X \in L$ is degenerate if $\underline{X} = \overline{X}$, i.e. $X=[x,x]$ for some $x \in [0,1]$. Given $X \in L$, we denote its standard complement $[1-\overline{X}, 1-\underline{X}]$ by X. A more general notion of complement (or negation) for L^* -values can be found in [8].

We can consider the following partial order on L,

$$X \leq_L Y \text{ iff } \underline{X} \leq \underline{Y} \text{ and } \overline{X} \leq \overline{Y}.$$

As it is well-known, $\langle L, \leq_L \rangle$ is a complete lattice and so it can be seen as a Goguen L-fuzzy set.

As pointed by Moore in [45], an interval has a dual nature: as a set of real numbers and as a new kind of number (an ordered pair of real numbers with the restriction that the first component is smaller than or equal to the second one). The order $\leq_L$ is an order which stresses the nature of ordered pair for elements in L whereas the inclusion of sets stresses the nature of set for elements in L. Nevertheless, the inclusion order on L can also be expressed using the ordered pair nature as follows:

$$X \not\subset Y \text{ iff } \underline{Y} \leq \underline{X} \leq \overline{X} \leq \overline{Y}.$$

The score and accuracy function for interval fuzzy values, i.e. of an arbitrary interval $X \in L$ are defined as follows:

$$s(X) = v(X) \text{ -1 and } h(X) = 1\text{-}w(x).$$

Where $v(X) = \underline{X} + \overline{X}$ and $w(X) = \overline{X} - \underline{X}$.

[1] These projections are particular cases of Atanassov's K_α-operator for intervals [19], [48].

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As it is well known, the lattices $\langle L^*, \leq_{L^*} \rangle$ and $\langle L, \leq_L \rangle$ are isomorphic. The map $\rho: L \to L^*$ defined by $\rho(X): (\underline{X}, 1-\overline{X})$ is a such an isomorphism. Although both lattices are algebraically equivalent, from a semantical point of view they are different [61].

Remark 1. Note that, the score and the accuracy indexes on L and L^* are related as follows: $s = s^* \ o \ \rho$ and $h = h^* \ o \ \rho$. Notice that the partial order $\leq_{XY}$ on L obtained from the partial order $\leq_{XY}$ in *Eq.(3)* by using this isomorphism, i.e. $X \leq_{XY} Y$ iff $\rho(X) \leq_{XY} \rho(Y)$, can be equivalently obtained as following:

$$X \leq_{XY} Y \text{ iff } s(X) \prec_{XY} s(Y) \text{ or } (s(X) = s(Y) \text{ and } h(X) \leq h(Y)). \tag{4}$$

Bustince et al. [22] introduced the notion of admissible orders in the context of interval-valued fuzzy functions in order to always be possible to compare intervals which is important in some kind of applications [23]. An order $\leq$ on L is admissible if it refines $\leq_L \leq_L$, i.e. $X \leq Y$ whenever $X \leq_L Y$. In particular $\leq_{XY}$ is an admissible order. Other examples of admissible orders can be found in [53]. In addition, when we translate the notion of intuitionistic fuzzy index for interval values, we get the interval-valued fuzzy index $\Pi(X) = \pi^*(\rho(X)) = \overline{X} - \underline{X} = w(X)$ for each $X \in L$. Thus, the length of an interval is a measure of their indeterminacy or imprecision.

2.1| The Best L and L* Representation of the OWA Operator

In [13], it was adapted the notion of interval representation of [11], [54] in the context of interval-valued fuzzy sets theory for the particular case of the intervalvalued t-norms. Interval representation captures, in a formal way, the property of correctness of interval functions in the sense of [37]. From then, interval representations of several other connectives and fuzzy constructions (see for example [8], [12], [49]) have been studied. Here we are interested in considering the case of n-ary increasing fuzzy functions. Let's start recalling some notions.

Definition 3. Let $f: [0,1]^n \to [0,1]$ be an n-ary function. A function $F: L^n \to L$ is an interval representation or *L*-representation of *f* if for each $X_1, \ldots, X_n \in L$ and $x_i \in X_i$ with $i=1,\ldots,n$ we have that $f(x_1, \ldots, x_n) \in F(X_1, \ldots, X_n)$.

Let $F, G: L^n \to L$. We write $F \subseteq \to_L G$, if for any $X_1, \ldots, X_n \in L$, $G(X_1, \ldots, X_n) \subseteq F(X_1, \ldots, X_n)$. Notice that if $X, Y \in L$ and $X \subseteq Y$ then $h(X) \geq h(Y)$. Thus, $F \subseteq \to_L G$ means that *G* is always more accurate than *F*, i.e. $h(F(X_1, \ldots, X_n)) \leq h(G(X_1, \ldots, X_n))$ for any $X_1, \ldots, X_n \in L$. Notice also that if *G* is an *L*-representation of a function *f* and $F \subseteq \to_L G$ then *F* is also an *L*-representation of *f*, but less accurate than *G*. Therefore, *G* is a better *L*-representation of *f* than *F*.

Proposition 1. [34]. Let $f: [0,1]^n \to [0,1]$ be an n-ary increasing fuzzy function. Then the function $\hat{f}: L^n \to L$ defined by

$$\hat{f}(X_1, \ldots, X_n) = [f(\underline{X}_1, \ldots, \underline{X}_n), f(\overline{X}_1, \ldots, \overline{X}_n)], \tag{5}$$

is an *L*-representation of *f*. Moreover, for any other *L*-representation *F* of *f*, $F \subseteq_L \hat{f}$.

$\hat{f}$ is therefore the more accurate L-representation of f, i.e. the best L-representation w.r.t. the $\subseteq_L$ order. So $\hat{f}$ has the property of optimality in the sense of [37].

Remark 2. [10]. An important characteristic of the best L-representation is that when we identify points and degenerate intervals, via the merging $m(x)=[x,x]$, f and $\hat{f}$ have the same behavior, i.e. $m(f(x_1,\dots,x_n))=\hat{f}(m(x_1),\dots,m(x_n))$. Another property of the best L-representation of some increasing function is that it is isotone with respect to both, the inclusion order and the $\leq L$ order, i.e. if $X_i, Y_i \in L$ and $i=1,\dots,n$ *then* $\hat{f}(X_1,\dots,X_n) \subseteq \hat{f}(Y_1,\dots,Y_n)$ *and, analogously, if* $X_i \leq_{\mathbb{L}} Y_i$ *for each* $i=1,\dots,n$ *then* $\hat{f}(X_1,\dots,X_n) \leq_{\mathbb{L}} \hat{f}(Y_1,\dots,Y_n)$.

Let $\Lambda=(\lambda_1,\dots,\lambda_n)\in[0,1]^n$ be an n-ary weighting vector, i.e. $\sum_{i=1}^{n}\lambda_i=1$. The weighted average (WA) operator is defined by

$$\mathrm{wa}_\Lambda(x_1,\dots,x_n)=\sum_{i=1}^{n}\lambda_i x_i.$$

The Ordered Weighted Averaging (OWA) operator introduced by Yager [69] is defined by

$$\mathrm{owa}_\Lambda(x_1,\dots,x_n)=\sum_{i=1}^{n}\lambda_i x_{\sigma(i)}.$$

Where $\sigma:\{1,\dots,n\}\to\{1,\dots,n\}$ is the permutation such that $x_{\sigma(i)}\geq x_{\sigma(i+1)}$ for any $i=1,\dots,n-1$, i.e. it orders in decreasing way a n-tuple of values in [0,1] and so $x_\sigma(i)$ is the ith greatest element of $\{x_1,\dots,x_n\}$. Notice that,

$$\mathrm{owa}_\Lambda(x_1,\dots,x_n)=\mathrm{wa}_\Lambda(x_{\sigma(1)},\dots,x_{\sigma(n)}). \tag{6}$$

Several interval-valued and Atanassov intuitionistic extensions of the OWA operator have been proposed (see for example [15], [44], [70]), but most of them are not $\mathbb{L}$ (L^*)-representations of the OWA operator and do not reduce to the fuzzy OWA operator when applied to degenerate intervals.

The best $\mathbb{L}$-representation of owa_Λ is the interval-valued function $\widehat{owa_\Lambda}:\mathbb{L}^n\to\mathbb{L}$ defined by

$$\widehat{\mathrm{owa}}_\Lambda(X_1,\dots,X_n)=\left[\mathrm{owa}_\Lambda\left(\underline{X_1},\dots,\underline{X_n}\right),\mathrm{owa}_\Lambda\left(\overline{X_1},\dots,\overline{X_n}\right)\right]=\sum_{i=1}^{n}\lambda_i X_{\tau(i)}.$$

Where $X_{\tau i}=[\underline{X}_{\tau_1(i)},\overline{X}_{\tau_2(i)}]$; $\tau_1,\tau_2:\{1,\dots,n\}\to\{1,\dots,n\}$ are permutations such that $\underline{X}_{\tau_1(i)}\geq\underline{X}_{\tau_1(i+1)}$ and $\overline{X}_{\tau_2(i)}\geq\overline{X}_{\tau_2(i+1)}$ for any $i=1,\dots,n-1$; the scalar product is the usual in interval mathematics (see [45]), i.e. for any $\lambda\in[0,1]$ and $X,Y\in\mathbb{L}$, $\lambda X=[\lambda\underline{X},\lambda\overline{X}]$ and the sum is w.r.t. the limited addition defined by $X[+]Y=[\min(\underline{X}+\underline{Y},1),\min(\overline{X}+\overline{Y},1)]$. Notice that, in this case, because $\sum_{i=1}^{n}\lambda_i=1$,

$$\left[\sum_{i=1}^{n}\right]\lambda_i X_i=\left[\min\left(\sum_{i=1}^{n}\lambda_i\underline{X_i},1\right),\min\left(\sum_{i=1}^{n}\lambda_i\overline{X_i},1\right)\right]=\left[\sum_{i=1}^{n}\lambda_i\underline{X_i},\sum_{i=1}^{n}\lambda_i\overline{X_i}\right]$$
$$=\sum_{i=1}^{n}\lambda_i X_i. \tag{7}$$

Where $[\sum_{i=1}^{n}]$ is the sommatory with respect to [+] and $\sum_{i=1}^{n}$ is the sommatory with respect the usual addition between intervals (see [45]).

Note that for each term in the sum above, lower and upper bounds from different intervals may be considered for a given weight λ_i. For example, for $\lambda_1=0.2$, $\lambda_2=0.3$, $\lambda_3=0.5$, $X_1=[0.6,0.8]$, $X_2=[0.7,0.9]$ and $X_3=[0.5,1]$ we have that $\left[\sum_{i=1}^{3}\right]\lambda_i X_i=[min(0.2\cdot0.6+0.3\cdot0.7+0.5\cdot0.5,1),min(0.2\cdot0.8+0.3\cdot0.9+0.5\cdot1,1)]=[0.58,0.93]=[0.2\cdot0.6+0.3\cdot0.7+0.5\cdot0.5,0.2\cdot0.8+0.3\cdot0.9+0.5\cdot1]=\sum_{i=1}^{3}\lambda_i X_i$.

Analogously, a function $F: (L^*)^n \to L^*$ is an L^*-representation of a function $f: [0,1]^n \to [0,1]$ if for each $\mathbf{x}_i \in L^*$ and $x_i \in [\mathbf{x}_i, 1-\widetilde{\mathbf{x}_i}]$, with $i = 1, \dots, n$,

$$F(x_1, \dots, x_n) \le f(x_1, \dots, x_n) \le 1 - F(\widetilde{x_1, \dots, x_n}). \tag{8}$$

Let $F, G: (L^*)^n \to L^*$. We denote by $F \sqsubseteq_{L^*} G$, if for any $x_1, \dots, x_n \in L^*$, $G(x_1, \dots, x_n) \subseteq_{L^*} F(x_1, \dots, x_n)$, where $x \subseteq_{L^*} y$ *if* $x \le y$ and $\tilde{x} \le \tilde{y}$. Notice that although of this order be the usual on R^2, considering the mathematical equivalence of L^* and $\mathbb{L}$, we have that $x \subseteq_{L^*} y$ iff $\rho^{-1}(X) \subseteq \rho^{-1}(Y)$. Thus, $F \sqsubseteq_{L^*} G$ means than the result of G is always more accurate than the result of F, i.e. $h^*(F(x_1, \dots, x_n)) \le h^*(G(x_1, \dots, x_n))$ for any $x_1, \dots, x_n \in L^*$.

Proposition 2. Let $f: [0,1]^n \to [0,1]$ be an increasing function. Then the function $f: (L^*)^n \to L^*$ defined by

$$f(x_1, \dots, x_n) = \left(f(x_1, \dots, x_n), 1 - f(1 - \widetilde{x_1}, \dots, 1 - \widetilde{x_n})\right).$$

is the greatest L^*-representation of f w.r.t. $\sqsubseteq_{L^*}$ order and so is the best one.

Proof. If $x_i \in [\mathbf{x}_i, 1-\widetilde{\mathbf{x}_i}]$ for each $i = 1, \dots, n$, then because f is increasing we have that $f(x_1, \dots, x_n) \le f(x_1, \dots, x_n) \le f(1 - \widetilde{x_1}, \dots, 1 - \widetilde{x_n})$ and therefore, $f(x_1, \dots, x_n) \le f(x_1, \dots, x_n) \le 1 - f(\widetilde{x_1, \dots, x_n})$. So, $f(x_1, \dots, x_n)$ is an L^*-representation of f.

Now, suppose that F is another L^*-representation of f, then by *Eq. (8)* and because f is increasing, we have that $F(x_1, \dots, x_n) \le f(x_1, \dots, x_n) \le f(1 - \widetilde{x_1}, \dots, 1 - \widetilde{x_n}) \le 1 - F(\widetilde{x_1, \dots, x_n})$. Therefore, $f(x_1, \dots, x_n) \subseteq_{L^*} F(x_1, \dots, x_n)$, *i.e.* $F \sqsubseteq^* f$.

Moreover, if f is an aggregation function then f is also an L^*-valued aggregation function [42] (*Lemma 1*). Clearly, $f = \rho \circ \hat{f} \circ \rho^{-1}$, or equivalently, $\hat{f} = \rho^{-1} \circ f \circ \rho$. Therefore, owa_Λ it is the best L^*-representation of owa_Λ .

Proposition 3. Let $f, g: [0,1]^n \to [0,1]$. If $f \le g$ then $\hat{f} \le \hat{g}$ and $f \le g$.

Proof. Straightforward.

Remark 3. owa as well as $\widehat{owa}$ are interval-valued and Atanassov intuitionistic aggregation functions in the sense of [42]. Moreover, both are symmetric and idempotent, and as a consequence of the above proposition, they are bounded by $\widehat{owa_{(0,\dots,0,1)}}$ ($owa_{(0,\dots,0,1)}$), i.e. $\widehat{min}$ (min) and $\widehat{owa_{(1,0,\dots,0)}}$ ($owa_{(1,0,\dots,0)}$), i.e. $\widehat{max}$ (max).

3 | Interval-Valued Atanassov's Intuitionistic Fuzzy Sets

Definition 4. [3]. An IVAIFS A over a nonempty set X is an expression given by

$$A = \{(x, \mu_A(x), \nu_A(x))/x \in X\},$$

where $\mu_A, \nu_A: X \to \mathbb{L}$ with the condition $\overline{\mu_A(x)} + \overline{\nu_A(x)} \le 1$.

Deschrijver and Kerre [33] provide an alternative approach for Atanassov intuitionistic fuzzy sets in term of L-fuzzy sets in the sense of Goguen [35]. Analogously, we can also see IVAIFS as a particular case of L-fuzzy set by considering the complete lattice $\langle \mathbb{L}^*, \le_{\mathbb{L}^*} \rangle$ where

$$\mathbb{L}^* = \{(X, Y) \in \mathbb{L} \times \mathbb{L} / \overline{X} + \overline{Y} \leq 1\}.$$

And

$$(X_1, X_2) \leq_{\mathbb{L}^*} (Y_1, Y_2) \text{ iff } X_1 \leq_{\mathbb{L}} Y_1 \text{ and } Y_2 \leq_{\mathbb{L}} X_2.$$

Notice that $0_{\mathbb{L}^*} = ([0,0], [1,1])$ and $1_{\mathbb{L}^*} = ([1,1], [0,0])$. Analogously to the case of L^*, we define the projections $l, r: \mathbb{L}^* \to \mathbb{L}$ by

$$l(X_1, X_2) = X_1 \text{ and } r(X_1, X_2) = X_2,$$

and for each $X \in \mathbb{L}^*$, we denote $l(X)$ and $r(X)$ by X and $\tilde{X}$, respectively.

Elements of $\mathbb{L}^*$ will be called $\mathbb{L}^*$-values. An $\mathbb{L}^*$-value X is a semi-diagonal element if X and $\tilde{X}$ are degenerate intervals. $X \in \mathbb{L}^*$ is a diagonal element if $X + \tilde{X} = [1,1]$ i.e. if $X = ([x, x], [1 - x, 1 - x])$ for some $x \in [0,1]$. We denote by $\mathscr{D}_S$ and $\mathscr{D}$ the sets of semi-diagonal and diagonal elements of $\mathbb{L}^*$, respectively. Clearly, $\mathscr{D} \subseteq \mathscr{D}_S$ and there is a bijection between $[0,1]$ and $\mathscr{D}$ ($\phi(x) = ([x, x], [1 - x, 1 - x])$), between L^* and $\mathscr{D}_S$ ($\psi(x) = ([x, x], [\tilde{x}, \tilde{x}])$) and between $\mathbb{L}$ and $\mathscr{D}_S$ ($\varphi(X) = ([\underline{X}, \underline{X}], [\overline{X}, \overline{X}]^c)$, i.e. $\varphi = \psi \circ \rho$) [29].

3.1 | Some indexes for $\mathbb{L}^*$-Values

In [50] the Atanassov intuitionistic fuzzy index was extended for IVAIFS, in order to provide an interval measure of the hesitation degree in IVAIFS. Let A be an IVAIFS over a set X. The interval-valued Atanassov intuitionistic fuzzy index of an element $x \in X$ for the IVAIFS A is determined by the expression $\Pi^*(x) = [1,1] - \mu_A(x) - \nu_A(x)$. In an analogous way the interval-valued Atanassov intuitionistic fuzzy index of an $(X, Y) \in \mathbb{L}^*$ is defined by

$$\Pi^*(X, Y) = [1,1] - X - Y. \tag{9}$$

The Chen and Tan score measure was extended for $\mathbb{L}^*$ in [66] [1] and [41].

In this paper we consider Xu's definition: Let $S: \mathbb{L}^* \to [-1,1]$ be defined by

$$S(X) = \frac{v(X) - v(\tilde{X})}{2}.$$

For each $X \in \mathbb{L}^*$, $S(X)$ is called the score of X.

Remark 4. S when applied to semi-diagonal elements is the same, up to an isomorphism ψ, as s^*, i.e. $S(\psi(x)) = s^*(x)$ for any $x \in L^*$. Analogously, S when applied to semi-diagonal elements is the same, up to an isomorphism φ, as s, i.e. $S(\varphi(X)) = s(X)$ for any $X \in \mathbb{L}$. Moreover, the range of S ($[-1,1]$) is the same as that of s^* and S can be obtained from s and s^*, as shown by the *Eq. (10)*.

$$S(X) = \frac{s^*\left(s(X), s(\tilde{X})\right)}{2}. \tag{10}$$

Since we can have two different $\mathbb{L}^*$-values with the same score, for example $S([0.2,0.3], [0.4,0.5]) = S([0.1,0.2], [0.3,0.4]) = -0.2$, the score determines just a pre-order on $\mathbb{L}^*$:

$$X \leq_S Y \text{ iff } S(X) \leq S(Y).$$

[1] Because this reference is in Chinese, we are based on the definition in [64].

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Since $\leq_S$ is a pre-order, it defines the following natural equivalence relation: $X \equiv_S Y$ *iff* $X \leq_S Y$ and $Y \leq_S X$

Another important index for $\mathbb{L}^*$-values is the extension of the accuracy function. Nevertheless, in the literature several non-equivalent such "extensions" have been proposes. In [29], [30], it was made an analysis of five of such proposals concluding that the more reasonable would be the new accuracy function proposed in that paper and the one proposed in [66]. Here we will consider Xu's accuracy function:

$$\mathrm{H(X)} = \frac{\mathrm{v(X)} + \mathrm{v}(\tilde{\mathrm{X}})}{2}.$$

because, analogously to the case of S, the Xu's accuracy function when applied to semi-diagonal elements is the same, up to an isomorphisms ψ and φ, as h^* and h, respectively, i.e. $H(\psi(x)) = h^*(x)$ for any $x \in L^*$ and $H(\varphi(X)) = h(X)$ for any $X \in \mathbb{L}$. In addition, the range of H, h and h^* are the same.

3.2| Order for $\mathbb{L}^*$-Values

In [56], it was introduced the notion of n-dimensional fuzzy interval and it was observed that 4-dimensional fuzzy sets are isomorphic to IVAIFS. The degrees in an n-dimensional fuzzy interval take values in $L_n([0,1]) = \{(x_1, \dots, x_n) \in [0,1]^n / x_i \leq x_{i+1}$ for each $i = 1, \dots, n-1\}$. In [9] the elements of $L_n([0,1])$ are called n-dimensional intervals and the bijection $\varrho: \mathbb{L}^* \to L_4([0,1])$ defined by $\varrho(X) = (\nabla(X), \triangle(X), 1 - \triangle(\tilde{X}), 1 - \nabla(\tilde{X}))$ was provided. One of the possible interpretations considered in [9] for the 4-dimensional intervals (x_1, x_2, x_3, x_4) is that the intervals $[x_1, x_2]$ and $[x_3, x_4]$ represent an interval uncertainty in the bounds of an interval-valued degree, i.e. of an element $[x, y] \in \mathbb{L}$, and so $x \in [x_1, x_2]$ and $y \in [x_3, x_4]$. Having it in mind, we introduce the notion of membership of $\mathbb{L}$-values in $\mathbb{L}^*$-values.

Definition 5. Let $\mathrm{X} \in \mathbb{L}$ and $X \in \mathbb{L}^*$. We say that $X \Subset X$ if $\underline{X} \in X$ and $\overline{X} \in \tilde{X}^c$.

Observe that this notion is strongly related to the notion of nesting given in [6], [7] and therefore also can be used as a representation of IVAIFS by pairs of AIFS.

Notice that, for each $X, Y, Z \in \mathbb{L}$,

I. if $X \subseteq Y \subseteq Z$ and $X, Z \Subset Y$ for some $Y \in \mathbb{L}^*$, then $Y \Subset Y$;
II. if $X \leq_{\mathbb{L}} Y \leq_{\mathbb{L}} Z$ and $X, Z \Subset Y$ for some $Y \in \mathbb{L}^*$, then $Y \Subset Y$;
III. $Y \Subset \varphi(X)$ iff $Y = \mathrm{X}$.

For any $X \in \mathbb{L}^*$ we will denote

$$\vec{\mathrm{X}} = [\nabla(\mathrm{X}), \nabla(\tilde{\mathrm{X}}^c)] \quad \text{and} \quad \overleftarrow{\mathrm{X}} = [\triangle(\mathrm{X}), \triangle(\tilde{\mathrm{X}}^c)], \tag{11}$$

i.e. $\vec{X} = [\nabla(X), 1 - \triangle(\tilde{X})]$ and $\overleftarrow{X} = [\triangle(X), 1 - \nabla(\tilde{X})]$. Notice that, the set $S_X = \{X \in \mathbb{L} / X \Subset X\}$ is bounded, i.e. for any $X \in S_X$, $\vec{X} \leq_{\mathbb{L}} X \leq_{\mathbb{L}} \overleftarrow{X}$ and $\vec{X}, \overleftarrow{X} \in S_X$. Thus, S_X is a closed interval $([\vec{X}, \overleftarrow{X}])$ of $\mathbb{L}$-values and hence, analogously to $\mathbb{L}$-values, $\mathbb{L}^*$-values also have a dual nature: as an ordered pair of $\mathbb{L}$-values with some condition and as a set (an interval) of $\mathbb{L}$-values.

3.2.1 | Subset order for $\mathbb{L}^*$-values

Since the usual membership relation is used to introduce the subset relation in set theory, the relation $\Subset$ will allow us to introduce a notion of subset between $\mathbb{L}^*$-values. Let $X, Y \in \mathbb{L}^*$, we say that $X \subseteq Y$ if for each $X \Subset X$ we have that $X \Subset Y$. Analogously to the case of $\mathbb{L}$-values, we can also define this inclusion relation via the bounds of the interval associated to $\mathbb{L}^*$-values.

Proposition 4. Let $X, Y \in \mathbb{L}^*$. Then the following expression are equivalents

I. $X \underline{\subseteq} Y$;
II. $S_X \subseteq S_Y$;
III. $\vec{Y} \leq_{\mathbb{L}} \vec{X} \leq_{\mathbb{L}} \overleftarrow{X} \leq_{\mathbb{L}} \overleftarrow{Y}$;
IV. $X \subseteq Y$ *and* $\tilde{X} \subseteq \tilde{Y}$.

Proof.

I. 1⇒2: If $X\_(\subseteq) Y$ then for each $X\_(\in) X$ also $X\_(\in) Y$, and so $S\_X \subseteq S\_Y$.
II. 2 ⇒ 3: Straightforward once that $S_X = [\vec{X}, \overleftarrow{X}]$.
III. 3 ⇒ 4: If $\vec{Y} \leq_{\mathbb{L}} \vec{X} \leq_{\mathbb{L}} \overleftarrow{X} \leq_{\mathbb{L}} \overleftarrow{Y}$ then by definition $[\nabla(Y), \nabla(\tilde{Y}^c)] \leq_{\mathbb{L}} [\nabla(X), \nabla(\tilde{X}^c)] \leq_{\mathbb{L}} [\triangle(X), \triangle(\tilde{X}^c)] \leq_{\mathbb{L}} [\triangle(Y), \triangle(\tilde{Y}^c)]$ So, $\nabla(Y) \leq \nabla(X) \leq \triangle(X) \leq \triangle(Y)$ and $\nabla(\tilde{Y}^c) \leq \nabla(\tilde{X}^c) \leq \triangle(\tilde{X}^c) \leq \triangle(\tilde{Y}^c)$, *i.e.* $1 - \triangle(\tilde{Y}) \leq 1 - \triangle(\tilde{X}) \leq 1 - \nabla(\tilde{X}) \leq 1 - \nabla(\tilde{Y})$. Therefore $X \subseteq Y$ *and* $\tilde{X} \subseteq \tilde{Y}$.
IV. 4 ⇒ 1: If $X \underline{\in} X$ then $\underline{X} \in X$ and $\overline{X} \in \tilde{X}^c$. So, because $X \subseteq Y$ and $\tilde{X} \subseteq \tilde{Y}$, then $\underline{X} \in Y$ and $\overline{X} \in \tilde{Y}^c$. Therefore, $X \underline{\in} Y$ and hence $X \underline{\subseteq} Y$.

Remark 5. Some properties of $\_(\subseteq)$:

i. It is a partial order on $\mathbb{L}^*$-values;

ii. For each $X, Y \in \mathbb{L}$, $\varphi(X) \underline{\subseteq} \varphi(Y)$ iff $X = Y$;

iii. For *each* $x, y \in [0,1]$, $\phi(x) \underline{\subseteq} \phi(y)$ iff $x = y$;

iv. Defining the complement of $\mathbb{L}^*$-values by $X^c = (\tilde{X}, X)$, then $X \underline{\subseteq} Y$ iff $X^c \underline{\subseteq} Y^c$.

3.2.2| Extension of $\leq_{XY}$ total order for $\mathbb{L}^*$-values

In order to rank any possible set of $\mathbb{L}^*$-values it is necessary to provide a total order on $\mathbb{L}^*$, as made in [68] for L^*-values which was based on the score and accuracy index. Following the same idea, we define the next binary relation on $\mathbb{L}^*$-values:

$$X \leq_{S,H} Y \ \text{ iff } \ \begin{cases} X <_S Y \ \text{ or} \\ X \equiv_S Y \ \text{ and } \ H(X) \leq H(Y) \end{cases} \tag{12}$$

for any $X, Y \in \mathbb{L}^*$, where $X <_S Y$ iff $X \leq_S Y$ and $X \not\equiv_S Y$.

Nevertheless, as noted in [64], this relation is not an order. However, in [64] it was provided the next total order[1] for $\mathbb{L}^*$:

$$X \preccurlyeq Y \ \text{ iff } \ \begin{cases} X <_S Y \ \text{ or} \\ X \equiv_S Y \ \text{ and } \ H(X) < H(Y) \ \text{ or} \\ X \equiv_S Y \ \text{ and } \ H(X) = H(Y) \ \text{ and } \ T(X) < T(Y) \ \text{ or} \\ X \equiv_S Y \ \text{ and } \ H(X) = H(Y) \ \text{ and } \ T(X) = T(Y) \ \text{ and } \ G(X) \leq G(Y) \end{cases} \tag{13}$$

for any $X, Y \in \mathbb{L}^*$, where $T(X) = w(X) - w(\tilde{X})$ and $G(X) = w(X) + w(\tilde{X})$.

[1] In [64] was not claimed this, but from Proposition 4.1. of [31], it is possible to conclude that this order is total.

In [29], it was defined a new total order for $\mathbb{L}^*$-values, denoted here by $\lesssim$, which is based on the total order for L^*-values of Xu and Yager given by *Eq. (3)*.

Theorem 1. [29] The binary relation $\lesssim$ on $\mathbb{L}^*$, defined for any $X, Y \in \mathbb{L}^*$ by

$$\mathrm{X} \lesssim \mathrm{Y} \quad \text{iff} \quad \mathrm{X} <_{\mathrm{XY}} \mathrm{Y} \quad \text{or} \quad \left(\mathrm{X} = \mathrm{Y} \quad \text{and} \quad \tilde{\mathrm{X}} \leq_{\mathrm{XY}} \tilde{\mathrm{Y}}\right), \tag{14}$$

is a total order.

Observe that the order $\lesssim$ is a particular instance of the admissible orders on $\mathbb{L}^*$ introduced in [30], [31] (see also [32]), i.e. is total and refines $\leq_{\mathbb{L}^*}$.

Here, we propose a new total order, with the same principle as (14), but by considering other intervals:

Theorem 2. The binary relation $\lessapprox$ on $\mathbb{L}^*$, defined for any $X, Y \in \mathbb{L}^*$, by

$$\mathrm{X} \lessapprox \mathrm{Y} \quad \text{iff} \quad \vec{\mathrm{X}} <_{\mathrm{XY}} \vec{\mathrm{Y}} \quad \text{or} \quad \left(\vec{\mathrm{X}} = \vec{\mathrm{Y}} \quad \text{and} \quad \overleftarrow{\mathrm{X}} \leq_{\mathrm{XY}} \overleftarrow{\mathrm{Y}}\right). \tag{15}$$

is a total order.

Proof. Trivially, $\lessapprox$ is reflexive and antisymmetric. The transitivity of $\lessapprox$ follows from the transitivity of $\leq_{XY}$ and equality. Analogously, the totallity of $\lessapprox$ follows from the totality of $\leq_{XY}$.

4| $\mathbb{L}^*$-Representation of OWA

4.1 | $\mathbb{L}^*$-Representations of $\mathbb{L}$-Functions

The notion of membership on $\mathbb{L}^*$-values also allows us to adapt the notion of interval representation for $\mathbb{L}^*$ in the following way.

Definition 6. Let $F: \mathbb{L}^n \to \mathbb{L}$ and $\mathscr{F}: (\mathbb{L}^*)^n \to \mathbb{L}^*$. $\mathcal{F}$ is an $\mathbb{L}^*$-representation of F if for each $\mathbf{X}_i \in \mathbb{L}^*$, and $X_i \underline{\in} \mathbf{X}_i$, with $i = 1, \dots, n$, $\mathrm{F}(X_1, \dots, X_n) \underline{\in} \mathscr{F}(\mathbf{X}_1, \dots, \mathbf{X}_n)$.

Let $\mathscr{G}, \mathscr{F}: (\mathbb{L}^*)^n \to \mathbb{L}^*$. We say that $\mathscr{F}$ is narrower than $\mathscr{G}$, denoted by $\mathscr{G} \sqsubseteq_{\mathbb{L}^*} \mathscr{F}$, if for any $\mathbf{X}_i \in \mathbb{L}^*$ with $i = 1, \dots, n$, $\mathscr{F}(\mathbf{X}_1, \dots, \mathbf{X}_n) \underline{\subseteq} \mathscr{G}(\mathbf{X}_1, \dots, \mathbf{X}_n)$. Analogously to the case of $\mathbb{L}$-representation, we say that an $\mathbb{L}^*$-representation $\mathscr{F}$ of a function $F: \mathbb{L}^n \to \mathbb{L}$ is better than another $\mathbb{L}^*$-representation $\mathscr{G}$ of F if $\mathscr{G} \sqsubseteq_{\mathbb{L}^*} \mathscr{F}$.

Theorem 3. Let $F: \mathbb{L}^n \to \mathbb{L}$ be an isotone function. Then $\ddot{F}: (\mathbb{L}^*)^n \to \mathbb{L}^*$ defined by

$$\ddot{\mathrm{F}}(\mathrm{X}_1, \dots, \mathrm{X}_n) = \left(\left[\underline{\mathrm{F}\left(\overrightarrow{\mathrm{X}_1}, \dots, \overrightarrow{\mathrm{X}_n}\right)}, \underline{\mathrm{F}\left(\overleftarrow{\mathrm{X}_1}, \dots, \overleftarrow{\mathrm{X}_n}\right)}\right], \left[1 - \overline{\mathrm{F}\left(\overleftarrow{\mathrm{X}_1}, \dots, \overleftarrow{\mathrm{X}_n}\right)}, 1 - \overline{\mathrm{F}\left(\overrightarrow{\mathrm{X}_1}, \dots, \overrightarrow{\mathrm{X}_n}\right)}\right]\right), \tag{16}$$

is an $\mathbb{L}^*$-representation of F. Moreover, if $\mathscr{F}$ is another $\mathbb{L}^*$-representation of F then $\mathscr{F} \sqsubseteq_{\mathbb{L}^*} \ddot{F}$.

Proof. Let $\mathbf{X}_i \in \mathbb{L}^*$ with $i = 1, \dots, n$. Since, F is isotone w.r.t. $\leq_{\mathbb{L}}$, then for each $X_i \in \mathbf{X}_i$ with $i = 1, \dots, n$, $F(\overrightarrow{X_1}, \dots, \overrightarrow{X_n}) \leq_{\mathbb{L}} F(X_1, \dots, X_n) \leq_{\mathbb{L}} F(\overleftarrow{X_1}, \dots, \overleftarrow{X_n})$ and so $\underline{F(\overrightarrow{X_1}, \dots, \overrightarrow{X_n})} \leq \underline{F(X_1, \dots, X_n)} \leq \underline{F(\overleftarrow{X_1}, \dots, \overleftarrow{X_n})}$ and $\overline{F(\overrightarrow{X_1}, \dots, \overrightarrow{X_n})} \leq \overline{F(X_1, \dots, X_n)} \leq \overline{F(\overleftarrow{X_1}, \dots, \overleftarrow{X_n})}$. Therefore,

$\underline{F(X_1,\dots,X_n)} \in [\underline{F(\overrightarrow{X_1},\dots,\overrightarrow{X_n})}, \underline{F(\overleftarrow{X_1},\dots,\overleftarrow{X_n})}] = \ddot{F}(X_1,\dots,X_n)$ and $\overline{F(X_1,\dots,X_n)} \in [\overline{F(\overrightarrow{X_1},\dots,\overrightarrow{X_n})}, \overline{F(\overleftarrow{X_1},\dots,\overleftarrow{X_n})}] = \ddot{F}(\widetilde{X_1,\dots,X_n})^c$

Hence, $F(X_1,\dots,X_n) \underline{\in} \ddot{F}(X_1,\dots,X_n)$.

If $\mathscr{F}: (\mathbb{L}^*)^n \to \mathbb{L}^*$ is another $\mathbb{L}^*$-representation of F, then for each $X_i \in \mathbb{L}^*$, and $X_i \underline{\in} X_i$, with $i = 1,\dots,n$, $F(X_1,\dots,X_n) \underline{\in} \mathscr{F}(X_1,\dots,X_n)$. In particular, $F(\overrightarrow{X_1},\dots,\overrightarrow{X_n}), F(\overleftarrow{X_1},\dots,\overleftarrow{X_n}) \underline{\in} \mathscr{F}(X_1,\dots,X_n)$. So, by definition of $\underline{\in}$, $\underline{F(\overrightarrow{X_1},\dots,\overrightarrow{X_n})}, \underline{F(\overleftarrow{X_1},\dots,\overleftarrow{X_n})} \in \mathscr{F}(X_1,\dots,X_n)$ and $\overline{F(\overleftarrow{X_1},\dots,\overleftarrow{X_n})}, \overline{F(\overrightarrow{X_1},\dots,\overrightarrow{X_n})} \in \mathscr{F}(\widetilde{X_1,\dots,X_n})^c$, i.e. $1 - \overline{F(\overleftarrow{X_1},\dots,\overleftarrow{X_n})}, 1 - \overline{F(\overrightarrow{X_1},\dots,\overrightarrow{X_n})} \in \mathscr{F}(\widetilde{X_1,\dots,X_n})$. Therefore, $\ddot{F}(X_1,\dots,X_n) \subseteq \mathscr{F}(X_1,\dots,X_n)$ and $\ddot{F}(\widetilde{X_1,\dots,X_n}) \subseteq \mathscr{F}(\widetilde{X_1,\dots,X_n})$ and so, by Proposition 4, $\ddot{F}(X_1,\dots,X_n) \underline{\subseteq} \mathscr{F}(X_1,\dots,X_n)$. Hence, $\mathscr{F} \sqsubseteq_{\mathbb{L}^*} \ddot{F}$.

Corollary 1. Let $\mathrm{F}: \mathbb{L}^n \to \mathbb{L}$ be an isotone function. Then

$$\overrightarrow{\ddot{\mathrm{F}}(\mathrm{X}_1,\dots,\mathrm{X}_n)} = \mathrm{F}\left(\overrightarrow{\mathrm{X}_1},\dots,\overrightarrow{\mathrm{X}_n}\right) \text{ and } \overleftarrow{\ddot{\mathrm{F}}(\mathrm{X}_1,\dots,\mathrm{X}_n)} = \mathrm{F}\left(\overleftarrow{\mathrm{X}_1},\dots,\overleftarrow{\mathrm{X}_n}\right). \tag{17}$$

Proof. Straightforward from *Theorem 3* and *Eq. (11).*

Corollary 2. Let $\mathrm{f}: [0,1]^n \to [0,1]$ be an isotone function. Then

$$\hat{\mathrm{f}}(\mathrm{X}_1,\dots,\mathrm{X}_n) = (\hat{\mathrm{f}}(\mathrm{X}_1,\dots,\mathrm{X}_n), \hat{\mathrm{f}}\left(\widetilde{\mathrm{X}_1}^c,\dots,\widetilde{\mathrm{X}_n}^c)^c\right). \tag{18}$$

Proof. Straightforward from *Theorem 3* and *eq. (11).*

Corollary 3. Let $f, g: [0,1]^n \to [0,1]$ be isotone functions such that $f \leq g$. Then, $\hat{f} \leq \hat{g}$, i.e. $\hat{f}(X_1,\dots,X_n) \leq_{\mathbb{L}^*} \hat{g}(X_1,\dots,X_n)$ for each $X_i \in \mathbb{L}^*$ with $i = 1,\dots,n$.

Proof. Straightforward from *Corollary 2* and definition of $\leq_{\mathbb{L}^*}$.

Proposition 5. Let $\mathrm{F}: \mathbb{L}^n \to \mathbb{L}$ be an isotone function. Then $\ddot{\mathrm{F}}(\mathscr{D}_S) \subseteq \mathscr{D}_S$ and $\ddot{\mathrm{F}}(\mathscr{D}) \subseteq \mathscr{D}$

Proof. For any $i = 1,\dots,n$, let $X_i \in \mathscr{D}_S$. Then $X_i = ([x_i, x_i],[y_i, y_i])$ for some $x_i, y_i \in [0,1]$ such that $x_i + y_i \leq 1$. Since $\overrightarrow{X_\iota} = [x_i, 1 - y_i] = \overleftarrow{X_\iota}$ then, by *Eq. (16)*, $\ddot{F}(X_1,\dots,X_n)$ and $\ddot{F}(\widetilde{X_1,\dots,X_n})$ are degenerate intervals and so $\ddot{F}(X_1,\dots,X_n) \in \mathscr{D}_S$.

For any $i = 1,\dots,n$, let $X_i \in \mathscr{D}$. Then $X_i = ([x_i, x_i],[1 - x_i, 1 - x_i])$ for some $x_i \in [0,1]$. Since $\overrightarrow{X_\iota} = [x_i, x_i] = \overleftarrow{X_\iota}$ then, by equation (16), $\ddot{F}(X_1,\dots,X_n)$ and $\ddot{F}(\widetilde{X_1,\dots,X_n})$ are degenerate intervals and $\ddot{F}(X_1,\dots,X_n) = \ddot{F}(\widetilde{X_1,\dots,X_n})^c$. So $\ddot{F}(X_1,\dots,X_n) \in \mathscr{D}$.

Lemma 1. Let $\mathrm{X}, \mathrm{Y} \in \mathbb{L}^*$. Then $\vec{\mathrm{X}} \subseteq \vec{\mathrm{Y}}$ and $\overleftarrow{\mathrm{X}} \subseteq \overleftarrow{\mathrm{Y}}$ iff $\mathrm{X} \leq \mathrm{Y}$ and $\tilde{\mathrm{X}} \leq \tilde{\mathrm{Y}}$. Dually, $\mathrm{X} \subseteq \mathrm{Y}$ and $\tilde{\mathrm{X}} \subseteq \tilde{\mathrm{Y}}$ iff $\vec{\mathrm{X}} \leq \vec{\mathrm{Y}}$ and $\overleftarrow{\mathrm{X}} \leq \overleftarrow{\mathrm{Y}}$.

Proof. $\vec{\mathrm{X}} \subseteq \vec{\mathrm{Y}}$ and $\overleftarrow{\mathrm{X}} \subseteq \overleftarrow{\mathrm{Y}}$ iff $\nabla(\mathrm{Y}) \leq \nabla(\mathrm{X})$, $\triangle(\mathrm{Y}) \leq \triangle(\mathrm{X})$, $\nabla(\tilde{\mathrm{X}}^c) \leq \nabla(\tilde{\mathrm{Y}}^c)$ and $\triangle(\tilde{\mathrm{X}}^c) \leq \triangle(\tilde{\mathrm{Y}}^c)$ iff $\mathrm{X} \leq \mathrm{Y}$ and $\tilde{\mathrm{X}}^c \leq \tilde{\mathrm{Y}}^c$ iff $\mathrm{X} \leq \mathrm{Y}$ and $\tilde{\mathrm{X}} \leq \tilde{\mathrm{Y}}$. The other case is analogous.

Proposition 6. Let $\mathrm{F}: \mathbb{L}^n \to \mathbb{L}$ be an isotone function. Then

$$\ddot{\mathrm{F}}(\mathrm{X}_1,\dots,\mathrm{X}_n) = \mathrm{F}(\mathrm{X}_1,\dots,\mathrm{X}_n) \text{ and } \ddot{\mathrm{F}}(\widetilde{\mathrm{X}_1,\dots,\mathrm{X}_n}) = \mathrm{F}\left(\widetilde{\mathrm{X}_1},\dots,\widetilde{\mathrm{X}_n}\right). \tag{19}$$

Proof. Straightforward from *Lemma 1* and *Corollary 1.*

Proposition 7. Let $\mathrm{F}, \mathrm{G}: \mathbb{L}^{\mathrm{n}} \rightarrow \mathbb{L}$ be isotone functions. If $\mathrm{F} \sqsubseteq_{\mathbb{L}} \mathrm{G}$ then $\ddot{\mathrm{F}} \sqsubseteq_{\mathbb{L}^*} \ddot{\mathrm{G}}$.

Proof. Let $\mathrm{X}_i \in \mathbb{L}^*$ for any $i = 1, \dots, n$. Since, $F \sqsubseteq_{\mathbb{L}} G$, then $G(\overrightarrow{\mathrm{X}_1}, \dots, \overrightarrow{\mathrm{X}_n}) \subseteq F(\overrightarrow{\mathrm{X}_1}, \dots, \overrightarrow{\mathrm{X}_n})$ and $G(\overleftarrow{\mathrm{X}_1}, \dots, \overleftarrow{\mathrm{X}_n}) \subseteq F(\overleftarrow{\mathrm{X}_1}, \dots, \overleftarrow{\mathrm{X}_n})$ and so $\underline{F(\overrightarrow{\mathrm{X}_1}, \dots, \overrightarrow{\mathrm{X}_n})} \leq \underline{G(\overrightarrow{\mathrm{X}_1}, \dots, \overrightarrow{\mathrm{X}_n})} \leq \overline{G(\overrightarrow{\mathrm{X}_1}, \dots, \overrightarrow{\mathrm{X}_n})} \leq \overline{F(\overrightarrow{\mathrm{X}_1}, \dots, \overrightarrow{\mathrm{X}_n})}$ and $\underline{F(\overleftarrow{\mathrm{X}_1}, \dots, \overleftarrow{\mathrm{X}_n})} \leq \underline{G(\overleftarrow{\mathrm{X}_1}, \dots, \overleftarrow{\mathrm{X}_n})} \leq \overline{G(\overleftarrow{\mathrm{X}_1}, \dots, \overleftarrow{\mathrm{X}_n})} \leq \overline{F(\overleftarrow{\mathrm{X}_1}, \dots, \overleftarrow{\mathrm{X}_n})}$. Therefore, by *Theorem 3*, $\ddot{F}(\mathrm{X}_1, \dots, \mathrm{X}_n) \subseteqq \ddot{\mathrm{G}}(\mathrm{X}_1, \dots, \mathrm{X}_n)$. Hence, $\ddot{F} \sqsubseteq_{\mathbb{L}^*} \ddot{G}$.

Note that, considering the interval point of view for $\mathbb{L}^*$-values, we have that

$$\ddot{\mathrm{F}}(\mathrm{X}_1, \dots, \mathrm{X}_\mathrm{n}) \simeq [\mathrm{F}(\overrightarrow{\mathrm{X}_1}, \dots, \overrightarrow{\mathrm{X}_\mathrm{n}}), \mathrm{F}(\overleftarrow{\mathrm{X}_1}, \dots, \overleftarrow{\mathrm{X}_\mathrm{n}})].$$

4.2 | $\mathbb{L}^*$-Representations of $[0,1]$-Functions

Let $x \in [0,1]$ and $\mathrm{X} \in \mathbb{L}^*$. Then $x \in^{**} \mathrm{X}$ if $\phi(x) \subseteqq \mathrm{X}$, i.e. if $1 - \nabla(\tilde{\mathrm{X}}) \leq x \leq \triangle(\mathrm{X})$. There is a close relation between $\underline{\in}$ and $\in^{**}$ as can we see in the next proposition.

Proposition 8. Let $\mathrm{X} \in \mathbb{L}^*$ and $X \in \mathbb{L}$. $X \underline{\in} \mathrm{X}$ if and only if $\underline{X} \in^{**} \mathrm{X}$ and $\overline{X} \in^{**} \mathrm{X}$

Proof. Since, trivially, $\overrightarrow{\phi(x)} = [x, x] = \overleftarrow{\phi(x)}$ for any $x \in \mathbb{L}$, then

$$\begin{array}{llll}
X \underline{\in} \mathrm{X} & iff & \vec{\mathrm{X}} \leq_{\mathbb{L}} X \leq_{\mathbb{L}} \overleftarrow{\mathrm{X}} & \\
 & iff & \vec{\mathrm{X}} \leq_{\mathbb{L}} [\underline{X}, \underline{X}] \leq_{\mathbb{L}} [\overline{X}, \overline{X}] \leq_{\mathbb{L}} \overleftarrow{\mathrm{X}} & \\
 & iff & \phi(\underline{X}) \subseteqq \mathrm{X} \;\; and \;\; \phi(\overline{X}) \subseteqq \mathrm{X} & by\ Prop.\ 4 \\
 & iff & \underline{X} \in^{**} \mathrm{X} \;\; and \;\; \overline{X} \in^{**} \mathrm{X} & by\ def.\ of\ \in^{**}
\end{array}$$

With this notion of membership, we can naturally extend the notion of $\mathbb{L}$-representation of fuzzy function for the $\mathbb{L}^*$-representation of fuzzy function and introduce a new notion of inclusion for $\mathbb{L}^*$-values.

Definition 7. Let $\mathrm{f}: [0,1]^{\mathrm{n}} \rightarrow [0,1]$ and $\mathcal{F}: (\mathbb{L}^*)^{\mathrm{n}} \rightarrow \mathbb{L}^*$. $\mathcal{F}$ is an $\mathbb{L}^*$-representation of f if for each $\mathrm{X}_\mathrm{i} \in \mathbb{L}^*$ and $\mathrm{x}_\mathrm{i} \in^{**} \mathrm{X}_\mathrm{i}$, with $\mathrm{i} = 1, \dots, \mathrm{n}$, we have that $\mathrm{f}(\mathrm{x}_1, \dots, \mathrm{x}_\mathrm{n}) \in^{**} \mathcal{F}(\mathrm{X}_1, \dots, \mathrm{X}_\mathrm{n})$

Let $\mathrm{X}, \mathrm{Y} \in \mathbb{L}^*$. Then $\mathrm{X} \subseteq^{**} \mathrm{Y}$ if for each $x \in^{**} \mathrm{X}$, also $x \in^{**} \mathrm{Y}$. However, $\subseteq^{**}$ is not a partial order (it is not antisymmetric – e.g. consider $\mathrm{X} = ([0.2,0.3], [0.4,0.5])$ and $\mathrm{Y} = ([0.1,0.3], [0.2,0.5])$). Therefore, we just consider $\subseteqq$ as the extension of inclusion order for $\mathbb{L}^*$.

Analogously to the case of $\mathbb{L}$-representation, we said that an $\mathbb{L}^*$-representation $\mathcal{F}$ of a function $f: [0,1]^n \rightarrow [0,1]$ is better than another $\mathbb{L}^*$-representation $\mathcal{G}$ of f if $\mathcal{G} \sqsubseteq_{\mathbb{L}^*} \mathcal{F}$.

Proposition 9. Let $\mathrm{f}: [0,1]^{\mathrm{n}} \rightarrow [0,1]$ and $\mathrm{F}: \mathbb{L}^{\mathrm{n}} \rightarrow \mathbb{L}$ be isotone functions. If F is an $\mathbb{L}$-representation of f then $\ddot{\mathrm{F}}$ is an $\mathbb{L}^*$-representation of f.

Proof. If $x_i \in^{**} \mathrm{X}_i$ for any $i = 1, \dots, n$, then $\phi(x_i) = ([x_i, x_i], [1 - x, 1 - x]) \subseteqq \mathrm{X}_i$ and so, by *Proposition 4*, $[x_i, x_i] \subseteq \mathrm{X}_i$ and $[x_i, x_i]^c \subseteq \widetilde{\mathrm{X}_\imath}$, or equivalently, $[x_i, x_i] \subseteq \widetilde{\mathrm{X}_\imath}^c$. Therefore, $x_i \in \widetilde{\mathrm{X}_\imath}$ and $x_i \in \widetilde{\mathrm{X}_\imath}^c$. Thus, since F is an $\mathbb{L}$-representation of f, $f(x_1, \dots, x_n) \in F(\mathrm{X}_1, \dots, \mathrm{X}_n)$ and so $[f(x_1, \dots, x_n), f(x_1, \dots, x_n)] \subseteq F(\mathrm{X}_1, \dots, \mathrm{X}_n)$ and $[f(x_1, \dots, x_n), f(x_1, \dots, x_n)]^c \subseteq F(\widetilde{\mathrm{X}_1}^c, \dots, \widetilde{\mathrm{X}_n}^c)^c$. Hence, by *Corollary 2*, $[f(x_1, \dots, x_n), f(x_1, \dots, x_n)] \subseteq \ddot{F}(\mathrm{X}_1, \dots, \mathrm{X}_n)$ and $[f(x_1, \dots, x_n), f(x_1, \dots, x_n)]^c \subseteq \ddot{F}(\widetilde{\mathrm{X}_1, \dots, \mathrm{X}_n})$. Therefore, $\phi(f(x_1, \dots, x_n)) \subseteqq \ddot{F}(\mathrm{X}_1, \dots, \mathrm{X}_n)$, i.e. $f(x_1, \dots, x_n) \in^{**} \ddot{F}(\mathrm{X}_1, \dots, \mathrm{X}_n)$. So, $\ddot{F}$ is an $\mathbb{L}^*$-representation of f.

Theorem 4. Let $f: [0,1]^n \to [0,1]$ be an isotone function. $\hat{f}$ is the best, w.r.t. $\sqsubseteq_{\mathbb{L}^*}$, $\mathbb{L}^*$-representation of f.

Proof. From *Propositions 1* and *9* and *Remark 2* it follows that $\hat{f}$ is an $\mathbb{L}^*$-representation of f. Thus, it only remains to prove that is the best one.

Let $\mathscr{F}: (\mathbb{L}^*)^n \to \mathbb{L}^*$ be another $\mathbb{L}^*$-representation of f and $X_i \in \mathbb{L}^*$ for $i = 1, \dots, n$. If $X_i \Subset X_i$, for any $i = 1, \dots, n$, then by *Proposition 8* $\underline{X_i} \in^{**} X_i$ and $\overline{X_i} \in^{**} X_i$. So, because $\mathscr{F}$ is $\mathbb{L}^*$-representation of f, $f(\underline{X_1}, \dots, \underline{X_n}) \in^{**} \mathscr{F}(X_1, \dots, X_n)$ and $f(\overline{X_1}, \dots, \overline{X_n}) \in^{**} \mathscr{F}(X_1, \dots, X_n)$. Thus, by equation (5), $\underline{\hat{f}(X_1, \dots, X_n)} \in^{**} \mathscr{F}(X_1, \dots, X_n)$ and $\overline{\hat{f}(X_1, \dots, X_n)} \in^{**} \mathscr{F}(X_1, \dots, X_n)$. Therefore, by Proposition 8, $\hat{f}(X_1, \dots, X_n) \Subset \mathscr{F}(X_1, \dots, X_n)$, i.e. $\mathscr{F}$ is an $\mathbb{L}^*$-representation of $\hat{f}$. Hence, by *Theorem 3*, $F \sqsubseteq_{\mathbb{L}^*} \hat{f}$, and so $\hat{f}$ is a better $\mathbb{L}^*$-representation of f than $\mathscr{F}$.

4.3| The Best $\mathbb{L}^*$-Representation of the OWA Operator

Aggregation functions play an important role in fuzzy sets theory, so it is natural to extend this definition for IVAIFS.

Definition 8. An n-ary function $\mathscr{A}: (\mathbb{L}^*)^n \to \mathbb{L}^*$ is an n-ary interval-valued Atanassov's intuitionistic aggregation function if

I. If $X_i \leq_{\mathbb{L}^*} Y_i$, for each $i = 1, \dots, n$, then $\mathscr{A}(X_1, \dots, X_n) \leq_{\mathbb{L}^*} \mathscr{A}(Y_1, \dots, Y_n)$;
II. $\mathscr{A}(0_{\mathbb{L}^*}, \dots, 0_{\mathbb{L}^*}) = 0_{\mathbb{L}^*}$ and $\mathscr{A}(1_{\mathbb{L}^*}, \dots, 1_{\mathbb{L}^*}) = 1_{\mathbb{L}^*}$.

Theorem 5. Let $A: [0,1]^n \to [0,1]$ be an n-ary aggregation function. Then $\widehat{A}$ is an n-ary interval-valued Atanassov's intuitionistic aggregation function. Moreover, if A is idempotent and/or symmetric, then $\widehat{A}$ is also idempotent and/or symmetric.

Proof. Straightforward from *Corollary 2* and *Remark 2*.

In order to motivate the next section, we will need some arithmetic operations on $\mathbb{L}^*$.

Scalar product. The multiplication $\odot$ of an scalar $\lambda \in [0,1]$ by $X \in \mathbb{L}^*$ is defined by

$$\lambda \odot X = \left(\lambda X, \lambda \tilde{X}\right). \tag{20}$$

Division by a positive integer. Let $n \in \mathbb{Z}^+$ be a positive integer, then $\frac{X}{n} = \frac{1}{n} \odot X$

Limited addition. Let $X, Y \in L^*$. Then

$$X \oplus Y = (X[+]Y, \tilde{X}[+]\tilde{Y}). \tag{21}$$

It is clear that these operations are well defined, i.e. they always provide an element of $\mathbb{L}^*$.

Definition 9. Let Λ be an n-ary weighting vector, i.e. $\Lambda = (\lambda_1, \dots, \lambda_n) \in [0,1]^n$ such that $\sum_{i=1}^{n} \lambda_i = 1$. The n-dimensional interval-valued intuitionistic weighted average $\mathbb{L}^* - WA_\Lambda$ is given by

$$\mathbb{L}^* - WA_\Lambda(X_1, \dots, X_n) = \sum_{i=1}^{n} \lambda_i \odot X_i, \tag{22}$$

where the sum is w.r.t. the limited addition.

Lemma 2. Let $X, Y \in \mathbb{L}^*$ and $\lambda_1, \lambda_2 \in [0,1]$ such that $\lambda_1 + \lambda_2 \leq 1$. Then $\lambda_1 \odot X \oplus \lambda_2 \odot Y = (\lambda_1 X + \lambda_2 Y, \lambda_1 \tilde{X} + \lambda_2 \tilde{Y})$.

Proof. Straightforward from *Eqs. (7)*, *(20)* and *(21)*.

Lemma 3. Let Λ be a weighting vector. Then, $\widehat{\mathrm{wa}}_\Lambda(\mathrm{X}_1^c, \dots, \mathrm{X}_n^c)^c = \widehat{\mathrm{wa}}_\Lambda(\mathrm{X}_1, \dots, \mathrm{X}_n)$.

Proof. Straightforward from *Proposition 1* and the fact that $1 - wa_\Lambda(1 - x_1, \dots, 1 - x_n) = wa_\Lambda(x_1, \dots, x_n)$.

Theorem 6. Let Λ be a weighting vector. Then $\mathbb{L}^* - WA_\Lambda = \widehat{wa}_\Lambda$, i.e. is the best $\mathbb{L}^*$-representation of the weighted average operator.

Proof. First note that by the monotonicity of the weighted average operator, $\widehat{wa}_\Lambda(X_1, \dots, X_n) = [wa_\Lambda(\underline{X_1}, \dots, \underline{X_n}), wa_\Lambda(\overline{X_1}, \dots, \overline{X_n})] = \sum_{i=1}^n \lambda_i X_i$. So,

$$\begin{aligned}
\widehat{wa}_\Lambda(\mathrm{X}_1, \dots, \mathrm{X}_n) &= (\widehat{wa}_\Lambda(\mathrm{X}_1, \dots, \mathrm{X}_n), \widehat{wa}_\Lambda(\widetilde{\mathrm{X}_1^c}, \dots, \widetilde{\mathrm{X}_n^c})^c) \quad by\ \ Cor.\ 2 \\
&= (\widehat{wa}_\Lambda(\mathrm{X}_1, \dots, \mathrm{X}_n), \widehat{wa}_\Lambda(\widetilde{\mathrm{X}_1}, \dots, \widetilde{\mathrm{X}_n})) \quad by\ \ Lemma\ \ 23 \\
&= (\textstyle\sum_{i=1}^n \lambda_i \mathrm{X}_i, \sum_{i=1}^n \lambda_i \widetilde{\mathrm{X}_i}) \quad by\ \ Prop.\ 231 \\
&= \textstyle\sum_{i=1}^n \lambda_i \odot \mathrm{X}_i \quad by\ \ Lemma\ \ 2312\,.
\end{aligned}$$

Definition 10. Let Λ be an n-ary weighting vector, i.e. $\Lambda = (\lambda_1, \dots, \lambda_n) \in [0,1]^n$ such that $\sum_{i=1}^n \lambda_i = 1$. The n-dimensional interval-valued intuitionistic weighted addition $\mathbb{L}^* - OWA_\Lambda$ is given by

$$\mathbb{L}^* - \mathrm{OWA}_{\Lambda}(\mathrm{X}_1, \dots, \mathrm{X}_n) = \sum_{i=1}^{n} \lambda_i \odot \mathrm{X}_{\gamma(i)}, \tag{23}$$

where the sum is w.r.t. the limited addition and

$$\mathrm{X}_{\gamma(i)} = ([\nabla(\mathrm{X}_{\gamma_1(i)}), \triangle(\mathrm{X}_{\gamma_2(i)})], [\nabla(\widetilde{\mathrm{X}_{\gamma_3(i)}}), \triangle(\widetilde{\mathrm{X}_{\gamma_4(i)}})]). \tag{24}$$

with $\gamma_j: \{0,1 \dots, n\} \to \{0,1 \dots, n\}$ for $j = 1, \dots, 4$, being permutations such that $\nabla(\mathrm{X}_{\gamma_1(i)}) \geq \nabla(\mathrm{X}_{\gamma_1(i+1)})$, $\triangle(\mathrm{X}_{\gamma_2(i)}) \geq \triangle(\mathrm{X}_{\gamma_2(i+1)})$, $\nabla(\widetilde{\mathrm{X}_{\gamma_3(i)}}) \leq \nabla(\widetilde{\mathrm{X}_{\gamma_3(i+1)}})$ and $\triangle(\widetilde{\mathrm{X}_{\gamma_4(i)}}) \leq \triangle(\widetilde{\mathrm{X}_{\gamma_4(i+1)}})$ for any $i = 1, \dots, n-1$.

Lemma 4. Let Λ be a weighting vector. Then, $\widehat{owa}_\Lambda(X_1^c, \dots, X_n^c)^c = \widehat{owa}_{\Lambda^r}(X_1, \dots, X_n)$ where $\Lambda^r = (\lambda_n, \dots, \lambda_1)$.

Proof. Straightforward from *Proppsition 1* and the fact that $1 - owa_\Lambda(1 - x_1, \dots, 1 - x_n) = owa_{\Lambda^r}(x_1, \dots, x_n)$.

Theorem 7. Let Λ be a weighting vector. Then $\mathbb{L}^* - OWA_\Lambda = \widehat{owa}_\Lambda$, i.e. is the best $\mathbb{L}^*$-representation of the ordered weighted average operator.

Proof.

$$
\begin{aligned}
& \mathbb{L}^* - \mathrm{OWA}_{\Lambda}(\mathrm{X}_1, \dots, \mathrm{X}_n) && \\
= \; & \sum_{i=1}^{n} \lambda_i \odot \mathrm{X}_{\gamma(i)} && \text{by eq. (23)} \\
= \; & \mathbb{L}^* - \mathrm{WA}_{\Lambda}(\mathrm{X}_{\gamma(1)}, \dots, \mathrm{X}_{\gamma(n)}) && \text{by eq. (2322)} \\
= \; & \widehat{\mathrm{wa}}_{\Lambda}(\mathrm{X}_{\gamma(1)}, \dots, \mathrm{X}_{\gamma(n)}) && \text{by Thm. 23226} \\
= \; & (\widehat{\mathrm{wa}}_{\Lambda}(\mathrm{X}_{\gamma(1)}, \dots, \mathrm{X}_{\gamma(n)}), \widehat{\mathrm{wa}}_{\Lambda}(\widetilde{\mathrm{X}_{\gamma(1)}}^{c}, \dots, \widetilde{\mathrm{X}_{\gamma(n)}}^{c})^{c}) && \text{by eq. (2322618)} \\
= \; & (\widehat{\mathrm{wa}}_{\Lambda}(\mathrm{X}_{\gamma(1)}, \dots, \mathrm{X}_{\gamma(n)}), \widehat{\mathrm{wa}}_{\Lambda}(\widetilde{\mathrm{X}_{\gamma(1)}}, \dots, \widetilde{\mathrm{X}_{\gamma(n)}})) && \text{by Lemma 23226183} \\
= \; & (\widehat{\mathrm{wa}}_{\Lambda}([\nabla(\mathrm{X}_{\gamma_1(1)}), \triangle(\mathrm{X}_{\gamma_2(1)})], \dots, [\nabla(\mathrm{X}_{\gamma_1(n)}), \triangle(\mathrm{X}_{\gamma_2(n)})], && \\
& \;\widehat{\mathrm{wa}}_{\Lambda}([\nabla(\widetilde{\mathrm{X}_{\gamma_3(1)}}), \triangle(\widetilde{\mathrm{X}_{\gamma_4(1)}})], \dots, [\nabla(\widetilde{\mathrm{X}_{\gamma_3(n)}}), \triangle(\widetilde{\mathrm{X}_{\gamma_4(n)}})])) && \text{by eq. (2322618324)} \\
= \; & ([\mathrm{wa}_{\Lambda}(\nabla(\mathrm{X}_{\gamma_1(1)}), \dots, \nabla(\mathrm{X}_{\gamma_1(n)})), \mathrm{wa}_{\Lambda}(\triangle(\mathrm{X}_{\gamma_2(1)}), \dots, \triangle(\mathrm{X}_{\gamma_2(n)}))], && \\
& \;[\mathrm{wa}_{\Lambda}(\nabla(\widetilde{\mathrm{X}_{\gamma_3(1)}}), \dots, \nabla(\widetilde{\mathrm{X}_{\gamma_3(n)}})), \mathrm{wa}_{\Lambda}(\triangle(\widetilde{\mathrm{X}_{\gamma_4(1)}}), \dots, \triangle(\widetilde{\mathrm{X}_{\gamma_4(n)}}))]) && \text{by eq. (23226183245)} \\
= \; & ([\mathrm{wa}_{\Lambda}(\nabla(\mathrm{X}_{\gamma_1(1)}), \dots, \nabla(\mathrm{X}_{\gamma_1(n)})), \mathrm{wa}_{\Lambda}(\triangle(\mathrm{X}_{\gamma_2(1)}), \dots, \triangle(\mathrm{X}_{\gamma_2(n)}))], && \\
& \;[\mathrm{wa}_{\Lambda^r}(\nabla(\widetilde{\mathrm{X}_{\gamma_3(n)}}), \dots, \nabla(\widetilde{\mathrm{X}_{\gamma_3(1)}})), \mathrm{wa}_{\Lambda^r}(\triangle(\widetilde{\mathrm{X}_{\gamma_4(n)}}), \dots, \triangle(\widetilde{\mathrm{X}_{\gamma_4(1)}}))]) && \\
= \; & ([\mathrm{owa}_{\Lambda}(\nabla(\mathrm{X}_{\gamma(1)}), \dots, \nabla(\mathrm{X}_{\gamma(n)})), \mathrm{owa}_{\Lambda}(\triangle(\mathrm{X}_{\gamma(1)}), \dots, \triangle(\mathrm{X}_{\gamma(n)}))], && \\
& \;[\mathrm{owa}_{\Lambda^r}(\nabla(\widetilde{\mathrm{X}_{\gamma(1)}}), \dots, \nabla(\widetilde{\mathrm{X}_{\gamma(n)}})), \mathrm{owa}_{\Lambda^r}(\triangle(\widetilde{\mathrm{X}_{\gamma(1)}}), \dots, \triangle(\widetilde{\mathrm{X}_{\gamma(n)}}))]) && \text{by eq. (232261832456)} \\
= \; & (\widehat{\mathrm{owa}}_{\Lambda}(\mathrm{X}_{\gamma(1)}, \dots, \mathrm{X}_{\gamma(n)}), \widehat{\mathrm{owa}}_{\Lambda^r}(\widetilde{\mathrm{X}_{\gamma(1)}}, \dots, \widetilde{\mathrm{X}_{\gamma(n)}})) && \text{by eq. (2322618324565)} \\
= \; & (\widehat{\mathrm{owa}}_{\Lambda}(\mathrm{X}_1, \dots, \mathrm{X}_n), \widehat{\mathrm{owa}}_{\Lambda}(\widetilde{\mathrm{X}_1}^{c}, \dots, \widetilde{\mathrm{X}_n}^{c})^{c}) && \text{by Lemma 23226183245654} \\
= \; & \widehat{\mathrm{owa}}_{\Lambda}(\mathrm{X}_1, \dots, \mathrm{X}_n). && \text{by eq. (2322618324565418)}
\end{aligned}
$$

Corollary 4. $\mathbb{L}^* - OWA_{\Lambda}$ is an idempotent and symmetric n-ary interval-valued Atanassov's intuitionistic aggregation function. In addition, $\mathbb{L}^* - OWA_{\Lambda}$ is bounded, i.e. $\widehat{min} \leq_{\mathbb{L}^*} \mathbb{L}^* - OWA_{\Lambda} \leq_{\mathbb{L}^*} \widehat{max}$

Proof. Straightforward from *Theorems 7* and *5* and *Corollary 3*.

5| A Method for Multi-attribute Group Decision Making Based Interval-Valued Atanassov's Intuitionistic Decision Matrices

Let $E = \{e_1, \dots, e_m\}$ be a set of experts, $X = \{x_1, \dots, x_n\}$ be a finite set of alternatives, and $A = \{a_1, \dots, a_p\}$ be a set of attributes or criteria. The decision makers determines a weighting vector $W = (w_1, \dots, w_p)^T$ for the attributes. A method for MAGDM based on IVAIDM is an algorithm which determines a ranking of the alternatives in X based in the opinion of each expert in E of how much the alternatives attend each attribute. In particular we consider the case where the evaluation of the experts contains imprecision and hesitation which is represented by interval-valued Atanassov's intuitionistic degrees.

We propose the next method (algorithm) to obtain such ranking:

X, W, and for every $l = 1, \dots, m$ an $\mathbb{L}^*$-valued decision matrix R^l of dimension $n \times p$ where each position (i, j) in R^l, denoted by R^l_{ij}, contains the interval-valued Atanassov's intuitionistic value which reflects how much the alternative x_i attends the attribute (or criterium[1]) a_j.

A ranking $r: X \to \{1, \dots, n\}$, denoting that an alternative $x \in X$ is better than an alternative $y \in X$ whenever $r(x) \leq r(y)$ and when $r(x) = r(y)$ meaning that the method is not able of determine if x is better or worst alternative than y[2].

[1] For the case of the cost criteria is considered the usal complement of these interval-valued Atanassov's intuitionistic values.

[2] The most decision making methods admits cases for which the method is unable of discriminate between two different alternatives which is better.

Step 1. Aggregate the IVAIDM of all experts in a single IVAIDM $\mathcal{RC}$, for each $i = 1, \dots, n$ and $j = 1, \dots, p$, as follows:

$$\mathscr{RC}_{\mathrm{ij}} = \widehat{\mathrm{owa}}_{\Lambda}\left(\mathrm{R}_{\mathrm{ij}}^{1}, \dots, \mathrm{R}_{\mathrm{ij}}^{\mathrm{m}}\right). \tag{25}$$

where $\Lambda = (\lambda_1, \dots, \lambda_m)$ is the following weighting vector:

1. Case m is even: $\lambda_i = \frac{1}{2^{\frac{m}{2}+2-i}} + \frac{1}{m2^{\frac{m}{2}}}$ for each $i = 1, \dots, \frac{m}{2}$, and $\lambda_i = \lambda_{m+1-i}$ for each $i = \frac{m}{2} + 1, \dots, m$.
2. Case m is odd: $\lambda_i = \frac{1}{2^{\frac{m+1}{2}+2-i}} + \frac{1}{m2^{\frac{m+1}{2}}} + \frac{1}{4m}$ for each $i = 1, \dots, \frac{m+1}{2}$, and $\lambda_i = \lambda_{m+1-i}$ for each $i = \frac{m+1}{2} + 1, \dots, m$.

Table 1. Assesses of expert $\mathbf{p_1}$.

$\mathbf{R^1}$	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_3}$	$\mathbf{a_4}$
A_1	([0.4,0.8],[0.0,0.1])	([0.3,0.6],[0.0,0.2])	([0.2,0.7],[0.2,0.3])	([0.3,0.4],[0.4,0.5])
A_2	([0.5,0.7],[0.1,0.2])	([0.3,0.5],[0.2,0.4])	([0.4,0.7],[0.0,0.2])	([0.1,0.2],[0.7,0.8])
A_3	([0.5,0.7],[0.2,0.3])	([0.6,0.8],[0.1,0.2])	([0.4,0.7],[0.1,0.2])	([0.6,0.8],[0.0,0.2])

$\mathscr{RC}$ is the IVAIDM of consensus of all expert opinions[1].

Step 2. For each alternative x_i, with $i = 1, \dots, n$, using $\widehat{wa}_W$, determine the collective overall index $\mathbb{L}^*$-value O_i as follows:

$$\mathrm{O_i} = \widehat{\mathrm{wa}}_{\mathrm{W}}(\mathscr{RC}_{\mathrm{i1}}, \dots, \mathscr{RC}_{\mathrm{in}}). \tag{26}$$

Step 3. Rank the alternatives by considering a total order on their collective overall index $\mathbb{L}^*$-values and choosing the greatest one. Thus, the output function $r: X \to \{1, \dots, n\}$ is defined by $r(x_i) = j$ iff O_i is the jth greatest collective overall index. Notice that if two or more alternatives, e.g. x and y, have the same collective overall index, then $r(x) = r(y)$.

Example 1. Consider the air-condition system selection problem used as example in [62]. This problem considers three air-condition systems (alternatives) $\{A_1, A_2, A_3\}$; four attributes: a_1 (economical), a_2 (function), a_3 (being operative) and a_4 (longevity); and three experts $\{p_1, p_2, p_3\}$. By using statistical methods, for each expert p_l, alternative A_i and atribute a_j an interval-valued membership degree and an interval-valued non-membership degree, i.e. an IVAIFV, is provided. These IVAIFV are summarized in the *Tables 1*, *2* and *3* (the same used in [62]). We consider the weighting vector $W = (0.2134, 0.1707, 0.2805, 0.3354)$ for the attributes[2] .

Since we have three experts ($m = 3$), then the weighting vector Λ is calculated as following:

$$\lambda_1 = \frac{1}{2^3} + \frac{1}{3 \cdot 2^2} + \frac{1}{4 \cdot 3} = \frac{1}{8} + \frac{1}{6} = 0.291\overline{6},$$

$$\lambda_2 = \frac{1}{2^2} + \frac{1}{3 \cdot 2^2} + \frac{1}{4 \cdot 3} = \frac{1}{4} + \frac{1}{6} = 0.41\overline{6},$$

$$\lambda_3 = \frac{1}{2^3} + \frac{1}{3 \cdot 2^2} + \frac{1}{4 \cdot 3} = \frac{1}{8} + \frac{1}{6} = 0.291\overline{6}.$$

[1] It is not hard of prove that when $n > 1$, Λ is a weighting vector.

[2] In [62] was considered the weights $\mathrm{V} = (0.35, 0.28, 0.46, 0.55)$ which not satisfy the condition that the sums of the weights must be equal to 1. W is the weighting vector obtained normalizing V in order to satisfy this condition.

Table 2. Assesses of expert p_2.

R^2	a_1	a_2	a_3	a_4
A_1	([0.5,0.9],[0.0,0.1])	([0.4,0.5],[0.3,0.5])	([0.5,0.8],[0.0,0.1])	([0.4,0.7],[0.1,0.2])
A_2	([0.7,0.8],[0.1,0.2])	([0.5,0.6],[0.2,0.3])	([0.5,0.8],[0.0,0.2])	([0.5,0.6],[0.3,0.4])
A_3	([0.5,0.6],[0.1,0.4])	([0.6,0.7],[0.1,0.2])	([0.4,0.8],[0.1,0.2])	([0.2,0.6],[0.2,0.3])

Table 3. Assesses of expert p_3.

R^3	a_1	a_2	a_3	a_4
A_1	([0.3,0.9],[0.0,0.1])	([0.2,0.5],[0.1,0.4])	([0.4,0.7],[0.1,0.2])	([0.3,0.6],[0.3,0.4])
A_2	([0.3,0.8],[0.1,0.2])	([0.5,0.6],[0.1,0.3])	([0.2,0.8],[0.0,0.2])	([0.3,0.5],[0.2,0.3])
A_3	([0.2,0.6],[0.1,0.2])	([0.2,0.6],[0.2,0.3])	([0.3,0.6],[0.1,0.3])	([0.4,0.7],[0.1,0.2])

The *Table 4* present the collective reflexive IvIFPR obtained from *Tables 1*, *2* and *3* by consider the *Eq. (25)*.

The collective overall preference obtained by using the calculation in *Eq. (26)*, is the following:

$$O_1 = ([0.3509555488, 0.6721], [0.140916, 0.2651]),$$

$$O_2 = ([0.3867014634, 0.6262], [0.180441, 0.3184]),$$

$$O_3 = ([0.4086795732, 0.6848], [0.111192, 0.2443]).$$

Thus, considering this collective overall preference and the total orders shows in section III.B, we have the ranking of the alternatives in the *Table 5*. Therefore, all the ranking obtained with this method, for the different the orders considered, agree with four of the five ranking obtained in [39], [62], [63], for this same illustrative example.

Example 2. Consider the investment choice problem used as example in [59], [60]. This problem considers an investment company which would like to invest a sum of money in the best option among the following five possible alternatives to invest the money: A_1 is a car company; A_2 is a food company; A_3 is a computer company; A_4 is an arms company; and A_5 is a TV company. The choice of the best investmente must be made taking into account the following four benefit criteria: c_1 is the profit ability; c_2 is the growth analysis; c_3 is the social-political impact; and c_4 is the enterprise culture. The five possible alternatives will be evaluated considering the interval-valued intuitionistic fuzzy information given by three decision makers e_1, e_2 and e_3, who evaluate how much the alternative satisfies each one of the criterias. These informations are summarized in the *Tables 6*, *7* and *8* (the same considered in [58], [59], [60]).

Since, in [59], [60] it was not considered a weight for the criteria, here we consider that all criteria have the same weight, i.e. we consider $W = (0.25, 0.25, 0.25, 0.25)$. The ranking obtained by using our method considering the four total orders and the obtained by [59], [60] is summarized in the *Table 9*.

Table 4. Collective reflexive IvIFPR.

RC	a_1	a_2	a_3	a_4
A_1	([0.4,0.871],[0.0,0.1])	([0.3,0.53],[0.13,0.371])	([0.371,0.73],[0.1,0.2])	([0.33,0.571],[0.271,0.371])
A_2	([0.5,0.771],[0.1,0.2])	[0.4416,0.571],[0.171,0.33])	[0.371,0.771],[0.0,0.2])	[0.3,0.4416],[0.3875,0.4874])
A_3	[0.4125,0.63],[0.13,0.3])	([0.4833,0.7],[0.13,0.23])	([0.371,0.7],[0.1,0.23])	([0.4,0.7],[0.1,0.23])

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Table 5. Ranking obtained for the alternatives considering several total orders and the obtained in [62], [63].

$\lessapprox$	$\precsim$	$\preccurlyeq$	[62]	[63] (a), (b) and (d)	[63] (c)
A_3	A_3	A_3	A_3	A_3	A_2
A_1	A_1	A_1	A_1	A_1	A_1
A_2	A_2	A_2	A_2	A_2	A_3

Table 6. Assesses of expert e_1.

R^1	a_1	a_2	a_3	a_4
A_1	([0.4,0.5],[0.3,0.4])	([0.4,0.6],[0.2,0.4])	([0.1,0.3],[0.5,0.6])	([0.3,0.4],[0.3,0.5])
A_2	([0.6,0.7],[0.2,0.3])	([0.6,0.7],[0.2,0.3])	([0.4,0.7],[0.1,0.2])	([0.5,0.6],[0.1,0.3])
A_3	([0.6,0.7],[0.1,0.2])	([0.5,0.6],[0.3,0.4])	([0.5,0.6],[0.1,0.3])	([0.4,0.5],[0.2,0.4])
A_4	([0.3,0.4],[0.2,0.3])	([0.6,0.7],[0.1,0.3])	([0.3,0.4],[0.1,0.2])	([0.3,0.7],[0.1,0.2])
A_5	([0.7,0.8],[0.1,0.2])	([0.3,0.5],[0.1,0.3])	([0.5,0.6],[0.2,0.3])	([0.3,0.4],[0.5,0.6])

Table 7. Assesses of expert e_2.

R^2	a_1	a_2	a_3	a_4
A_1	([0.3,0.4],[0.4,0.5])	([0.5,0.6],[0.1,0.3])	([0.4,0.5],[0.3,0.4])	([0.4,0.6],[0.2,0.4])
A_2	([0.3,0.6],[0.3,0.4])	([0.4,0.7],[0.1,0.2])	([0.5,0.6],[0.2,0.3])	([0.6,0.7],[0.2,0.3])
A_3	([0.6,0.8],[0.1,0.2])	([0.5,0.6],[0.1,0.2])	([0.5,0.7],[0.2,0.3])	([0.1,0.3],[0.5,0.6])
A_4	([0.4,0.5],[0.3,0.5])	([0.5,0.8],[0.1,0.2])	([0.2,0.5],[0.3,0.4])	([0.4,0.7],[0.1,0.2])
A_5	([0.6,0.7],[0.2,0.3])	([0.6,0.7],[0.1,0.2])	([0.5,0.7],[0.2,0.3])	([0.6,0.7],[0.1,0.3])

Table 8. Assesses of expert e_3.

R^3	a_1	a_2	a_3	a_4
A_1	([0.2,0.5],[0.3,0.4])	([0.4,0.5],[0.1,0.2])	([0.3,0.6],[0.2,0.3])	([0.3,0.7],[0.1,0.3])
A_2	([0.2,0.7],[0.2,0.3])	([0.3,0.6],[0.2,0.4])	([0.4,0.7],[0.1,0.2])	([0.5,0.8],[0.1,0.2])
A_3	([0.5,0.6],[0.3,0.4])	([0.7,0.8],[0.1,0.2])	([0.5,0.6],[0.2,0.3])	([0.4,0.5],[0.3,0.4])
A_4	([0.3,0.6],[0.2,0.4])	([0.4,0.6],[0.2,0.3])	([0.1,0.4],[0.3,0.6])	([0.3,0.7],[0.1,0.2])
A_5	([0.6,0.7],[0.1,0.3])	([0.5,0.6],[0.3,0.4])	([0.5,0.6],[0.2,0.3])	([0.5,0.6],[0.2,0.4])

Table 9. Ranking obtained for the alternatives considering several total orders and the obtained in [62].

Proposed Method			The methods proposed in [59], [60]				
$\lessapprox$	$\precsim$	$\preccurlyeq$	[60]	[59] $\gamma < 0.378$	[59] $\gamma = 0.378$	[59] $0.378 < \gamma < 1$	[59] $\gamma = 1$
A_5	A_5	A_2	A_5	A_3	$A_3 \sim A_5$	A_5	A_5
A_2	A_2	A_5	A_2	A_5		A_3	$A_3 \sim A_2$
A_3	A_3	A_3	A_3	A_2	A_2	A_2	
A_4	A_4	A_4	A_4	A_4	A_4	A_4	A_4
A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1

Thus, making an analysis of these rankings of the alternatives we have that there is an absolute consensus that the worst alternative is *A1* and the second worst alternative is *A4*. On the other hand, if we consider, for the other alternatives, the amount of times that an alternative was better ranked than the others (which is summarized in the *Table 10*) we can conclude that the more rasonable ranking of the alternatives would be$A_5 > A_2 > A_3 > A_4 > A_1$

which agrees with the ranking obtained in [60] and also for the proposed method with the orders $\lessapprox$ and $\precsim$.

This way of aggregate or fuses many rankings of a set of alternatives corresponds to the ranking fusion function M2 of [20].

6| Final remarks

This paper proposes a new extension of the OWA and WA operators in the context of interval-valued intuitionistic fuzzy values, which has as main characteristic by the best $\mathbb{L}^*$-representation of the usual OWA and WA operators. Therefore, when applied to the diagonal elements these new operators have the same behaviour as the OWA and WA. This paper also extended the notion of interval representations introduced in [54] for $\mathbb{L}^*$-representations, and has introduced a new notion of inclusion for $\mathbb{L}^*$-values which is based in a notion of membership. Besides, we introduced a new total order for $\mathbb{L}^*$-values and provide new extensions of the OWA operator for $\mathbb{L}$ and L^*-values.

Table 10. Comparing based on the Table 9.

R^3	A_2	A_3	A_5
A_2	–	5	1
A_3	3	–	1
A_5	7	5	–

We have shown the validity of our theoretical develpments by means of an illustrative decision-making example. In [32] was introduced an interval-valued Atanassov's intuitionistic extension of OWA's where the weights are assigned by decreasingly ordering the inputs with respect to an admissible order. The problem with this OWA is that in general it is not increasing with respect to the admissible order. So, as future work we intend to investigate OWAs on $\mathbb{L}^*$ which are increasing with respect to a fixed admissible order. In addition, based on [17], we will use such OWAs in a method to select the most important vertice of an Interval-Valued Intuitionistic Fuzzy Graph [7].

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References

[1] da Cruz Asmus, T., Dimuro, G. P., Bedregal, B., Sanz, J. A., Pereira Jr, S., & Bustince, H. (2020). General interval-valued overlap functions and interval-valued overlap indices. *Information sciences*, *527*, 27-50.

[2] Atanassov, K. (2016). Intuitionistic fuzzy sets. *International journal bioautomation*, *20*, 1.

[3] Atanassov, K. T. (1999). Interval valued intuitionistic fuzzy sets. In *Intuitionistic fuzzy sets* (pp. 139-177). Physica, Heidelberg.

[4] Atanassov, K. T. (1994). Operators over interval valued intuitionistic fuzzy sets. *Fuzzy sets and systems*, *64*(2), 159-174.

[5] Atanassov, K. T. (1999). Intuitionistic fuzzy sets. In *Intuitionistic fuzzy sets* (pp. 1-137). Physica, Heidelberg.

[6] Atanassov, K., & Rangasamy, P. (2018). Intuitionistic fuzzy sets and interval valued intuitionistic fuzzy sets. *Advanced studies in contemporary mathematics*, *28*(2), 167-176.

[7] Nayagam, V. L. G., Muralikrishnan, S., & Sivaraman, G. (2011). Multi-criteria decision-making method based on interval-valued intuitionistic fuzzy sets. *Expert systems with applications*, *38*(3), 1464-1467.

[8] Bedregal, B. C. (2010). On interval fuzzy negations. *Fuzzy sets and systems*, *161*(17), 2290-2313.

[9] Bedregal, B., Beliakov, G., Bustince, H., Calvo, T., Mesiar, R., & Paternain, D. (2012). A class of fuzzy multisets with a fixed number of memberships. *Information sciences*, *189*, 1-17.

[10] Bedregal, B. C., Dimuro, G. P., Santiago, R. H. N., & Reiser, R. H. S. (2010). On interval fuzzy S-implications. *Information sciences*, *180*(8), 1373-1389.

[11] Bedregal, B., & Santiago, R. (2013). Some continuity notions for interval functions and representation. *Computational and applied mathematics*, *32*(3), 435-446.

[12] Bedregal, B. C., & Santiago, R. H. N. (2013). Interval representations, Łukasiewicz implicators and Smets–Magrez axioms. *Information sciences*, *221*, 192-200.

[13] Bedregal, B. R. C., & Takahashi, A. (2005, May). Interval t-norms as interval representations of t-norms. *The 14th IEEE international conference on fuzzy systems, 2005. FUZZ'05.* (pp. 909-914). IEEE.

[14] Bedregal, B. R., Dimuro, G. P., Reiser, R. H. S., Carvalho, J. P., Dubois, D., Kaymak, U., & da Costa Sousa, J. M. (2009). An approach to interval-valued r-implications and automorphisms. In *IFSA/EUSFLAT Conf.* (pp. 1-6). Lisbon, Portugal.

[15] Beliakov, G., Bustince, H., Goswami, D. P., Mukherjee, U. K., & Pal, N. R. (2011). On averaging operators for Atanassov's intuitionistic fuzzy sets. *Information sciences*, *181*(6), 1116-1124.

[16] Beliakov, G., Pradera, A., & Calvo, T. (2007). *Aggregation functions: A guide for practitioners* (Vol. 221). Heidelberg: Springer.

[17] Bello Lara, R., González Espinosa, S., Martín Ravelo, A., & Leyva Vázquez, M. Y. (2015). Model for static analysis in fuzzy graphs based on composite indicators of centrality. *Revista cubana de ciencias informáticas*, *9*(2), 52-65. (**In Spanish**). http://scielo.sld.cu/scielo.php?pid=S2227-18992015000200004&script=sci_arttext&tlng=en

[18] Bustince Sola, H., Barrenechea Tartas, E., Pagola Barrio, M., Fernández Fernández, F. J., Xu, Z., Bedregal, B., ... & Baets, B. D. (2016). A historical account of types of fuzzy sets and their relationships. *IEEE transactions on fuzzy systems, 24(1).* https://hdl.handle.net/2454/32795

[19] Bustince, H., Barrenechea, E., & Pagola, M. (2008). Generation of interval-valued fuzzy and atanassov's intuitionistic fuzzy connectives from fuzzy connectives and from Kα operators: Laws for conjunctions and disjunctions, amplitude. *International journal of intelligent systems*, *23*(6), 680-714.

[20] Bustince, H., Bedregal, B., Campion, M. J., Da Silva, I., Fernandez, J., Indurain, E., ... & Santiago, R. H. N. (2020). Aggregation of individual rankings through fusion functions: criticism and optimality analysis. *IEEE transactions on fuzzy systems*. DOI: 10.1109/TFUZZ.2020.3042611

[21] Bustince, H., & Burillo, P. (1995). Correlation of interval-valued intuitionistic fuzzy sets. *Fuzzy sets and systems*, *74*(2), 237-244.

[22] Bustince, H., Fernández, J., Kolesárová, A., & Mesiar, R. (2013). Generation of linear orders for intervals by means of aggregation functions. *Fuzzy sets and systems*, *220*, 69-77.

[23] Bustince, H., Galar, M., Bedregal, B., Kolesarova, A., & Mesiar, R. (2013). A new approach to interval-valued Choquet integrals and the problem of ordering in interval-valued fuzzy set applications. *IEEE transactions on fuzzy systems*, *21*(6), 1150-1162.

[24] Chiclana, F., Herrera-Viedma, E., Alonso, S., Alberto, R., & Pereira, M. (2008). Preferences and consistency issues in group decision making. In *Fuzzy sets and their extensions: representation, aggregation and models* (pp. 219-237). Springer, Berlin, Heidelberg.

[25] da Costa, C. G., Bedregal, B. C., & Neto, A. D. D. (2011). Relating De Morgan triples with Atanassov's intuitionistic De Morgan triples via automorphisms. *International journal of approximate reasoning*, *52*(4), 473-487.

[26] Deschrijver, G., Cornelis, C., & Kerre, E. E. (2004). On the representation of intuitionistic fuzzy t-norms and t-conorms. *IEEE transactions on fuzzy systems*, *12*(1), 45-61.

[27] Chen, S. M., & Tan, J. M. (1994). Handling multicriteria fuzzy decision-making problems based on vague set theory. *Fuzzy sets and systems*, *67*(2), 163-172.

[28] da Silva, I. A., Bedregal, B., da Costa, C. G., Palmeira, E., & da Rocha, M. P. (2015). Pseudo-uninorms and Atanassov's intuitionistic pseudo-uninorms. *Journal of intelligent & fuzzy systems*, *29*(1), 267-281.

[29] da Silva, I. A., Bedregal, B., Santiago, R. H., & Neto, A. D. (2012). A new method for interval-valued intuitionistic group decision making. In *Segundo congresso brasileiro de sistemas fuzzy* (pp. 282-294).

[30] Da Silva, I. A., Bedregal, B., & Santiago, R. H. N. (2016). On admissible total orders for interval-valued intuitionistic fuzzy membership degrees. *Fuzzy information and engineering*, *8*(2), 169-182.

[31] De Miguel, L., Bustince, H., Fernández, J., Induráin, E., Kolesárová, A., & Mesiar, R. (2016). Construction of admissible linear orders for interval-valued Atanassov intuitionistic fuzzy sets with an application to decision making. *Information fusion*, *27*, 189-197.

[32] De Miguel, L., Bustince, H., Pekala, B., Bentkowska, U., Da Silva, I., Bedregal, B., ... & Ochoa, G. (2016). Interval-valued Atanassov intuitionistic OWA aggregations using admissible linear orders and their application to decision making. *IEEE transactions on fuzzy systems*, *24*(6), 1586-1597.

[33] Deschrijver, G., & Kerre, E. E. (2003). On the relationship between some extensions of fuzzy set theory. *Fuzzy sets and systems*, *133*(2), 227-235.

[34] Dimuro, G. P., Bedregal, B. C., Santiago, R. H. N., & Reiser, R. H. S. (2011). Interval additive generators of interval t-norms and interval t-conorms. *Information sciences*, *181*(18), 3898-3916.
[35] Goguen, J. A. (1967). L-fuzzy sets. *Journal of mathematical analysis and applications*, *18*(1), 145-174.
[36] Grattan-Guinness, I. (1976). Fuzzy membership mapped onto intervals and many-valued quantities. *Mathematical logic quarterly*, *22*(1), 149-160.
[37] Hickey, T., Ju, Q., & Van Emden, M. H. (2001). Interval arithmetic: From principles to implementation. *Journal of the ACM (JACM)*, *48*(5), 1038-1068.
[38] Hong, D. H., & Choi, C. H. (2000). Multicriteria fuzzy decision-making problems based on vague set theory. *Fuzzy sets and systems*, *114*(1), 103-113.
[39] Kim, S. H., & Han, C. H. (1999). An interactive procedure for multi-attribute group decision making with incomplete information. *Computers & operations research*, *26*(8), 755-772.
[40] Jahn, K. U. (1975). Intervall-wertige Mengen. *Mathematische nachrichten*, *68*(1), 115-132.
[41] Lee, W. (2009, August). A novel method for ranking interval-valued intuitionistic fuzzy numbers and its application to decision making. *2009 international conference on intelligent human-machine systems and cybernetics* (Vol. 2, pp. 282-285). IEEE.
[42] Mesiar, R., & Komorníková, M. (2010). Aggregation functions on bounded posets. In *35 years of fuzzy set theory* (pp. 3-17). Springer, Berlin, Heidelberg.
[43] Milfont, T., Mezzomo, I., Bedregal, B., Mansilla, E., & Bustince, H. (2021). Aggregation functions on n-dimensional ordered vectors equipped with an admissible order and an application in multi-criteria group decision-making. https://arxiv.org/abs/2101.12030
[44] Mitchell, H. B. (2004). An intuitionistic OWA operator. *International journal of uncertainty, fuzziness and knowledge-based systems*, *12*(06), 843-860.
[45] Moore, R. E. (1979). *Methods and applications of interval analysis*. Society for Industrial and Applied Mathematics.
[46] Paternain, D., Jurio, A., Barrenechea, E., Bustince, H., Bedregal, B., & Szmidt, E. (2012). An alternative to fuzzy methods in decision-making problems. *Expert systems with applications*, *39*(9), 7729-7735.
[47] Pedrycz, W., Ekel, P., & Parreiras, R. (2011). *Fuzzy multicriteria decision-making: models, methods and applications*. John Wiley & Sons.
[48] Reiser, R. H. S., & Bedregal, B. (2014). K-operators: An approach to the generation of interval-valued fuzzy implications from fuzzy implications and vice versa. *Information sciences*, *257*, 286-300.
[49] Reiser, R. H. S., Bedregal, B. C., & Dos Reis, G. A. A. (2014). Interval-valued fuzzy coimplications and related dual interval-valued conjugate functions. *Journal of computer and system sciences*, *80*(2), 410-425.
[50] Reiser, R. H. S., Bedregal, B., & Visintin, L. (2013). Index, expressions and properties of interval-valued intuitionistic fuzzy implications. *TEMA (São Carlos)*, *14*, 193-208.
[51] Reiser, R. H. S., & Bedregal, B. (2017). Correlation in interval-valued Atanassov's intuitionistic fuzzy sets—conjugate and negation operators. *International journal of uncertainty, fuzziness and knowledge-based systems*, *25*(05), 787-819.
[52] Sambuc, R. (1975) Φ-fuzzy functions: Application to diagnostic aid in thyroid pathology (Phd Thesis, University of Marseille, France). (**In French**). https://www.scirp.org/(S(i43dyn45teexjx455qlt3d2q))/reference/ReferencesPapers.aspx?ReferenceID=1235361
[53] Santana, F., Bedregal, B., Viana, P., & Bustince, H. (2020). On admissible orders over closed subintervals of [0, 1]. *Fuzzy sets and systems*, *399*, 44-54.
[54] Santiago, R. H. N., Bedregal, B. R. C., & Acióly, B. M. (2006). Formal aspects of correctness and optimality of interval computations. *Formal aspects of computing*, *18*(2), 231-243.
[55] Santiago, R., Bedregal, B., Dimuro, G. P., Fernandez, J., Bustince, H., & Fardoun, H. M. (2021). Abstract homogeneous functions and Consistently influenced/disturbed multi-expert decision making. *IEEE Transactions on fuzzy systems*. DOI: 10.1109/TFUZZ.2021.3117438
[56] Shang, Y. G., Yuan, X. H., & Lee, E. S. (2010). The n-dimensional fuzzy sets and Zadeh fuzzy sets based on the finite valued fuzzy sets. *Computers & mathematics with applications*, *60*(3), 442-463.
[57] Sirbiladze, G. (2021). New view of fuzzy aggregations. part I: general information structure for decision-making models. *Journal of fuzzy extension and applications*, 2(2), 130-143. DOI: 10.22105/jfea.2021.275084.1080

[58] Tan, C. (2011). A multi-criteria interval-valued intuitionistic fuzzy group decision making with Choquet integral-based TOPSIS. *Expert systems with applications*, *38*(4), 3023-3033.

[59] Tan, C., & Chen, X. (2013). Interval-valued intuitionistic fuzzy multicriteria group decision making based on VIKOR and Choquet integral. *Journal of applied mathematics*. https://doi.org/10.1155/2013/656879

[60] Tan, C. Y., Deng, H., & Cao, Y. (2010). A multi-criteria group decision making procedure using interval-valued intuitionistic fuzzy sets. *Journal of computational information systems*, *6*(3), 855-863.

[61] Tizhoosh, H. R. (2008). Interval-valued versus intuitionistic fuzzy sets: Isomorphism versus semantics. *Pattern recognition*, *41*(5), 1812-1813.

[62] Wan, S., & Dong, J. (2014). A possibility degree method for interval-valued intuitionistic fuzzy multi-attribute group decision making. *Journal of computer and system sciences*, *80*(1), 237-256.

[63] Wan, S., & Dong, J. (2020). *Decision making theories and methods based on interval-valued intuitionistic fuzzy sets*. Springer Nature, Singapore Pte Ltd.

[64] Wang, Z., Li, K. W., & Wang, W. (2009). An approach to multiattribute decision making with interval-valued intuitionistic fuzzy assessments and incomplete weights. *Information sciences*, *179*(17), 3026-3040.

[65] Wei, G. (2010). Some induced geometric aggregation operators with intuitionistic fuzzy information and their application to group decision making. *Applied soft computing*, *10*(2), 423-431.

[66] Ze-Shui, X. (2007). Methods for aggregating interval-valued intuitionistic fuzzy information and their application to decision making [J]. *Control and decision*, *2*, 019.

[67] Xu, Z. S., & Jian, C. H. E. N. (2007). Approach to group decision making based on interval-valued intuitionistic judgment matrices. *Systems engineering-theory & practice*, *27*(4), 126-133.

[68] Xu, Z., & Yager, R. R. (2006). Some geometric aggregation operators based on intuitionistic fuzzy sets. *International journal of general systems*, *35*(4), 417-433.

[69] YagerR, P. (1988). On ordered weighted averaging operators in multicritefia decision making. *IEEE transactions on systems, man and cybernetics*, *18*, 183-190.

[70] Yager, R. R. (2009). OWA aggregation of intuitionistic fuzzy sets. *International journal of general systems*, *38*(6), 617-641.

[71] Zadeh, L. A. (1965). Fuzzy sets. *Information and control*, *8*(3), 338-353.

[72] Zadeh, L. A. (1975). The concept of a linguistic variable and its application to approximate reasoning—I. *Information sciences*, *8*(3), 199-249.

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An Application of Neutrosophic Logic in the Confirmatory Data Analysis of the Satisfaction with Life Scale

Volkan Duran[1], Selçuk Topal[2,*], Florentin Smarandache[3]

[1] Iğdır University, Faculty of Science and Arts, Iğdır, Turkey; volkan.duran@igdir.edu.tr.
[2] Bitlis Eren University, Faculty of Science and Arts, Bitlis, Turkey; s.topal@beu.edu.tr.
[3] Department of Mathematics, University of New Mexico, 705 Gurley Avenue, Gallup, NM 87301, USA; smarand@unm.edu.

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Abstract

The main concept of neutrosophy is that any idea has not only a certain degree of truth but also a degree of falsity and indeterminacy in its own right. Although there are many applications of neutrosophy in different disciplines, the incorporation of its logic in education and psychology is rather scarce compared to other fields. In this study, the Satisfaction with Life Scale was converted into the neutrosophic form and the results were compared in terms of confirmatory analysis by convolutional neural networks. To sum up, two different formulas are proposed at the end of the study to determine the validity of any scale in terms of neutrosophy. While the Lawshe methodology concentrates on the dominating opinions of experts limited by a one-dimensional data space analysis, it should be advocated that the options can be placed in three-dimensional data space in the neutrosophic analysis. The effect may be negligible for a small number of items and participants, but it may create enormous changes for a large number of items and participants. Secondly, the degree of freedom of Lawshe technique is only 1 in 3D space, whereas the degree of freedom of neutrosophical scale is 3, so researchers have to employ three separate parameters of 3D space in neutrosophical scale while a resarcher is restricted in a 1D space in Lawshe technique in 3D space. The third distinction relates to the analysis of statistics. The Lawhe technical approach focuses on the experts' ratio of choices, whereas the importance and correlation level of each item for the analysis in neutrosophical logic are analysed. The fourth relates to the opinion of experts. The Lawshe technique is focused on expert opinions, yet in many ways the word expert is not defined. In a neutrosophical scale, however, researchers primarily address actual participants in order to understand whether the item is comprehended or opposed to or is imprecise. In this research, an alternative technique is presented to construct a valid scale in which the scale first is transformed into a neutrosophical one before being compared using neural networks. It may be concluded that each measuring scale is used for the desired aim to evaluate how suitable and representative the measurements obtained are so that it's content validity can be evaluated.

Keywords: Convolutional neural network, Neutrosophic logic, Scale development, Neutrosophic social science, Validity.

1 | Introduction

Scale development is an important part of computational social science research, especially for quantitative research. Therefore, this research mostly relies on psychometric research. Usually, psychometricians assess human differences by administering test batteries that have been found to have accurate measuring properties. Effects from these tests are then evaluated by factor analysis and multidimensional scaling to classify latent variables or factors responsible for similar trends of correlations. Specific differences for aimed cognitive skills are generally represented in terms of factors in those studies [1]. The main objective of those who support the psychometric strategy is to allow for the assessment to be made objective. From this standpoint, assessment should be based on objective determinations. For this reason, the psychometric approach emphasizes scales based on

Corresponding Author: s.topal@beu.edu.tr
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statistical methods such as factor analysis, item analysis, and test analysis, and tests its validity and reliability with scientific methods [2].

Neutrosophical set is a potent field of study that has shown its efficiency and strength in various applications. In the meantime, most contributions were theoretical and only validated using mathematical examples or limited data sets and did not use other applications in general [37]. When the literature is reviewed, although it has many applications in natural sciences, recent works focus on the applications of the neutrosophic logic in social sciences [38]. Neutrosophic sets are even more suitable than fuzzy sets to represent the possible responses to questionnaires. The former enables the individual polled to communicate their genuine ideas and emotions even more precisely, thanks to the indeterminacy function of their membership. The benefit of the neutrosophical method is that responders may describe their ideas and emotions more correctly, since both indeterminacy and an independent membership function of falsehood are taken into account [39], [40]. In this respect, this research aims to use the application of the neutrosophic philosophy in social sciences especially in education and assessment and evaluation methods of scale development.

2| Preliminaries

The numerical properties obtained depending on the group to which a test is applied are generally called test statistics. Some of the test statistics can be calculated based on item statistics. In general, the test statistics like the average of the test, the average difficulty of the test, the variance of the test, and other test statistics are highly useful [3]. Researchers want to show whether there is harmony in an instrument's responses. Factor analysis is one of the multivariate approaches that social scientists use to validate psychological aspects. When several independent variables are grouped in a single study, statistical analysis can become rather challenging. It is often advantageous to group together those variables that are correlated with one another. Factor analysis is a technique that allows researchers to see whether many variables can be portrayed as a few factors [4]. Factor analysis seeks to identify some new specific factors by putting together a small number of factors that aren't connected (a p-dimensional space) [5]. It is recommended that the scale of the explanatory factor analysis process should be tested through confirmatory factor analysis [6]. Confirmatory factor analysis could be considered as a way to verify the validity of factor structures. Using this method, it is attempted to prove that the observed variables are connected with the hidden variables and hidden variables are connected. To investigate these relationships, measurement models were built [7].

There are three types of factors for developing a more grounded scale: (i) reliability; (ii) validity; and (iii) sensitivity. Reliability refers to the extent to which a measurement of a phenomenon produces consistent results as given in *Fig. 1* [8]. Therefore, reliability means consistency or stability. Consistency of any measurement scale is important for objective scientific research and this concept is related to ‘agreement’, ‘reproducibility’, and ‘repeatability’ of any measurement. The agreement is the closeness of two measurements made on the same subject as opposed to one another. Reliability includes repeatability. Repeatability means measuring accurately the same variable again and again for the same circumstances [9]. A test or measure is said to be reliable if there are always identical results using the same testing procedure [10]. This means that regardless of how many times the measurement has been taken or by whom it has been performed, you will always obtain the same value. This means two things: first of all, you should get the same result each time you use the measure, and secondly, you should use the measure as many times as possible. This can be an issue in data collection when several people are involved [11]. Reproducibility referred to variations in test results while tests were performed on subjects on different occasions. The changed circumstances may be due to the use of various methods of measurement or instruments, measurement by several observers or raters, or measurements during a period in which the variable's error-free level may undergo a non-negligible change [9].

Reliability is, therefore, the level of error-free. As the amount of error decreases as a result of measurement, reliability increases, and as the number of errors increases, reliability decreases. Reliability levels of measurement tools are determined by reliability analysis. Reliability is best expressed with the reliability coefficient (r) ranging from 0.00 to +1.00. The closer the reliability coefficient of the measurement tool is to 1, the higher the reliability, the closer to 0, the lower the reliability [12].

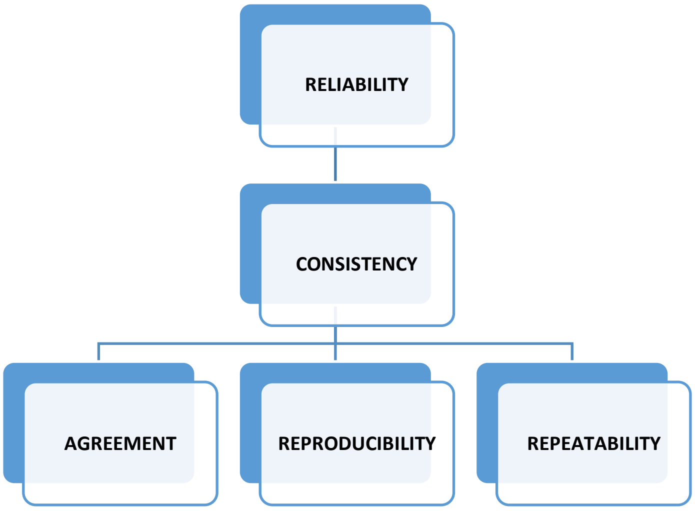

Fig. 1. Reliability and its components.

Validity simply means "measure what is intended to be measured" [13]. There are different types of validity in social sciences (*Fig. 2*). Face validity is a subjective judgment on the operationalization of a construct whether it is appropriate, unambiguous, simple, and proper [14]. Content validity refers to how appropriate and representative the measurements collected are for the desired assessment purpose. The representativeness criterion may have two definitions. Quantifying the extent of sampling is one of them. The second is the extent to which items reflect the structures of the whole scale [15]. Construct is a pattern formed by certain elements that are thought to be related to each other or by the relationships between them. The construct validity measurement tool shows to what extent it can accurately measure the structure and concept that it claims to measure [12]. Construct validity refers to how well you translated or transformed a concept, idea, or behavior that is a construct into a functioning and operating reality, the operationalization [14]. Construct validity is used when trying to quantify a hypothetical construct, like fear. Convergent and discriminant validity should be used to determine the validity of a construct by suggesting that the new measurements are correlated with other measurements of that construct and that the dimensions proposed are inappropriate to the construct unrelated, respectively [16]. Discriminant validity is the extent to which latent variable a discriminates from other latent variables. The Convergent Validity is the degree to which two measurements of a construct are connected theoretically [14]. The validity of the criterion is also divided into concurrent and predictive validity, where the validity of the criterion deals with the correlation between the current measurement and the criterion measurement (such as the gold standard) [16]. Content and construct validity in social sciences are defined as credibility/internal validity. Internal validity is related to the question of whether the research findings fit with reality in the external world. Internal validity is determined by experimenting with specific characteristics and no specific biases. For example, the question of "can we recognize people by looking at their faces?" can be examined. This question is answered by asking two more questions. First, is the independent variable the cause of the dependent variable? Second, can other possible explanations for the relationship between independent variable and dependent be logically eliminated? If the answer to these questions is yes, the researchers can claim that the experiment has internal validity [17]. Criterion validity is the degree to which it is empirically relevant to the outcome. This is something that calculates how well one measure predicts another measure. There are three types of criterion validity namely; concurrent validity, predictive and postdictive validity [14].

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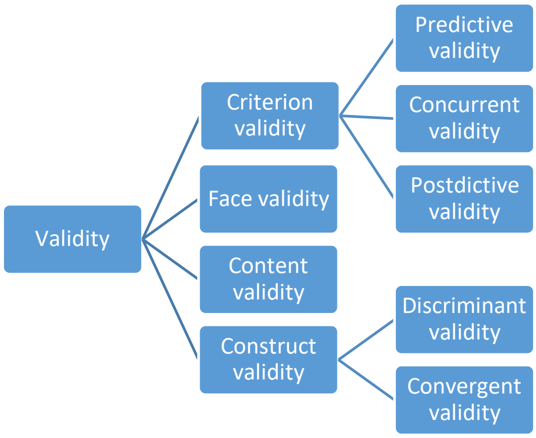

Fig. 2. Subtypes of various forms of validity tests.

Fig. 3. illustrates how reliability and validity are related. In the first target, the shots reached the same spot, but none were effective in reaching the same point. The second target can be regarded as valid but not reliable since the points are expanding over the entire place. The third target did not present reliability or validity, since they hit spread points. The fourth target stands as an indicator of reliability and validity; the shots landed right in the target center and were consistent, right in the target center [18].

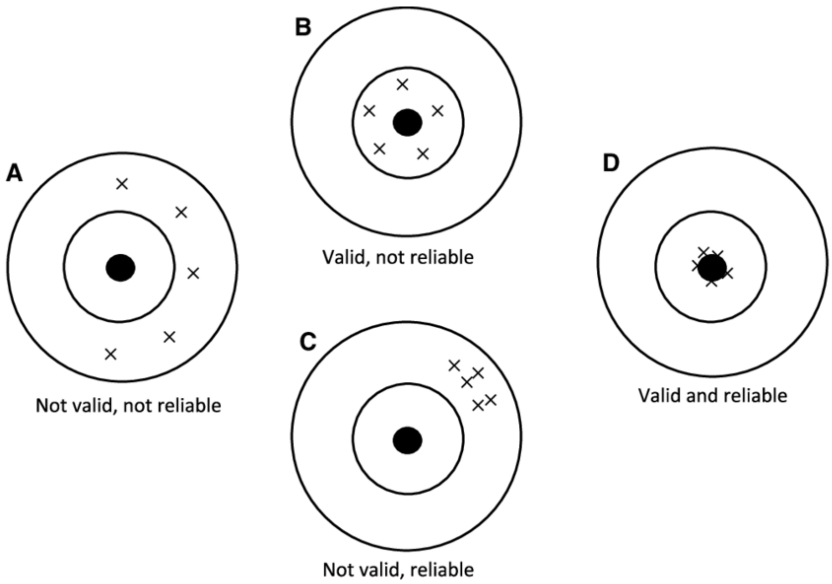

Fig. 3. Possible combinations of validity and reliability of measurement instruments [18].

Sensitivity is defined as the consensus closeness between randomly selected individual measurements or results. It is therefore concerned with the variance of repeated measurements. A measurement tool with low variance is more sensitive than those with a higher variance. For example, as a researcher, one wants to know what is the smallest sample you can use that will take into consideration the variability in the dependent measure and yet be sensitive enough to notice a statistically meaningful difference, whether there is one. Our capacity to distinguish significant differences between groups is defined in part by the variability of individuals in our sample and how much variability occurs among them. Therefore, less variability may contribute to greater sensitivity, and more variability results in less sensitivity [19].

As mentioned above, the key aim of developing questionnaires or scales is to collect correct and appropriate data. The reliability and validity of scale or questionnaire formats is an important feature of testing methodology [14]. The reliable and accurate measurement may, in the simplest intuitive terms, indicate that the current measurement is equal with, or follows, the truth. However, it is often impractical to require the new measurement to be identical to the truth, either because 1) we accept the measurement of a tolerable (or acceptable) error or 2) the truth is simply impossible for us (either because it is not measurable or because it is only measurable with some degree of error) [16]. In this regard, data space and data range are the important dimensions of developing scales because it also changes the data type, the logical space of the analysis, methodology, and validity and reliability of the results (*Fig. 4*).

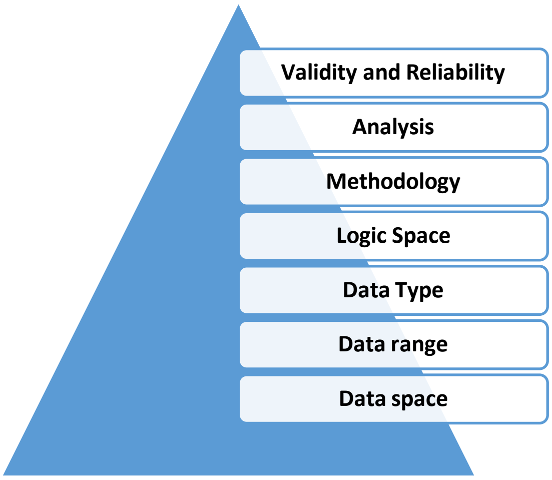

Fig. 4. Data space and data range determines the validity and reliability of any scale.

Data space in measurement tools like scale refers to the set of independent options regarding the particular item of the scale. For example, on any Likert-type scale, the participant can express only one option, so the data space is 1d, whereas on the neutrosophic scale, there are three independent dimensions regarding any item as undecided, agree, and disagree.As it can be seen, data space is 1d in any Likert-type scale and 3d in neutrosophic space and if our measurement tools become more qualitative, like having items requiring free opinions in a paragraph like choices, it has more dimensions, even in ideal cases it has infinite dimensions. However, although n-dimensional space is more appropriate for better valid and reliable results, less dimensional spaces have less vagueness in terms of the interpretation of the data and they can be more easily statistically handled. Additionally, as the dimension of space increases, the objectivity of the measurement tool in terms of measuring common characteristics decreases. The advantage of the 3-dimensional neutrosophic scale is that it both seeks the agreement, disagreement, and confusion levels of the participants. In daily life, many items are encountered to give an opinion about them and we are not restricted within a 1-dimensional space where we can only choose one answer regarding whether we agree, disagree or express uncertainly about a particular case. However, in the three-dimensional neutrosophic space, participants express both their agreement and disagreement level as well as the uncertainty in the items or dimensions of the scale. People sometimes think that they understand a statement, but one word in the statement makes us uncertain whether it is the "right meaning" intended by the source. Similarly, people sometimes agree on some propositions, but just because of the source of the message itself, they also disagree with the item. Therefore, the neutrosophic scale is different from the classical Likert-type scales in terms of data space (*Fig. 5*).

The second important point that distinguishes any measurement tool from each other is the data range. The range of a set of data is the difference between the highest and lowest values in the set. Likert-type scales are commonly arranged in terms of data, ranging from 3 point Likert-type scales to 10 point Likert-type scales. However, the range of the neutrosophic scale is broader than the Likert-type scales. It includes any rational number in a range between 0 and 100. As a result, neutrosophic scales have continuous variable types, whereas Likert-type scales have discrete value types in terms of rational numbers, so data analysis may differ as a result. This can contribute to increasing the sensitivity of the measurement tool in this respect. This is actually what is called as neutrosophic data in some recent researches is the piece of information that contains some indeterminacy. Similar to the classical statistics, it can be classified as [39]:

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– *Discrete neutrosophic data, if the values are isolated points.*

– *Continuous neutrosophic data, if the values form one or more intervals.*

– *Quantitative (numerical) neutrosophic data; for example: a number in the interval [2, 5] (we do not know exactly), 47, 52, 67 or 69 (we do not know exactly).*

– *Qualitative (categorical) neutrosophic data; for example: blue or red (we do not know exactly), white, black or green or yellow (not knowing exactly).*

– *The univariate neutrosophic data is a neutrosophic data that consists of observations on a neutrosophic single attribute.*

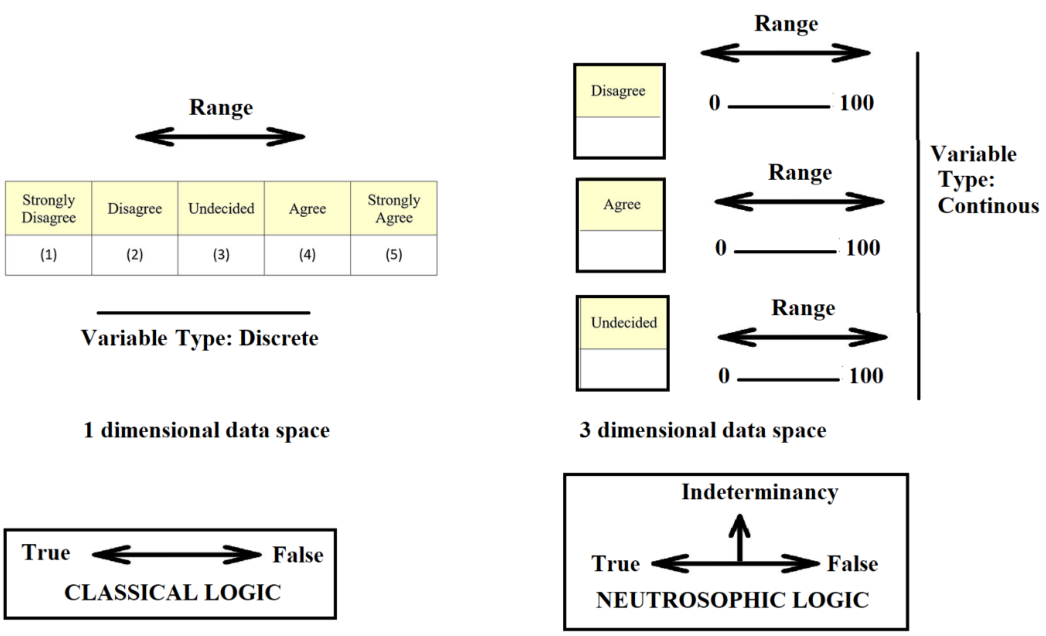

Fig. 5. Data space of classical Likert-type scale, neutrosophic scale.

The third important point of any measurement tool is its logic space. Logic space is important because "in any field of knowledge, each structure is composed from two parts: a space, and a set of axioms (or laws) acting (governing) on it. If the space, or at least one of its axioms (laws), has some indeterminacy of the form (t, i, f) ≠ (1, 0, 0), that structure is a (t, i, f)-Neutrosophic Structure" [41]. Therefore the logic which is in our focus, Neutrosophic Logic, is an emerging field where each proposition is reckoned to have the proportion (percentage) of truth in a subset T, the proportion of indeterminacy in a subset I, and the proportion of falsity in a subset F. A subset of truth (or falsity or indeterminacy) here is considered, rather than just a number, since in many situations can not be precisely determined the proportions of truth and falsity but we can only approach them. For example, suppose that a statement (or proposition) is between 32% and 48% true and 59% to 73% false; worse: 32% to 39% or 41 to 52% true (according to various observers) and 57% or 62% to 71% false. Subsets are not basic intervals but are any set (open or closed or semi-open/semi-closed intervals, discrete, continuous, intersections or unions of previous sets, etc.) following the given proposition. The adventure of gaining meaning and mathematical results from situations of uncertainty was initiated by Zadeh [20]. Fuzzy sets added a new wrinkle to the concept of classical set theory. Elements of the sets have degrees of belongingness (in other words, membership) according to the underlying sets. Atanassov defined intuitionistic fuzzy sets including belongingness and non-belongingness degrees [21], [32]-[34]. Smarandache suggested neutrosophy as a computational solution to the idea of neutrality [22]. Neutrosophic sets consider belongingness, non-belongingness, and indeterminacy degrees. Intuitionistic fuzzy sets are defined by the degree of belongingness and non-belongingness and uncertainty degrees by the 1-(membership degree plus non-membership degree), while the degree of uncertainty is assessed independently of the degree of belongingness and non-belongingness in neutrosophic sets. Here, belongingness, non-belongingness, and degree of uncertainty (uncertainty), like degrees of truth and falsity, can be assessed according to the interpretation of the places to be utilized. This indicates a difference between the neutrosophic set and the intuitionistic fuzzy set. The definition of neutrosophy is, in this sense, a potential solution and representation of problems in different fields. Two detailed and mathematical fundamental differences between relative truth (IFL) and absolute truth (NL) are as follows:

I. NL can distinguish absolute truth (truth in all possible worlds, according to Leibniz) from the relative truth (truth in at least one world) because NL (absolute truth) = 1^+ while IFL (relative truth) = 1. This has been practiced in philosophy and linguistics (see the Neutrosophy). The standard interval [0, 1] used in IFL has been extended to the unitary non-standard interval $]^-0, 1^+[$ in NL. Parallel distinctiveness for absolute or relative falsehood and absolute or relative indeterminacy are allowed to consider in NL.

II. There do not exist any limits on T, I, F apart from they are subsets of $]^-0, 1^+[$, thus: $^-0 \le \inf T + \inf I + \inf F \le \sup T + \sup I + \sup F \le 3^+$ in NL. This permission allows dialetheist, paraconsistent, and incomplete information to be identified in NL, while these situations impossible to be identified in IFL since F (falsehood), T (truth), I (indeterminacy) are restricted either to $t + i + f = 1$ or to $t2 + f2 \le 1$, if T, I, F are all reduced to the points t, i, f respectively, or to sup $T + \sup I + \sup F = 1$ if T, I, F are subsets of [0, 1] in IFL.

Although there are usually three options in Likert-type scales: agreement, disagreement, and vagueness, its logic is based on one valued option located on the opposite sides of true and false values. However, the neutrosophic set has three independent components, giving more freedom for analysis so that it brings different logical operations as well. Therefore, the methodology of the analysis of the data should be changed based on the logical structure of the scale. For instance, while factor analysis is used for classical Likert-type scales, neural networks are more appropriate for the analysis of the data of the neutrosophic scales. Nevertheless, it should be noted that classical analysis and methods can indeed be used for neutrosophic scales based on different analysis procedures. To sum up, "a space with an item, it means an opinion, another element induces another opinion, another element in turn induces another opinion, and so on. The opinion of each element of the structure must be respected. In this way it builds a neutrosophic social structure. The result is a very large socio-neutrosophic structure that is intended to be filtered, evaluated, analyzed by scientific algorithms" [42]. Hence, we can conclude that the validity and reliability of the measurement tools can change based on the logical structure of the scale. As a result, in this study, we take The Satisfaction with Life Scale developed by Diener et al. [23] and adapted in Turkish by Dağlı and ve Baysal [24] and convert it into neutrosophic form, compare the results, and use this analysis to propose new type confirmatory analysis procedures and develop neutrosophic scales. There are many ways to evaluate and interpret data. Some recent studies reveal important developments based on the interpretation and effective use of data [42]-[44].

2.1| The Difference between Lawshe Technique and Neutrosophic Scale

Some argue that the well-known Lawshe technique is very similar to neutrosophic analysis and propose what is the reason behind the logic of neutrosophic forms. Initially suggested in a seminal 1975 paper in Lawshe [25], the method of Lawshe was common in various areas including health care, education, organizational development, personnel psychology, and market research for determining and quantifying content validity [26], [27].

Lawshe [25] has proposed a quantitative measure to evaluate validity of the content termed as the Content Validity Ratio (CVR). The validity ratio of content provides information about validity of items. The approach includes the use of an expert panel to evaluate items based on their relevance to the scale domain. Each item on a scale is classified as a three-point rating system (1) point is irrelevant, 2) item is important, but not essential, and 3) item is essential). The percent of experts considering items significant or essential for the substantive content of the scale is calculated for every element of a CVR. Also a overall measurement of the validity of the content of the scale may be created. The index is calculated as a mean of the CVR scores for items [36].

A quantitative criteria is necessary in the Lawshe approach for determining the validity of content. The Content Validity Index and CVR are the criterion for validity used by experts. In order for each item to be included in the Scale, the content validity ratio is an internationally accepted standard. For all finished items, the Content Validity Index is the average CVR. The CVR should assess whether or not each item is essential, and the Content Validity Index should identify the relationships between the scale items and scale . The Content Validity Index is calculated by using the degree of agreement of the experts on the relevance and clarity of the items. According to CVR values,

– *If all the experts in the panel answered "not necessary" for any item, that item is completely unnecessary.*
– *If all of the experts on the panel gave the answer "useful but not necessary" for any item, that item is significantly necessary.*
– *If the number of experts who give the answer "required" for any item is more than half, it can be commented that the item has a certain validity value, and the validity value of the item will increase as the number of experts who give the answer "required" increases [35].*

First of all, the main difference between those two techniques is in their data space. Although there are three choices in the Lawshe technique for each item as an a-Essential? b-Useful but not essential? Why? c-Not necessary? Why, while membership in neutrosophic logic is very similar to Truth T, indeterminacy I, and falsity F, their dimensions are different from each other because there is only one option regarding each item, which corresponds to one-dimensional data space, but there are three independent data spaces in the neutrosophic form where each data represents a different. According to this, whether all participants agree that the information or ability that has been tested is necessary, or whether none says it is relevant, we are sure that the component has been added or omitted. If there is no majority, the dilemma emerges. There are two hypotheses, both compatible with existing psychophysical principles [28].

– *Every item for which more than half of the experts consider any item to be "essential" has content validity.*
– *The wider the extent or degree of its validity is the more experts (above 50 percent) who view an item as "essential."*

Therefore, the Lawshe technique focuses on the dominant opinions of the experts which are restricted by one-dimensional data space so that it might hide their indeterminacy or disagreement because they are weak compared to the other options. It should be pointed out that altough Lawshe technique is not strictly restricted by the one dimensional options for experts because it also take their suggestions, in the statistical analysis process it focuses on only one options. For a small number of items, the effect of this can be negligible, but for a huge number of items, it can make huge differences.

There is one parameter in the Lawshe technique. Researcher can only choose one option among agreement, disagreement, and indeterminacy based on his/her dominant view. Hence it is actually a 1d dimensional function in a 3-dimensional space. There are three parameters in the Neutrosophic scale. A researcher must choose three options among agreement, disagreement, and indeterminacy. Hence it is actually a 3d dimensional function in a 3-dimensional space. Therefore, the degree of freedom of the Lawshe technique is 1 in 3-d space whereas the degree of freedom of the neutrosophic scale is 3, that is, a researcher is restricted in 1-d space in 3d space of possibilities in Lawshe technique whereas researchers must use three independent parameters of 3d space in neutrosophic scale (*Fig. 6*).

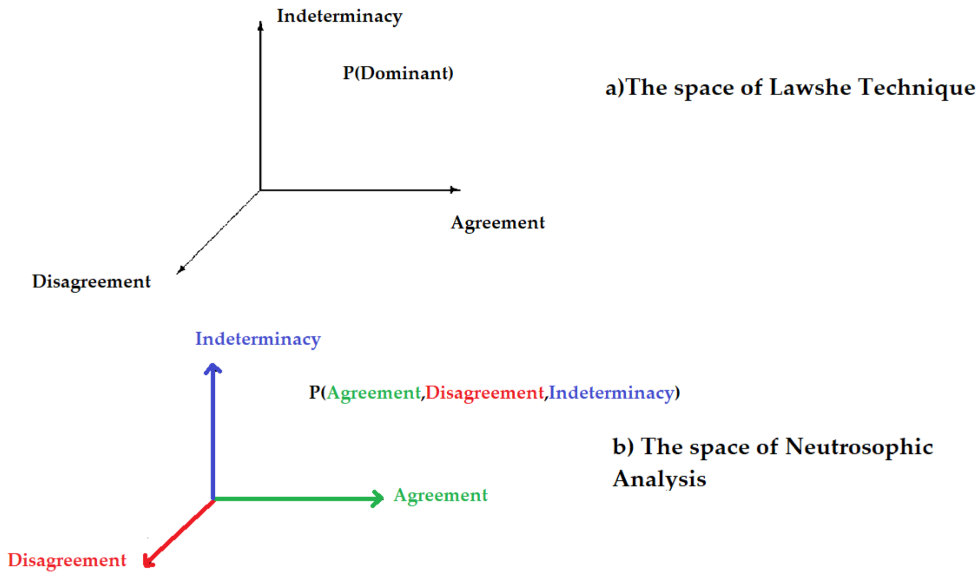

Fig. 6. The difference between the space and parameters of the Lawshe technique with neutrosophic scale.

a) There is one parameter in the Lawshe technique. The analysis focus on one option among agreement, disagreement and indeterminacy based on the dominant view. Hence it is actually a 1d function in 3d space b) there are three parameters in the Neutrosophic scale. The analysis focuses on three options among agreement, disagreement and indeterminacy. Therefore, it is a 3D function in the 3D space.

Therefore, for the participation of a huge number of researchers, the dominant view of the researcher restricted within 1d space in the Lawshe technique may dismiss the other two parameters that cannot be ignored in the actual case. These hidden variables can lead to huge differences especially in the case of the analysis of the options of a huge number of participants and even this cannot be realized. However, in neutrosophic logic, it is impossible to dismiss three parameters since the researchers must give their opinions on them (*Fig. 7*).

The second difference is related to the data range. The Lawshe technique is limited by discrete data that can be manipulated with qualitative comments. Although qualitative comments make the item better, in terms of generalizability we may not be confident that the item is suitable for its content. Opinions of the experts may indicate different content, but the understanding of common participants may indicate different content in this respect.

The third difference is related to statistical analysis. In the Lawshe technique, it is focused on the ratio of decisions of the experts, whereas in the neutrosophic logic we focus on the importance and correlation level of each item for the analysis. In the Lawshe technique, there is no distinction between the importance level and correlation, so it means that the item that is seen as important by experts might not be correlated with the content in the actual applications (*Fig. 8*). In daily life, we wonder about particular features and we seek them in particular sets, but the items of the set can be seen as important but are not relevant to what we want to seek. For example, we may meet a close relative whom we have not seen in a long time and look for him/her in a specific location, and the individuals resembling our relative are important to us, but the importance is diminished when we discover that there is no correlation between the actual close relative and the similar person resembling him/her.

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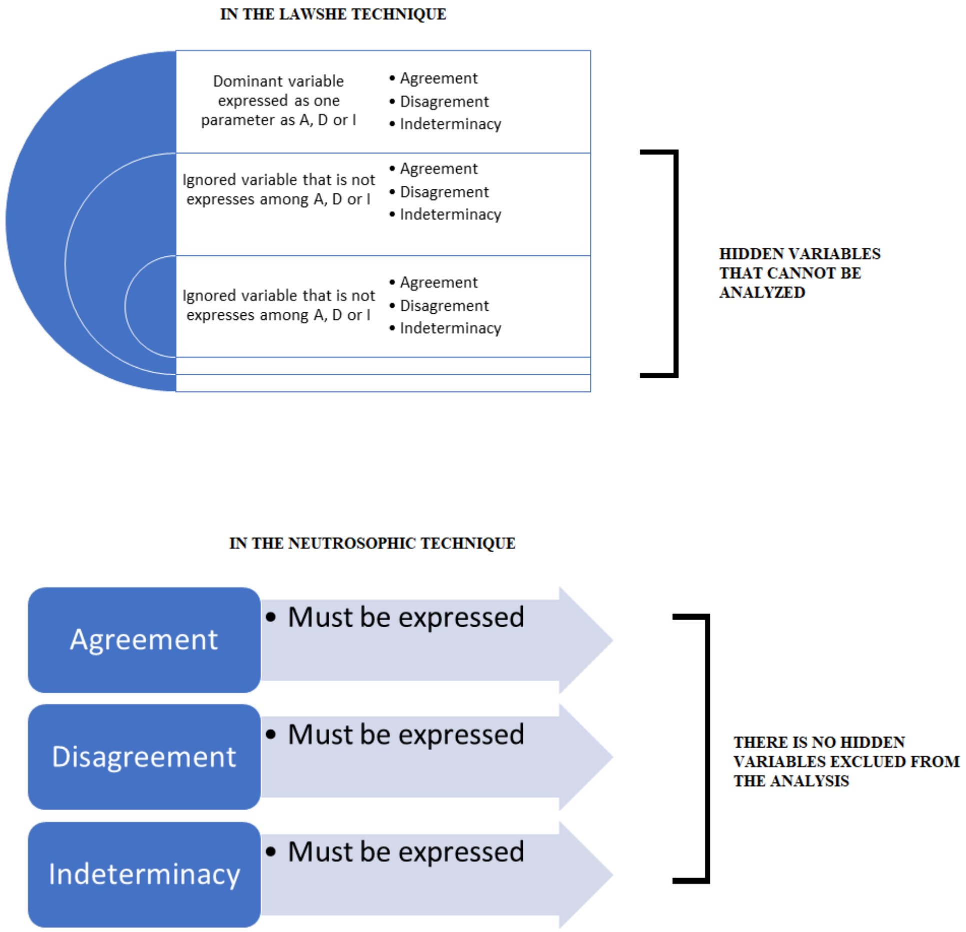

Fig. 7. There is no hidden variable in the neutrosophic technique but there are hidden variables in the Lawshe technique.

Actually Sartre's vivid description [29] regarding his hypothetical appointment with Pierre can be given as a more explicit example for the importance and correlation as follows:

I have an appointment with Pierre at four o'clock. I arrive at the cafe a quarter of an hour late. Pierre is always punctual. Will he have waited for me? I look at the room, the patrons, and I say, "he is not here." Is there an intuition of Pierre's absence, or does negation indeed enter in only with judgment? At first sight it seems absurd to speak here of intuition since to be exact there could not be an intuition of nothing and since the absence of Pierre is this nothing…..

Similarly Pierre's actual presence in a place which I do not know is also a plenitude of being. We seem to have found fullness everywhere. But we must observe that in perception there is always the construction of a figure on a ground. No one object, no group of objects is especially designed to be organized as specifically either ground or figure; all depends on the direction of my attention. When i enter this cafe to search for PIerre, there is formed a synthetic organization of all the objects in the cafe, on the ground of which Pierre is given as about to appear. This organization of the cafe as the ground is an original nihilation. Each element of the setting, a person, a table, a chair, attempts to isolate itself, to lift itself upon the ground constituted by the totality of the other objects, only to fall back once more into the undifferentiation of this ground; it melts into the ground. For the ground is that which is seen only in addition, that which is the object of a purely marginal attention. Thus the original nihilation of all the figures which appear and are swallowed up in the total neutrality of a ground is the necessary condition for the appearance of the principle figure, which is here the person of Pierre. This nihilation is given to my intuition; i am witness to the successive disappearance of all the objects which i look at-in particular of the faces, which detain me for an instant (could this be Pierre?) and which as quickly decompose precisely because they "are not" the face of Pierre. Nevertheless, if i should finally discover Pierre, my intuition would be filled by a solid element, i should be suddenly arrested by his face and the whole cafe would organize itself around him as a discrete presence.

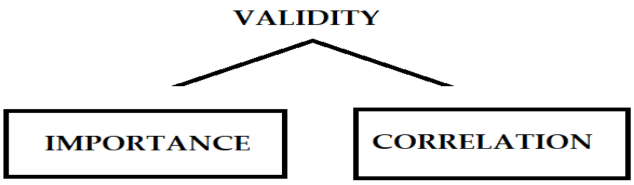

Fig. 8. There is a distinction between the concept of importance and correlation in neutrosophic logic.

Therefore, when experts make a decision, there is no clear distinction between their decision-making process in terms of importance or correlation.

The fourth one is related to expert opinion. Lawshe technique focuses on expert opinion, but the term expert is not clear in many respects. For example, if somebody studies a novel concept that has not been studied previously, how an expert decides whether the item is suitable or not besides deciding on its grammar or meaning. Furthermore, we need different experts for decision-making about the suitability of the item, but the ratio of those experts shouldn't be equal in the proportion of the decision-making process. For example, on some scales, the opinion of a psychologist might be more important than the other experts and their contribution should vary by this. However, in the neutrosophic scales, we mainly aim at the real participants so that we can understand to the extent whether the item is understood or objected or vague.

3| Methodology

In the methodology, first, the items of the Satisfaction with Life Scale were converted into the neutrosophic form where each item has three independent components referring to the agreement, disagreement, and indeterminacy. However, to compare the neutrosophic scale, the classical scale were also used as well. Secondly, each item of neutrosophic scale were analyzed in terms of classical scale in terms of neural networks and Spearman correlation constant. In the second part of the study, the Neutrosophic Life Satisfaction Scale were analyzed in terms of whole structure for confirmatory factor analysis. Finally, the decision-making formula were created to decide to remove or keep the items on the neutrosophic scale (*Fig. 9*).

In this analysis var1 refers to the variable number and a (such as var1a) stands for agreement b stands for indeterminacy and c refers to disagreement. In the neural network analysis for the study, for the level of the analysis of each item, the input variables are three sub-items of each item on the neutrosophic scale and the output variable is each classical scale. Similarly, for the whole structure for confirmatory factor analysis, the input variables are all the items on the neutrosophic scale and output variables are the classical items of the classical scale. The activation function both for the hidden and output layer was chosen as the sigmoid function. The number of hidden layers in each analysis was chosen to be two (*Fig. 10*). Criteria training=batch optimization=gradientdescent was chosen as the criterion. In the analysis of the data, independent variable importance analysis was used.

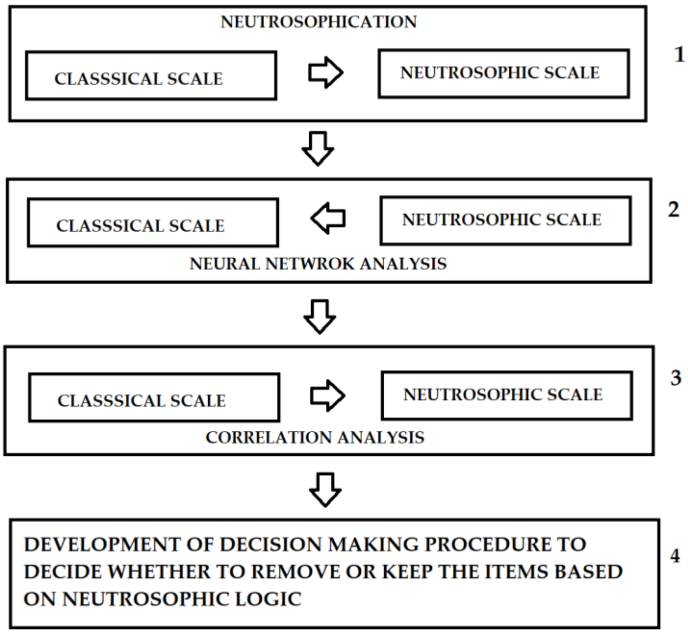

Fig. 9. The procedure for the development of neutrosophic scale.

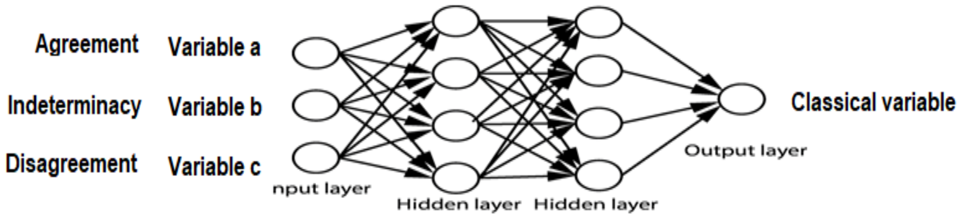

Fig. 10. The general structure of the Convolutional Neural Network (CNN) we used in this study is a three-layer neural network with three input neurons, two hidden layers of four neurons each, and one output layer [30].

Independent variable importance analysis performs a sensitivity analysis, which computes the importance of each predictor in determining the neural network. The importance of an independent variable is a measure of how much the network's model-predicted value varies with different values of the independent variable. Normalized importance is just the importance values that are grouped by and represented as percentages of importance values. In another words, the importance of an independent variable is a measure of how much the network's model-predicted value changes for different values of the independent variable. Normalized importance is simply the importance values divided by the largest importance values and expressed as percentages. However, it should be underlined that you cannot tell is the "direction" of the relationship between these variables and the predicted probability of default" [31], [41]. The importance chart is simply a bar chart of the values in the importance table, sorted in descending value of importance. It allows to guess that a larger amount of debt indicates a greater likelihood of default, but to be sure, you would need to use a model with more easily interpretable parameters [41]. Therefore, the spearman correlations between the variables are examined to see the direction and relationship of the items to decide whether they are suitable or not.

3.1| Measurement Tools

In this study, the satisfaction with Life Scale adapted into Turkish by Dağlı and ve Baysal [24] which was developed by Diener et al. [23] was converted into the neutrosophic form and the results were compared in terms of confirmatory analysis by convolutional neural networks. One might ask why an adapted version of a scale was chosen rather than adapting or developing a new scale in the neutrosophic form. The first reason for this is that the method based on neutrosophic logic is a very new one so that in more grounded levels it must be tested rather than directly using it to assess and develop scales. Secondly, the neutrosophic form could be compared with the classical one and infer the advantageous and disadvantageous sides of the neutrosophic scale in terms of its different aspects. Thirdly, this study is aimed at conducting confirmatory analysis so that a particular measurement tool must be used to assess whether the neutrosophic form can be used for the analysis. In classical confirmatory analysis, similar measurement tools can be used to analyze this, but in this article, the main aim is to use the neutrosophic form to conduct confirmatory analysis.

4| Findings

In this section, we give our findings.

4.1| Analysis of Neutrosophic Life Satisfaction Scale in terms of Reliability

Before using the neutrosophic scale it can be wondered about its reliability before comparing it with the classical one. Cronbach's Alpha constant can be used for the neutrosophic scale. However, it should be noted that Cronbach's Alpha constant should be used three times for three independent factors as given in *Table 1* below.

Table 1. Cronbach's Alpha constant for three dimensions.

Cronbach's Alpha Constant	Variables
0.863	VAR1a VAR2a VAR3a VAR4a VAR5a
0.777	VAR1b VAR2b VAR3b VAR4b VAR5b
0.792	VAR1c VAR2c VAR3c VAR4c VAR5c

Results show that our neutrosophic scale is also reliable which also supports the reliability of the classical scale because Cronbach's Alpha constant is an acceptable level for three dimensions.

4.2| Analysis of Neutrosophic Life Satisfaction Scale in terms of Items of Validity

According to Spearman's rho correlation coefficient, classical variable 1 has a high positive significant correlation with var1a which is related to the agreeing level of the participants and it has an average level negative significant level of correlation variable 1c which is related to the disagreeing level of the participants. Both correlations can be related to the points of a participant who has either a high level of life satisfaction or not. Besides, no correlation between vagueness and classical items shows that there is no indeterminacy about this item.

Table 2. Correlation among neutrosophic item 1 and classical item 1.

	VAR1a	VAR1b	VAR1c
VAR1 Correlation Coefficient	0.678**	-0.022	-0.417**
Sig. (2-tailed)	0.000	0.768	0.000
N	189	189	189

Neural network analysis of the items reveals that participants with positive life satisfaction for item 1a contribute 100% to classical variable 1 and participants with negative life satisfaction for item 1c contribute 26.4% to classical variable 1. This might be related to the differentiation of the number of participants having high-level life satisfaction and a low level of life satisfaction. However, it should be noted that the vagueness of this item is 57.5% implies that there is a moderate level of confusion about this article either because of meaning or the usage of the words or some unknown parameters, although there is no correlation between var1b and classical variable.

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Table 3. Independent variable ımportance for classical item 1 in terms of neutrosophic items.

Independent Variable Importance		
	Importance	Normalized Importance
VAR1a	0.544	100,0%
VAR1b	0.313	57,5%
VAR1c	0.143	26,4%

According to Spearman's rho correlation coefficient, classical variable 2a has a significant positive correlation with var2a, which is related to the participants' agreeing level, and variable 2c has a negative significant low level of correlation, which is related to the participants' disagreeing level. Both correlations can be related to the points of participants who have either a high level of life satisfaction or not. Besides, no correlation between vagueness and classical items shows that there is no indeterminacy about this item.

Table 4. Correlation among neutrosophic item 2 and classical item 2.

		VAR2a	VAR2b	VAR2c
VAR2	Correlation Coefficient	0.732**	0.120	-0.277**
	Sig. (2-tailed)	0.000	0.099	0.000
	N	189	189	189

Neural network analysis of the items reveals that participants with positive life satisfaction for item 2a contribute 100% to classical variable 2 and participants with negative life satisfaction for item 2c contribute 26.6% to classical variable 2. This might be related to the differentiation of the number of participants having high-level life satisfaction and a low level of life satisfaction. However, it should be noted that the vagueness of this item is 31.7% implies that there is a weak level of confusion about this article either because of meaning or the usage of the words or some unknown parameters, although there is no correlation between var1b and classical variable.

Table 5. Independent variable importance for classical item 2 in terms of neutrosophic items.

Independent Variable Importance		
	Importance	Normalized Importance
VAR2a	0.632	100,0%
VAR2b	0.200	31,7%
VAR2c	0.168	26,6%

According to Spearman's rho correlation coefficient classical variable 3 has a moderate positive significant correlation with var3a which is related to the agreeing level of the participants and it has a negative significant moderate level of correlation which is related to the disagreeing level of the participants. Both correlations can be related to the points of participants who have either a high level of life satisfaction or not. However, the weak level of significant correlation between vagueness and classical item shows that there is an indeterminacy about this item.

Table 6. Correlation among neutrosophic item 3 and classical item 3.

		VAR3a	VAR3b	VAR3c
VAR3	Correlation Coefficient	0.474**	-0.178*	-0.430**
	Sig. (2-tailed)	0.000	0.014	0.000
	N	189	189	189

According to the results of the neural network analysis for the items, participants with positive life satisfaction for item 3a have a 100% contribution to classical variable 3, while participants with negative life satisfaction for item 3c have a 38, 0% contribution to classical variable 3. This might be related to the differentiation of the number of participants having high-level life satisfaction and a low level of life satisfaction. However, it should be noted that the vagueness of this item 3c, which is 21,7%, implies that there is a weak level of confusion about this article either because of meaning or the usage of the words or some unknown parameters. It should be noted that there is also a weak level significant correlation between item 3b and item 3.

Table 7. Independent variable importance for classical item 3 in terms of neutrosophic items.

Independent Variable Importance		
	Importance	Normalized Importance
VAR3a	0.626	100,0%
VAR3b	0.136	21,7%
VAR3c	0.238	38,0%

According to Spearman's rho correlation coefficient classical variable 4 has a high-level significant correlation with var4a which is related to agreeing on the level of the participants and it has a negative moderate level significant correlation which is related to the disagreeing level of the participants. Both correlations can be related to the points of participants who have either a high level of life satisfaction or not. Besides, no correlation between vagueness and classical items shows that there is no indeterminacy about this item (*Table 8*).

Table 8. Correlation among neutrosophic item 4 and classical item 4.

		VAR4a	VAR4b	VAR4c
VAR4	Correlation Coefficient	0.715**	-0.115	-0.475**
	Sig. (2-tailed)	0.000	0.115	0.000
	N	189	189	189

Neural network analysis of the items reveals that participants with positive life satisfaction for item 4a contribute 95.8% to classical variable 4 and participants with negative life satisfaction for item 4c contribute 100.0% to classical variable 4. This might be related to the differentiation of the number of participants having high-level life satisfaction and a low level of life satisfaction. However, it should be noted that the vagueness of this item 4c is 27,0%, implies that there is a weak level of confusion about this article either because of meaning or the usage of the words or some unknown parameters, although there is no correlation between variable 4b and classical variable (*Table 9*).

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Table 9. Independent variable importance for classical item 4 in terms of neutrosophic items.

Independent Variable Importance		
	Importance	Normalized Importance
VAR4a	0.430	95,8%
VAR4b	0.121	27,0%
VAR4c	0.449	100,0%

According to Spearman's rho correlation coefficient classical variable 5 has a high level of positive significant correlation with var5a which is related to the agreeing level of the participants and it has a weak level of negative significant correlation which is related to the disagreeing level of the participants. Both correlations can be related to the points of participants who have either a high level of life satisfaction or not. Besides, there is a weak level significant correlation between variable 5 and variable 5b. Therefore, the weak level significant correlation between vagueness and classical item shows that there is an indeterminacy about this item (*Table 10*).

Table 10. Correlation among neutrosophic item 5 and classical item 5.

		VAR5a	VAR5b	VAR5c
VAR5	Correlation Coefficient	0.706**	0.149*	-0.347**
	Sig. (2-tailed)	0.000	0.040	0.000
	N	189	189	189

The results of the neural network analysis for the items show that participants with positive life satisfaction for item 5a have a 100% contribution to the classical variable 4 and participants with negative life satisfaction for item 5c have an 84.2% contribution to the classical variable 4. This might be related to the differentiation of the number of participants having high-level life satisfaction and a low level of life satisfaction. However, it should be noted that the vagueness of this item 4c is 39,6%, implies that there is a weak level of confusion about this article either because of the meaning of the usage of the words or some unknown parameters (*Table 11*).

Table 11. Correlation among neutrosophic item 5 and classical item 5.

Independent Variable Importance		
	Importance	Normalized Importance
VAR5a	0.447	100,0%
VAR5b	0.177	39,6%
VAR5c	0.376	84,2%

4.3| Analysis of Neutrosophic Life Satisfaction Scale in terms of whole Structure for Confirmatory Factor Analysis

Neural network analysis results for two scales can be given as follows. It seems that variable 2 and variable 5 might be problematic when considering the overall contribution of the items for the whole scale since variable …b items are related to the vagueness of the participants. (*Table 12*).

Table 12. Independent variable importance for the whole scales.

Independent Variable Importance		
	Importance	Normalized Importance
VAR5c	0.162	100.00%
VAR2a	0.133	82.30%
VAR5a	0.121	74.70%
VAR3a	0.1	61.50%
VAR1c	0.096	59.30%
VAR2b	0.09	55.70%
VAR5b	0.083	51.10%
VAR4a	0.075	46.60%
VAR3c	0.035	21.50%
VAR1a	0.032	20.00%
VAR2c	0.022	13.30%
VAR4b	0.018	11.20%
VAR4c	0.015	9.00%
VAR1b	0.013	7.80%
VAR3b	0.005	2.90%

5| Discussion and Conclusion

Content validity refers to how appropriate and representative the measurements collected are for the desired assessment purpose. Content validity refers to how appropriate and representative the measurements obtained are for the desired assessment purpose. The representativeness criterion may have two definitions. Quantifying the extent of sampling is one of them. The second is the extent to which items reflect the structures of the whole scale [15]. In this regard, the most obviating factor in determining whether an item should be removed or not is to use the participants' vagueness choices for each item. In this respect, we have two kinds of variables to formalize our decision-making as correlation constant and importance level. If the decision function is labelled as d where r stands for correlation constant and I stands for importance level, the function for decision making can be written as like this:

$$D=R*I. \tag{1}$$

The interpretation of this formula can be given in *Table 1*. It should be noted that the correlation constant is the absolute value of r as |*R*|.

Table 13. The interpretation of the formula D=R*I.

The Interpretation of The Correlation Coefficient (r)	**The Interpretation of The Importance Level**	**Decision Criteria for Accepting or Rejecting The Item where 0<cc<1** **Decision=[correlation coefficient for vagueness (r)]*[Importance level for vagueness]**
Very weak correlation or no correlation if r <0.2	Very weak importance level if <20%	if 0≤cc≤20, item acceptable
Weak correlation between 0.2-0.4	Weak importance level 20%-40%	if 20<cc≤40, item acceptable
A moderate correlation between 0.4-0.6	Moderate importance level 40%-60%	if 40<cc≤60, the item should be modified or removed
The high correlation between 0.6-0.8	High importance level 60%-80%	if 60<cc≤80, the item should be modified or removed
If r>0.8, it is interpreted that there is a very high correlation	If 80%>, it is interpreted that there is a very high importance level	if 80<cc≤100, the item should be removed

The formula 5.1 can be applied for the findings of the items of the neutrosophic Life Satisfaction Scale for confirmatory analysis. Let's look at our findings based on item levels with the *Eq. (1)* as given in *Table 14*. The results show that this scale is valid because all the items are at an acceptable level.

Table 14. Application of the Eq. (1) for each item.

	Importance Level (i)	Correlation Constant (r)	Decision Result (d=i*r)	
Var1	57.5	0.22	12.65	Acceptable
Var2	31.7	0.12	3.804	Acceptable
Var3	21.7	0.178	3.8626	Acceptable
Var4	27	0.115	3.105	Acceptable
Var5	39	0.149	5.811	Acceptable

In *Table 11*, independent variable importance for the whole scale shows that variable 2 and variable 5 might be problematic when considering the overall contribution of the items for the whole scale since variable …b items are related to the vagueness of the participants. However, formula 4.1 shows that although the importance level is high, it is not significant, so that all the items on the scale are valid. Finally, one might ask that if the item related to vagueness is only focused on, why do we need the other two items regarding agreement and disagreement ? Although on this scale such a conflict is not seen, this data can be used to evaluate the validity and reliability of the scale. For instance, if both agreement and disagreement items have a similar sign to the target item, it can be concluded that this item is also problematic because it reflects both agreement and disagreement at the same time, implying that there is confusion about it for determining the aimed question. Let label that the correlation of agreement item is α and the correlation of disagreement item is β since these items are opposite to each other their correlation should naturally be opposite to each other so that $\alpha*\beta=-1$. If $\alpha*\beta=+1$ it can be concluded that there is a contradiction in this item. If the *Eq. (1)* is modified for these values where i_1 is the importance level of the first item and i_2 is the importance level of the second item as follows:

$$i_1 *\alpha* i_2* \beta)/100= d. \tag{2}$$

Because we don't want to deal with huge numbers in all the importance levels 100 and correlations 1 or-1, the multiplication is divided by 100 simply by scaling the value into a more simple form.Let apply the rule of our correlation constants in the finding section for each item in *Table 3*. An opposite sign indicates that our data is consistent. Otherwise, the effect of the correlations can be examined and evaluated to be whether the item should be removed or not just as in the classification given in *Table 13*.

Table 15. Decision matrix evaluating the consistency of the items in terms of agreement and disagreement items of the neutrosophic scale.

	i_1	A	i_2	β	$i_1 *\alpha* i_2* \beta$	Decision
Variable 1	100	0.678	26.4	-0.417	-7.4639664	Acceptable
Variable 2	100	0.732	26.6	-0.277	-5.3935224	Acceptable
Variable 3	100	0.474	38	-0.430	-7.74516	Acceptable
Variable 4	95.8	0.715	100	-0.475	-32.536075	Acceptable
Variable 5	100	0.706	84.2	-0.347	-20.6274844	Acceptable

5.1| Future Directions

A neutrosophic scale can be used to confirm the reliability of the classical one because the neutrosophic scale is just an extended form of the classical one. The results show that our neutrosophic scale is also reliable, which also supports the reliability of the classical scale because Cronbach's Alpha constant is an acceptable level for three dimensions. In this respect, it can be understood the Agreement dimension of reliability because the classical scale can be extended into the neutrosophic one and assess the closeness of two measurements made on the same subject as opposed to one another. The repeatability of the scale can be also assessed because the same variable can be measured again and again for the same circumstances [9]. The reproducibility of the scale can be also tested because the variations in test results can also be tested while tests are performed on subjects on different occasions.

Validity simply means "measure what is intended to be measured" [13]. To decide whether a scale is valid or not, its validity can be compared by comparing similar scales or decisions based on expert opinion can be made. In this study, it is offered an alternative method for developing a valid scale where first the scale is converted into a neutrosophic one and then they are compared through neural networks. It can be inferred that any scale to assess how appropriate and representative the measurements collected are for the desired assessment purpose so that its content validity can be evaluated. It can bee can understood how well a concept, idea, or behavior is translated or transformed that is a construct into a functioning and operating reality, the operationalization [14] on any scale so that its construct validity can be understood. This method can also be used for criterion validity because how well one measure predicts another measure can also be calculated.

This research is limited by Three-Valued Logic but it can be extended higher n-valued logics as well. It is limited by classical statistics such as correlation or neural networks but neutrosophic statistics can be also used or the whole data. It is limited by investigating the validity in terms of neutrosophy but this research can be extended into more broader concepts in education. Additionally, more sophisticated formulas can be also developed for subsequent analysis.

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Conflicts of Interest

The authors declare that they have no conflict of interest.

Authorship Contributions

The authors declare that they contribute equally to the study.

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References

[1] Eliot. J. (1987). *The Psychometric Approach. In: Models of Psychological Space*. Springer, New York, NY. https://doi.org/10.1007/978-1-4612-4788-3_3

[2] Özgüven, İ. E. (2011). *Psychological tests*. Ankara: Pdrem Yayınları. (**In Torkish**). https://www.nobelyayin.com/kitap_bilgileri/dosyalar/psiko_test_jen_133413.pdf

[3] Atılgan, H. (2009). *Measurement and evaluation in education*. Ankara: Anı Yayıncılık. (**In Torkish**).

[4] Fraenkel, J. R. Wallen, N. E. & Hyun, H. H. (2012*). How to design and evaluate research in education*. USA: The McGraw-Hill Companies.

[5] Tavşancıl, E. (2010). *Measuring attitudes and data analysis with SPSS*. Ankara: Nobel Yayıncılık. (**In Torkish**). https://www.nobelyayin.com/kitap_bilgileri/dosyalar/tutum_olclmesi_jen_171512.pdf

[6] Hinkin, T. (1995). A review of scale development practices in the study of organizations. *Journal of management*, 21(5), 967-988.

[7] Şencan, H. (2005). *Reliability and validity*. Ankara: Seçkin Yayıncılık. (**In Torkish**).

https://books.google.com/books?id=_AdLDwAAQBAJ&lpg=PA288&ots=ULVqWcWYjr&dq=Sosyal%20ve%20Davran%C4%B1%C5%9Fsal%20%C3%961%C3%A7%C3%BCmlerde%20G%C3%BCvenilirlik%20ve%20Ge%C3%A7erlilik&lr&pg=PA288#v=onepage&q=Sosyal%20ve%20Davran%C4%B1%C5%9Fsal%20%C3%961%C3%A7%C3%BCmlerde%20G%C3%BCvenilirlik%20ve%20Ge%C3%A7erlilik&f=false

[8] Carmines, E. G. & Zeller, R. A. (1979). *Reliability and validity assessment*. Newbury Park, Ca, Sage.

[9] Bartlett, J. W., & Frost, C. (2008). Reliability, repeatability and reproducibility: analysis of measurement errors in continuous variables. *Ultrasound in obstetrics and gynecology: the official journal of the international society of ultrasound in obstetrics and gynecology*, *31*(4), 466-475.

[10] Moser, C. A. & Kalton, G. (1989). *Survey methods ın social ınvestigation*, Aldershot, Gower.

[11] Rosen, A. M. (2019). *Effective research methods for any project*. USA: Teaching Company.

[12] Hergüner, S. (2010). *Basic concepts of the use of measurement tools.* Türkiye Çocuk ve Genç Psikiyatrisi Derneği, İstanbul. (**In Torkish**).

https://www.researchgate.net/profile/Sabri-Herguener-2/publication/281107511_OLCME_ARACLARININ_KULLANIMI_ILE_ILGILI_TEMEL_KAVRAMLAR/links/55d5ca9508aec156b9a51e75/OeLCME-ARACLARININ-KULLANIMI-ILE-ILGILI-TEMEL-KAVRAMLAR.pdf

[13] Field, A. P. (2005). *Discovering statistics using SPSS*. Sage Publications Inc.

[14] Taherdoost, H. (2016). Validity and reliability of the research instrument; how to test the validation of a questionnaire/survey in a research. *How to test the validation of a questionnaire/survey in a research*. https://ssrn.com/abstract=3205040

[15] Slavec, A., & Drnovšek, M. (2012). A perspective on scale development in entrepreneurship research. *Economic and business review*, *14*(1).

[16] Barnhart, H. X., Haber, M. J., & Lin, L. I. (2007). An overview on assessing agreement with continuous measurements. *Journal of biopharmaceutical statistics*, *17*(4), 529-569.

[17] Vanderstoep, S. W., & Johnson, D. D. (2008). *Research methods for everyday life: Blending qualitative and quantitative approaches* (Vol. 32). John Wiley & Sons.

[18] Souza, A. C. D., Alexandre, N. M. C., & Guirardello, E. D. B. (2017). Psychometric properties in instruments evaluation of reliability and validity. *Epidemiologia e serviços de saúde*, *26*, 649-659.

[19] Mertens, D. M.(2010). *Research and evaluation in education and psychology: integrating diversity with quantitative, qualitative, and mixed methods*. USA: Sage.

[20] Zadeh, L. A. (1965). Fuzzy sets. *Inf. Control*, 8, 338–353.

[21] Atanassov, K. T. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets Syst*. 20, 87–96.

[22] Smarandache, F. (1998). *Neutrosophy: neutrosophic probability, set, and logic : analytic synthesis & synthetic analysis*. American Research Press.

[23] Diener, E. D., Emmons, R. A., Larsen, R. J., & Griffin, S. (1985). The satisfaction with life scale. *Journal of personality assessment*, *49*(1), 71-75.

[24] Dağlı, A. ve Baysal, N. (2016). Adaptation of satisfaction with life scale into turkish: validity and reliability study. *Elektronik sosyal bilimler dergisi* , *15*(59), 1250-1262.

[25] Lawshe, C. H. (1975). A quantitative approach to content validity. *Personnel psychology*, *28*(4), 563-575.

[26] Wilson, F. R., Pan, W., & Schumsky, D. A. (2012). Recalculation of the critical values for Lawshe's content validity ratio. *Measurement and evaluation in counseling and development*, *45*(3), 197-210.

[27] Ayre, C., & Scally, A. J. (2014). Critical values for Lawshe's content validity ratio: revisiting the original methods of calculation. *Measurement and evaluation in counseling and development*, *47*(1), 79-86.

[28] Gilbert, G. & Prion, S. (2016). Making sense of methods and measurement: Lawshe's content validity ındex. *Clinical simulation in nursing*, *12*(12), 530-531. https://doi.org/10.1016/j.ecns.2016.08.002

[29] Sartre, J. P. (2015). Being and nothingness. *Central works of philosophy: the twentieth century: Moore to popper*, *4*, 155.

[30] Nait Aicha, A., Englebienne, G., Van Schooten, K. S., Pijnappels, M., & Kröse, B. (2018). Deep learning to predict falls in older adults based on daily-life trunk accelerometry. *Sensors*, *18*(5), 1654. https://doi.org/10.3390/s18051654

[31] IBM SPSS Neural Networks 21. (2011). *IBM SPSS neural networks 21 manual*, USA: IBM. http://www.sussex.ac.uk/its/pdfs/SPSS_Neural_Network_21

[32] Çevik, A., Topal, S., & Smarandache, F. (2018). Neutrosophic logic based quantum computing. *Symmetry*, *10*(11), 656.

[33] Smarandache, F. (2005). *A unifying field in logics: neutrosophic logic. Neutrosophy, neutrosophic set, neutrosophic probability: neutrsophic logic. Neutrosophy, neutrosophic set, neutrosophic probability*. Infinite Study.

[34] Taş, F., Topal, S., & Smarandache, F. (2018). Clustering neutrosophic data sets and neutrosophic valued metric spaces. *Symmetry*, *10*(10), 430.

[35] Doğan, İ. Doğan, N. (2019). An Overview evaluation of the content validity used in scale development studies. *Turkiye Klinikleri J Biostat*, *11*(2), 143-51. DOI: 10.5336/biostatic.2019-65953

[36] Bertea, E., & Zaiţ, P. A. (2013). Scale validity in exploratory stages of research. *Management and marketing journal*, *11*(1), 38-46.

[37] AboElHamd, E., Shamma, H. M., Saleh, M., & El-Khodary, I. (2021). Neutrosophic logic theory and applications. *Neutrosophic sets and systems*, *41*, 30-51.

[38] Smarandache, F. (2019). *Introduction to neutrosophic sociology (neutrosociology)*. Infinite Study.

[39] Martínez, C. R., Hidalgo, G. A., Matos, M. A., & Smarandache, F. (2020). *Neutrosophy for survey analysis in social sciences* (Vol. 37). Infinite Study.

[40] Leyva-Vázquez, M. (2018). *Neutrosophy: New advances in the treatment of uncertainty*. (**In Spanish**). Pons Publishing House / Pons asbl. https://www.ibm.com/docs/en/spss-statistics/23.0.0?topic=overtraining-independent-variable-importance

[41] Smarandache, F. (2015). Neutrosophic social structures specificities. *Social sciences and education research review*, *2*(1), 3-10.

[42] Khan, Z., Gulistan, M., Kadry, S., Chu, Y., & Lane-Krebs, K. (2020). On scale parameter monitoring of the Rayleigh distributed data using a new design. *IEEE access*, *8*, 188390-188400.

[43] Khan, Z., Gulistan, M., Hashim, R., Yaqoob, N., & Chammam, W. (2020). Design of S-control chart for neutrosophic data: An application to manufacturing industry. *Journal of intelligent & fuzzy systems*, *38*(4), 4743-4751.

[44] Khan, Z., Gulistan, M., Chammam, W., Kadry, S., & Nam, Y. (2020). A new dispersion control chart for handling the neutrosophic data. *IEEE access*, *8*, 96006-96015.

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Quadripartitioned Neutrosophic Pythagorean Lie Subalgebra

R. Radha [1], **A. Stanis Arul Mary**[2]

[1] Research Scholar, Department of Mathematics, Nirmala College for Women, Coimbatore, Tamil Nadu, India; radharmat2020@gmail.com.

[2] Assistant Professor, Department of Mathematics, Nirmala College for Women, Coimbatore, Tamil Nadu, India; stanisarulmary@gmail.com.

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Abstract

A Quadripartitioned Neutrosophic Pythagorean (QNP) set is a powerful general format framework that generalizes the concept of Quadripartitioned Neutrosophic Sets and Neutrosophic Pythagorean Sets. In this paper, we apply the notion of quadripartitioned Neutrosophic Pythagorean sets to Lie algebras. We develop the concept of QNP Lie subalgebras and QNP Lie ideals. We describe some interesting results of QNP Lie ideals.

Keywords: QNP Lie ideal; QNP Lie subalgebra; Lie ideal; Lie subalgebra.

1 | Introduction

The concept of Lie groups was first introduced by Sophus Lie in nineteenth century through his studies in geometry and integration methods for differential equations. Lie algebras were also discovered by him when he attempted to classify certain smooth subgroups of a general linear group. The importance of Lie algebras in mathematics and physics has become increasingly evident in recent years. In mathematics, Lie theory remains a robust tool for studying differential equations, special functions and perturbation theory. It's noted that Lie theory has applications not only in mathematics and physics but also in diverse fields like continuum mechanics, cosmology and life sciences. Lie algebra has been utilized by electrical engineers, mainly within the mobile robot control [5].

Lie algebra has also been accustomed solve the problems of computer vision. Fuzzy structures are related to theoretical soft computing, especially Lie algebras and their different classifications, have numerous applications to the spectroscopy of molecules, atoms and nuclei. One amongst the key concepts within the applying of Lie algebraic method in physics is that of spectrum generating algebras and their associated dynamic symmetries. The most important advancements within the fascinating world of fuzzy sets started with the work of renowned scientist Zulqarnain et al. [14] with new directions and ideas.

Corresponding Author: radharmat2020@gmail.com

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Wang et al. [6] defined SVN sets as a generalization of fuzzy sets and intuitionistic fuzzy sets [4]. Algebraic structures have a major place with vast applications in various disciplines.

Neutrosophic set has been applied to algebraic structures. Fuzzification of Lie algebras has been discussed in [1]–[3]. The idea of single valued neutrosophic Lie algebra was investigated by Akram et al. [7]. Quadripartitioned Neutrosophic Set and its properties were introduced by Smarandache [12]. During this case, indeterminacy is split into two components: contradiction and ignorance membership function. The Quadripartitioned Neutrosophic Set is a particular case of Refined Neutrosophic Set. Smarandache [12] extended the Neutrosophic Set to refined [n-valued] neutrosophic set, and to refined neutrosophic logic, and to refined neutrosophic probability, i.e. the truth value T is refined/split into types of sub-truths such as T1, T2, …, similarly indeterminacy I is refined/split into types of sub-indeterminacies I1, I2, …, and the falsehood F is refined/split into sub-falsehood F1, F2 ,...

We've now extended our research during this Pentapartitioned neutrosophic set as a space. Also we introduced the concept of Penta partitioned neutrosophic Pythagorean set [8]-[14] and establish variety of its properties in our previous work. During this paper, we apply the notion of Quadripartitioned Neutrosophic Pythagorean (QNP) sets to Lie algebras.

In this paper, we develop the concepts of QNP Lie subalgebras and investigated some of its properties. Furthermore, we have also studied the concept of QNP Lie ideals. We describe some interesting results of QNP Lie ideals.

2| Preliminaries

Lie algebra [1] is a vector space L over a field F (equal to R or C) on which $L \times L \to L$ denoted

by (x, y) ⟶ [x, y] is defined satisfying the following axioms:

(L1) [x, y] is bilinear,

(L2) [x, x] = 0 for all x ∈L,

(L3) [[x, y], z] + [[y, z], x] + [[z, x], y] = 0 for all x, y, z ∈L (Jacobi identity).

Throughout this paper, L is a Lie algebra and F is a field. We note that the multiplication

in a Lie algebra is not associative, i.e., it is not true in general that [[x, y], z] = [x, [y, z]]. But it

is anti commutative, i.e., [x, y] = −[y, x]. A subspace H of L closed under [· , ·] will be called a

Lie subalgebra.

A fuzzy set μ: L ⟶ [0, 1] is called a fuzzy Lie ideal [1] of L if

I. $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$,
II. $\mu(\alpha x) \geq \mu(x)$,
III. $\mu([x, y]) \geq \mu(x)$,

hold for all x, y ∈L and α∈F.

Definition 1. [9]. Let R be a space of points (objects). A QNP set on a non-empty R is characterized by truth membership function A1: R → [0, 1], contradiction membership function A2: R→ [0, 1], ignorance membership function A4: R → [0, 1] and false membership function A5: R → [0, 1].

Thus, R = { <r, A1(r) ,A2(r), A4 (r),A5(r)>} satisfies with the following conditions $A1+A2+A4+A5\leq 2$, $A1 + A5 \leq 1$,$A2+A4 \leq 1$.Here A1,A2, A4, A5 are dependent neutrosophic components.

Definition 2. [7]. An SVN set N = (TN, IN, FN) on Lie algebra L is called an SVN Lie subalgebra if the following conditions are satisfied:

I. $TN(x + y) \geq min(TN(x), TN(y)), IN(x + y) \geq min(IN(x), IN(y))$ and $FN(x + y) \leq max(FN(x), FN(y)),$
II. $TN(\alpha x) \geq TN(x), IN(\alpha x) \geq IN(x)$ and $FN(\alpha x) \leq FN(x),$
III. $TN ([x, y]) \geq min \{TN(x), TN(y)\}, IN ([x, y]) \geq min \{IN(x), IN(y)\}$ and $FN ([x, y]) \leq max \{FN(x), FN(y)\}$ for all x, $y \in L$ and $\alpha \in F$.

Definition 3. [7]. A SVN set N = (TN, IN, FN) onL is called an SVN Lie ideal if it satisfies the *Conditions (I), (II)* and the following additional condition:

Single-valued Neutrosophic Lie algebras

IV. $TN([x, y]) \geq TN(x), IN([x, y]) \geq IN(x)$ and $FN([x, y]) \leq FN(x)$

for all x, y ∈L.

From *Condition (2)* it follows that:

V. $TN(0) \geq TN(x), IN(0) \geq IN(x), FN(0) \leq FN(x),$
VI. $TN (-x) \geq TN(x), IN (-x) \geq IN(x), FN (-x) \leq FN(x).$

3 | Quadripartitioned Neutrosophic Pythagorean Lie Subalgebra

We define here QNP Lie subalgebras and QNP Lie ideal.

Definition 4. A QNP set R = ($A1_R$, $A2_R$, $A4_R$, $A5_R$) on is called a QNP Lie subalgebra $\mathcal{L}$ if the following conditions are satisfied:

I. $A1_R (a + b) \geq min (A1_R (a), A1_R (b)), A2_R (a + b) \geq min (A2_R (a), A2_R (b)), A4_R (a + b) \leq max (A4_R (a), A4_R (b)), A5_R (a + b) \leq max (A5_R (a), A5_R (b)),$
II. $A1_R (\beta a) \geq A1_R (a), A2_R (\beta a) \geq A2_R (a), A4_R (\beta a) \leq A4_R (a)$ and $A5_R (\beta a) \leq A5_R (a).$
III. $A1_R ([a, b]) \geq min (A1_R (a), A1_R (b)), A2_R ([a, b]) \geq min (A2_R (a), A2_R (b)), A4_R ([a, b]) \leq max (A4_R (a), A4_R (b)), A5_R ([a, b]) \leq max (A5_R (a), A5_R (b)).$

For all $a, b \in \mathscr{L}$ and $\in \mathscr{F}$.

Definition 5. A QNP set R = ($A1_R$, $A2_R$, $A3_R$, $A4_R$,$A5_R$) on $\mathcal{L}$ is called an QNP Lie ideal if it satisfies the following *Conditions (I)* and *(II)* and the following additional conditions:

IV. $A1_R ([a, b]) \geq A1_R (a), A2_R ([a, b]) \geq A2_R (a), A4_R ([a, b]) \leq A4_R (a), A5_R ([a, b]) \leq A5_R (a),$

From (II), it follows that:

V. $A1_R (0) \geq A1_R (a), A2_R (0) \geq A2_R (a), A4_R (0) \leq A4_R (a)$ and $A5_R (0) \leq A5_R (a),$
VI. $A1_R (- a) \geq A1_R (a), A2_R (- a) \geq A2_R (a), A4_R (- a) \leq A4_R (a)$ and $A5_R (- a) \leq A5_R (a).$

Proposition 1. Every QNP Lie ideal is a QNP Lie subalgebra.

We note here that the converse of the above proposition does not hold in general as it can be seen in the following example.

Example 1. Consider $\mathscr{F} = \mathbb{R}$. Let $\mathscr{L} = \Re^3 = \{(a, b, c): a, b, c \in \mathbb{R}\}$ be the set of all three dimensional real vectors which forms a QNP Lie algebra and define

$$\Re^3 x\, \Re^3 \rightarrow \Re^3$$

$$[a, b] \rightarrow a \text{ x } b,$$

Where x is the usual cross product. We define an QNP set $R = (A1_R, A2_R, A4_R, A5_R): \Re^3 \rightarrow [0,1] \text{ x } [0,1] \text{ x } [0,1] \text{ x } [0,1]$ by

$$A1_R\,(a, b, c) = \begin{cases} 1, & \text{if } a = b = c = 0, \\ 0.3, & \text{if } a \neq 0, b = c = 0, \\ 0, & \text{otherwise} \end{cases}$$

$$A2_R\,(a, b, c) = \begin{cases} 1, & \text{if } a = b = c = 0, \\ 0.2, & \text{if } a \neq 0, b = c = 0, \\ 0, & \text{otherwise} \end{cases}$$

$$A4_R\,(a, b, c) = \begin{cases} 0, & \text{if } a = b = c = 0, \\ 0.3, & \text{if } a \neq 0, b = c = 0, \\ 1, & \text{otherwise} \end{cases}$$

$$A5_R\,(a, b, c) = \begin{cases} 0, & \text{if } a = b = c = 0, \\ 0.5, & \text{if } a \neq 0, b = c = 0. \\ 1, & \text{otherwise} \end{cases}$$

Then $R = (A1_R, A2_R, A4_R, A5_R)$ is an QNP Lie subalgebra of $\mathscr{L}$ but $R = (A1_R, A2_R, A4_R, A5_R)$ is not an QNP Lie ideal of $\mathscr{L}$ since

$$A1_R\,([1,0,0)\ (1,1,1)]) = A1_R\,(0, -1, 1) = 0,$$

$$A2_R\,([1,0,0)\ (1,1,1)]) = A2_R\,(0, -1, 1) = 0,$$

$$A4_R\,([1,0,0)\ (1,1,1)]) = A4_R\,(0, -1, 1) = 1,$$

$$A5_R\,([1,0,0)\ (1,1,1)]) = A5_R\,(0, -1, 1) = 1,$$

$$A1_R\,(1,0,0) = 0.2,\ A2_R\,(1,0,0) = 0.3,\ A4_R\,(1,0,0) = 0.3,\ A5_R\,(1,0,0) = 0.5.$$

That is,

$$A1_R\,([1,0,0)\ (1,1,1)]) \ngeq A1_R\,(1,0,0),$$

$$A2_R\,([1,0,0)\ (1,1,1)]) \ngeq A2_R\,(1,0,0),$$

$$A4_R\,([1,0,0)\ (1,1,1)]) \nleq A4_R\,(1,0,0),$$

$$A5_R\,([1,0,0)\ (1,1,1)]) \nleq A5_R\,(1,0,0).$$

Proposition 2. If R is an QNP Lie ideal of $\mathscr{L}$, then

I. $A1_R\,(0) \geq A1_R\,(a)$, $A2_R\,(0) \geq A2_R\,(a)$, $A4_R\,(0) \leq A4_R\,(a)$ and $A5_R\,(0) \leq A5_R\,(a)$,
II. $A1_R\,([a, b]) \geq \max\,\{A1_R\,(a), A1_R\,(b)\}$,

III. $A2_R([a,b]) \geq \max\{A2_R(a), A2_R(b)\}$,
IV. $A4_R([a,b]) \leq \min\{A4_R(a), A4_R(b)\}$,
V. $A5_R([a,b]) \leq \min\{A5_R(a), A5_R(b)\}$,
VI. $A1_R([a,b]) = A1_R(-[b,a]) = A1_R([b,a])$,
VII. $A2_R([a,b]) = A2_R(-[b,a]) = A2_R([b,a])$,
VIII. $A4_R([a,b]) = A4_R(-[b,a]) = A4_R([b,a])$,
IX. $A5_R([a,b]) = A5_R(-[b,a]) = A5_R([b,a])$.

For all $a, b \in \mathscr{L}$.

Proof. The proof follows from *Definition 5*.

Proposition 3. If $\{R_i : i \in J\}$ is a family of QNP Lie algebra of $\mathscr{L}$, then $\bigcap R_i = (\wedge A1_{Ri} \wedge A2_{Ri} \vee A4_{Ri} \vee A5_{Ri})$ is an QNP Lie ideal of $\mathscr{L}$ where,

$$\wedge A1_{Ri}(a) = \inf\{\wedge A1_{Ri}(a) : i \in J,\ a \in \mathscr{L}\},$$

$$\wedge A2_{Ri}(a) = \inf\{\wedge A2_{Ri}(a) : i \in J,\ a \in \mathscr{L}\},$$

$$\vee A4_{Ri}(a) = \sup\{\vee A4_{Ri}(a) : i \in J,\ a \in \mathscr{L}\},$$

$$\vee A5_{Ri}(a) = \sup\{\vee A5_{Ri}(a) : i \in J,\ a \in \mathscr{L}\}.$$

Proof. The proof follows from *Definition 5*.

Definition 6. Let R = ($A1_R$, $A2_R$, $A4_R$, $A5_R$) be an QNP Lie subalgebra of $\mathscr{L}$ and let $(\alpha, \beta, \delta, \vartheta)$ [0,1] X [0,1] X [0,1] X [0,1] with $\alpha + \beta + \delta + \vartheta \leq 2$. Then level subset of R is defined as

$$R^{(\alpha,\beta,\delta,\vartheta)} = \{\ a \in \mathscr{L}: A1(a) \geq \alpha, A2(a) \geq \beta, A4(a) \leq \delta, A5(a) \leq \vartheta\},$$

are called $(\alpha, \beta, \gamma, \delta, \vartheta)$ level subsets of QNP set *R*. The set of all $(\alpha, \beta, \delta, \vartheta) \in$ Im($A1_R$) X Im($A2_R$) X Im($A4_R$) X Im($A5_R$) such that $\alpha + \beta + \delta + \vartheta \leq 2$ is known as image of R = ($A1_R$, $A2_R$, $A4_R$, $A5_R$).

Note:

$$R^{(\alpha,\beta,\delta,\vartheta)} = \{a \in \mathscr{L}: A1(a) \geq \alpha, A2(a) \geq \beta, A4(a) \leq \delta, A5(a) \leq \vartheta\},$$

$$R^{(\alpha,\beta,\delta,\vartheta)} = \{a \in \mathscr{L}: A1(a) \geq \alpha\} \cap \{a \in \mathscr{L}: A2(a) \geq \beta\} \cap \{a \in \mathscr{L}: A4(a) \leq \delta\} \cap \{a \in \mathscr{L}: A5(a) \leq \vartheta\},$$

$$R^{(\alpha,\beta,\delta,\vartheta)} = U(A1(a), \alpha) \cap U'(A2(a), \beta) \cap \gamma) \cap L'(A4(a), \delta) \cap L''(A5(a), \vartheta).$$

Theorem 1. An QNP set R = ($A1_R$, $A2_R$, $A4_R$, $A5_R$) of $\mathscr{L}$ is an QNP lie ideal of $\mathscr{L}$ *iff* $R^{(\alpha,\beta,\delta,\vartheta)}$ is a QNP Lie ideal of $\mathscr{L}$ for every $(\alpha, \beta, \gamma, \delta, \vartheta)$ [0,1] X [0,1] X [0,1] X [0,1] with $\alpha + \beta + \delta + \vartheta \leq 3$.

Proposition 4. Let R = ($A1_R$, $A2_R$, $A4_R$, $A5_R$) be an QNP Lie ideal of $\mathscr{L}$ and (r_1, s_1, u_1, v_1), (r_2, s_2, u_2, v_2) $\in$ Im(A1R) X Im($A2_R$) X Im($A4_R$) X Im($A5_R$) with ri+ si+ ui +vi≤ 3 for i = 1,2. Then $\mathscr{L}_R^{(r1,s1,u1,v1)} = \mathscr{L}_R^{(r2,s2,u2,v2)}$ if and only if (r1, s1, u1, v1) = (r2, s2, u2, v2)

Theorem 2. Let $K_0 \subset K_1 \subset K_2 \subset K_3 \ldots\ldots\ldots \subset K_n = L$ be a chain of QNP Lie ideals of a QNP Lie algebra $\mathscr{L}$. Then there exists an QNP ideal $A1_R$ of $\mathscr{L}$ for which level subsets U (A1 (a),α), U$'$(A2(a), β), L$'$(A 4(a), δ) and L”(A5(a),ϑ) coincide with this chain.

Proof. Let { r_k: k = 0,1,2…,n}, {s_k:k = 0,1,….n} , {u_k: k = 0,1,2…n} and {v_k: k = 0,1,2…n} be finite decreasing and increasing sequences in [0,1]. Let Let R = ($A1_R$, $A2_R$, $A4_R$, $A5_R$) be a QNP set in $\mathscr{L}$ defined by $A1_R(K_0) = r_0$, $A2_R(K_0) = s_0$, $A4_R(K_0) = u_0$, $A5_R(K_0) = v_0$, $A1_R(K_l{:}K_{l-1}) = r_l$, $A2_R(K_l\backslash K_{l-1}) = s_l$, $A4_R(K_l\backslash K_{l-1}) = u_l$,$A5_R(K_l\backslash K_{l-1}) = v_l$, for $0 < l \leq n$. Let a, b $\in \mathscr{L}$. If a ,b$\in K_l\backslash K_{l-1}$, then a +b, βa, [a,b] $\in K_l$

$$A1_R(a + b) \geq r_k = \min\{A1_R(a), A1_R(b)\},$$

$$A2_R(a + b) \geq s_k = \min\{A2_R(a), A2_R(b)\},$$

$$A4_R(a + b) \leq u_k = \max\{A4_R(a), A4_R(b)\},$$

$$A5_R(a + b) \leq v_k = \max\{A5_R(a), A5_R(b)\},$$

$$A1_R(\alpha a) \geq r_k = A1_R(a),\ A2_R(\alpha a) \geq s_k = A2_R(a),$$

$$A4_R(\alpha a) \leq u_k = A4_R(a),\ A5_R(\alpha a) \leq v_k = A5_R(a),$$

$$A1_R([a, b]) \geq r_k = A1_R(a),\ A2_R([a, b]) \geq s_k = A2_R(a),$$

$$A4_R([a, b]) \leq u_k = A4_R(a),\ A5_R([a, b]) \leq v_k = A5_R(a).$$

For i> j, if a $\in K_i \setminus K_{i-1}$ and b $\in K_j \setminus K_{j-1}$, then $A1_R(a) = r_i = A1_R(b)$, $A2_R(a) = s_i = A2_R(b)$,

$A4_R(a) = u_j = A4_R(b)$, $A5_R(a) = v_j = A5_R(b)$ and a +b, αa, [a, b] $\in K_I$. Thus

$$A1_R(a + b) \geq r_i = \min\{A1_R(a), A1_R(b)\},$$

$$A2_R(a + b) \geq s_i = \min\{A2_R(a), A2_R(b)\},$$

$$A4_R(a + b) \leq u_j = \max\{A4_R(a), A4_R(b)\},$$

$$A5_R(a + b) \leq v_j = \max\{A5_R(a), A5_R(b)\},$$

$$A1_R(\alpha a) \geq r_i = A1_R(a),\ A2_R(\alpha a) \geq s_i = A2_R(a),$$

$$A4_R(\alpha a) \leq u_j = A4_R(a),\ A5_R(\alpha a) \leq v_j = A5_R(a),$$

$$A1_R([a, b]) \geq r_i = A1_R(a),\ A2_R([a, b]) \geq s_i = A2_R(a),$$

$$A4_R([a, b]) \leq u_j = A4_R(a),\ A5_R([a, b]) \leq v_j = A5_R(a).$$

Thus, we conclude that R = ($A1_R$, $A2_R$, $A4_R$, $A5_R$) is a QNP Lie ideal of a QNP Lie algebra $\mathscr{L}$ and all its non-empty level subsets are QNP Lie ideals.

Since Im ($A1_R$) = {r_0, r_1, r_2…..,r_n }, Im ($A2_R$) = {s_0, s_1, s_2…..,s_n },

Im ($A4_R$) = {u_0, u_1, u_2…..,u_n }, Im ($A5_R$) = {v_0, v_1, v_2…..,v_n }, level subsets of R forms chains:

$$U(A1_R, r_0) \subset U(A1_R, r_1) \subset \ldots.. \subset U(A1_R, r_n) = L,$$

$$U'(A2_R, s_0) \subset U'(A2_R, s_1) \subset \ldots.. \subset U'(A2_R, s_n) = L,$$

$L'(A4_R, u_0) \subset L'(A4_R, u_1) \subset \ldots.. \subset L'(A4_R, u_n) = L$,

$L''(A5_R, v_0) \subset L''(A5_R, v_1) \subset \ldots.. \subset L''(A5_R, v_n) = L$.

Respectively. Indeed

$U(A1_R, r_0) = \{a \in \mathcal{L}: A1_R(a) \geq r_0\} = K_0$,

$U'(A2_R, s_0) = \{a \in \mathcal{L}: A2_R(a) \geq s_0\} = K_0$,

$L'(A4_R, u_0) = \{a \in \mathcal{L}: A4_R(a) \leq u_0\} = K_0$,

$L''(A5_R, v_0) = \{a \in \mathcal{L}: A5_R(a) \leq v_0\} = K_0$.

We prove that $U(A1_R, r_l) = U'(A2_R, s_l) = L'(A4_R, u_l) = L''(A5_R, v_l) = K_l$ for $0 \leq l \leq n$.

Clearly, $K_l \subseteq U(A1_R, r_l)$, $K_l \subseteq U'(A2_R, s_l)$, $K_l \subseteq L'(A4_R, u_l)$, $K_l \subseteq L''(A5_R, v_l)$.

If $a \in U(A1_R, r_l)$, then $A1_R(a) \geq r_l$ and for $a \notin K_j$, for $j > l$. Hence $A1_R(a) \in \{r_0, r_1, r_2 \ldots.., r_l\}$,

Which implies $a \in {}_j$ for some $j \leq l$. Since $K_j \subset K_l$, it follows that $a \in K_l$ .Consequently, $U(A1_R, r_l) = K_l$ for some $0 < l \leq n$.

If $a \in U'(A2_R, s_l)$, then $A2_R(a) \geq s_l$ and for $a \notin K_j$, for $j > l$. Hence $A2_R(a) \in \{s_0, s_1, s_2 \ldots.., s_l\}$,

Which implies $a \in K_j$ for some $j \leq l$. Since $K_j \subset K_l$,it follows that $a \in K_l$ .Consequently, $U'(A2_R, s_l) = K_l$ for some $0 < l \leq n$.

If $a \in L'(A4_R, u_l)$, then $A4_R(a) \leq u_l$ and for $a \notin K_m$,for $m > l$. Hence $A4_R(a) \in \{u_0, u_1, u_2 \ldots.., u_l\}$,

Which implies $a \in K_m$ for some $m \leq l$. Since $K_m \subset K_l$,it follows that $a \in K_l$.

Consequently, $L'(A4_R, u_l) = K_l$ for some $0 < l \leq n$.

If $a \in L''(A5_R, v_l)$,then $A5_R(a) \leq v_l$ and for $a \notin K_m$,for $m > l$. Hence $A5_R(a) \in \{v_0, v_1, v_2 \ldots.., v_l\}$,

Which implies $a \in K_m$ for some $m \leq l$. Since $K_m \subset K_l$,it follows that $a \in K_l$.

Consequently, $L''(A5_R, v_l) = K_l$ for some $0 < l \leq n$. This completes the proof.

Theorem 3. If $R = (A1_R, A2_R, A4_R, A5_R)$ is an QNP Lie ideal of a QNP Lie algebra $\mathscr{L}$, then

$A1_R(a) = \sup \{r \in [0,1] \setminus a \in U(A1_R, r)\}$,

$A2_R(a) = \sup \{s \in [0,1] \setminus a \in U'(A2_R, s)\}$,

$A4_R(a) = \inf \{u \in [0,1] \setminus a \in L'(A4_R, u)\}$,

$A5_R(a) = \inf \{v \in [0,1] \setminus a \in U(A5_R, v)\}$,

for every $a \in \mathscr{L}$.

Proof. The proof follows from *Definition 5*.

Definition 7. Let f be a map from a set $\mathscr{L}_1$ to a set $\mathscr{L}_2$. If R = ($A1_R$, $A2_R$, $A4_R$, $A5_R$) and

R = ($A1_R$, $A2_R$, $A4_R$, $A5_R$) are QNP sets in $\mathscr{L}_1$ and $\mathscr{L}_2$ respectively, then the preimage of R2 under f, denoted by f^{-1}(R2), is a QNP set defined by

$f^{-1}(R2) = (f^{-1}(A1_{R2}), f^{-1}(A2_{R2}), f^{-1}(A4_{R2}), f^{-1}(A5_{R2}))$.

Theorem 4. Let f : $\mathscr{L}_1 \rightarrow \mathscr{L}_2$ be an onto homomorphisms of Lie algebras. If R = ($A1_R$, $A2_R$, $A4_R$, $A5_R$) is a QNP Lie ideal of $\mathscr{L}_2$, then the preimage

$f^{-1}(R2) = (f^{-1}(A1_{R2}), f^{-1}(A2_{R2}), f^{-1}(A4_{R2}), f^{-1}(A5_{R2}))$ under f is a QNP Lie ideal of $\mathcal{L}_1$.

Proof. The proof follows from *Definitions 5* and *7*.

Theorem 5. Let f : $\mathscr{L}_1 \rightarrow \mathscr{L}_2$ be an epimorphisms of QNP Lie algebras. If R = ($A1_R$, $A2_R$, $A4_R$, $A5_R$) is a QNP Lie ideal of $\mathscr{L}_2$, then the preimage $f^{-1}((R1)^C) = (f^{-1}(R1))^C$

Proof. The proof follows from *Definitions 5* and *7*.

Theorem 6. Let f : $\mathscr{L}_1 \rightarrow \mathscr{L}_2$ be an epimorphisms of QNP Lie algebras. If R = ($A1_R$, $A2_R$, $A4_R$, $A5_R$) is a QNP Lie ideal of $\mathscr{L}_2$ and R = ($A1_R$, $A2_R$, $A4_R$, $A5_R$) is the preimage of R = ($A1_R$, $A2_R$, $A4_R$, $A5_R$) under f. Then R2 is a QNP Lie ideal of $\mathscr{L}_1$.

Proof. The proof follows from *Definitions 5* and *7*.

Definition 8. Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be two QNP Lie algebras and f be a mapping of $\mathscr{L}_1$ into $\mathscr{L}_2$. If R = ($A1_R$, $A2_R$, $A4_R$, $A5_R$) is a QNP set of $\mathscr{L}_1$, then the image of R1 under f is the QNP set in $\mathcal{L}_2$ defined by

$$f(A1_{R1})(b) = \begin{cases} \sup_{a\in f^{-1}(b)} A1_{R1}(a), & \text{if } f^{-1}(b) \neq 0, \\ 0, & \text{otherwise} \end{cases}$$

$$f(A2_{R1})(b) = \begin{cases} \sup_{a\in f^{-1}(b)} A2_{R1}(a), & \text{if } f^{-1}(b) \neq 0, \\ 0, & \text{otherwise} \end{cases}$$

$$f(A4_{R1})(b) = \begin{cases} \inf_{a\in f^{-1}(b)} A4_{R1}(a), & \text{if } f^{-1}(b) \neq 0, \\ 1, & \text{otherwise} \end{cases}$$

$$f(A5_{R1})(b) = \begin{cases} \inf_{a\in f^{-1}(b)} A5_{R1}(a), & \text{if } f^{-1}(b) \neq 0, \\ 1, & \text{otherwise} \end{cases}$$

for each $b \in \mathscr{L}_2$

Theorem 7. Let f : $\mathscr{L}_1 \rightarrow \mathscr{L}_2$ be an epimorphisms of QNP Lie algebras. If R = ($A1_R$, $A2_R$, $A4_R$, $A5_R$) is a QNP Lie ideal of $\mathscr{L}_1$, then f(R1) is a QNP Lie ideal of $\mathscr{L}_2$.

Proof. The proof follows from *Definitions 5* and *8*.

Definition 9. Let f : $\mathscr{L}_1 \rightarrow \mathscr{L}_2$ be an homomorphisms of QNP Lie algebras, For any QNP set, If R = ($A1_R$, $A2_R$, $A4_R$, $A5_R$) is a QNP Lie ideal of $\mathscr{L}_2$ we define a PNP set $R^f = (A1^f_R, A2^f_R, A4^f_R, A5^f_R)$ in $\mathscr{L}_1$ by

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$A1^f_R$ (a) = $A1_R$ (f(a)), $A2^f_R$(a) = $A2_R$ (f(a)), $A4^f_R$(a) = $A4_R$ (f(a)), $A5^f_R$(a) = $A5_R$ (f(a)), for all a ∈ $\mathscr{L}_1$.

Lemma 1. Let f :$\mathscr{L}_1 \rightarrow \mathscr{L}_2$ be an homomorphisms of QNP Lie algebras, If R = ($A1_R$, $A2_R$, $A4_R$, $A5_R$) is a QNP Lie ideal of $\mathscr{L}_2$, then R^f = ($A1^f_R$, $A2^f_R$, $A4^f_R$, $A5^f_R$) is a QNP Lie ideal in $\mathscr{L}_1$.

Proof. Let a, b ∈ $\mathscr{L}_1$ *and* $\beta \in \mathscr{F}$. Then

$A1^f_R$ (a + b) = $A1_R$ (f(a + b)) = $A1_R$ (f(a) + f(b)) ≥ min{ $A1_R$ (f(a)), $A1_R$ (f(b))} = min{$A1^f_R$(a), $A1^f_R$(b)},

$A2^f_R$ (a + b) = $A2_R$ (f(a + b)) = $A2_R$ (f(a) + f(b)) ≥ min{ $A2_R$ (f(a)), $A2_R$ (f(b))} = min{$A2^f_R$ (a), $A2^f_R$(b)},

$A4^f_R$ (a + b) = $A4_R$ (f(a + b)) = $A4_R$ (f(a) + f(b)) ≤ min{ $A4_R$ (f(a)), $A4_R$ (f(b))} = min{$A4^f_R$(a), $A4^f_R$(b)},

$A5^f_R$ (a + b) = $A5_R$ (f(a + b)) = $A5_R$ (f(a) + f(b)) ≤ min{ $A5_R$ (f(a)), $A5_R$ (f(b))} = min{$A5^f_R$(a), $A5^f_R$(b)},

$A1^f_R$ (βa) = $A1_R$ (f(βa)) = $A1_R$ (βf(a)) ≥ $A1_R$ (f(a)) = $A1^f_R$(a),

$A2^f_R$ (βa) = $A2_R$ (f(βa)) = $A2_R$ (βf(a)) ≥ $A2_R$ (f(a)) = $A2^f_R$(a),

$A4^f_R$ (βa) = $A4_R$ (f(βa)) = $A4_R$ (βf(a)) ≤ $A4_R$ (f(a)) = $A4^f_R$(a),

$A5^f_R$ (βa) = $A5_R$ (f(βa)) = $A5_R$ (βf(a)) ≤ $A5_R$ (f(a)) = $A5^f_R$(a).

Similarly,

$A1^f_R$ ([a, b]) = $A1_R$ (f[a, b]) = $A1_R$ ([f(a), f(b]) ≥ $A1_R$ (f(a)) = $A1^f_R$(a),

$A2^f_R$ ([a, b]) = $A2_R$ (f([a,b]) = $A2_R$ ([f(a), f(b)]) ≥ $A2_R$ (f(a)) = $A2^f_R$(a),

$A4^f_R$ ([a, b]) = $A4_R$ (f([a, b]) = $A4_R$ ([f(a), f(b)]) ≤ $A4_R$ (f(a)) = $A4^f_R$(a),

$A5^f_R$ ([a, b]) = $A5_R$ (f([a, b]) = $A5_R$ ([f(a), f(b)]) ≤ $A5_R$ (f(a)) = $A5^f_R$(a).

This proves that $R^f = (A1^f_R, A2^f_R, A4^f_R, A5^f_R)$ is a QNP Lie ideal in $\mathscr{L}_1$.

We now characterize the QNP Lie ideals of Lie algebras.

Theorem 8. Let f :$\mathscr{L}_1 \rightarrow \mathscr{L}_2$ be an epimorphisms of QNP Lie algebras. Then $R^f = (A1^f_R, A2^f_R, A4^f_R, A5^f_R)$ is a QNP Lie ideal in $\mathscr{L}_1$ iff R = ($A1_R$, $A2_R$, $A4_R$, $A5_R$)is a QNP Lie ideal of $\mathscr{L}_2$.

Definition 10. Let R = ($A1_R$, $A2_R$, $A4_R$, $A5_R$) be a QNP Lie ideal in $\mathscr{L}$. Define a inductively a sequences of QNP Lie ideals in $\mathscr{L}$ by $R^0 = R$, $R^1 = [R^0, R^0]$, $R^2 = [R^1, R^1]$,..... $R^n = [R^{n-1}, R^{n-1}]$.

R^n is called the n th derived QNP Lie ideal of $\mathscr{L}$. A series $R^0 \supseteq R^1 \supseteq R^2 \supseteq .....\supseteq R^n \supseteq \cdots$

is called derived series of a QNP Lie ideal R in $\mathscr{L}$.

Definition 11. A QNP Lie ideal R in is called a solvable QNP Lie ideal, if there exists a positive integer n such that $R^0 \supseteq R^1 \supseteq R^2 \supseteq .....\supseteq R^n = (0,0,0)$

Theorem 9. Homomorphic images of solvable QNP Lie ideals are solvable QNP Lie ideals.

Proof. Let f $:\mathcal{L}_1 \rightarrow \mathscr{L}_2$ be homomorphisms of QNP Lie algebras. Suppose that R = ($A1_R$, $A2_R$, $A4_R$, $A5_R$) is a QNP Lie ideal of $\mathscr{L}_1$. We prove by induction on n that $f(R^n) \supseteq [f(R)]^n$, where n is any positive integer. First we claim that f([R, A])⊇ [f(R), f(R)]. Let y ∈ $\mathscr{L}_2$. Then

$$f(<<A1_R, A1_R>>)(y) = \sup \{<<A1_R, A1_R>>(y)\backslash f(x) = y\}$$

$$= \sup\{\sup\{\min(A1_R(a), A1_R(b))\backslash a,b \in \mathscr{L}_1,[a, b] = x, f(x) = y\}\}$$

$$= \sup\{\min(A1_R(a), A1_R(b))\backslash a, b \in \mathscr{L}_1,,[a, b] = x, f(x) = y\}\}$$

$$= \sup\{\min(A1_R(a), A1_R(b))\backslash a,b \in \mathscr{L}_1,,[f(a), f(b)] = x\}$$

$$= \sup\{\min(A1_R(a), A1_R(b))\backslash a,b \in \mathscr{L}_1,f(a) = u, f(b) = v,[u, v] = y\}\}$$

$$\geq \sup\{\min(\sup_{a\in f^{-1}(u)} A1_R(a), \min(\sup_{b\in f^{-1}(v)} A1_R(b)\backslash[u, v] = y\}$$

$$= \sup\{\min\{f(A1_R)(u),f(A1_R)(v))\backslash[u, v] = y\}$$

$$= <<f(A1_R), f(A1_R)>>(y),$$

$$f(<<A2_R, A2_R>>)(y) = \sup \{ <<A2_R, A2_R>>(y)\backslash f(x) = y\}$$

$$= \sup\{\sup\{\min(A2_R(a), A2_R(b))\backslash a, b \in \mathscr{L}_1,[a, b] = x, f(x) = y\}\}$$

$$= \sup\{\min(A2_R(a), A2_R(b))\backslash a, b \in \mathscr{L}_1,,[a, b] = x, f(x) = y\}\}$$

$$= \sup\{\min(A2_R(a), A2_R(b))\backslash a, b \in \mathscr{L}_1,,[f(a), f(b)] = x\}$$

$$= \sup\{\min(A2_R(a), A2_R(b))\backslash a, b \in \mathscr{L}_1,,f(a) = u, f(b) = v,[u, v] = y\}\}$$

$$\geq \sup\{\min(\sup_{a\in f^{-1}(u)} A2_R(a), \min(\sup_{b\in f^{-1}(v)} A2_R(b)\backslash[u, v] = y\}$$

$$= \sup\{\min\{f(A2_R)(u),f(A2_R)(v))\backslash[u, v] = y\}$$

$$= <<f(A2_R), f(A2_R)>>(y),$$

$$f(<<A4_R, A4_R>>)(y) = \inf \{ <<A4_R, A4_R>>(y)\backslash f(x) = y\}$$

$$= \inf\{\inf\{\max(A4_R(a), A4_R(b))\backslash a, b \in \mathcal{L}_1,[a, b] = x, f(x) = y\}\}$$

$$= \inf\{\max(A4_R(a), A4_R(b))\backslash a, b \in \mathcal{L}_1,[a, b] = x, f(x) = y\}\}$$

$$= \inf \{\max(A4_R(a), A4_R(b))\backslash a, b \in \mathcal{L}_1,[f(a), f(b)] = x\}$$

$$= \inf\{\max(A4_R(a), A4_R(b))\backslash a, b \in \mathcal{L}_1,f(a) = u, f(b) = v,[u, v] = y\}\}$$

$$\leq \inf\{\max(\inf_{a\in f^{-1}(u)} A4_R(a), \min(\inf_{b\in f^{-1}(v)} A4_R(b)\backslash[u, v] = y\}$$

$= \inf\{\max\{f(A4_R)(u), f(A4_R)(v))\backslash[u, v] = y\}$

$= \langle\langle f(A4_R), f(A4_R)\rangle\rangle(y)$,

$f(\langle\langle A5_R, A5_R\rangle\rangle)(y) = \inf\{\langle\langle A5_R, A5_R\rangle\rangle(y)\backslash f(x) = y\}$

$= \inf\{\inf\{\max(A5_R(a), A5_R(b))\backslash a, b \in \mathcal{L}_1, [a, b] = x, f(x) = y\}\}$

$= \inf\{\max(A5_R(a), A5_R(b))\backslash a, b \in \mathcal{L}_1, [a, b] = x, f(x) = y\}\}$

$= \inf\{\max(A5_R(a), A5_R(b))\backslash a, b \in \mathcal{L}_1, [f(a), f(b)] = x\}$

$= \inf\{\max(A5_R(a), A5_R(b))\backslash a, b \in \mathcal{L}_1, f(a) = u, f(b) = v, [u, v] = y\}\}$

$\leq \inf\{\max(\inf_{a\in f^{-1}(u)} A5_R(a), \min(\inf_{b\in f^{-1}(v)} A5_R(b)\backslash[u, v] = y\}$

$= \inf\{\max\{f(A5_R)(u), f(A5_R)(v))\backslash[u, v] = y\}$

$= \langle\langle f(A5_R), f(A5_R)\rangle\rangle(y)$.

Thus $f([R, R]) \supseteq f(\langle\langle A, A\rangle\rangle) \supseteq \langle\langle f(R), f(R)\rangle\rangle = [f(R), f(R)]$.

Now for $n > 1$, we get $f(R^n) = f([R^{n-1}, R^{n-1}]) \supseteq [f(R^{n-1}), f(R^{n-1})]$.

This completes the proof

Definition 12. Let $R = (A1_R, A2_R, A4_R, A5_R)$ be a QNP Lie ideal in $\mathscr{L}$. We define a inductively a sequences of QNP Lie ideals in $\mathscr{L}$ by $R_0 = R, R_1 = [R, R_0], R_2 = [R, R_1] \ldots . R_n = [R, R_{n-1}]$.

A series $R_0 \supseteq R_1 \supseteq R_2 \supseteq \ldots \supseteq R_n \supseteq \cdots$

is called descending central series of a QNP Lie ideal R in $\mathscr{L}$.

Definition 13. An QNP Lie ideal R is called a nilpotent QNP Lie ideal in $\mathscr{L}$, if there exists a positive integer n such that $R_0 \supseteq R_1 \supseteq R_2 \supseteq \ldots \supseteq R_n = (0,0,0)$.

Theorem 10. Homomorphic image of a nilpotent QNP Lie ideal is a nilpotent QNP Lie ideal.

Proof. It is obvious

Theorem 14. Let K be a QNP Lie ideal of a QNP Lie algebra $\mathscr{L}$. If $R = (A1_R, A2_R, A4_R, A5_R)$ is a QNP Lie ideal of $\mathscr{L}$, then the QNP set $^*R = (^*A1_R, ^*A2_R, ^*A4_R, ^*A5_R)$ of $\mathscr{L}/K$ defined by

$^*A1_R(a + K) = \sup_{x\in K} A1_R(a + x)$,

$^*A2_R(a + K) = \sup_{x\in K} A2_R(a + x)$,

$^*A4_R(a + K) = \inf_{x\in K} A4_R(a + x)$,

$^*A5_R(a + K) = \inf_{x\in K} A5_R(a + x)$,

is a QNP Lie ideal of the quotient QNP Lie algebra $\mathscr{L}/K$ of $\mathscr{L}$ with respect to K.

Proof. Clearly,*R is defined. Let x + K, y + K ∈ $\mathscr{L}$/ K. Then

$*A1_R((x + K) + (y + K)) = *A1_R((x + y) + K)$

$$= \sup_{z\in K} A1_R\big((x+y)+z\big),$$

$$= \sup_{z=s+t\in K} A1_R\big((x+y)+(s+t)\big),$$

$$\geq \sup_{s,t\in K} \min\{A1_R(x+s), A1_R(y+t)\},$$

$$= \min\{\sup_{s\in K} A1_R(x+s), \sup_{t\in K} A1_R(y+t)\},$$

$$= \min\{* A1_R(x+s), * A1_R(y+t)\},$$

$*A1_R(\beta(x + K) = *A1_R(\beta x + K) = \sup_{z\in K} A1_R(\beta x + z) \geq \sup_{z\in K} A1_R(x+z) = *A1_R(x + K)$.

$*A1_R([x + K, *A1_R(a + K) = \sup_{x\in K} A1_R(a + x)$,

$y + K]) = *A1_R([x , y] + K) = \sup_{z\in K} A1_R([x,y]+z) \geq \sup_{z\in K} A1_R([x,y]+z) = *A1_R(x + K)$.

Thus $*A1_R$ is a PNP Lie ideal of $\mathscr{L}$/ K. In a similar way, we can verify that $*A2_R$, $*A4_R$ and $*A5_R$ PNP Lie ideals of $\mathscr{L}$ /K. Hence $*R = (*A1_R, *A2_R, *A4_R, *A5_R)$ is a QNP Lie ideal of $\mathscr{L}$ /K.

Theorem 15. Let K be a QNP Lie ideal of a QNP Lie algebra $\mathscr{L}$. Then there is a one-to=one correspondence between the set of QNP Lie ideals $R = (A1_R, A2_R, A4_R, A5_R)$ of $\mathscr{L}$ such that R(0) = A(s) for all s ∈ K and the set of all QNP Lie ideals $*R = (*A1_R, *A2_R, *A4_R, *A5_R)$ of $\mathscr{L}$ /K.

Proof. Let R = (A1 R, A2 R, A4R ,A5 R) be QNP Lie ideal of $\mathscr{L}$. Using Theorem 3.27, we prove that

*A1 R, *A2 R, *A4R ,*A5 R defined by

$*A1R(a + K) = \sup_{x\in K} A1\ R(a + x)$,

$*A2R(a + K) = \sup_{x\in K} A2\ R(a + x)$,

$*A4R(a + K) = \inf_{x\in K} A4\ R(a + x)$,

$*A5R(a + K) = \inf_{x\in K} A5\ R(a + x)$,

are QNP Lie ideals of $\mathscr{L}$ /K. Since A1R(0) = A1 R(s), A2 R(0) = A2 R(s),

A4R(0) = A4 R(s), A5 R(0) = A5 R(s) for all s ∈ K,

A1R(a + s) ≥ min(A1 R(a), A1 R(s)) = A1 R(a),

A2R(a + s) ≥ min(A2 R(a), A2 R(s)) = A2 R(a),

A4R(a + s) ≤ max(A4 R(a), A4 R(s)) = A4 R(a),

A5R(a + s) ≤ min(A5 R(a), A5 R(s)) = A5 R(a).

Again,

$$A1\ R(a) = A1R(a + s - s) \geq \min(A1\ R(a + s), A1\ R(s)) = A1\ R(a + s),$$

$$A2\ R(a) = A2R(a + s - s) \geq \min(A2\ R(a + s), A2\ R(s)) = A2\ R(a + s),$$

$$A4\ R(a) = A4R(a + s - s) \leq \max(A4\ R(a + s), A4\ R(s)) = A4\ R(a + s),$$

$$A5\ R(a) = A5R(a + s - s) \leq \max(A5\ R(a + s), A5\ R(s)) = A5\ R(a + s).$$

Thus $R(a + s) = R(a)$ for all $s \in K$. Hence the correspondence $R \to {}^*R$ is one- to -one. Let $^*R$ be a QNP Lie ideal of $\mathscr{L}$ / K and define a PNP set R = (A1 R, A2 R, A4R ,A5 R) in $\mathscr{L}$ by

$A1\ R(a) = {}^*A1R(a + K)$, $A2\ R(a) = {}^*A2\ R(a + K)$, $A4\ R(a) = {}^*A4\ R(a + K)$, $A5\ R(a) = {}^*A5\ R(a + K)$.

For $a,b \in \mathscr{L}$, we have

$$A1R(a + b) = {}^*A1\ R((a + b) + K) = {}^*A1\ R((a + K) + (b + K)),$$

$$\geq \min\{{}^*A1\ R(a + K), {}^*A1\ R(b + K)\},$$

$$= \min\{A1\ R(a), A1\ R(b)\},$$

$$A1\ R(\beta a) = {}^*A1R(\beta a + K) \geq {}^*A1\ R(a + K) = A1\ R(a),$$

$$A1R([a, b]) = {}^*A1\ R([a,b] + K) = {}^*A1\ R([a + K, b + K]),$$

$$\geq {}^*A1R(a + K) = A1\ R(a).$$

Thus A1 R is a QNP lie ideal of $\mathscr{L}$. In a similar way, we can verify that A2 R, A4 R and A5R are QNP Lie ideals of $\mathscr{L}$. Hence R = (A1 R, A2 R, A4R ,A5 R) is a QNP Lie ideal of $\mathscr{L}$.

Note that $A1\ R(a) = {}^*A1R(a + K)$, $A2\ R(a) = {}^*A2\ R(a + K)$, $A4\ R(a) = {}^*A4\ R(a + K)$, $A5\ R(a) = {}^*A5R(a + K)$.

This completes the proof.

4| Conclusion

In this article, we have discussed above QNP Lie subalgebra and QNP Lie ideals of a QNP Lie Algebra. We have also investigated some of its properties of Quadripartitioned Neutrosophic Pythagorean Lie ideals. In future, we are planned to study on Lie rings. We may also develop for heptapartitioned neutrosophic sets and other hybrid sets.

References

[1] Akram, M. (2006). Anti fuzzy Lie ideals of Lie algebras. *Quasigroups and related systems*, *14*(2), 123-132.
[2] Akram, M. (2007). Intuitionistic (S, T)-fuzzy Lie ideals of Lie algebras. *Quasigroups and related systems*, *15*(2), 201-218.
[3] Akram, M. (2008). Generalized fuzzy Lie subalgebras. *Journal of generalized lie theory and applications*, *2*(4), 261-268.
[4] Atanassov, K. (2016). Intuitionistic fuzzy sets. *International journal bioautomation*, *20*, 1.

[5] Coelho, P., & Nunes, U. (2003). Lie algebra application to mobile robot control: a tutorial. *Robotica*, *21*(5), 483-493.

[6] Wang, H., Smarandache, F., Zhang, Y., & Sunderraman, R. (2010). *Single valued neutrosophic sets*. Infinite study.

[7] Akram, M., Gulzar, H., & Shum, K. P. (2018). Single–valued neutrosophic lie algebra. *Journal of Mathematical research with applications*. DOI: 10.3770/j.issn:2095-2651.2019.02.003

[8] Radha, R., & Mary, A. S. A. (2021). *Pentapartitioned neutrosophic pythagorean soft set*. Infinite Study.

[9] Mary, S. A. (2021). Pentapartitioned neutrosophic pythagorean set. *International research journal on advanced science hub*, *3*, 62-68.

[10] Radha, R., & Mary, A. S. A. (2021). Heptapartitioned neutrosophic sets. *International journal of creative research thoughts (IJCRT)*, 9(2), 222-230.

[11] Radha, R., Mary, A. S. A., & Smarandache, F. (2021). Quadripartitioned neutrosophic pythagorean soft set. *International journal of neutrosophic science (IJNS)*, *14*(1), 9-23.

[12] Smarandache, F. (1998). *Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis*. Amer Res Press, Rehoboth, USA.

[13] Smarandache, F. (2015). n – valued refined neutrosophic logic and its applications in physics. In *Unmatter plasma, relativistic oblique-length contraction factor, neutrosophic diagram and neutrosophic degree of paradoxicity* (pp. 40-53). Pons Publishing.

[14] Zulqarnain, R. M., Xin, X. L., Siddique, I., Khan, W. A., & Yousif, M. A. (2021). TOPSIS method based on correlation coefficient under pythagorean fuzzy soft environment and its application towards green supply chain management. *Sustainability*, *13*(4), 1642.

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Pythagorean Fuzzy Weak Bi-Ideals of Γ- Near Ring

J. Marynirmala[1,*], D. Sivakumar[1]

[1] Department of Mathematics, Annamalai University,Annamalainagar, Tamilnadu, India; marynirla@gmail.com.

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Abstract

We present the notion of Pythagorean Fuzzy Weak Bi-Ideals (PFWBI) and interval valued Pythagorean fuzzy weak bi-ideals of Γ-near-rings and studies some of its properties. We present the notion of interval valued Pythagorean fuzzy weak bi-ideal and establish some of its properties. We study interval valued Pythagorean fuzzy weak bi-ideals of Γ-near-ring using homomorphism.

Keywords: Pythagorean fuzzy, Fuzzy ideals, Homomorphism, Near ring.

1 | Introduction

Zadeh [26] defined Fuzzy Set (FS) to deal with uncertainty. Atanassov [2] presented the notion of Intuitionistic FS (IFS) and studied some of its properties. Later, Yager [24], [25] defined and studied the properties of Pythagorean Fuzzy Set (PFS) and also used PFS to solve Multi-Criteria Decision-Making (MCDM) problems. Booth [3] presented the properties of Γ-near-rings. Chinnadurai and Kadalarasi [7] studied the near-ring properties of Fuzzy Weak Bi-Ideals (FWBI). Chinnadurai et al. [4],[5] studied the Γ-near-rings characterization of fuzzy weak bi-ideal and interval-valued fuzzy weak bi-ideal. Later, Chinnadurai et al. [6] discussed the Γ-near-rings properties of interval-valued fuzzy ideals.

Akram [1] established the properties of fuzzy lie algebras. Kim and Kim [13] studied the near-rings concept of fuzzy ideals. Kaviyarasu et al. [10]-[12] studied the different type of ideals in INK- algebras. Jun et al. [9] presented the notion of fuzzy ideals and studied their properties in Γ-near-rings. Manikantan [14] defined and studied some of the near-rings properties of fuzzy bi-ideals. Meenakumari and Chelvam [15] presented the Γ-near-ringsproperties of fuzzy bi-ideals. Narayanan and Manikatan [16] introduced the near-rings notions of fuzzy subnear-ring, fuzzy ideal, and fuzzy quasi-ideal.

Corresponding Author: marynirla@gmail.com

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Pilz [17] introduced the concept of anti fuzzy soft gamma rings and studied their properties. Rao and Swaminathan [20] presented the notion of anti-homomorphism between two fuzzy rings and established its properties. Rao and Venkateswarlu [21] studied the properties of anti fuzzy ideal and pre-image of fuzzy ideal. Satyanarayana [18] dealt with the theory of near-rings. Salah Abou-Zaid [19] studied fuzzy ideals of a near-ring. Chelvam and Meenakumari [22] obtained the characterization for gamma near-fields. Thillaigovindan et al. [23] introduced the notion of generalized T-fuzzy bi-ideals of a gamma-semigroup. Cho et al. [8] presented the notion of bi-ideals in near-rings and used it in near-fields.

We introduce the notion of Pythagorean fuzzy weak bi-ideal of Γ-near-rings and interval valued Pythagorean fuzzy weak bi-ideal of Γ-near-rings. We discuss and present some properties of homomorphism of Pythagorean fuzzy weak bi-ideal and homomorphism of an interval valued Pythagorean fuzzy weak bi-ideal in gamma near-ring.

2| Preliminaries

Definition 1. [16]. A fuzzy set π of a Γ-near-ring M is called a fuzzy left (resp. right) ideal of M if

I. $\pi(k-l) \geq min\{\pi(k), \pi(l)\}$, *for all* $k, l \in M$,
II. $\pi(y+x-y) \geq \pi(x)$, *for all* $x, y \in M$,
III. $\pi(u\alpha(x+v) - u\alpha v) \geq \pi(x)$, (*resp.* $\pi(x\alpha u) \geq \pi(x)$) *for all* $x, u, v \in M$ *and* $\alpha \in \Gamma$.

Definition 2. [15]. A fuzzy set π of M is called a fuzzy bi-ideal of M if

I. $\pi(x-y) \geq min\{\pi(x), \pi(y)\}$ for all $x, y \in M$,
II. $\pi(x\alpha y\beta z) \geq min\{\pi(x), \pi(z)\}$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

Definition 3. [2]. An intuitionistic fuzzy set A is a nonempty set X is an object having the form $A = \{x, (\pi_A(x), \vartheta_A(x))$: $x \in X\}$ where the functions $\pi_A : X \to [0,1]$ and $\vartheta_A : X \to [0,1]$ define the degree of membership and non-membership of the element $x \in X$ to the set A, which is a subset of X respectively $0 \leq \pi_A(x) + \vartheta_A(x) \leq 1$ we use the simple $A = (\pi_A, \vartheta_A)$.

Definition 4. [25]. A Pythagorean fuzzy subset P is a nonempty set X is an object having the form $P = \{(x, \pi_P(x), \vartheta_P(x))/x \in X\}$, where the functions $\pi_P: X \to [0,1]$ and $\vartheta_P: X \to [0,1]$ denote the degree of membership and non membership of each element $x \in X$ to the set P, respectively, and $0 \leq (\pi_P(x))^2 + (\vartheta_P(x))^2 \leq 1$ for all $x \in X$. For the sake of simplicity, for the Pythagorean fuzzy subset $P = \{(x, \pi_P(x), \vartheta_P(x)) /x \in X\}$.

3| Pythagorean Fuzzy Weak Bi-Ideals of Γnear Ring

In this section, we initiate the notion of Pythagorean fuzzy weak bi-ideal of M and discuss some of its properties.

Definition 4. A subgroup W of $(M, +)$ is said to be a weak bi-ideal of M if $W\Gamma W\Gamma W \subseteq W$.

Definition 5. A Pythagorean fuzzy set$P = (\pi_P, \vartheta_P)$ of M is called a Pythagorean fuzzy weak bi-ideal of M, if

I. $\pi_P(x-y) \geq min\{\pi_P(x), \pi_P(y)\}$.
II. $\vartheta_P(x-y) \leq max\{\vartheta_P(x), \vartheta_P(y)\}$.
III. $\pi_P(x\gamma y\gamma z) \geq min\{\pi_P(x), \pi_P(y), \pi_P(z)\}$.
IV. $\vartheta_P(x\gamma y\gamma z) \leq max\{\vartheta_P(x), \vartheta_P(y), \vartheta_P(z)\}$ *for all* $x, y, z \in M$ *and* $\alpha, \beta \in \Gamma$.

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Example 1. Let $M = \{w,\ x,\ y,\ z\}$ be a nonempty set with binary operation `$+$' and $\Gamma = \{\gamma\}$ be a nonempty set of binary operations as the following tables:

+	w	x	y	z
w	w	x	y	z
x	x	w	z	y
y	y	z	w	x
z	z	y	x	w

and

γ	w	x	y	z
w	w	x	w	x
x	w	x	w	x
y	w	x	y	z
z	w	x	y	z

Let $\pi_P : M \to [0,1]$ be a Pythagorean fuzzy subset defined by $\pi_P(w) = 0.7, \pi_P(x) = 0.6, \pi_P(y) = \pi_P(z) = 0.5$. and $\vartheta_P(w) = 0.3, \vartheta_P(x) = 0.5, \vartheta_P(y) = 0.8 = \vartheta_P(z)$. Then $P = (\pi_P, \vartheta_P)$ is a Pythagorean fuzzy weak bi-ideal of M.

Theorem 1. Let $P = (\pi_P, \vartheta_P)$ be a Pythagorean fuzzy subgroup of M. Then $P = (\pi_P, \vartheta_P)$ is a Pythagorean fuzzy weak bi-ideal of M if and only if $\pi_P \star \pi_P \star \pi_P \subseteq \pi_P$ and $\vartheta_P \star \vartheta_P \star \vartheta_P \supseteq \vartheta_P$.

Proof. Assume that $P = (\pi_P, \vartheta_P)$ be a Pythagorean fuzzy weak bi-ideal of M. Let $x, y, z, y_1, y_2 \in M$ and $\alpha, \beta \in \Gamma$ such that $x = y\alpha z$ and $y = y_1\beta y_2$. Then

$$(\pi_P \star \pi_P \star \pi_P)(x) = \sup_{x=y\alpha z} \{\min\{(\pi_P \star \pi_P)(y), \pi_P(z)\}\}$$

$$= \sup_{x=y\alpha z} \{\min\{ \sup_{y=y_1\beta y_2} \min\{\pi_P(y_1), \pi_P(y_2)\}, \pi_P(z)\}\}$$

$$= \sup_{x=y\alpha z} \sup_{y=y_1\beta y_2} \{\min\{\min\{\pi_P(y_1), \pi_P(y_2)\}, \pi_P(z)\}\}$$

$$= \sup_{x=y_1\beta y_2\alpha z} \left\{\min\left\{\pi_P(y_1), \pi_P(y_2), \pi_P(z)\right\}\right\},$$

since π_P is a fuzzy weak bi-ideal of M,

$$\pi_P(y_1\beta y_2\alpha z) \geq \min\{\pi_P(y_1), \pi_P(y_2), \pi_P(z)\}$$

$$\leq \sup_{x=y_1\beta y_2\alpha z} \pi_P(y_1\beta y_2\alpha z)$$

$$= \pi_P(x).$$

And

$$(\vartheta_P \star \vartheta_P \star \vartheta_P)(x) = \inf_{x=y\alpha z} \{\min\{(\vartheta_P \star \vartheta_P)(y), \vartheta_P(z)\}\},$$

$$= \inf_{x=y\alpha z} \{\max\{ \inf_{y=y_1\beta y_2} \min\{\vartheta_P(y_1), \vartheta_P(y_2)\}, \vartheta_P(z)\}\}$$

$$= \inf_{x=y\alpha z} \sup_{y=y_1\beta y_2} \{\max\{\max\{\vartheta_P(y_1), \vartheta_P(y_2)\}, \vartheta_P(z)\}\}$$

$$= \inf_{x=y_1\beta y_2\alpha z} \left\{\max\left\{\vartheta_P(y_1), \vartheta_P(y_2), \vartheta_P(z)\right\}\right\},$$

since ϑ_P is a fuzzy weak bi-ideal of M,

$$\vartheta_P(y_1\beta y_2\alpha z) \leq \max\{\vartheta_P(y_1), \vartheta_P(y_2), \vartheta_P(z)\}$$

$$\geq \inf_{x=y_1\beta y_2\alpha z} \vartheta_P(y_1\beta y_2\alpha z)$$

$$= \vartheta_P(x).$$

If x can not be expressed as $x = y\alpha z$, then $(\pi_P \star \pi_P \star \pi_P)(x) = 0 \leq \pi_P(x)$ and

$(\vartheta_P \star \vartheta_P \star \vartheta_P)(x) = 0 \geq \vartheta_P(x)$. In both cases $\pi_P \star \pi_P \star \pi_P \subseteq \pi_P$, and $\vartheta_P \star \vartheta_P \star \vartheta_P \supseteq \vartheta_P$.

Conversely, assume that $\pi_P \star \pi_P \star \pi_P \subseteq \pi_P$. For $x\prime, x, y, z \in M$ and $\alpha, \beta, \alpha_1, \beta_1 \in \Gamma$.

Let $x\prime$ be such that $x\prime = x\alpha y\beta z$.

Then $\pi_P(x\alpha y\beta z) = \pi_P(x\prime) \geq (\pi_P \star \pi_P \star \pi_P)(x\prime)$

$$= \sup_{x\prime=p\alpha_1 q} \{\min\{(\pi_P \star \pi_P)(p), \pi_P(q)\}\}$$

$$= \sup_{x\prime=p\alpha_1 q} \{\min\{\sup_{p=p_1\beta_1 p_2} \min\{\pi_P(p_1), \pi_P(p_2)\}, \pi_P(q)\}\}$$

$$= \sup_{x'=p_1\beta_1 p_2\alpha_1 q} \left\{\min\left\{\pi_P(p_1), \pi_P(p_2), \pi_P(q)\right\}\right\}$$

$$\geq \min\{\pi_P(x), \pi_P(y), \pi_P(z)\}.$$

$$\vartheta_P(x\alpha y\beta z) = \vartheta_P(x\prime) \leq (\vartheta_P * \vartheta_P * \vartheta_P)(x\prime)$$

$$= \inf_{x\prime=p\alpha_1 q} \{\max\{(\vartheta_P * \vartheta_P)(p), \vartheta_P(q)\}\}$$

$$= \inf_{x\prime=p\alpha_1 q} \{\max\{\inf_{p=p_1\beta_1 p_2} \min\{\vartheta_P(p_1), \vartheta_P(p_2)\}, \vartheta_P(q)\}\}$$

$$= \inf_{x\prime=p_1\beta_1 p_2\alpha_1 q} \{\max\{\vartheta_P(p_1), \vartheta_P(p_2), \vartheta_P(q)\}\}$$

$$\leq \max\{\vartheta_P(x), \vartheta_P(y), \vartheta_P(z)\}.$$

Hence $\pi_P(x\alpha y\beta z) \geq min\{\pi_P(x), \pi_P(y), \pi_P(z)\}$, and $\vartheta_P(x\alpha y\beta z) \leq max\{\vartheta_P(x), \vartheta_P(y), \vartheta_P(z)\}$.

Lemma 1. Let $\pi_P = (\pi_{P1}, \pi_{P2})$ and $\vartheta_P = (\vartheta_{P1}, \vartheta_{P2})$ be Pythagorean fuzzy weak bi-ideals of M. Then the products $\pi_P * \vartheta_P$ and $\vartheta_P * \pi_P$ are also Pythagorean fuzzy weak bi-ideals of M.

Proof. Let π_P and ϑ_P be a Pythagorean fuzzy weak bi-ideals of M and let $\alpha, \alpha_1, \alpha_2 \in \Gamma$. Then

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$$(\pi_{P1} \star \pi_{P2})(x-y) = \sup_{x-y=a\alpha b} \min\{\pi_{P1}(a), \pi_{P2}(b)\}$$

$$\geq \sup_{x-y=a_1\alpha_1 b_1 - a_2\alpha_2 b_2 < (a_1-a_2)(b_1-b_2)} \min\{\pi_{P1}(a_1-a_2), \pi_{P2}(b_1-b_2)\}$$

$$\geq \operatorname{supmin}\{\min\{\pi_{P1}(a_1), \pi_{P1}(a_2)\}, \min\{\pi_{P2}(b_1), \pi_{P2}(b_2)\}\}$$

$$= \operatorname{supmin}\{\min\{\pi_{P1}(a_1), \pi_{P2}(b_1)\}, \min\{\pi_{P1}(a_2), \pi_{P2}(b_2)\}\}$$

$$\geq \min\{\sup_{x=a_1\alpha_1 b_1} \min\{\pi_{P1}(a_1), \pi_{P2}(b_1)\}, \sup_{y=a_2\alpha_2 b_2} \min\{\pi_{P1}(a_2), \pi_{P2}(b_2)\}\}$$

$$= \min\{(\pi_{P1} \star \pi_{P2})(x), (\pi_{P1} \star \pi_{P2})(y)\}.$$

$$(\vartheta_{P1} \star \vartheta_{P2})(x-y) = \inf_{x-y=a\alpha b} \max\{\vartheta_{P1}(a), \vartheta_{P2}(b)\}$$

$$\leq \inf_{x-y=a_1\alpha_1 b_1 - a_2\alpha_2 b_2 < (a_1-a_2)(b_1-b_2)} \max\{\vartheta_{P1}(a_1-a_2), \vartheta_{P2}(b_1-b_2)\}$$

$$\leq \operatorname{infmax}\{\max\{\vartheta_{P1}(a_1), \vartheta_{P1}(a_2)\}, \max\{\vartheta_{P2}(b_1), \vartheta_{P2}(b_2)\}\}$$

$$= \operatorname{infmax}\{\max\{\vartheta_{P1}(a_1), \vartheta_{P2}(b_1)\}, \max\{\vartheta_{P1}(a_2), \vartheta_{P2}(b_2)\}\}$$

$$\leq \max\{\inf_{x=a_1\alpha_1 b_1} \max\{\vartheta_{P1}(a_1), \vartheta_{P2}(b_1)\}, \inf_{y=a_2\alpha_2 b_2} \max\{\vartheta_{P1}(a_2), \vartheta_{P2}(b_2)\}\}$$

$$= \max\{(\vartheta_{P1} \star \vartheta_{P2})(x), (\vartheta_{P1} \star \vartheta_{P2})(y)\}.$$

It follows that $\pi_P * \vartheta_P$ is a Pythagorean fuzzy subgroup of M. Further,

$$(\pi_P \star \vartheta_P) \star (\pi_P \star \vartheta_P) \star (\pi_P \star \vartheta_P) = \pi_P \star \vartheta_P \star (\pi_P \star \vartheta_P \star \pi_P) \star \vartheta_P$$

$$\subseteq \pi_P \star \vartheta_P \star (\vartheta_P \star \vartheta_P \star \vartheta_P) \star \vartheta_P$$

$$\subseteq \pi_P \star (\vartheta_P \star \vartheta_P \star \vartheta_P),$$

since P is a Pythagorean fuzzy weak bi-ideal of $M \subseteq \pi_P \star \vartheta_P$.

Therefore $\pi_P * \vartheta_P$ is a Pythagorean fuzzy weak bi-ideal of M. Similarly $\vartheta_P * \pi_P$ is a Pythagorean fuzzy weak bi-ideal of M.

Lemma 2. Every Pythagorean fuzzy ideal of M is a Pythagorean fuzzy bi-ideal of M.

Proof. Let $P = (\pi_P, \vartheta_P)$ be a Pythagorean fuzzy ideal of M. Then

$$\pi_P \star \mathrm{M} \star \pi_P \subseteq \pi_P \star \mathrm{M} \star \mathrm{M} \subseteq \pi_P \star \mathrm{M} \subseteq \pi_P,$$

$$\vartheta_P \star \mathrm{M} \star \vartheta_P \supseteq \vartheta_P \star \mathrm{M} \star \mathrm{M} \supseteq \vartheta_P \star \mathrm{M} \supseteq \vartheta_P,$$

since $P = (\pi_P, \vartheta_P)$ be a Pythagorean fuzzy ideal of M.

This implies that $\pi_P \star \mathrm{M} \star \pi_P \subseteq \pi_P$ and $\vartheta_P \star \mathrm{M} \star \vartheta_P \supseteq \vartheta_P$.

Therefore $P = (\pi_P, \vartheta_P)$ be a Pythagorean fuzzy bi-ideal of M.

Theorem 2. Every Pythagorean fuzzy bi-ideal of M is a Pythagorean fuzzy weak bi-ideal of M.

Proof. Assume that $P = (\pi_P, \vartheta_P)$ be a Pythagorean fuzzy bi-ideal of M.

Then $\pi_P \star \mathrm{M} \star \pi_P \subseteq \pi_P$ and $\vartheta_P \star \mathrm{M} \star \vartheta_P \supseteq \vartheta_P$.

We have $\pi_P \star \pi_P \star \pi_P \subseteq \pi_P \star \mathrm{M} \star \pi_P$ and $\vartheta_P \star \vartheta_P \star \vartheta_P \supseteq \vartheta_P \star \mathrm{M} \star \vartheta_P$.

This implies that $\pi_P \star \pi_P \star \pi_P \subseteq \pi_P \star \mathrm{M} \star \pi_P \subseteq \pi_P$

and $\vartheta_P \star \vartheta_P \star \vartheta_P \supseteq \vartheta_P \star \mathrm{M} \star \vartheta_P \supseteq \vartheta_P$.

Therefore $P = (\pi_P, \vartheta_P)$ is a Pythagorean fuzzy weak bi-ideal of M.

Example 2. Let $M = \{w,\ x,\ y,\ z\}$ be a nonempty set with binary operation$+$ and $\Gamma = \{\alpha\}$ be a nonempty set of binary operations as the following tables:

+	w	x	y	z
w	w	x	y	z
x	x	w	z	y
y	y	z	w	x
z	z	y	x	w

and

α	w	x	y	z
w	w	w	w	w
x	w	x	w	x
y	w	w	y	y
z	w	x	y	z

Let $\pi_P : M \to [0,1]$ be a fuzzy set defined by $\pi_P(w) = 0.9,\ \pi_P(x) = 0.4 = \pi_P(y)$ and $\pi_P(z) = 0.6$, and $\vartheta_P(w) = 0.1, \vartheta_P(x) = 0.5 = \vartheta_P(y), \vartheta_P(z) = 0.3$. Then π_P is a fuzzy weak bi-ideal of M. But π_P is not a fuzzy ideal and bi-ideal of M and $\pi_P(z\gamma y\gamma z) = \pi_P(y) = 0.4 > 0.6 = min\{\pi_P(z), \pi_P(z)\}$ and $\vartheta_P(x\alpha(z+w) - x\alpha w) \leq \vartheta_P(z) = 0.5 \not\leq 0.3$ and $\vartheta_P(z\gamma x\gamma z) = \vartheta_P(x) = 0.5 \not\leq 0.3 = \min\{\vartheta_P(z), \vartheta_P(z)\}$.

Theorem 3. Let $\{(\pi_{P_i}, \vartheta_{P_i}) | i \in \Omega\}$ be family of Pythagorean fuzzy weak bi-ideals of a near-ring M, then $\bigcap_{i\in\Omega} \pi_{P_i}$and $\cup_{i\in\Omega} \vartheta_{P_i}$are also a Pythagorean fuzzy weak bi-idea of M, where Ω is any index set.

Proof. Let $\{\pi_i\}_{i\in\Omega}$ be a family of Pythagorean fuzzy weak bi-ideals of M.

Let $x, y, z \in M, \alpha, \beta \in \Gamma$ and $\pi = \bigcap_{i\in\Omega} \pi_i$.

Then,$\bigcap_{i\in\Omega} \pi_{P_i}(x) = \bigcap_{i\in\Omega} \pi_{P_i}(x) = (\inf_{i\in\Omega} \pi_{P_i})(x) = \inf_{i\in\Omega} \pi_{P_i}(x)$and

$\cup_{i\in\Omega} \vartheta_{P_i}(x) = \cup_{i\in\Omega} \vartheta_{P_i}(x) = (\sup_{i\in\Omega} \vartheta_{P_i})(x) = \sup_{i\in\Omega} \vartheta_{P_i}(x)$.

$$\bigcap_{i\in\Omega} \pi_{P_i}(x-y) = \inf_{i\in\Omega} \pi_{P_i}(x-y)$$

$$\geq \inf_{i\in\Omega}\min\{\pi_{P_i}(x), \pi_{P_i}(y)\}$$

$$= \min\{\inf_{i\in\Omega}\pi_{P_i}(x), \inf_{i\in\Omega}\pi_{P_i}(y)\}$$

$$= \min\{\bigcap_{i\in\Omega} \pi_{P_i}(x), \bigcap_{i\in\Omega} \pi_{P_i}(y)\}$$

$$\cup_{i\in\Omega} \vartheta_{P_i}(x-y) = \sup_{i\in\Omega}\pi_{P_i}(x-y)$$

$$\leq \sup_{i\in\Omega}\max\{\vartheta_{P_i}(x), \vartheta_{P_i}(y)\}$$

$$= \max\{\sup_{i\in\Omega}\vartheta_{P_i}(x), \sup_{i\in\Omega}\vartheta_{P_i}(y)\}$$

$$= \max\{\cup_{i\in\Omega} \vartheta_{P_i}(x), \cup_{i\in\Omega} \vartheta_{P_i}(y)\}.$$

And,

$$\bigcap_{i\in\Omega} \pi_{P_i}(x\alpha y\beta z) = \inf_{i\in\Omega}\pi_{P_i}(x\alpha y\beta z)$$

$$\geq \inf_{i\in\Omega}\min\{\pi_{P_i}(x), \pi_{P_i}(y), \pi_{P_i}(z)\}$$

$$= \min\{\inf_{i\in\Omega}\pi_{P_i}(x), \inf_{i\in\Omega}\pi_{P_i}(y), \inf_{i\in\Omega}\pi_{P_i}(z)\}$$

$$= \min\{\bigcap_{i\in\Omega} \pi_{P_i}(x), \bigcap_{i\in\Omega} \pi_{P_i}(y), \bigcap_{i\in\Omega} \pi_{P_i}(z)\}$$

$$\cup_{i\in\Omega} \vartheta_{P_i}(x\alpha y\beta z) = \sup_{i\in\Omega}\vartheta_{P_i}(x\alpha y\beta z)$$

$$\leq \sup_{i\in\Omega}\max\{\vartheta_{P_i}(x), \vartheta_{P_i}(y), \vartheta_{P_i}(z)\}$$

$$= \max\{\sup_{i\in\Omega}\vartheta_{P_i}(x), \sup_{i\in\Omega}\vartheta_{P_i}(y), \sup_{i\in\Omega}\vartheta_{P_i}(z)\}$$

$$= \max\{\cup_{i\in\Omega} \vartheta_{P_i}(x), \cup_{i\in\Omega} \vartheta_{P_i}(y), \cup_{i\in\Omega} \vartheta_{P_i}(z)\}.$$

Hence the set $\bigcap_{i\in\Omega} \pi_{P_i}$ and $\cup_{i\in\Omega} \vartheta_{P_i}$ are also a family of Pythagorean fuzzy weak bi-ideals of M.

Theorem 4. Let $P = (\pi_P, \vartheta_P)$ be a Pythagorean fuzzy subset of M. Then $U(\pi_P;t)$ and $L(\vartheta_P;s)$ is a Pythagorean fuzzy weak bi-ideal of M if and only if π_{P_t} is a weak bi-ideal of M, for all $t \in [0,1]$.

Proof. Assume that $P = (\pi_P, \vartheta_P)$ is a Pythagorean fuzzy weak bi-ideal of M.

Let $s, t \in [0,1]$ such that $x, y \in U(\pi_P;t)$.

Then $\pi_P(x) \geq t$ and $\pi_P(y) \geq t$,

Then $\pi_P(x-y) \geq min\{\pi_P(x), \pi_P(y)\} \geq min\{t,t\} = t$ and

$\vartheta_P(x-y) \leq max\{\vartheta_P(x), \vartheta_P(y)\} \leq max\{s,s\} = s.$

Thus $x-y \in U(\pi_P\ \ t)$.Let $x,y,z \in \pi_{Pt}$ and $\alpha,\beta \in \Gamma$.

This implies that $\pi_P(x\alpha y\beta z) \geq min\{\pi_P(x),\pi_P(y),\pi_P(z)\} \geq min\{t,t,t\} = t$, and

$\vartheta_P(x\alpha y\beta z) \leq max\{\vartheta_P(x),\vartheta_P(y),\vartheta_P(z)\} \leq max\{s,\ s,\ s\} = s$.

Therefore $x\alpha y\beta z \in U(\pi_P;s)$.

Hence $U\ (\pi_P;t)$ and $(\vartheta_P;s)$ is a weak bi-ideal of M.

Conversely, assume that $U\ (\pi_P;t)$ and $(\vartheta_P;s)$ is a weak bi-ideal of M, for all $s,t \in [0,1]$.

Let $x,y \in M$. Suppose

$\pi_P(x-y) < min\{\pi_P(x),\pi_P(y)\} and\ \vartheta_P(x-y) > max\{\vartheta_P(x),\vartheta_P(y)\}$.

Choose t such that $\pi(x-y) < t < min\{\pi_P(x),\pi_P(y)\}$ and $\vartheta_P(x-y) > s > max\{\vartheta_P(x),\vartheta_P(y)\}$.

This implies that $\pi_P(x) > t, \pi_P(y) > t$ and $\pi_P(x-y) < t$.

Then we have $x,y \in \pi_{Pt}$ but $x-y \not\in \pi_{Pt}$ and $\vartheta_P(x) < s, \vartheta_P(y) < s$ and $\vartheta_P(x-y) > s$, we have $x,y \in \vartheta_{Ps}$ but $x-y \not\in \vartheta_{Ps}$ a contradiction.

Thus $\pi_P(x-y) \geq min\{\pi_P(x),\pi_P(y)\}$ and $\vartheta_P(x-y) \leq max\{\vartheta_P(x),\vartheta_P(y)\}$.

If there exist $x,y,z \in M$ and $\alpha,\beta \in \Gamma$ such that $\pi_P(x\alpha y\beta z) < min\{\pi_P(x),\pi_P(y),\pi_P(z)\}$ and $\vartheta_P(x\alpha y\beta z) > max\{\vartheta_P(x),\vartheta_P(y),\vartheta_P(z)\}$.

Choose t such that $\pi_P(x\alpha y\beta z) < t < min\{\pi_P(x),\pi_P(y),\pi_P(z)\}$.

Choose s such that $\vartheta_P(x\alpha y\beta z) > s > max\{\vartheta_P(x),\vartheta_P(y),\vartheta_P(z)\}$.

Then $\pi_P(x) > t, \pi_P(y) > t, \pi_P(z) > t$ and $\vartheta_P(x) < s, \vartheta_P(y) < s, \vartheta_P(z) < s$ and $\pi_P(x\alpha y\beta z) < t$.

So, $x,y,z \in \pi_P >$ but $x\alpha y\beta z \not\in \pi_{Pt}$, and $x\alpha y\beta z \not\in \vartheta_{Ps}$, which is a contradiction.

Hence $\pi_P(x\alpha y\beta z) \geq min\{\pi_P(x),\pi_P(y),\pi_P(z)\},\ \vartheta_P(x\alpha y\beta z) \leq max\{\vartheta_P(x),\vartheta_P(y),\vartheta_P(z)\}$.

Therefore $P = (\pi_P,\vartheta_P)$ is a Pythagorean fuzzy weak bi-ideal of M.

Theorem 5. Let $P = (\pi_P,\vartheta_P)$ be a Pythagorean fuzzy weak bi-ideal of M then the set $M_{\pi,\vartheta} = \{x \in M|\pi_P(x) = \pi_P(0) = \vartheta_P(x)\}$ is aweak bi-ideal of M.

Proof. Let $x,y \in M_{(\pi_P,\vartheta_P)}$. Then $\pi_P(x) = \pi_P(0), \pi_P(y) = \pi_P(0), \vartheta_P(x) = 0, \vartheta_P(y) = 0$ and

$$\pi_P(x-y) \geq min\{\pi_P(x),\pi_P(y)\}$$

$$= \min\{\pi_P(0),\pi_P(0)\}$$

$$= \pi_P(0), \text{ and}$$

$$\vartheta_P(x-y) \leq \max\{\vartheta_P(x),\vartheta_P(y)\}$$

$$= \max\{\vartheta_P(0),\vartheta_P(0)\}$$

$= \vartheta_P(0)$.

So $\pi_P(x-y) = \pi_P(0), \vartheta_P(x-y) = \vartheta_P(0)$.

Thus $x-y \in M_{\pi_P},\ x-y \in M_{\vartheta_P}$. For every $x,y,z \in M_{\pi_P}$ and $\alpha,\beta \in \Gamma$. We have

$$\pi_P(x\alpha y\beta z) \geq \min\{\pi_P(x), \pi_P(y), \pi_P(z)\}$$

$$= \min\{\pi_P(0), \pi_P(0), \pi_P(0)\}$$

$$= \pi_P(0).$$

And

$$\vartheta_P(x\alpha y\beta z) \leq \max\{\vartheta_P(x), \vartheta_P(y), \vartheta_P(z)\}$$

$$= \max\{\vartheta_P(0), \vartheta_P(0), \vartheta_P(0)\}$$

$$= \vartheta_P(0).$$

Thus $x\alpha y\beta z \in M_{\pi_P},\ x\alpha y\beta z \in M_{\vartheta_P}$. Hence $M_{(\pi_P,\vartheta_P)}$ is a weak bi-ideal of M.

4| Homomorphism of Pythagorean Fuzzy Weak Bi-Ideals of Γ-Near-Rings

In this section, we characterize Pythagorean fuzzy weak bi-ideals of Γ-near-rings using homomorphism.

Definition 6. Let f be a mapping from a set M to a set S. Let $f = (\pi_P, \vartheta_P)$ be a Pythagorean fuzzy subsets of M and S, resp. then f is image of π_P and ϑ_P under f is a fuzzy subset of S defined by

$$f(\pi_P)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \pi_P(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

$$f(\vartheta_P)(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \vartheta_P(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 1 & \text{otherwise} \end{cases}$$

And the pre-image of π_P and ϑ_P under f is a fuzzy subset of M defined by

$$f^{-1}(\pi_P(x)) = \pi_P(f(x)),\ f^{-1}(\vartheta_P(x)) = \vartheta_P(f(x))\ \textit{for all}\ x \in M\ \textit{and}\ f^{-1}(y) = \{x \in M | f(x) = y\}.$$

Theorem 6. Let $f: M \to S$ be a homomorphism between Γ-near-rings M and S. If $P = (\pi_P, \vartheta_P)$ is a Pythagorean fuzzy weak bi-ideal of S, then $f^{-1}(P) = [f^{-1}(\pi_P, \vartheta_P)]$ is a Pythagorean fuzzy weak bi-ideal of M.

Proof. Let f be a Pythagorean fuzzy weak bi-ideal of M. Let $x,y,z \in M$ and $\alpha,\beta \in \Gamma$. Then

$$f^{-1}(\pi_P)(x-y) = \pi_P(f(x-y))$$

$$= \pi_P(f(x) - f(y))$$

$\geq \min\{\pi_P(f(x)), \pi_P(f(y))\}$

$= \min\{f^{-1}(\pi_P(x)), f^{-1}(\pi_P(y))\}.$

$f^{-1}(\vartheta_P)(x-y) = \vartheta_P(f(x-y))$

$= \vartheta_P(f(x) - f(y))$

$\leq \max\{\vartheta_P(f(x)), \vartheta_P(f(y))\}$

$= \max\{f^{-1}(\vartheta_P(x)), f^{-1}(\vartheta_P(y))\}.$

$f^{-1}(\pi_P)(x\alpha y\beta z) = \pi_P(f(x\alpha y\beta z))$

$= \pi_P(f(x)\alpha f(y)\beta f(z))$

$\geq \min\{\pi_P(f(x)), \pi_P(f(y)), \pi_P(f(z))\}$

$= \min\{f^{-1}(\pi_P(x)), f^{-1}(\pi_P(y)), f^{-1}(\pi_P(z))\}.$

$f^{-1}(\vartheta_P)(x\alpha y\beta z) = \vartheta_P(f(x\alpha y\beta z))$

$= \vartheta_P(f(x)\alpha f(y)\beta f(z))$

$\leq \max\{\vartheta_P(f(x)), \vartheta_P(f(y)), \vartheta_P(f(z))\}$

$= \max\{f^{-1}(\vartheta_P(x)), f^{-1}(\vartheta_P(y)), f^{-1}(\vartheta_P(z))\}.$

Therefore $f^{-1}(P) = [f^{-1}(\pi_P, \vartheta_P)]$ is a Pythagorean fuzzy weak bi-ideal of M.

We can also state the converse of the *Theorem 7* by strengthening the condition on f as follows.

Theorem 7. Let $f: M \to S$ be an onto homomorphism of Γ-near-rings M and S. Let $P = (\pi_P, \vartheta_P)$ be a Pythagorean fuzzy subset of S. If $f^{-1}(P) = [f^{-1}(\pi_P), f^{-1}(\vartheta_P)$ is a Pythagorean fuzzy weak bi-ideal of M, then $P = (\pi_P, \vartheta_P)$ is a Pythagorean fuuzy weak bi-ideal of S.

Proof. Let $x, y, z \in S$. Then $f(j) = x$, $f(k) = y$ and $f(l) = z$ for some $j, k, l \in M$ and $\alpha, \beta \in \Gamma$. It follows that

$\pi_P(x-y) = \pi_P(f(j) - f(k))$

$= \pi_P(f(j-k))$

$= f^{-1}(\pi_P)(j-k)$

$\geq \min\{f^{-1}(\pi_P)(j), f^{-1}(\pi_P)(k)\}$

$= \min\{\pi_P(f(j)), \pi_P(f(k))\}$

$= \min\{\pi_P(x), \pi_P(y)\}.$

$\vartheta_P(x-y) = \vartheta_P(f(j) - f(k))$

$$= \vartheta_P(f(j-k))$$

$$= f^{-1}(\vartheta_P)(j-k)$$

$$\leq \max\{f^{-1}(\vartheta_P)(j), f^{-1}(\vartheta_P)(k)\}$$

$$= \max\{\vartheta_P(f(j)), \vartheta_P(f(k))\}$$

$$= \max\{\vartheta_P(x), \vartheta_P(y)\}.$$

And

$$\pi_P(x\alpha y\beta z) = \pi_P(f(j)\alpha f(k)\beta f(l))$$

$$= \pi_P(f(jkl))$$

$$= f^{-1}(\pi_P)(jkl)$$

$$\geq \min\{f^{-1}(\pi_P)(j), f^{-1}(\pi_P)(k), f^{-1}(\pi_P)(l)\}$$

$$= \min\{\pi_P(f(j)), \pi_P(f(k)), \pi_P(f(l))\}$$

$$= \min\{\pi_P(x), \pi_P(y), \pi_P(z)\}.$$

$$\vartheta_P(x\alpha y\beta z) = \vartheta_P(f(j)\alpha f(k)\beta f(l))$$

$$= \vartheta_P(f(jkl))$$

$$= f^{-1}(\vartheta_P)(jkl)$$

$$\leq \max\{f^{-1}(\vartheta_P)(j), f^{-1}(\vartheta_P)(k), f^{-1}(\vartheta_P)(l)\}$$

$$= \max\{\vartheta_P(f(j)), \vartheta_P(f(k)), \vartheta_P(f(l))\}$$

$$= \max\{\vartheta_P(x), \vartheta_P(y), \vartheta_P(z)\}.$$

Hence P is a Pythagorean fuzzy weak bi-ideal of *S*.

Theorem 8. Let $f: M \to S$ be an onto Γ-near-ring homomorphism. If $P = (\pi_P, \vartheta_P)$ is a Pythagorean fuzzy weak bi-ideal of M, then $f(P) = f(\pi_P, \vartheta_P)$ is a Pythagorean fuzzy weak bi-ideal of M.

Proof. Let P be a Pythagorean fuzzy weak bi-ideal of M. Since $f(\pi_P)(x\prime) = \sup_{f(x)=x'} (\pi_P(x))$, for $x\prime \in S$ and $f(\vartheta_P)(x\prime) = \inf_{f(x)=x'} (\vartheta_P(x))$, for $x\prime \in S$hence $f(P)$ is nonempty. Let $x\prime, y\prime \in S$ and $\alpha, \beta \in \Gamma$. Then we have $\{x|x \in f^{-1}(x\prime - y) \supseteq \{x - y|x \in f^{-1}(x\prime)$ and $y \in f^{-1}(y\prime)\}$ and $\{x|x \in f^{-1}(x\prime y\prime)\} \supseteq \{x\alpha y|x \in f^{-1}(x\prime)$ and $y \in f^{-1}(y\prime)\}$.

$$f(\pi_P)(x\prime - y\prime) = \sup_{f(z)=(x\prime - y\prime)} \{\pi_P(z)\}$$

$$\geq \sup_{f(x)=x\prime, f(y)=y\prime} \{\pi_P(x-y)\}$$

$$\geq \sup_{f(x)=x',f(y)=y'} \{\min\{\pi_P(x), \pi_P(y)\}\}$$

$$= \min\{\sup_{f(x)=x'} \{\pi_P(x)\}, \sup_{f(y)=y'} \{\pi_P(y)\}\}$$

$$= \min\{f(\pi_P)(x'), \quad f(\pi_P)(y')\}.$$

And

$$f(\vartheta_P)(x' - y') = \inf_{f(z)=x'-y'} \{\vartheta_P(z)\}$$

$$\leq \inf_{f(x)=x',f(y)=y'} \{\vartheta_P(x - y)\}$$

$$\leq \inf_{f(x)=x',f(y)=y'} \{\max\{\vartheta_P(x), \vartheta_P(y)\}\}$$

$$= \max\{\inf_{f(x)=x'} \{\vartheta_P(x)\}, \inf_{f(y)=y'} \{\vartheta_P(y)\}\}$$

$$= \max\{f(\vartheta_P)(x'), \quad f(\vartheta_P)(y')\}.$$

Next,

$$f(\pi_P)(x'\alpha y'\beta z') = \sup_{f(w)=x'\alpha y'\beta z'} \{\pi_P(w)\}$$

$$\geq \sup_{f(x)=x',f(y)=y',f(z)=z'} \{\pi_P(x\alpha y\beta z)\}$$

$$\geq \sup_{f(x)=x',f(y)=y',f(z)=z'} \{\min\{\pi_P(x), \pi_P(y), \pi_P(z)\}\}$$

$$= \min\{\sup_{f(x)=x'} \{\pi_P(x)\}, \sup_{f(y)=y'} \{\pi_P(y)\}, \sup_{f(z)=z'} \{\pi_P(z)\}\}$$

$$= \min\{f(\pi_P)(x'), f(\pi_P)(y'), f(\pi_P)(z')\}.$$

And

$$f(\vartheta_P)(x'\alpha y'\beta z') = \inf_{f(w)=x'\alpha y'\beta z'} \{\vartheta_P(w)\}$$

$$\leq \inf_{f(x)=x',f(y)=y',f(z)=z'} \{\vartheta_P(x\alpha y\beta z)\}$$

$$\leq \inf_{f(x)=x',f(y)=y',f(z)=z'} \{\max\{\vartheta_P(x), \vartheta_P(y), \vartheta_P(z)\}\}$$

$$= \max\{\inf_{f(x)=x'} \{\vartheta_P(x)\}, \inf_{f(y)=y'} \{\vartheta_P(y)\}, \inf_{f(z)=z'} \{\vartheta_P(z)\}\}$$

$$= \max\{f(\vartheta_P)(x'), \quad f(\vartheta_P)(y'), \quad f(\vartheta_P)(z')\}.$$

Therefore $f(P)$ is a Pythagorean fuzzy weak bi-ideal of S.

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5| Interval Valued Pythagorean Fuzzy Weak Bi-Ideals of Γ-Near-Rings

In this section, we initiate the notion of interval valued Pythagorean fuzzy weak bi-ideal of M and discuss some of its properties.

Definition 7. An interval valued Pythagorean fuzzy set $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ of M is called an interval valued Pythagorean fuzzy weak bi-ideal of M, if

I. $\overline{\pi}_P(x-y) \geq min\{\overline{\pi}_P(x), \overline{\pi}_P(y)\}$.

II. $\overline{\vartheta}_P(x-y) \leq max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y)\}$.

III. $\overline{\pi}_P(x\gamma y\gamma z) \geq min\{\overline{\pi}_P(x), \overline{\pi}_P(y), \overline{\pi}_P(z)\}$.

IV. $\overline{\vartheta}_P(x\gamma y\gamma z) \leq max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y), \overline{\vartheta}_P(z)\}$ *for all* $x, y, z \in M$ *and* $\alpha, \beta \in \Gamma$.

Example 3. Let $M = \{w,\ x,\ y,\ z\}$ be a nonempty set with binary operation $`+'$ and $\Gamma = \{\gamma\}$ be a nonempty set of binary operations as the following tables:

+	w	x	y	z
w	w	x	y	z
x	x	w	z	y
y	y	z	w	x
z	z	y	x	w

and

γ	w	x	y	z
w	w	x	w	x
x	w	x	w	x
y	w	x	y	z
z	w	x	y	z

Let $\overline{\pi}_P : M \to D[0,1]$ and $\overline{\vartheta}_P : M \to D[0,1]$ be aninterval valued fuzzy subsets defined by $\overline{\pi}_P(w) = [0.6,0.7]$, $\overline{\pi}_p(x) = [0.5,0.6]$, $\overline{\pi}_P(y) = \overline{\pi}_P(z) = [0.4,0.5]$. And $\overline{\vartheta}_P(w) = [0.2,0.3]$, $\overline{\vartheta}_P(x) = [0.4,0.5]$, $\overline{\vartheta}_P(y) = [0.7,0.8] = \overline{\vartheta}_P(z)$. Then $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ is an interval valued Pythagorean fuzzy weak bi-ideal of M.

Theorem 9. Let $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ be an interval valued Pythagorean fuzzy subgroup of M. Then $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ is an interval valued Pythagorean fuzzy weak bi-ideal of M if and only if $\overline{\pi}_P \star \overline{\pi}_P \star \overline{\pi}_P \subseteq \overline{\pi}_P$ and $\overline{\vartheta}_P \star \overline{\vartheta}_P \star \overline{\vartheta}_P \supseteq \overline{\vartheta}_P$.

Proof. Assume that $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ be an interval valued Pythagorean fuzzy weak bi-ideal of M. Let $x, y, z, y_1, y_2 \in M$ and $\alpha, \beta \in \Gamma$ such that $x = y\alpha z$ and $y = y_1\beta y_2$. Then

$$(\overline{\pi}_P \star \overline{\pi}_P \star \overline{\pi}_P)(x) = \sup_{x=y\alpha z}\{\min\{(\overline{\pi}_P \star \overline{\pi}_P)(y), \overline{\pi}_P(z)\}\}$$

$$= \sup_{x=y\alpha z}\{\min\{\sup_{y=y_1\beta y_2}\min\{\overline{\pi}_P(y_1), \overline{\pi}_P(y_2)\}, \overline{\pi}_P(z)\}\}$$

$$= \sup_{x=y\alpha z}\sup_{y=y_1\beta y_2}\{\min\{\min\{\overline{\pi}_P(y_1), \overline{\pi}_P(y_2)\}, \overline{\pi}_P(z)\}\}$$

$$= \sup_{x=y_1\beta y_2\alpha z} \{\min\{\overline{\pi}_P(y_1), \overline{\pi}_P(y_2), \overline{\pi}_P(z)\}\}$$

$$\overline{\pi}_P(y_1\beta y_2\alpha z) \geq \min\{\overline{\pi}_P(y_1), \overline{\pi}_P(y_2), \overline{\pi}_P(z)\}$$

$$\leq \sup_{x=y_1\beta y_2\alpha z} \overline{\pi}_P(y_1\beta y_2\alpha z)$$

$$= \overline{\pi}_P(x).$$

And

$$(\overline{\vartheta}_P \star \overline{\vartheta}_P \star \overline{\vartheta}_P)(x) = \inf_{x=y\alpha z} \{\min\{(\overline{\vartheta}_P \star \overline{\vartheta}_P)(y), \overline{\vartheta}_P(z)\}\}$$

$$= \inf_{x=y\alpha z} \{\max\{ \inf_{y=y_1\beta y_2} \min\{\overline{\vartheta}_P(y_1), \overline{\vartheta}_P(y_2)\}, \overline{\vartheta}_P(z)\}\}$$

$$= \inf_{x=y\alpha z y=y_1\beta y_2} \inf \{\max\{\max\{\overline{\vartheta}_P(y_1), \overline{\vartheta}_P(y_2)\}, \overline{\vartheta}_P(z)\}\}$$

$$= \inf_{x=y_1\beta y_2\alpha z} \{\max\{\overline{\vartheta}_P(y_1), \overline{\vartheta}_P(y_2), \overline{\vartheta}_P(z)\}\}$$

$$\overline{\vartheta}_P(y_1\beta y_2\alpha z) \leq \max\{\overline{\vartheta}_P(y_1), \overline{\vartheta}_P(y_2), \overline{\vartheta}_P(z)\}$$

$$\geq \inf_{x=y_1\beta y_2\alpha z} \overline{\vartheta}_P(y_1\beta y_2\alpha z)$$

$$= \overline{\vartheta}_P(x).$$

Since $\overline{P}$ is an interval valued Pythagorean fuzzy weak bi-ideal of M,

If x can not be expressed as $x = y\alpha z$, then $(\overline{\pi}_P \star \overline{\pi}_P \star \overline{\pi}_P)(x) = 0 \leq \overline{\pi}_P(x)$ and

$(\overline{\vartheta}_P \star \overline{\vartheta}_P \star \overline{\vartheta}_P)(x) = 0 \geq \overline{\vartheta}_P(x)$. In both cases $\overline{\pi}_P \star \overline{\pi}_P \star \overline{\pi}_P \subseteq \overline{\pi}_P$ and $\overline{\vartheta}_P \star \overline{\vartheta}_P \star \overline{\vartheta}_P \supseteq \overline{\vartheta}_P$.

Conversely, assume that $\overline{\pi}_P \star \overline{\pi}_P \star \overline{\pi}_P \subseteq \overline{\pi}_P$. For $x', x, y, z \in M$ and $\alpha, \beta, \alpha_1, \beta_1 \in \Gamma$.

Let x' be such that $x' = x\alpha y\beta z$.

Then $\overline{\pi}_P(x\alpha y\beta z) = \overline{\pi}_P(x') \geq (\overline{\pi}_P \star \overline{\pi}_P \star \overline{\pi}_P)(x')$

$$= \sup_{x'=p\alpha_1 q} \{\min\left\{(\overline{\pi}_P \star \overline{\pi}_P)(p), \overline{\pi}_P(q)\right\}\}$$

$$= \sup_{x'=p\alpha_1 q} \{\min\{ \sup_{p=p_1\beta_1 p_2} \min\{\overline{\pi}_P(p_1), \overline{\pi}_P(p_2)\}, \overline{\pi}_P(q)\}\}$$

$$= \sup_{x'=p_1\beta_1 p_2\alpha_1 q} \{\min\{\overline{\pi}_P(p_1), \overline{\pi}_P(p_2), \overline{\pi}_P(q)\}\}$$

$$\geq \min\{\overline{\pi}_P(x), \overline{\pi}_P(y), \overline{\pi}_P(z)\}.$$

$$\overline{\vartheta}_P(x\alpha y\beta z) = \overline{\vartheta}_P(x') \leq (\overline{\vartheta}_P * \overline{\vartheta}_P * \overline{\vartheta}_P)(x')$$

$$= \inf_{x\prime=p\alpha_1 q}\{\max\{(\overline{\vartheta}_P * \overline{\vartheta}_P)(p), \overline{\vartheta}_P(q)\}\}$$

$$= \inf_{x\prime=p\alpha_1 q}\{\max\{\inf_{p=p_1\beta_1 p_2}\min\{\overline{\vartheta}_P(p_1), \overline{\vartheta}_P(p_2)\}, \overline{\vartheta}_P(q)\}\}$$

$$= \inf_{x\prime=p_1\beta_1 p_2\alpha_1 q}\{\max\{\overline{\vartheta}_P(p_1), \overline{\vartheta}_P(p_2), \overline{\vartheta}_P(q)\}\}$$

$$\leq \max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y), \overline{\vartheta}_P(z)\}.$$

Hence $\overline{\pi}_P(x\alpha y\beta z) \geq min\{\overline{\pi}_P(x), \overline{\pi}_P(y), \overline{\pi}_P(z)\}$ and $\overline{\vartheta}_P(x\alpha y\beta z) \leq max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y), \overline{\vartheta}_P(z)\}$.

Lemma 3. Let $\overline{\pi}_P = (\overline{\pi}_{P1}, \overline{\pi}_{P2})$ and $\overline{\vartheta}_P = (\overline{\vartheta}_{P1}, \overline{\vartheta}_{P2})$ be an interval valued Pythagorean fuzzy weal bi-ideals of *M*. Then the products $\overline{\pi}_P * \overline{\vartheta}_P$ *and* $\overline{\vartheta}_P * \overline{\pi}_P$ are also interval valued Pythagorean fuzzy weak bi-ideals of *M*.

Proof. Let $\overline{\pi}_P$ and $\overline{\vartheta}_P$ be an interval valued Pythagorean fuzzy weak bi-ideals of M and let $\alpha, \alpha_1, \alpha_2 \in \Gamma$. Then

$$(\overline{\pi}_1 \star \overline{\pi}_2)(x-y) = \sup_{x-y=a\alpha b}\min\{\overline{\pi}(a), \overline{\pi}(b)\}$$

$$\geq \sup_{x-y=a_1\alpha_1 b_1 - a_2\alpha_2 b_2 < (a_1-a_2)(b_1-b_2)}\min\{\overline{\pi}(a_1-a_2), \overline{\pi}(b_1-b_2)\}$$

$$\geq \sup\min\{\min\{\overline{\pi}(a_1), \overline{\pi}(a_2)\}, \min\{\overline{\pi}(b_1), \overline{\pi}(b_2)\}\}$$

$$= \sup\min\{\min\{\overline{\pi}(a_1), \overline{\pi}(b_1)\}, \min\{\overline{\pi}(a_2), \overline{\pi}(b_2)\}\}$$

$$\geq \min\{\sup_{x=a_1\alpha_1 b_1}\min\{\overline{\pi}(a_1), \overline{\pi}(b_1)\}, \sup_{y=a_2\alpha_2 b_2}\min\{\overline{\pi}(a_2), \overline{\pi}(b_2)\}\}$$

$$= \min\{(\overline{\pi} \star \overline{\pi})(x), (\overline{\pi} \star \overline{\pi})(y)\}.$$

$$(\overline{\vartheta}_1 \star \overline{\vartheta}_2)(x-y) = \inf_{x-y=a\alpha b}\max\{\overline{\vartheta}(a), \overline{\vartheta}(b)\}$$

$$\leq \inf_{x-y=a_1\alpha_1 b_1 - a_2\alpha_2 b_2 < (a_1-a_2)(b_1-b_2)}\max\{\overline{\vartheta}(a_1-a_2), \overline{\vartheta}(b_1-b_2)\}$$

$$\leq \inf\max\{\max\{\overline{\vartheta}(a_1), \overline{\vartheta}(a_2)\}, \max\{\overline{\vartheta}(b_1), \overline{\vartheta}(b_2)\}\}$$

$$= \inf\max\{\max\{\overline{\vartheta}(a_1), \overline{\vartheta}(b_1)\}, \max\{\overline{\vartheta}(a_2), \overline{\vartheta}(b_2)\}\}$$

$$\leq \max\{\inf_{x=a_1\alpha_1 b_1}\max\{\overline{\vartheta}(a_1), \overline{\vartheta}(b_1)\}, \inf_{y=a_2\alpha_2 b_2}\max\{\overline{\vartheta}(a_2), \overline{\vartheta}(b_2)\}\}$$

$$= \max\{(\overline{\vartheta} \star \overline{\vartheta})(x), (\overline{\vartheta} \star \overline{\vartheta})(y)\}.$$

It follows that $\overline{\pi} \star \overline{\vartheta}$ is an interval valued Pythagorean fuzzy subgroup of M. Further,

$(\overline{\pi} \star \overline{\vartheta}) \star (\overline{\pi} \star \overline{\vartheta}) \star (\overline{\pi} \star \overline{\vartheta}) = \overline{\pi} \star \overline{\vartheta} \star (\overline{\pi} \star \overline{\vartheta} \star \overline{\pi}) \star \overline{\vartheta} \subseteq \overline{\pi} \star \overline{\vartheta} \star (\overline{\vartheta} \star \overline{\vartheta} \star \overline{\vartheta}) \star \overline{\vartheta} \subseteq \overline{\pi} \star (\overline{\vartheta} \star \overline{\vartheta} \star \overline{\vartheta})$, since $\overline{\vartheta}$ is an interval valued Pythagorean fuzzy weak bi-ideal of $M \subseteq \overline{\pi} \star \overline{\vartheta}$.

Therefore $\overline{\pi}_P \star \overline{\vartheta}_P$ is an interval valued Pythagorean fuzzy weak bi-ideal of M. Similarly $\overline{\vartheta}_P \star \overline{\pi}_P$ is an interval valued Pythagorean fuzzy weak bi-ideal of M.

Lemma 4. Every interval valued Pythagorean fuzzy ideal of M is an interval valued Pythagorean fuzzy bi-ideal of M.

Proof. Let $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ be an interval valued Pythagorean fuzzy ideal of M. Then

$$\overline{\pi}_P \star \mathrm{M} \star \overline{\pi}_P \subseteq \overline{\pi}_P \star \mathrm{M} \star \mathrm{M} \subseteq \overline{\pi}_P \star \mathrm{M} \subseteq \overline{\pi}_P,$$

$$\overline{\vartheta}_P \star \mathrm{M} \star \overline{\vartheta}_P \supseteq \overline{\vartheta}_P \star \mathrm{M} \star \mathrm{M} \supseteq \overline{\vartheta}_P \star \mathrm{M} \supseteq \overline{\vartheta}_P,$$

since $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ be an interval valued Pythagorean fuzzy ideal of M.

This implies that $\overline{\pi}_P \star M \star \overline{\pi}_P \subseteq \overline{\pi}_P$ and $\overline{\vartheta}_P \star M \star \overline{\vartheta}_P \supseteq \overline{\vartheta}_P$.

Therefore $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ be an interval valued Pythagorean fuzzy bi-ideal of M.

Theorem 10. Every interval valued Pythagorean fuzzy bi-ideal of M is an interval valued Pythagorean fuzzy weak bi-ideal of M.

Proof. Assume that $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ be an interval valued Pythagorean fuzzy bi-ideal of M.

Then $\overline{\pi}_P \star M \star \overline{\pi}_P \subseteq \overline{\pi}_P$ and $\overline{\vartheta}_P \star M \star \overline{\vartheta}_P \supseteq \overline{\vartheta}_P$.

We have $\overline{\pi}_P \star \overline{\pi}_P \star \overline{\pi}_P \subseteq \overline{\pi}_P \star M \star \overline{\pi}_P$ and $\overline{\vartheta}_P \star \overline{\vartheta}_P \star \overline{\vartheta}_P \supseteq \overline{\vartheta}_P \star M \star \overline{\vartheta}_P$.

This implies that $\overline{\pi}_P \star \overline{\pi}_P \star \overline{\pi}_P \subseteq \overline{\pi}_P \star M \star \overline{\pi}_P \subseteq \overline{\pi}_P$ and $\overline{\vartheta}_P \star \overline{\vartheta}_P \star \overline{\vartheta}_P \supseteq \overline{\vartheta}_P \star M \star \overline{\vartheta}_P \supseteq \overline{\vartheta}_P$.

Therefore $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ is an interval valued Pythagorean fuzzy weak bi-ideal of M.

Example 4. Let $M = \{w, x, y, z\}$ be a nonempty set with binary operation$+$ and $\Gamma = \{\alpha\}$ be a nonempty set of binary operations as the following tables:

+	w	x	y	z
w	w	x	y	z
x	x	w	z	y
y	y	z	w	x
z	z	y	x	w

and

α	w	x	y	z
w	w	w	w	w
x	w	x	w	x
y	w	w	y	y
z	w	x	y	z

Let $\overline{P} : M \to D[0,1]$ be aninterval valued Pythagorean fuzzy set defined by $\overline{\pi}_P(w) = [0.8,0.9], \overline{\pi}_P(x) = [0.3,0.4] = \overline{\pi}_P(y)$ and $\overline{\pi}_P(z) = [0.5,0.6]$, and $\overline{\vartheta}_P(w) = [0,0.1], \overline{\vartheta}_P(x) = [0.4,0.5] = \overline{\vartheta}_P(y), \overline{\vartheta}_P(z) = [0.2,0.3]$. Then $\overline{\pi}_P$ is an interval valued fuzzy weak bi-ideal of M. But $\overline{\pi}_P$ is not a fuzzy ideal and bi-ideal of M and

$\overline{\pi}_P(z\gamma y\gamma z) = \overline{\pi}_P(y) = [0.3,0.4] > [0.5,0.6] = min\{\overline{\pi}_P(z), \overline{\pi}_P(z)\}$ and $\overline{\vartheta}_P(x\alpha(z+w) - x\alpha w) \leq \overline{\vartheta}_P(z) = [0.4,0.5] \not\leq [0.2,0.3]$ and $\overline{\vartheta}_P(z\gamma x\gamma z) = \overline{\vartheta}_P(x) = [0.4,0.5] \not\leq [0.2,0.3] = \min\{\overline{\vartheta}_P(z), \overline{\vartheta}_P(z)\}$.

Theorem 11. Let $\{(\overline{\pi}_{P_i}, \overline{\vartheta}_{P_i}) | i \in \Omega\}$ be family of interval valued Pythagorean fuzzy weak bi-ideals of a near-ring M, then $\bigcap_{i\in\Omega} \overline{\pi}_{P_i}$ and $\cup_{i\in\Omega} \overline{\vartheta}_{P_i}$ are also an interval valued Pythagorean fuzzy weak bi-ideal of M, where Ω is any index set.

Proof. Let $\{\overline{\pi}_{P_i}, \overline{\vartheta}_{P_i}\}_{i\in\Omega}$ be a family of interval valued Pythagorean fuzzy weak bi-ideals of M.

Let $x, y, z \in M, \alpha, \beta \in \Gamma$ and $\overline{\pi}_{P_i} = \bigcap_{i\in\Omega} \overline{\pi}_i$, $\overline{\vartheta}_{P_i} = \cup_{i\in\Omega} \overline{\vartheta}_{P_i}$

Then, $\overline{\pi}_{P_i}(x) = \bigcap_{i\in\Omega} \overline{\pi}_{P_i}(x) = (\inf_{i\in\Omega} \overline{\pi}_{P_i})(x) = \inf_{i\in\Omega} \overline{\pi}_{P_i}(x)$

and $\overline{\vartheta}_{P_i}(x) = \cup_{i\in\Omega} \overline{\vartheta}_{P_i}(x) = (\sup_{i\in\Omega} \overline{\vartheta}_{P_i})(x) = \sup_{i\in\Omega} \overline{\vartheta}_{P_i}(x)$.

$$\overline{\pi}_{P_i}(x-y) = \inf_{i\in\Omega} \overline{\pi}_{P_i}(x-y)$$

$$\geq \inf_{i\in\Omega} \min\{\overline{\pi}_{P_i}(x), \overline{\pi}_{P_i}(y)\}$$

$$= \min\{\inf_{i\in\Omega} \overline{\pi}_{P_i}(x), \inf_{i\in\Omega} \overline{\pi}_{P_i}(y)\}$$

$$= \min\{\bigcap_{i\in\Omega} \overline{\pi}_{P_i}(x), \bigcap_{i\in\Omega} \overline{\pi}_{P_i}(y)\}$$

$$= \min\{\overline{\pi}_{P_i}(x), \overline{\pi}_{P_i}(y)\}.$$

$$\overline{\vartheta}_{P_i}(x-y) = \sup_{i\in\Omega} \overline{\vartheta}_{P_i}(x-y)$$

$$\leq \sup_{i\in\Omega} \max\{\overline{\vartheta}_{P_i}(x), \overline{\vartheta}_{P_i}(y)\}$$

$$= \max\{\sup_{i\in\Omega} \overline{\vartheta}_{P_i}(x), \sup_{i\in\Omega} \overline{\vartheta}_{P_i}(y)\}$$

$$= \max\{\cup_{i\in\Omega} \overline{\vartheta}_{P_i}(x), \cup_{i\in\Omega} \overline{\vartheta}_{P_i}(y)\}$$

$$= \max\{\overline{\vartheta}_{P_i}(x), \overline{\vartheta}_{P_i}(y)\}.$$

And,

$$\overline{\pi}_{P_i}(x\alpha y\beta z) = \inf_{i\in\Omega} \overline{\pi}_{P_i}(x\alpha y\beta z)$$

$$\geq \inf_{i\in\Omega} \min\{\overline{\pi}_{P_i}(x), \overline{\pi}_{P_i}(y), \overline{\pi}_{P_i}(z)\}$$

$$= \min\{\inf_{i\in\Omega} \overline{\pi}_{P_i}(x), \inf_{i\in\Omega} \overline{\pi}_{P_i}(y), \inf_{i\in\Omega} \overline{\pi}_{P_i}(z)\}$$

$$= \min\{\bigcap_{i\in\Omega} \overline{\pi}_{P_i}(x), \bigcap_{i\in\Omega} \overline{\pi}_{P_i}(y), \bigcap_{i\in\Omega} \overline{\pi}_{P_i}(z)\}$$

$= \min\{\overline{\pi}_{P_i}(x), \overline{\pi}_{P_i}(y), \overline{\pi}_{P_i}(z)\}.$

$\overline{\vartheta}_{P_i}(x\alpha y\beta z) = \sup_{i\in\Omega} \overline{\vartheta}_{P_i}(x\alpha y\beta z)$

$\leq \sup_{i\in\Omega} \max\{\overline{\vartheta}_{P_i}(x), \overline{\vartheta}_{P_i}(y), \overline{\vartheta}_{P_i}(z)\}$

$= \max\{\sup_{i\in\Omega} \overline{\vartheta}_{P_i}(x), \sup_{i\in\Omega} \overline{\vartheta}_{P_i}(y), \sup_{i\in\Omega} \overline{\vartheta}_{P_i}(z)\}$

$= \max\{\cup_{i\in\Omega} \overline{\vartheta}_{P_i}(x), \cup_{i\in\Omega} \overline{\vartheta}_{P_i}(y), \cup_{i\in\Omega} \overline{\vartheta}_{P_i}(z)\}$

$= \max\{\overline{\vartheta}_{P_i}(x), \overline{\vartheta}_{P_i}(y), \overline{\vartheta}_{P_i}(z)\}.$

Theorem 12. Let $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ be an interval valued Pythagorean fuzzy subset of M. Then $U(\overline{\pi}_P; t)$ and $L(\overline{\vartheta}_P; s)$ is an interval valued Pythagorean fuzzy weak bi-ideal of M if and only if $\overline{\pi}_{Pt}$ is a weak bi-ideal of M, for all $t \in [0,1]$.

Proof. Assume that $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ is an interval valued Pythagorean fuzzy weak bi-ideal of R.

Let $s, t \in [0,1]$ such that $x, y \in U(\overline{\pi}_P; t)$.

Then $\overline{\pi}_P(x) \geq t$ and $\overline{\pi}_P(y) \geq t$,then $\overline{\pi}_P(x-y) \geq \min\{\overline{\pi}_P(x), \overline{\pi}_P(y)\} \geq \min\{t, t\} = t$ and

$\overline{\vartheta}_P(x-y) \leq \max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y)\} \leq max\{s, s\} = s.$

Thus $x - y \in U(\overline{\pi}_P\ \ t)$.Let $x, y, z \in \overline{\pi}_{Pt}$ and $\alpha, \beta \in \Gamma$.

This implies that $\overline{\pi}_P(x\alpha y\beta z) \geq \min\{\overline{\pi}_P(x), \overline{\pi}_P(y), \overline{\pi}_P(z)\} \geq \min\{t, t, t\} = t$, and

$\overline{\vartheta}_P(x\alpha y\beta z) \leq \max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y), \overline{\vartheta}_P(z)\} \leq \max\{s,\ s,\ s\} = s.$

Therefore $x\alpha y\beta z \in U(\overline{\pi}_P; s)$.

Hence U $(\overline{\pi}_P;$ t) and $(\overline{\vartheta}_P; s)$ is an interval valued Pythagorean fuzzy weak bi-ideal of M.

Conversely, assume that U $(\overline{\pi}_P;$ t) and $(\overline{\vartheta}_P; s)$ is an interval valuedPythagorean fuzzy weak bi-ideal of M, for all $s, t \in [0,1]$.

Let $x, y \in M$. Suppose $\overline{\pi}_P(x-y) < min\{\overline{\pi}_P(x), \overline{\pi}_P(y)\}$and $\overline{\vartheta}_P(x-y) > max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y)\}$.

Choose t such that $\overline{\pi}_P(x-y) < t < min\{\overline{\pi}_P(x), \overline{\pi}_P(y)\}$ and $\overline{\vartheta}_P(x-y) > s > max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y)\}$.

This implies that $\overline{\pi}_P(x) > t, \overline{\pi}_P(y) > t$ and $\overline{\pi}_P(x-y) < t$.

Then we have $x, y \in \overline{\pi}_{Pt}$ but $x - y \notin \overline{\pi}_{Pt}$ and $\overline{\vartheta}_P(x) < s, \overline{\vartheta}_P(y) < s$ and $\overline{\vartheta}_P(x-y) > s$, we have $x, y \in \overline{\vartheta}_{Ps}$ but $x - y \notin \overline{\vartheta}_{Ps}$ a contradiction.

Thus $\overline{\pi}_P(x-y) \geq \min\{\overline{\pi}_P(x), \overline{\pi}_P(y)\}$ and $\overline{\vartheta}_P(x-y) \leq \max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y)\}$.

If there exist $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ such that $\overline{\pi}_P(x\alpha y\beta z) < min\{\overline{\pi}_P(x), \overline{\pi}_P(y), \overline{\pi}_P(z)\}$ and $\overline{\vartheta}_P(x\alpha y\beta z) > max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y), \overline{\vartheta}_P(z)\}$.

Choose t such that $\overline{\pi}_P(x\alpha y\beta z) < t < min\{\overline{\pi}_P(x), \overline{\pi}_P(y), \overline{\pi}_P(z)\}$.

Choose s such that $\overline{\vartheta}_P(x\alpha y\beta z) > s > max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y), \overline{\vartheta}_P(z)\}$.

Then $\overline{\pi}_P(x) > t, \overline{\pi}_P(y) > t, \overline{\pi}_P(z) > t$ and $\overline{\vartheta}_P(x) < s, \overline{\vartheta}_P(y) < s, \overline{\vartheta}_P(z) < s$ and $\overline{\pi}_P(x\alpha y\beta z) < t$.

So, $x, y, z \in \overline{\pi}_P >$ but $x\alpha y\beta z \notin \overline{\pi}_{Pt}$, and $x\alpha y\beta z \notin \overline{\vartheta}_{Ps}$, which is a contradiction.

Hence $\overline{\pi}_P(x\alpha y\beta z) \geq \min\{\overline{\pi}_P(x), \overline{\pi}_P(y), \overline{\pi}_P(z)\}, \overline{\vartheta}_P(x\alpha y\beta z) \leq \max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y), \overline{\vartheta}_P(z)\}$.

Therefore $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ is an interval valued Pythagorean fuzzy weak bi-ideal of M.

Theorem 13. Let $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ be an interval valued Pythagorean fuzzy weak bi-ideal of M then the set $M_{\overline{\pi}_P, \overline{\vartheta}_P} = \{x \in M | \overline{\pi}_P(x) = \overline{\pi}_P(0) = \overline{\vartheta}_P(x)\}$ is a interval valued Pythagorean fuzzy weak bi-ideal of M.

Proof. Let $x, y \in M_{(\overline{\pi}_P, \overline{\vartheta}_P)}$.

Then $\overline{\pi}_P(x) = \overline{\pi}_P(0), \overline{\pi}_P(y) = \overline{\pi}_P(0), \overline{\vartheta}_P(x) = 0, \overline{\vartheta}_P(y) = 0$ and

$\overline{\pi}_P(x - y) \geq \min\{\overline{\pi}_P(x), \overline{\pi}_P(y)\} = \min\{\overline{\pi}_P(0), \overline{\pi}_P(0)\} = \overline{\pi}(0)$, and

$\overline{\vartheta}_P(x - y) \leq \max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y)\} = \max\{\overline{\vartheta}_P(0), \overline{\vartheta}_P(0)\} = \overline{\vartheta}_P(0)$.

So $\overline{\pi}_P(x - y) = \overline{\pi}_P(0), \overline{\vartheta}_P(x - y) = \overline{\vartheta}_P(0)$.

Thus $x - y \in M_{\overline{\pi}_P}$, $x - y \in M_{\overline{\vartheta}_P}$. For every $x, y, z \in M_{\overline{\pi}_P}$ and $\alpha, \beta \in \Gamma$. We have

$$\overline{\pi}_P(x\alpha y\beta z) \geq \min\{\overline{\pi}_P(x), \overline{\pi}_P(y), \overline{\pi}_P(z)\},$$

$$= \min\{\overline{\pi}_P(0), \overline{\pi}_P(0), \overline{\pi}_P(0)\} = \overline{\pi}_P(0),$$

and

$$\overline{\vartheta}_P(x\alpha y\beta z) \leq \max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y), \overline{\vartheta}_P(z)\}$$

$$= \max\{\overline{\vartheta}_P(0), \overline{\vartheta}_P(0), \overline{\vartheta}_P(0)\} = \overline{\vartheta}_P(0).$$

Thus $x\alpha y\beta z \in M_{\overline{\pi}_P}, x\alpha y\beta z \in M_{\overline{\vartheta}_P}$. Hence $M_{(\overline{\pi}_P, \overline{\vartheta}_P)}$ is an interval valuedPythagorean fuzzy weak bi-ideal of M.

6 | Homomorphism of Interval Valued Pythagorean Fuzzy Weak Bi-Ideals of Γ-Near-Rings

In this section, we characterize interval valued Pythagorean fuzzy weak bi-ideals of Γ-near-rings using homomorphism.

Definition 8. Let f be a mapping from a set M to a set S. Let $f = (\overline{\pi}_P, \overline{\vartheta}_P)$ be an interval valued Pythagorean fuzzy subsets of M and S, resp. then f is image of $\overline{\pi}_P$ and $\overline{\vartheta}_P$ under f is a fuzzy subset of S defined by

$$f(\overline{\pi}_P)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \overline{\pi}_P(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

$$f(\overline{\vartheta}_P)(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \overline{\vartheta}_P(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 1 & \text{otherwise} \end{cases}$$

And the pre-image of $\overline{\pi}_P$ and $\overline{\vartheta}_P$ under f is a fuzzy subset of M defined by

$$f^{-1}(\overline{\pi}_P(x)) = \overline{\pi}_P(f(x)), f^{-1}(\overline{\vartheta}_P(x)) = \overline{\vartheta}_P(f(x)) \textit{ for all } x \in M \textit{ and } f^{-1}(y) = \{x \in M | f(x) = y\}.$$

Theorem 14. Let $f: M \to S$ be a homomorphism between Γ-near-rings M and S. If $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ is an interval valued Pythagorean fuzzy weak bi-ideal of S, then $f^{-1}(\overline{P}) = [f^{-1}(\overline{\pi}_P, \overline{\vartheta}_P)]$ is an interval valued fuzzy weakbi-ideal of M.

Proof. Let f be an interval valued Pythagorean fuzzy weak bi-ideal of S. Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then

$$f^{-1}(\overline{\pi}_P)(x - y) = \overline{\pi}_P(f(x - y))$$

$$= \overline{\pi}_P(f(x) - f(y))$$

$$\geq \min\{\overline{\pi}_P(f(x)), \overline{\pi}_P(f(y))\}$$

$$= \min\{f^{-1}(\overline{\pi}_P(x)), f^{-1}(\overline{\pi}_P(y))\}.$$

$$f^{-1}(\overline{\vartheta}_P)(x - y) = \overline{\vartheta}_P(f(x - y))$$

$$= \overline{\vartheta}_P\big(f(x) - f(y)\big)$$

$$\leq \max\{\overline{\vartheta}_P(f(x)), \overline{\vartheta}_P(f(y))\}$$

$$= \max\{f^{-1}(\overline{\vartheta}_P(x)), f^{-1}(\overline{\vartheta}_P(y))\}.$$

$$f^{-1}(\overline{\pi}_P)(x\alpha y\beta z) = \overline{\pi}_P(f(x\alpha y\beta z))$$

$$= \overline{\pi}_P(f(x)\alpha f(y)\beta f(z))$$

$$\geq \min\{\overline{\pi}_P(f(x)), \overline{\pi}_P(f(y)), \overline{\pi}_P(f(z))\}$$

$$= \min\{f^{-1}(\overline{\pi}_P(x)), f^{-1}(\overline{\pi}_P(y)), f^{-1}(\overline{\pi}_P(z))\}.$$

$$f^{-1}(\overline{\vartheta}_P)(x\alpha y\beta z) = \overline{\vartheta}_P(f(x\alpha y\beta z))$$

$$= \overline{\vartheta}_P(f(x)\alpha f(y)\beta f(z))$$

$\leq \max\{\overline{\vartheta}_P(f(x)), \overline{\vartheta}_P(f(y)), \overline{\vartheta}_P(f(z))\}$

$= \max\{f^{-1}(\overline{\vartheta}_P(x)), f^{-1}(\overline{\vartheta}_P(y)), f^{-1}(\overline{\vartheta}_P(z))\}.$

Therefore $f^{-1}(\overline{P}) = [f^{-1}(\overline{\pi}_P, \overline{\vartheta}_P)]$ is an interval valued Pythagorean fuzzy weak bi-ideal of M.

We can also state the converse of the *Theorem 7* by strengthening the condition on f as follows.

Theorem 15. Let $f: M \to S$ be an onto homomorphism of Γ-near-rings M and S. Let $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ be an interval valued Pythagorean fuzzy subset of S. If $f^{-1}(\overline{P}) = [f^{-1}(\overline{\pi}_P, \overline{\vartheta}_P)]$ is an interval valued Pythagorean fuzzy weak bi-ideal of M, then $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ is a Pythagorean fuzzy weak bi-ideal of S.

Proof. Let $x, y, z \in S$. Then $f(j) = x, f(k) = y$ and $f(l) = z$ for some $j, k, l \in M$ and $\alpha, \beta \in \Gamma$. It follows that

$\overline{\pi}_P(x - y) = \overline{\pi}_P(f(j) - f(k))$

$= \overline{\pi}_P(f(j - k))$

$= f^{-1}(\overline{\pi}_P)(j - k)$

$\geq \min\{f^{-1}(\overline{\pi}_P)(j), f^{-1}(\overline{\pi}_P)(k)\}$

$= \min\{\overline{\pi}_P(f(j)), \overline{\pi}_P(f(k))\}$

$= \min\{\overline{\pi}_P(x), \overline{\pi}_P(y)\}.$

$\overline{\vartheta}(x - y) = \overline{\vartheta}(f(j) - f(k))$

$= \overline{\vartheta}(f(j - k))$

$= f^{-1}(\overline{\vartheta})(j - k)$

$\leq \max\{f^{-1}(\overline{\vartheta})(j), f^{-1}(\overline{\vartheta})(k)\}$

$= \max\{\overline{\vartheta}(f(j)), \overline{\vartheta}(f(k))\}$

$= \max\{\overline{\vartheta}(x), \overline{\vartheta}(y)\}.$

And

$\overline{\pi}_P(x\alpha y\beta z) = \overline{\pi}_P(f(j)\alpha f(k)\beta f(l))$

$= \overline{\pi}_P(f(jkl))$

$= f^{-1}(\overline{\pi}_P)(jkl)$

$\geq \min\{f^{-1}(\overline{\pi}_P)(j), f^{-1}(\overline{\pi}_P)(k), f^{-1}(\overline{\pi}_P)(l)\}$

$= \min\{\overline{\pi}_P(f(j)), \overline{\pi}_P(f(k)), \overline{\pi}_P(f(l))\}$

$= \min\{\overline{\pi}_P(x), \overline{\pi}_P(y), \overline{\pi}_P(z)\}.$

$\overline{\vartheta}(x\alpha y\beta z) = \overline{\vartheta}(f(j)\alpha f(k)\beta f(l))$

$= \overline{\vartheta}(f(jkl))$

$= f^{-1}(\overline{\vartheta})(jkl)$

$\leq \max\{f^{-1}(\overline{\vartheta})(j), f^{-1}(\overline{\vartheta})(k), f^{-1}(\overline{\vartheta})(l)\}$

$= \max\{\overline{\vartheta}(f(j)), \overline{\vartheta}(f(k)), \overline{\vartheta}(f(l))\}$

$= \max\{\overline{\vartheta}(x), \overline{\vartheta}(y), \overline{\vartheta}(z)\}.$

Hence $\overline{P}$ is an interval valued Pythagorean fuzzy weak bi-ideal of S.

Theorem 16. Let $f: M \rightarrow S$ be an onto Γ-near-ring homomorphism. If $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ is an interval valued Pythagorean fuzzy weak bi-ideal of M, then $f(\overline{P}) = [f(\overline{\pi}_P, \overline{\vartheta}_P)]$ is an interval valued Pythagorean fuzzy weak bi-ideal of S.

Proof. Let $\overline{P}$ be an interval valued Pythagorean fuzzy weak bi-ideal of M. Since $f(\overline{\pi}_P)(x\prime) = \sup\limits_{f(x)=x'} (\overline{\pi}_P(x))$ and $f(\overline{\vartheta}_P)(x\prime) = \inf\limits_{f(x)=x'} (\overline{\vartheta}_P(x))$, for $x\prime \in S$ and hence $f(\overline{P})$ is nonempty. Let $x\prime, y\prime \in S$ and $\alpha, \beta \in \Gamma$. Then we have $\{x|x \in f^{-1}(x\prime - y) \supseteq \{x - y|x \in f^{-1}(x\prime)$ and $y \in f^{-1}(y\prime)\}$ and $\{x|x \in f^{-1}(x\prime y\prime)\} \supseteq \{x\alpha y|x \in f^{-1}(x\prime)$ *and* $y \in f^{-1}(y\prime)\}$.

$f(\overline{\pi}_P)(x\prime - y\prime) = \sup\limits_{f(z)=(x\prime-y\prime)} \{\overline{\pi}_P(z)\}$

$\geq \sup\limits_{f(x)=x\prime, f(y)=y\prime} \{\overline{\pi}_P(x - y)\}$

$\geq \sup\limits_{f(x)=x', f(y)=y'} \{\min\{\overline{\pi}_P(x), \overline{\pi}_P(y)\}\}$

$= \min\{\sup\limits_{f(x)=x\prime} \{\overline{\pi}_P(x)\}, \sup\limits_{f(y)=y\prime} \{\overline{\pi}_P(y)\}\}$

$= \min\{f(\overline{\pi}_P)(x\prime), \ f(\overline{\pi}_P)(y\prime)\}.$

And

$f(\overline{\vartheta}_P)(x\prime - y\prime) = \inf\limits_{f(z)=x\prime-y\prime} \{\overline{\vartheta}_P(z)\}$

$\leq \inf\limits_{f(x)=x\prime, f(y)=y\prime} \{\overline{\vartheta}_P(x - y)\}$

$\leq \inf\limits_{f(x)=x\prime, f(y)=y\prime} \{\max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y)\}\}$

$= \max\{\inf\limits_{f(x)=x\prime} \{\overline{\vartheta}_P(x)\}, \inf\limits_{f(y)=y\prime} \{\overline{\vartheta}_P(y)\}\}$

$$= \max\{f(\overline{\vartheta}_P)(x'), \ f(\overline{\vartheta}_P)(y')\}.$$

Next,

$$f(\overline{\pi})(x'\alpha y'\beta z') = \sup_{f(w)=x'\alpha y'\beta z'} \{\overline{\pi}(w)\}$$

$$\geq \sup_{f(x)=x', f(y)=y', f(z)=z'} \{\overline{\pi}(x\alpha y\beta z)\}$$

$$\geq \sup_{f(x)=x', f(y)=y', f(z)=z'} \{\min\{\overline{\pi}(x), \overline{\pi}(y), \overline{\pi}(z)\}\}$$

$$= \min\{\sup_{f(x)=x'} \{\overline{\pi}(x)\}, \sup_{f(y)=y'} \{\overline{\pi}(y)\}, \sup_{f(z)=z'} \{\overline{\pi}(z)\}\}$$

$$= \min\{f(\overline{\pi})(x'), f(\overline{\pi})(y'), f(\overline{\pi})(z')\}.$$

And

$$f(\overline{\vartheta}_P)(x'\alpha y'\beta z') = \inf_{f(w)=x'\alpha y'\beta z'} \{\overline{\vartheta}_P(w)\}$$

$$\leq \inf_{f(x)=x', f(y)=y', f(z)=z'} \{\overline{\vartheta}_P(x\alpha y\beta z)\}$$

$$\leq \inf_{f(x)=x', f(y)=y', f(z)=z'} \{\max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y), \overline{\vartheta}_P(z)\}\}$$

$$= \max\{\inf_{f(x)=x'} \{\overline{\vartheta}_P(x)\}, \inf_{f(y)=y'} \{\overline{\vartheta}_P(y)\}, \inf_{f(z)=z'} \{\overline{\vartheta}_P(z)\}\}$$

$$= \max\{f(\overline{\vartheta}_P)(x'), f(\overline{\vartheta}_P)(y'), f(\overline{\vartheta}_P)(z')\}.$$

Therefore $f(B)$ is an interval valued Pythagorean fuzzy weak bi-ideal of S.

7| Conclusion

In this paper, we discuss Pythagorean fuzzy weak ideal, Pythagorean fuzzy weak bi-ideal, Homomorphism of Pythagorean fuzzy weak ideal and weak bi-ideal. An interval valued Pythagorean fuzzy ideal, interval valued Pythagorean fuzzy weak bi-ideal, Homomorphism of interval valued Pythagorean fuzzy weak ideal and bi-ideal in gamma near ring are studied and investigated some properties with suitable examples.

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References

[1] Akram, M. (2018). *Fuzzy lie algebras, infosys science foundation series in mathematical sciences*. Springer.

[2] Atanassov, K. (2016). Intuitionistic fuzzy sets. *International journal bioautomation*, *20*, 1.

[3] Booth, G. L. (1988). A note on Γ-near-rings. *Stud. Sci. Math. Hung*, *23*, 471-475.

[4] Chinnadurai, V., Arulmozhi, K. R. I. S. H. N. A. N., & Kadalarasi, S. (2017). Characterization of fuzzy weak bi-ideals of Γ-near-rings. *International journal of algebra and statistics*, *6*(1-2), 95-104.

[5] Chinnadurai, V., Arulmozhi, K., & Kadalarasi, S. (2017). Interval valued fuzzy weak bi-ideals of $ Gamma $-near-rings. *Journal of linear and topological algebra (JLTA)*, *6*(03), 223-236.

[6] Chinnadurai, V., Arulmozhi, K., & Kadalarasi, S. (2018). Interval valued fuzzy ideals of Γ-near-rings. *Bulletin of the international journal mathematics virtual institute*, *8*, 301-314.

[7] Chinnadurai, V., & Kadalarasi, S. (2016). Fuzzy weak bi-ideals of near-rings. *Annals of fuzzy mathematics and informatics*, *11*(5), 799-808.

[8] Cho, Y. U., Chelvam, T. T., & Jayalakshmi, S. (2007). Weak bi-ideals of near-rings. *The pure and applied mathematics*, *14*(3), 153-159.

[9] Jun, Y. B., SAPANCI, M., & ÖZTÜRK, M. A. (1999). Fuzzy ideals in gamma near-rings. *Turkish journal of mathematics*, *22*(4), 449-460.

[10] Kaviyarasu, M., Indhira, K., & Chandrasekaran, V. M. (2017). Fuzzy sub algebras and Fuzzy K-ideals in INK-algebras. *International journal of pure and applied mathematics*, *113*(6), 47-55.

[11] Kaviyarasu, M., Indhira, K., & Chandrasekaran, V. M. (2020). Fuzzy p-ideal in INK-Algebra. *Journal of Xi'an University of architecture and technology*, *12*(3), 4746-4752.

[12] Kaviyarasu, M., Indhira, K., & Chandrasekaran, V. M. (2019). Direct product of intuitionistic fuzzy K-ideals of INK-algebras. *Materials today: proceedings*, *16*, 449-455.

[13] Kim, S. D., & Kim, H. S. (1996). On fuzzy ideals of near-rings. *Bulletin of the Korean mathematical society*, *33*(4), 593-601.

[14] Manikantan, T. (2009). Fuzzy bi-ideals of near-rings. *Journal of fuzzy mathematics*, *17*(3), 659-671.

[15] Meenakumari, N., & Chelvam, T. T. (2011). Fuzzy bi-ideals in gamma near-rings. *Journal of algebra discrete structures*, *9*(1-2), 43-52.

[16] Narayanan, A., & Manikantan, T. (2005). (∈,∈∨ q)-fuzzy subnear-rings and (∈,∈∨ q)-fuzzy ideals of near-rings. *Journal of applied mathematics and computing*, *18*(1), 419-430.

[17] Pilz, G. (2011). *Near-rings: the theory and its applications*. Elsevier.

[18] Satyanarayana, B. (1984). *Contributions to near-ring theory* (Doctrol Thesis, Nagarjuna University, India).

[19] Abou-Zaid, S. (1991). On fuzzy subnear-rings and ideals. *Fuzzy sets and systems*, *44*(1), 139-146.

[20] Rao, C., & Swaminathan, V. (2010). Anti-homomorphism in fuzzy ideals. *International scholarly and scientific research and innovation*, *4*(8), 1211-1214.

[21] Rao, M. M. K., & Venkateswarlu, B. (2015). Anti-fuzzy k-ideals and anti-homomorphism of Γ-semirings. *J. Int. Math. Virtual Inst*, *5*, 37-54.

[22] Chelvam, T. T., & Meenakumari, N. (2004). Bi-Ideals of gamma near-rings. *Southeast Asian bulletin of mathematics*, *27*(6), 983-988.

[23] Thillaigovindan, N., Chinnadurai, V., & Coumaressane, S. (2016). T-fuzzy subsemigroups and T-fuzzy ideals of regular Γ-semigroups. *Annals of fuzzy mathematics and informatics*, *11*(4), 669-680.

[24] Yager, R. R. (2013, June). Pythagorean fuzzy subsets. In *2013 joint IFSA world congress and NAFIPS annual meeting (IFSA/NAFIPS)* (pp. 57-61). IEEE.

[25] Yager, R. R. (2013). Pythagorean membership grades in multicriteria decision making. *IEEE Transactions on fuzzy systems*, *22*(4), 958-965.

[26] Zadeth, L. A. (1965). Fuzzy sets. *Information and control*, *8*(3), 338-353.

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