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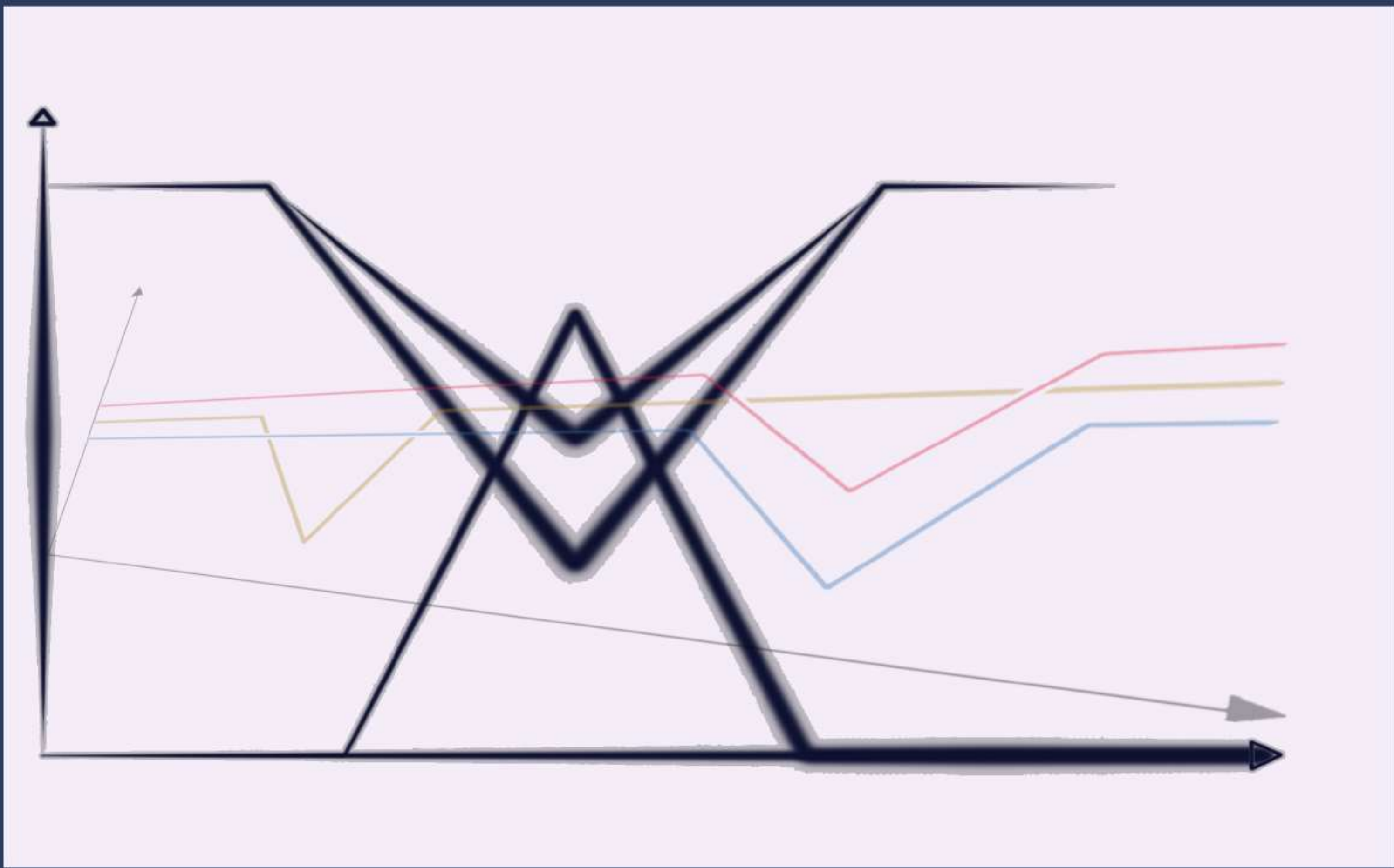

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Fuzzy Simple Linear Regression Using Gaussian Membership Functions Minimization Problem

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Abstract

Under the additional assumption that the errors are normally distributed, the Ordinary Least Squares (OLS) method is the maximum likelihood estimator. In this paper, we propose, for a simple regression, an estimation method alternative to the OLS method based on a so-called Gaussian membership function, one that checks the validity of the verbal explanation suggested by the observer. The fuzzy estimation approach demonstrated here is based on a suitable framework for a natural behavior observed in nature. An application based on a group of MENA countries in 2015 is presented to estimate the employment poverty relationship.

Keywords: Mathematical modeling, Fuzzy regression, Gaussian fuzzy responses, Gaussian membership function.

1 | Introduction

The simple linear regression concerns sample points with one independent variable and one dependent variable and finds a linear function (a non-vertical straight line) that, as accurately as possible, predicts the dependent variable values as a function of the independent variable. One of the most used regression methods is the Ordinary Least Squares (OLS) method [4].

OLS is a type of linear least squares method for estimating the unknown parameters in a linear regression model. The OLS estimator is consistent when the regressors are exogenous and, by the Gauss Markov theorem, optimal in the class of linear unbiased estimators when the errors are homoscedastic and serially uncorrelated. Under these conditions, the method of OLS provides minimum-variance mean-unbiased estimation when the errors have finite variances. Under the additional assumption that the errors are normally distributed, OLS is the maximum likelihood estimator. The errors after modeling, however, should be normal to draw a valid conclusion by hypothesis testing.

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Our data can be normal or not; we can check data distributions to understand their behaviors. Our data might not be normal for a reason. Usually, in such cases, you may want to transform it or use other analysis methods (e.g., generalized linear models or nonparametric methods).

Belhadj [2] proves using the theory of fuzzy subsets that any dependent or independent variable follows a particular real behavior. In particular, according to this author, poverty can have a trapezoidal behavior. However, the analysis and estimation of regression must change depending on the behavior of the regression variables. For example, Belhadj and Kaabi [3] proposed a method for estimating a simple regression where the behavior of the dependent variable is trapezoidal.

This paper aims to develop further and refine another case study on the same strand of research of [3]. First, we assume our data is normal and set the so-called Gaussian membership functions. They are fuzzy versions of the classic Gaussian distribution, those who verify the validity of the verbal explanation suggested by the observer. Secondly, we construct fuzzy mathematical modelings of a simple linear regression model using these Gaussian membership functions. Fuzzy modeling approaches demonstrated here are based on mimicking a natural behavior observed in nature. Fuzzy modeling approaches demonstrated here change according to the distribution of our data. It keeps reality, unlike the OLS and any other estimation method, in its entirety.

This paper is structured as follows: Section 2 presents some unique properties of Gaussian distribution and briefly presents the OLS method using the normal distribution. Section 3 shows the difference between the Gaussian distribution and membership function. Section 4 puts forward an alternative estimation method called fuzzy regression using Gaussian membership functions. In this section, estimators of a linear fuzzy regression model are constructed, and the consistency of these estimators is established. Section 5 illustrates the use of this proposed method to estimate the employment poverty relationship in MENA in 2015. Section 6 concludes.

2 | Gaussian Distribution, OLS Method

A Gaussian distribution, said normally distributed, is a continuous probability distribution for a real-valued random variable. Its probability density function [4] is

$$(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-1}{2}\left(\frac{x-m}{\sigma}\right)^2}, x\in R .$$

m is the mean and also its median and mode, while the parameter σ is its standard deviation.

The normal distribution is important in analytic studies. It is the only distribution with zero cumulants beyond the mean and variance. It is also the continuous distribution with the maximum entropy for a specified mean and variance. And is the only distribution where the mean and variance calculated from a set of independent draws are independent of each other. The normal distribution is a subclass of the elliptical distributions. It is symmetric about its mean and is non-zero over the entire real line. It is one of the few stable distributions with probability density functions that can be expressed analytically. Many results and methods, such as propagation of uncertainty and least squares parameter fitting, can be derived analytically in explicit form when the relevant variables are normally distributed.

Let the model $y_i = a + bx_i + \varepsilon_i$. The variable that is supposed to be normally distributed is just the prediction error ε_i. Prediction error should follow a normal distribution with a mean of 0. The confidence interval and variable significance calculation are based on this assumption [4]. For example, the effect of unemployment on poverty based on a 5% significance level requires following a normal

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distribution with a mean of 0. If the error distribution significantly deviates from the mean 0 normal distribution, the effect may not actually be significant enough to explain poverty.

The least squares estimates in the case $y_i = a + bx_i + \varepsilon_i$ are given by simple formulas [4]:

$$\hat{b} = \frac{n\sum_i x_i y_i - \sum_i x_i \sum_i y_i}{n\sum_i x_i^2 - \left(\sum_i x_i\right)^2},$$

$$\hat{a} = \overline{y} - \hat{b}\overline{x}.$$

The OLS estimators $\hat{a}$ and $\hat{b}$ are Best Linear Unbiased Estimator (BLUE) and Consistent.

3 | Gaussian Membership Function

Fuzzy sets (see, e.g. [11], [21] for application of this approach) are extensions of the classical sets whose elements have degrees of membership [23]. In fuzzy set theory, classical bivalent sets are usually called crisp sets. The fuzzy set theory allows describing situations in which the data are imprecise. Fuzzy sets handle such situations by attributing degrees to which elements belong to a set, called membership function, having intervals [0,1].

The membership function fully defines the fuzzy set. It measures the degree of similarity of an element to a fuzzy set. Membership functions can: either be chosen by the user arbitrarily, based on the user's experience. Or be designed using machine learning methods (e.g., artificial neural networks, genetic algorithms). Membership functions have different shapes, mainly Triangular, Trapezoidal, and Gaussian forms [14].

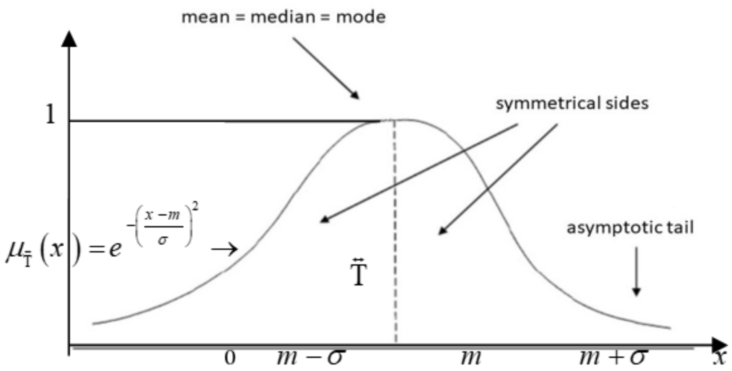

Fig. 1. Gaussian membership function.

In fuzzy logic, the Gaussian membership function is a generalization of the Gaussian distribution for classical sets. It represents the degree of truth often confused with probability. However, it is conceptually distinct because fuzzy truth represents membership in vaguely defined sets, not the likelihood of some event or condition. The general form of the Gaussian membership function is $\mu_{\tilde{T}}(x) = e^{-\left(\frac{x-m}{\sigma}\right)^2}$ (*Fig. 1*). For an element x of X, $\mu_{\tilde{T}}(x)$ quantifies the grade of membership of the element x to the fuzzy set $\tilde{T}$ (*Fig. 1*). The value 0 means that x is not a member of the fuzzy set; the value 1 indicates that x is fully a member of the fuzzy set. The values between 0 and 1 characterize fuzzy members, which belong to the fuzzy set only partially. The Gaussian membership function is employed in several domains (e.g., [16]-[18]).

4 | Fuzzy Linear Regression Models

A fuzzy linear regression model was first introduced by [21]. He formulated a linear regression model with fuzzy response data, crisp predictor data, and fuzzy parameters as a mathematical programming problem. Their approach was later improved to give birth to many other methods: linear-programming-based methods said possibilistic approach (e.g. [1], [6], [9], [10], [19], [20]), fuzzy least-squares methods (e.g. [5], [7], [8], [22]) and fuzzy minimizing method said dissemblance method [3].

Fuzzy simple linear regression is different from simple linear regression in the sense that it is a no statistical method [12]. However, in some cases, we may need to consider that the relationship expressed as $y_i = a + bx_i + \varepsilon_i$ may be fuzzy. Indeed, three cases are possible:

In the case where the predictor variable is fuzzy, but the parameters are crisp:

$$\tilde{y}_i = a + b\tilde{x}_i + \tilde{\varepsilon}_i. \tag{1}$$

The case of a crisp predictor and fuzzy parameters:

$$\tilde{y}_i = \tilde{a} + \tilde{b}x_i + \tilde{\varepsilon}_i. \tag{2}$$

And finally, the case of a fuzzy predictor and fuzzy parameters:

$$\tilde{y}_i = \tilde{a} + \tilde{b}\tilde{x}_i + \tilde{\varepsilon}_i, \tag{3}$$

where $ith, i = 1,\ldots,n,$ are fuzzy responses.

In this section, we retain a curve representing the modification of the confidence interval from 0 to 1. It may have a deflection in its slope, resulting in a flat region, as shown in *Fig. 2*. This curve is an L-R Gaussian membership function with a flat.

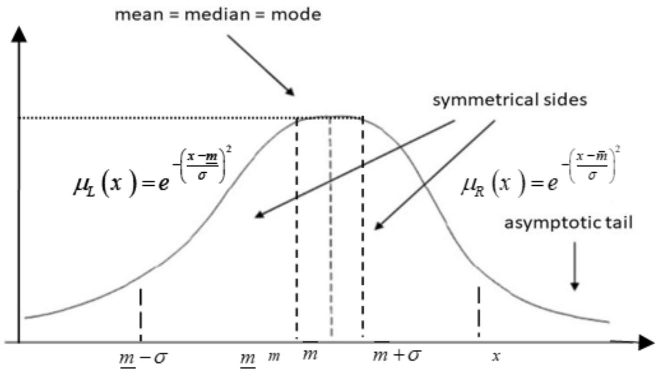

Fig. 2. L-R Gaussian fuzzy number with a flat.

We estimate *Model (2)* parameters when unknown parameters are L-R Gaussian fuzzy numbers with a flat. The *Model (2)* was also treated by [3] using Trapezoidal fuzzy numbers.

We choose four significant numbers to represent the L-R Gaussian fuzzy number nonunimodally. Let $\tilde{S} = \left(\underline{m}-\sigma, \underline{m}, \bar{m}, \bar{m}+\sigma\right)$ an L-R Gaussian fuzzy number with a Flat where $\underline{m}-\sigma$ and $\bar{m}+\sigma$ are the left and right "end" points of the corresponding bell, and $\underline{m}$ and $\bar{m}$ are the left and right "middle" points (*Fig. 2*).

The membership of the fuzzy response $\tilde{y}_i$ of the *Model (2)* is as follows:

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$$\mu_{\tilde{y}_i}(u_i)=\begin{cases} e^{-\left(\frac{u_i-\underline{m}}{\sigma}\right)^2}, & -\infty \le u_i < \underline{m}, \\ 1, & \underline{m} \le u_i < \bar{m}, \\ e^{-\left(\frac{u_i-\bar{m}}{\sigma}\right)^2}, & \bar{m} \le u_i < +\infty. \end{cases} \tag{4}$$

We suppose that the parameters of the *Model (2)*, $\tilde{a}$ and $\hat{b}$ are L-R Gaussian fuzzy numbers with a flat, where $\tilde{a}=(\underline{a}-\sigma_a,\underline{a},\bar{a},\bar{a}+\sigma_a)$ and $\tilde{b}=(\underline{b}-\sigma_b,\underline{b},\bar{b},\bar{b}+\sigma_b)$. Then according to the [13]:

$$a(+)bx=\left((\underline{a}-\sigma_a)+(\underline{b}-\sigma_b)x,\underline{a}+\underline{b}x,\bar{a}+\bar{b}x,(\bar{a}+\sigma_a)+(\bar{b}+\sigma_b)x\right),$$

from which we have

$$\forall u\in R:$$

$$\begin{aligned} \mu_{\tilde{a}(+)\tilde{b}x_i}(u_i) &= e^{-\left(\frac{u_i-((\underline{a}-\sigma_a)+(\underline{b}-\sigma_b)x_i)}{\sigma_a+\sigma_b}\right)^2} && -\infty < u_i < (\underline{a}-\sigma_a)+(\underline{b}-\sigma_b)x_i \\ &= 1 && \underline{a}+\underline{b}x_i \le u_i < \bar{a}+\bar{b}x_i \\ &= e^{-\left(\frac{u_i-((\bar{a}-\sigma_a)+(\bar{b}-\sigma_b)x_i)}{\sigma_a+\sigma_b}\right)^2} && (\bar{a}-\sigma_a)+(\bar{b}-\sigma_b)x_i \le u_i < +\infty. \end{aligned} \tag{5}$$

4.1 | Error's Gaussian Membership Function

Starting from *Model (2)*, we can write:

$$\mu_{\tilde{\varepsilon}_i}(\varepsilon_i)=\mu_{\tilde{y}_i(-)\,\tilde{a}+\tilde{b}x_i}(\varepsilon_i). \tag{6}$$

for the term to the right of *Eq. (6)*, it's about the subtraction of Gaussian fuzzy numbers, which is written as follows:

$$\mu_{\tilde{y}_i(-)\,\tilde{a}+\tilde{b}x_i}(\varepsilon_i)=\bigvee_{\varepsilon_i}\left(\mu_{\tilde{a}+\tilde{b}x_i}(a+bx_i)\wedge\mu_{\tilde{y}_i}(y_i)\right), \tag{7}$$

which corresponds to

$$\tilde{y}_i(-)\tilde{a}+\tilde{b}x_i=\left(\underline{m}-\sigma-(\bar{a}+\sigma_a)-(\bar{b}+\sigma_b)x_i,\underline{m}-\bar{a}-\bar{b}x_i,\bar{m}-\underline{a}-\underline{b}x_i,\bar{m}+\sigma-(\underline{a}-\sigma_a)-(\underline{b}-\sigma_b)x_i\right).$$

Now using *Eq. (4)*, let $\forall\alpha\in[0,1]$, $\alpha=\left(e^{-\left(\frac{u_i-\underline{m}}{\sigma}\right)^2}\right)^{(\alpha)}$, $\alpha=\underline{m}^{(\alpha)}$, $\alpha=\bar{m}^{(\alpha)}$, $\alpha=\left(e^{-\left(\frac{u_i-\bar{m}}{\sigma}\right)^2}\right)^{(\alpha)}$, from which

$$\begin{aligned} \breve{y}_{i\alpha} &= \left(y_{1i}^{(\alpha)},y_{2i}^{(\alpha)},y_{3i}^{(\alpha)},y_{4i}^{(\alpha)}\right) \\ &= \left[\alpha-e^{-\left(\frac{u_i-\underline{m}}{\sigma}\right)^2},\underline{m},\bar{m},\alpha-e^{-\left(\frac{u_i-\bar{m}}{\sigma}\right)^2}\right]. \end{aligned} \tag{8}$$

Now using *Eq. (5)*, we obtain $\alpha=\left(e^{-\left(\frac{u_i-\left(\left(\underline{a}-\sigma_a\right)+\left(\underline{b}-\sigma_b\right)x_i\right)}{\sigma_a+\sigma_b}\right)^2}\right)^{(\alpha)}$, $\alpha=\left(\underline{a}+\underline{b}x_i\right)^{(\alpha)}$, $\alpha=\left(\bar{a}+\bar{b}x_i\right)^{(\alpha)}$, $\alpha=\left(e^{-\left(\frac{u_i-\left(\left(\bar{a}-\sigma_a\right)+\left(\bar{b}-\sigma_b\right)x_i\right)}{\sigma_a+\sigma_b}\right)^2}\right)^{(\alpha)}$, from which

$$\begin{aligned}\left(\tilde{a}+\tilde{b}x_i\right)_\alpha&=\left(v_{1i}^{(\alpha)},v_{2i}^{(\alpha)},v_{3i}^{(\alpha)},v_{4i}^{(\alpha)}\right)\\&=\left[-\alpha+e^{-\left(\frac{u_i-\left(\left(\underline{a}-\sigma_a\right)+\left(\underline{b}-\sigma_b\right)x_i\right)}{\sigma_a+\sigma_b}\right)^2},\underline{a}+\underline{b}x_i,\bar{a}+\bar{b}x_i,-\alpha+e^{-\left(\frac{u_i-\left(\left(\bar{a}-\sigma_a\right)+\left(\bar{b}-\sigma_b\right)x_i\right)}{\sigma_a+\sigma_b}\right)^2}\right].\end{aligned} \tag{9}$$

We must then have

$$\breve{y}_{i\alpha}(-)\left(\breve{a}+\breve{b}x_i\right)_\alpha=\left(y_{1i}^{(\alpha)},y_{2i}^{(\alpha)},y_{3i}^{(\alpha)},y_{4i}^{(\alpha)}\right)(-)\left(v_{1i}^{(\alpha)},v_{2i}^{(\alpha)},v_{3i}^{(\alpha)},v_{4i}^{(\alpha)}\right)=\left[y_{1i}^{(\alpha)}-v_{4i}^{(\alpha)},y_{2i}^{(\alpha)}-v_{3i}^{(\alpha)},y_{3i}^{(\alpha)}-v_{2i}^{(\alpha)},y_{4i}^{(\alpha)}-v_{1i}^{(\alpha)}\right].$$

Thus if we define $y_{1i}^{(\alpha)}-v_{4i}^{(\alpha)}=\alpha-e^{-\left(\frac{u_i-\bar{m}}{\sigma}\right)^2}+\alpha-e^{-\left(\frac{u_i-\left(\left(\bar{a}-\sigma_a\right)+\left(\bar{b}-\sigma_b\right)x_i\right)}{\sigma_a+\sigma_b}\right)^2}$, $y_{4i}^{(\alpha)}-v_{1i}^{(\alpha)}=\alpha-e^{-\left(\frac{u_i-\underline{m}}{\sigma}\right)^2}+\alpha-e^{-\left(\frac{u_i-\left(\left(\underline{a}-\sigma_a\right)+\left(\underline{b}-\sigma_b\right)x_i\right)}{\sigma_a+\sigma_b}\right)^2}$, from which we have

$$\begin{aligned}&\forall u\in R:\\&\mu_{\tilde{\varepsilon}}\left(u_i\right)=e^{-\left(\frac{u_i-\left(\underline{m}-\sigma-\left(\underline{a}-\sigma_a\right)-\left(\underline{b}-\sigma_b\right)x_i\right)}{\sigma_a+\sigma_b+\sigma}\right)^2} && -\infty<u_i<\underline{m}-\sigma-\left(\underline{a}-\sigma_a\right)-\left(\underline{b}-\sigma_b\right)x_i\\&\qquad=1 && \underline{m}-\underline{a}-\underline{b}x_i\le u_i<\bar{m}-\bar{a}-\bar{b}x_i\\&\qquad=e^{-\left(\frac{u_i-\left(\bar{m}+\sigma-\left(\bar{a}+\sigma_a\right)-\left(\bar{b}+\sigma_b\right)x_i\right)}{\sigma_a+\sigma_b+\sigma}\right)^2} && \bar{m}+\sigma-\left(\bar{a}+\sigma_a\right)-\left(\bar{b}+\sigma_b\right)x_i\le u_i<+\infty\end{aligned} \tag{10}$$

The estimate of a and b of *Model (2)* consists of solving the following minimization problem:

$$\min\mu_{\tilde{\varepsilon}_i}\left(\tilde{a},\tilde{b}\right). \tag{11}$$

Using *Eq. (7)*, *Eq. (11)* becomes

$$\min\mu_{\tilde{\varepsilon}_i}\left(\tilde{a},\tilde{b}\right)=\min\bigvee_{\varepsilon_i}\left(\mu_{\tilde{a}+\tilde{b}x_i}\left(u_i\right)\wedge\mu_{\tilde{y}_i}\left(u_i\right)\right). \tag{12}$$

We show that to solve *Eq. (12)* is to solve the following system

$$\partial/\partial\underline{a},\partial\underline{b}\frac{\sum_{i=1}^{n}\left(u_i-\left(\underline{m}-\sigma-\left(\underline{a}+\sigma_a\right)-\left(\underline{b}+\sigma_b\right)x_i\right)\right)^2}{\left(\sigma_a+\sigma_b+\sigma\right)^2}=0\ . \tag{13}$$

$$\partial/\partial\bar{a},\partial\bar{b}\frac{\sum_{i=1}^{n}\left(u_i-\left(\bar{m}-\sigma-\left(\bar{a}+\sigma_a\right)-\left(\bar{b}+\sigma_b\right)x_i\right)\right)^2}{\left(\sigma_a+\sigma_b+\sigma\right)^2}=0. \tag{14}$$

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We prove that the error's membership function $\mu_{\tilde{\varepsilon}_i}\left(\tilde{a},\tilde{b}\right)$ is also the (L)-R hatched area of *Fig. 3*, given by the difference between the membership function of the observed value, $\mu_{\tilde{y}_i}\left(u_i\right)$, and the membership function of the estimated value of y_i, $\mu_{\tilde{a}+\tilde{b}x_i}\left(u_i\right)$. Therefore, solving *Eq. (12)* is equivalent to minimizing the hatched areas of *Fig. 3*, hence the following method.

4.2 | Fuzzy Numbers Gaps Square Method

Starting from *Eqs. (8)* and *(9)*, we define $\forall i = 1,\dots,n$ two Gaussian fuzzy numbers A and B, as follows: $A=\left[y_{1i},y_{2i},y_{3i},y_{4i}\right]$ and $B=\left[v_{1i},v_{2i},v_{3i},v_{4i}\right]$. Or, from *Eq. (4)*, $\forall\alpha\in\left[0,1\right]$

$\alpha = e^{-\left(\frac{u_i-\underline{m}}{\sigma}\right)^2} \Rightarrow u_i = \sigma\sqrt{-\log\alpha}+\underline{m}$, $\alpha = e^{-\left(\frac{u_i-\bar{m}}{\sigma}\right)^2} \Rightarrow u_i = \sigma\sqrt{-\log\alpha}+\bar{m}$, $A_\alpha=\left[\sigma\sqrt{-\log\alpha}+\underline{m},\sigma\sqrt{-\log\alpha}+\bar{m}\right]$. And from *Eq. (5)*, $B_\alpha=\left[\left(\left(\underline{a}-\sigma_a\right)+\left(\underline{b}-\sigma_b\right)x_i\right)+\left(\sigma_a+\sigma_b\right)\sqrt{-\log\alpha},\left(\left(\bar{a}-\sigma_a\right)+\left(\bar{b}-\sigma_b\right)x_i\right)+\left(\sigma_a+\sigma_b\right)\sqrt{-\log\alpha}\right]$.

Note that, in the right of *Fig. 2*, A and B intersect at the following two points (*Fig. 3*) $\sigma\sqrt{-\log\alpha}+\bar{m}=\left(\left(\bar{a}-\sigma_a\right)+\left(\bar{b}-\sigma_b\right)x_i\right)+\left(\sigma_a+\sigma_b\right)\sqrt{-\log\alpha}$, gives

$$\alpha_1 = e^{-\left(\frac{\left(\left(\bar{a}-\sigma_a\right)+\left(\bar{b}-\sigma_b\right)x_i\right)-\bar{m}}{\left(\sigma-\left(\sigma_a+\sigma_b\right)\right)}\right)^2}. \tag{15}$$

$\bar{m}=\bar{a}+\bar{b}x_i$ gives

$$\alpha_2 = 1. \tag{16}$$

We then have

$$\begin{aligned} y_{4i}^{(\alpha)} - v_{4i}^{(\alpha)} &= \sigma\sqrt{-\log\alpha}+\bar{m}-\left(\left(\bar{a}-\sigma_a\right)+\left(\bar{b}-\sigma_b\right)x_i\right)-\left(\sigma_a+\sigma_b\right)\sqrt{-\log\alpha} \\ &= \bar{m}-\left(\left(\bar{a}-\sigma_a\right)+\left(\bar{b}-\sigma_b\right)x_i\right)+\left(\sigma-\left(\sigma_a+\sigma_b\right)\right)\sqrt{-\log\alpha}. \end{aligned}$$

Always on the right, if we proceed to integrate $R\delta\left(A_\alpha,B_\alpha\right)$ from $\alpha=0$ to $\alpha=1$, we obtain a distance $R\delta\left(A,B\right)$ by the summation of distances: $R\delta\left(A,B\right)=\int_{\alpha=0}^{1}R\delta\left(A_\alpha,B_\alpha\right)d\alpha.$

It is a distance between two Gaussian fuzzy numbers A and B.

$$\begin{aligned} R\delta\left(A,B\right) &= \int_{\alpha=0}^{\alpha_1}\left[\bar{m}-\left(\left(\bar{a}-\sigma_a\right)+\left(\bar{b}-\sigma_b\right)x_i\right)+\left(\sigma-\left(\sigma_a+\sigma_b\right)\right)\sqrt{-\log\alpha}\right]d\alpha \\ &+\int_{\alpha_1}^{1}\left[\bar{m}-\left(\left(\bar{a}-\sigma_a\right)+\left(\bar{b}-\sigma_b\right)x_i\right)+\left(\sigma-\left(\sigma_a+\sigma_b\right)\right)\sqrt{-\log\alpha}\right]d\alpha \end{aligned} \tag{17}$$

The calculation of *Eq. (17)* shows

$$R\delta\left(A,B\right)=\left[\bar{m}-\left(\left(\bar{a}-\sigma_a\right)+\left(\bar{b}-\sigma_b\right)x_i\right)\right]\left(\left[\alpha\right]_{\alpha=0}^{\alpha_1}+\left[\alpha\right]_{\alpha_1}^{1}\right)+\left(\sigma-\left(\sigma_a+\sigma_b\right)\right)\int_0^1\left[\sqrt{-\log\alpha}\right]d\alpha.$$

And,

$$R\delta\left(A,B\right)=\left[\bar{m}-\left(\left(\bar{a}-\sigma_a\right)+\left(\bar{b}-\sigma_b\right)x_i\right)\right]\left(\left[\alpha\right]_{\alpha=0}^{\alpha_1}+\left[\alpha\right]_{\alpha_1}^{1}\right)+\frac{\sqrt{\pi}}{2}\left(\sigma-\left(\sigma_a+\sigma_b\right)\right). \tag{18}$$

For Gaussian fuzzy numbers A and B, the distance $R\delta(A,B)$ is the sum of areas hatched in *Fig. 3*.

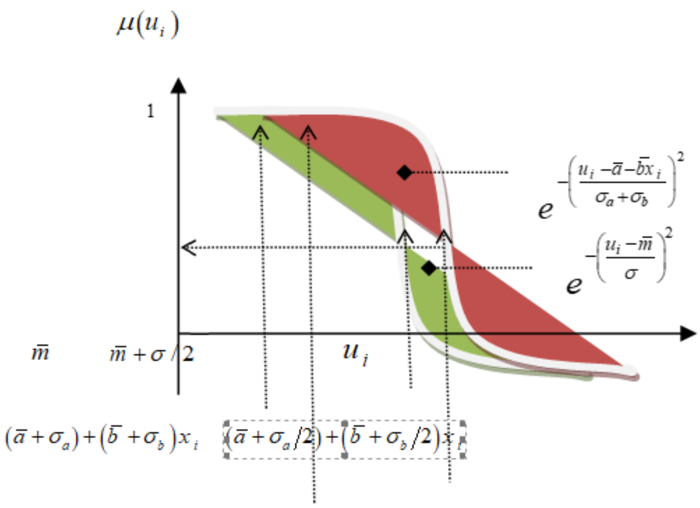

Fig. 3. R Gaussian fuzzy number with a flat.

The estimate of a and b of *Model (2)* consists of solving the following minimization problem:

$$\min\sum_i R\delta^2(A,B)=\min\sum_i R\delta^2(\tilde{a},\tilde{b}). \tag{19}$$

Problem (19) is equivalent to the *Problem (14)*. Its resolution consists of solving the *System (20)*.

$$\begin{cases} \dfrac{\partial R\delta^2(\tilde{a},\tilde{b})}{\partial\bar{a}}=0, \\ \dfrac{\partial R\delta^2(\tilde{a},\tilde{b})}{\partial\bar{b}}=0. \end{cases} \tag{20}$$

The solution of *Eq. (20)* is

$$\begin{cases} \bar{a}+\bar{b}\bar{X}=\dfrac{\sqrt{\pi}}{2}\left(\sigma-\sigma_a-\sigma_b\right)+\bar{m}+n\sigma_a+\bar{X}\sigma_b, \\ \bar{a}\bar{X}+\bar{b}\overline{X^2}=\dfrac{\sqrt{\pi}}{2}\left(\sigma-\sigma_a-\sigma_b\right)\bar{X}-\bar{m}\bar{X}+\bar{X}\sigma_a+\overline{X^2}\sigma_b. \end{cases} \tag{21}$$

Which gives

$$\bar{b}=\frac{-2\bar{m}\bar{X}+(1-n)\bar{X}\sigma_a}{\sigma_X^2}+\sigma_b. \tag{22}$$

And

$$\bar{a}=\frac{\sqrt{\pi}}{2}\left(\sigma-\sigma_b\right)+\frac{\bar{m}\left(\sigma_X^2+2\bar{X}^2\right)+\sigma_a\left[\left(n-\sqrt{\pi}/2\right)\sigma_X^2-(1-n)\bar{X}^2\right]}{\sigma_X^2}. \tag{23}$$

This paper's fuzzy estimators of the linear regression proposed are solutions to reality modeling. They may be of some use because of the complexity of phenomena, such as the social economy and the influence of various uncertain factors existing in the boundary environment around the phenomena. As justification for the proposed fuzzy modeling, we estimate the employment poverty relationship among a group of MENA countries in the following section in 2015. This estimation justifies the relationship unclear between poverty and unemployment.

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5 | Employment Poverty Relationship Using Fuzzy Numbers Gaps Square Method

We start from the observation that employment and the access that the poor have to decent earning opportunities will be crucial determinants of poverty reduction. This observation brings us back to estimating the relationship between the poverty headcount ratio at $1.90 and the unemployment rate in 2015 for a group of developing countries from the MENA region.

The estimation of this relation by the OLS method shows the following results

$$\begin{aligned} & P_i = 17{,}37 - 0{,}81U_i \\ & \quad (2{,}89)(-1{,}94) \\ & (R^2 = 0{,}201;\ N = 23;\ \text{t values in parentheses}). \end{aligned} \tag{24}$$

Looking at the results of *Eq. (24)*, we find that the application of the OLS method show, as [23], a negative relationship which can be explained by: the unemployment rate is generally higher in developing middle-income countries. In comparison, the poverty rate is higher in developing low-income countries [23]. But we show below that by changing the method, the result changes.

We retain the same sample from *Eq. (24)* and apply the fuzzy numbers gaps square method. We calculate, however, the solutions *Eq. (22)* and *Eq. (23)*. We, therefore, obtain the following results.

Table 1. Employment poverty relationship using the fuzzy numbers gaps square method.

	a	b	R^2
The left-Gaussian-fuzzy-number-estimates	23,45 (3,06)	-0,58 (-2,03)	0,879
The right-Gaussian-fuzzy-number-estimates	14,06 (3,97)	0,43 (3,77)	0,901

$R^2 = SSR / SST$ where SSR is the sum of squares due to regression, SST is the total sum of squares

As shown in *Table 1*, the value of the constant term depicts that when the unemployment rate is assumed to be zero, the incidence of poverty will still be visible, thus insinuating that the unemployment rate is not the only factor responsible for the incidence of poverty.

We also find that the parameter measuring unemployment is not always negative as in *Eq. (24)*. The relationship between the incidence of poverty and unemployment is positive in the right-Gaussian-fuzzy-number *Fig. 2*, but in the left-Gaussian-fuzzy-number, the relationship is negative. These results justify the relationship unclear between poverty and unemployment. However, being unemployed usually results in falling in one's living standard due to the absence of income, and one can be employed and still be poor.

The relationship between the incidence of poverty and unemployment is, then, subject to the linked labor force and income statistics path tracing. For example, the problem of interpretation becomes complicated among workers who experience an employment problem but whose family income does not fall below the poverty level.

However, the realistic depiction of the phenomenon is critical. For many natural effects, including fog, steam, smoke, and fire, a recent survey by [25] gives an overview of the best-known methods for importance sampling. Our approach illustrates an alternative importance sampling strategy that can yield more robust estimators. It is based on variable distribution to calculate the relationship between them validly.

Both positive and negative relationship between unemployment and poverty then returns to the distribution of each variable. A possible situation shows that MENA's 'working poor' still live in poverty.

The study results show that the employment rate has a statistically significant impact on reducing non-extreme poverty for the MENA sample. The results confirm that, in the right part of the distribution, a one-unit increase in the employment rate results in a fall in the poverty headcount ratio by about 43% for the MENA sample. Therefore, in the right part of the distribution, employment is a vector to lift out of poverty because job quality is sufficient, including adequate earnings, job security, and safe working environments. In the left part of the distribution, the working poverty rate reveals the proportion of employed people living in poverty despite being employed, implying that their employment-related incomes are insufficient to lift out of poverty and ensure decent living conditions.

The relationship between employment and poverty depends significantly on the extent to which decent work is ensured in the labor market. This means that having a job is not enough to keep out of poverty, pointing to job quality issues, particularly the inadequacy of earnings.

6 | Conclusion

In this paper, we propose new non-statistical methods - fuzzy regressions - as alternative methods of ordinary regression analysis. These methods deal with fuzziness; they estimate the parameters of a fuzzy simple regression model for the case of fuzzy parameters using Gaussian membership functions nonunimodally.

These methods are desirable since they are based on an integral abstraction of reality while keeping the same process as the classical linear regression. Moreover, the calculation technique for these methods keeps a Gaussian distribution in a real context. We can, in future work, extend our methodology for the case where predictor and parameters are fuzzy.

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Competing Interests

The authors have no relevant financial or non-financial interests to disclose.

Author Contributions

The author contributed to the study's conception and design. The author read and approved the final manuscript.

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Fuzzy Cognitive Study on Post Pandemic Impact on Occupational Shift in Rural Areas

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Abstract

The pandemic has created a wide range of impacts on the livelihood of the people especially in their occupation and income generation. The horrific pandemic impacts have caused the populace to switch their occupations for the sake of their livelihood sustainability. This research works aims in determining the impacts of the occupational shifts especially in case of rural populace. The decision-making method of Fuzzy Cognitive Maps (FCM) is used in combinations with the statistical data collection methods of survey methodology, participatory approach and multi stage purposive sampling. It is observed that a significant percentage of people have shifted from their occupation and the occupational shifts have impacts on the personal, economic, social and health dimensions of the rural populace. The factors under each dimension and their inter associational impacts are also determined using the method of FCM and FCM Expert software. Based on the findings of the research work, it is very evident that the occupational shifts have created a lot of impacts on the livelihood of the rural populace and also each of the person has experienced the impacts more personally. The societal contribution of the research lies in communicating the results and inferences to the concerned administrators so as to facilitate the affected rural populace in getting back to their primary occupation.

Keywords: Fuzzy cognitive maps, Occupational shift, Pandemic impact.

1 | Introduction

Fuzzy Cognitive Maps (FCM) developed by Kosko [1] is an extension of cognitive maps. FCMs are basically directed graphs with fuzzy weights that are widely applied in making optimal decisions. The four primary functions of FCM are explanatory, prediction, reflective, strategic. In a decision-making environment involving various factors, the FCM modeling considers these factors as the nodes and the relationship between the factors are represented using edges. FCM is explanatory as it builds the cause and impact relationship representations; it is predictive as it forecasts the impacts of new occurrences; it is reflective as it always possess space for making changes and it is strategic as it handles the complex situation with its precise description.

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Inspite of several decision-making methods, FCM is the choice to handle a multifaceted decision-making situation. FCM models are developed using deductive and inductive modelling approaches. Expert based methods under deductive modelling and computational methods under inductive modeling are used to construct FCM models. The development of learning algorithms of FCM has labelled FCM models as supervised learning fuzzy neural systems in the point of view of Artificial Intelligence.

FCM has been extended to accommodate and to increase the magnitude of flexibility and adaptability in solving highly complex problems. Researchers in FCM have contributed to the extensions of FCM. Carvalho and Tome [4] developed Rule based FCM, Salmeron [5] introduced fuzzy grey cognitive maps, Iakovidis and Papageorgiou [6] developed Intuitionistic FCM, Miao et al. [3]. developed Dynamical Cognitive Networks, Aguilar [16] proposed dynamic random FCM, Cai et al. [8]. constructed evolutionary FCM, Wei et al. [9]. built fuzzy time cognitive maps, Song et al. [17] developed fuzzy rules incorporated in FCM, Ruan et al. [10]. developed belief-degree distributed FCM, Chunying et al. [18] developed rough cognitive maps, Acampora et al. [11]. developed Time Automara-based FCM. Kandasamy and Smarandache [7] introduced neutrosophic cognitive maps, Martin and Smarandache [13] developed Plithogenic Cognitive Maps. FCM models are extended to increase the reliability and feasibility of decision making.

FCMs are widely applied in many fields of science and technology. The bountiful applications of FCM in behavioural sciences, medicine, telecommunication, engineering, production systems, information and technology management, education business and management are highly noteworthy. The FCM models are not only used in handling scientific issues or problems in the domain areas of science, but also commonly applied to deal with political, social, economic and strategic issues. FCMs are used to model the social problems and social researchers have applied FCM modelling approaches. Vasantha et al. [12] has dealt the issues of unemployment, socio-economic distress and its impacts on the life of mankind using FCM models. The other FCM models dealing with social aspects are associated with the dimensions of education, climate change and other related facets. In all these FCM models the associational impacts between the factors of the problems are analysed and also the factors are not grouped.

Presently the pandemic situation has accelerated the construction of FCM models to determine the cause and effect of COVID 19 Peter Groumpos [14] has developed FCM model to determine the cause and effect impact of the symptoms of COVID. The FCM model is dealt with symptom-disease aspect. Goswami et al. [15]. have applied FCM approach in determining the impact of COVID on small holder agricultural systems and to develop new strategies. It is inferred from the models that FCM is applied both in scientific and social sense to handle COVID issues and impacts. But in the model developed by Goswami et al. [15], FCM is used only as a tool to find the impact between the factors of the decision-making problem and the sub factors were not discussed. Based on the social utility of COVID FCM models, in this research work the impacts of occupational shift on the rural populace caused by pandemic is modelled using FCM. The researchers have made an extensive study on the impacts of COVID on employment but the literature on their occupational shifts of rural populace are sparse. This has motivated the authors to model the impacts of occupational shifts using FCM deductive modelling approach.

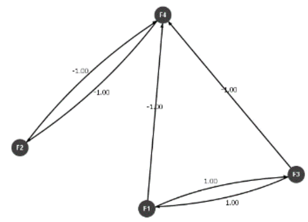
The paper is segmented into the following sections, Section 2 presents the fundamentals of FCM, Section 3 consists of FCM model, Section 4 discusses the results and the last sections concludes the work.

2 | Fundamentals of FCM

FCM is a directed graph comprising of nodes and edges representing the factors and their associations respectively. FCM are the extensions of cognitive maps in which the weights of the relationship assume fuzzy values rather than crisp values. In a simple FCM, the weights assumes values from the set {-1,0,1}. The value -1 signifies the negative associational impact between the factors, the value signifies the positive associational impact between the factors and the value 0 represent the null associations. But in reality the values -1, 0, and 1 alone cannot be used as benchmark to represent the relational impacts as these values are used only at times of pure existence of associational or relational impacts, but where as in real life situation, the chances of complete relational impacts are very less as the dominance of somewhat existence of relational impacts exist. In almost every time the existence of associational impact may differ in magnitude. At some instances it may be high, very high, moderate, less, very less. To handle such instances, the weighted FCM that assumes the relational weight in the interval range [-1,1] shall be used.

Let us consider a real life example of a decision making environment characterized by the factors contributing to weight management. The prime factors that are taken into account are F1 balanced diet, F2 physical exercises and F3 high hydration levels, F4 reduction of cholesterol levels. Let us try to determine the inter associational impacts between these four factors intuitively. The graphical representation of FCM is the respective connection matric of the above directed graph as shown below.

	F1	F2	F3	F4
F1	0	0	1	-1
F2	0	0	0	-1
F3	1	0	0	-1
F4	0	-1	0	0

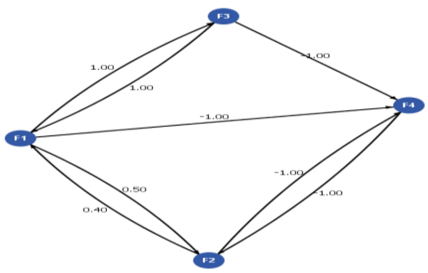


If balanced diet is maintained then the cholesterol levels will be reduced, so F1 has negative relational impacts on F4. Similarly F2 and F3 have also negative associational impacts on F4. Also F1 has positive associational impact on F3, as a good balance diet will certainly enhance high hydration levels. At a quick glance, the associational impacts between the factors F1 and F3 on F2 seems to be nil, but on profound analysis, there lies associational impacts as taking balanced diet and maintaining high hydration levels, the stamina is sustained at times of physical exercises. So in this case the direct relational impact is not represented but the indirect association is represented using fuzzy values.

It is inferred from the connection matrix that the representations using crisp weights -1,0 and 1 have focussed only on few inter associational impacts. Some of the factors and their inter associational impacts are not taken into account.

The representations using fuzzy weights have facilitated to accommodate more number of factors and their inter associations, but not all.

	F1	F2	F3	F4
F1	0	0.5	1	-1
F2	0.4	0	0	-1
F3	1	0	0	-1
F4	0	-1	0	0



The same representations shall be made using linguistic variable. In this case all the factors have been taken into account and the linguistic variables shall be quantified using fuzzy numbers.

To determine the associational impact between the factors of decision-making, let us assume the factor F1 in on position, The vector obtained is called as instantaneous vector which will be of the form (1 0 0 0). This shows that the first factor is in ON position and other factors are in OFF position. The most generalized form of representing an instantaneous vector is (a1, a2, ... an) whereaitakes the value either 0 or 1 indicating the ON and OFF position respectively. On passing the initial vector to either of the connection matrices of crisp, fuzzy and linguistic a new vector is obtained which on after applying the threshold values,the new updated vector is obtained. By repeating in the same fashion, the fixed point is attained which is the limit cycle of the dynamical fuzzy system.

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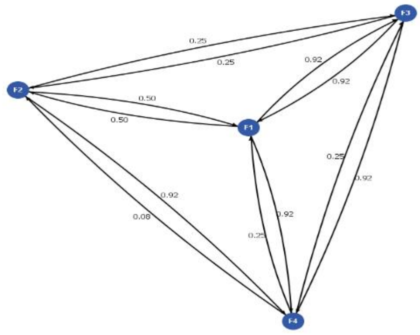
	F1	F2	F3	F4
F1	0	M	VH	VH
F2	M	0	L	VH
F3	VH	L	0	VH
F4	L	VL	L	0



3 | Methodology

In this section, the pandemic impacts on the occupational shifts are determined by applying suitable statistical methods of data collection such as survey, participatory research and multi-stage purposive sampling. Also the method of FCM is applied to determine the inter associational effects between the pandemic impacts. The place chosen for study is Usillampatti block in Madurai district in Tamil Nadu state of Indian nation. Usilampatti is a Panchayat Union consisting of 57 villages and a population of around one lakh people. The literacy rate is 63.17%. The total percentage of agricultural farmers is 11.21% and the labour percentage is 24.68%.

Among 57 villages nearly 30 villages were chosen for the study based on the percentage of working population and also on the feasibility of data collection. The data was collected from a minimum of 15 people from each village and the total respondents were 465. The method of structured interview was used to collect the data on occupational shifts at times of pandemic impact and also semi-structured interview method along with discussion method were used for convenience to collect data from the target groups on their adaptability and adoptability of new occupations.

From the data collected, it is inferred that the occupational shift at times of pandemic period has caused impacts on the dimensions of personal, social, economic and health of rural populace and it is represented in *Table 1*.

Table 1. The dimensions of personal, social, economic and health of rural populace.

Dimension	Sub-Factors
Personal	C_1 Self-satisfaction
	C_2 Acceptance of occupational change by the family members
	C_3 Adaptability to the new working environment
	C_4 Mutual support from the peer employees
	C_5 Creating flexible workplace
Social	C_6 Change in the social status
	C_7 Recognition gain in the society
	C_8 Disruption to self-identity
	C_9 Declination of self-respect and dignity
Economic	C_{10} Difficulty in accommodating the financial needs
	C_{11} Increase in Financial constraints
Health	C_{12} Physical stresses
	C_{13} Mental ailments
	C_{14} Emotional ill health

The initial linguistic connection matrix, based on the deductive approach of expert based method is shown in *Table 2*.

Table 2. The initial linguistic connection matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}
C_1	0	VH	H	VL	VL	L	H	VL	VL	VL	VL	VL	VL	L
C_2	VH	0	H	VL	M	VL	H	L	VL	VL	VL	L	VL	VL
C_M	VH	VH	0	VL	VL	VL	H	VL	VL	VL	VL	VL	VL	VL
C_4	VH	VH	H	0	VL	L	H	VL	VL	VL	VL	VL	VL	VL
C_5	VH	VH	H	VL	0	L	H	VL	L	VL	L	VL	L	L
C_6	VH	VH	H	VL	VL	0	H	VL	VL	VL	VL	VL	VL	VL
C_7	VH	VH	H	L	L	M	0	L	L	VL	L	L	L	L
C_8	VH	VH	H	VL	VL	VL	H	0	VL	VL	VL	VL	VL	VL
C_9	VH	VH	H	M	L	VL	H	VL	0	VL	VL	VL	VL	VL
C_{10}	VH	VH	H	VL	VL	VL	L	VL	VL	0	VL	VL	VL	VL
C_{11}	VH	VH	H	VL	M	VL	H	VL	VL	VL	0	VL	VL	VL
C_{12}	VH	VH	H	VL	VL	VL	H	VL	VL	VL	VL	0	VL	VL
C_{13}	VH	VH	H	VL	VL	VL	L	VL	VL	VL	VL	VL	0	VL
C_{14}	VH	VH	H	VL	VL	VL	H	VL	VL	VL	VL	VL	VL	0

The linguistic terms are quantified using triangular fuzzy numbers of the form $A = (a_1,a_2,a_3)$ where $a_1 \le a_2 \le a_3$ and the triangular fuzzy number is defuzzified using average method of $(a_1+a_2+a_3)/3$ (*Table 3*).

Table 3. The quantified triangular fuzzy numbers.

Linguistic Variable	Very Low (VL)	Low (L)	Medium (M)	High (H)	Very High (VH)
Triangular Fuzzy number Quantification	(0, 0, 0.25)	(0, 0.25, 0.50)	(0.25, 0.50, 0.75)	(0.50, 0.75, 1)	(0.75, 1,1)
Defuzzified Value	0.08	0.25	0.5	0.75	0.92

The modified connection matrix is

Table 4. The modified connection matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}
C_1	0	0.92	0.75	0.08	0.08	0.25	0.75	0.08	0.08	0.08	0.08	0.08	0.08	0.25
C_2	0.92	0	0.75	0.08	0.5	0.08	0.75	0.25	0.08	0.08	0.08	0.25	0.08	0.08
C_3	0.92	0.92	0	0.08	0.08	0.08	0.75	0.08	0.08	0.08	0.08	0.08	0.08	0.08
C_4	0.92	0.92	0.75	0	0.08	0.25	0.75	0.08	0.08	0.08	0.08	0.08	0.08	0.08
C_5	0.92	0.92	0.75	0.08	0	0.25	0.75	0.08	0.25	0.08	0.25	0.08	0.25	0.25
C_6	0.92	0.92	0.75	0.08	0.08	0	0.75	0.08	0.08	0.08	0.08	0.08	0.08	0.08
C_7	0.92	0.92	0.75	0.25	0.25	0.5	0	0.25	0.25	0.08	0.25	0.25	0.25	0.25
C_8	0.92	0.92	0.75	0.08	0.08	0.08	0.75	0	0.08	0.08	0.08	0.08	0.08	0.08
C_9	0.92	0.92	0.75	0.5	0.25	0.08	0.75	0.08	0	0.08	0.08	0.08	0.08	0.08
C_{10}	0.92	0.92	0.75	0.08	0.08	0.08	0.25	0.08	0.08	0	0.08	0.08	0.08	0.08
C_{11}	0.92	0.92	0.75	0.08	0.5	0.08	0.75	0.08	0.08	0.08	0	0.08	0.08	0.08
C_{12}	0.92	0.92	0.75	0.08	0.08	0.08	0.75	0.08	0.08	0.08	0.08	0	0.08	0.08
C_{13}	0.92	0.92	0.75	0.08	0.08	0.08	0.25	0.08	0.08	0.08	0.08	0.08	0	0.08
C_{14}	0.92	0.92	0.75	0.08	0.08	0.08	0.75	0.08	0.08	0.08	0.08	0.08	0.08	0

This is the connection matrix relating the associational impacts between the sub-factors of all the factors. The graphical representation of the associations is presented in *Fig. 1*.

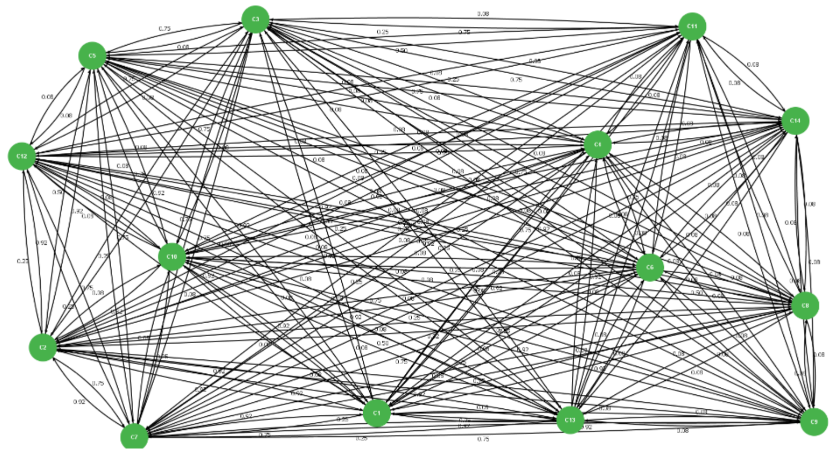


Fig. 1. Graphical representation of overall factors of FCM.

By using FCM expert software, the inference process is obtained and it is represented in *Fig. 2*

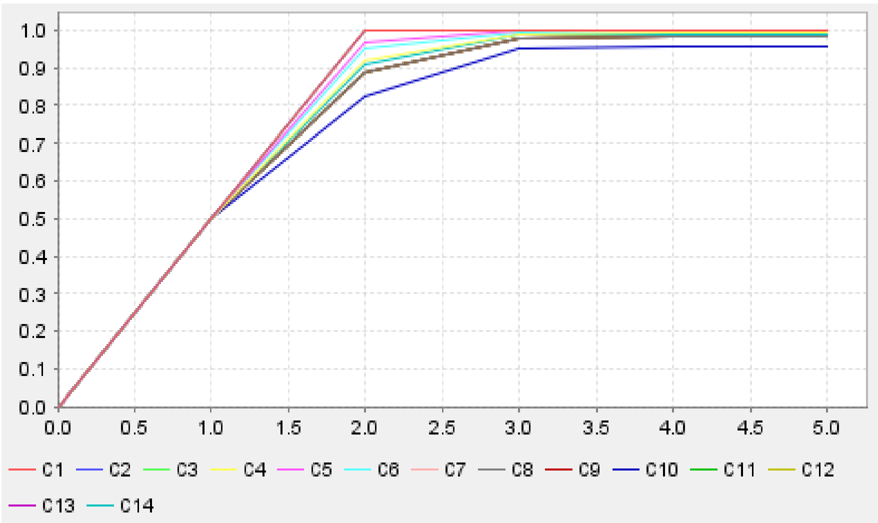


Fig. 2. Overall FCM inference process.

The interrelational impacts between the core factors are analyzed. The graphical representation of the interrelational impacts between Personal (P) and Social (S) factors and the respective inference process are presented in *Fig. 3* and *Fig. 4*, respectively.

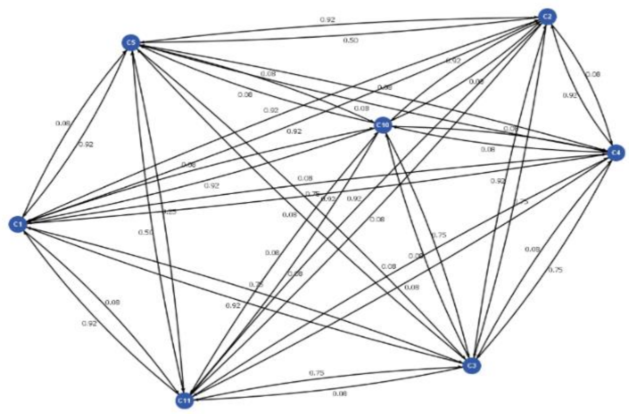


Fig. 3. Graphical representation of P&S factors of FCM.

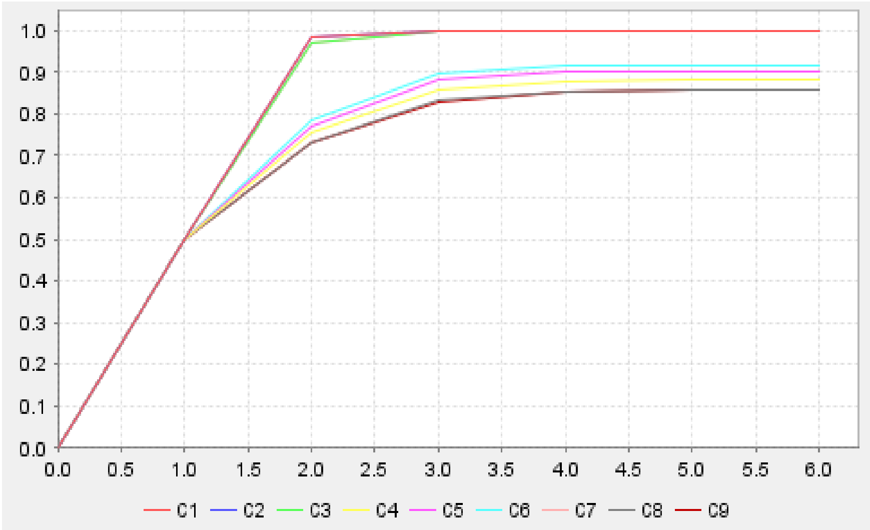


Fig. 4. FCM of P&S inference process.

The graphical representation of the interrelational impacts between Personal (P) and Economic (E) factors and the respective inference process are presented in *Fig. 5* and *Fig. 6*, respectively.

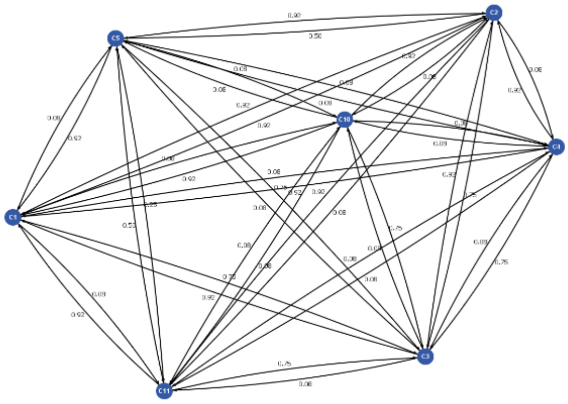


Fig. 5. Graphical representation of P&E factors of FCM.

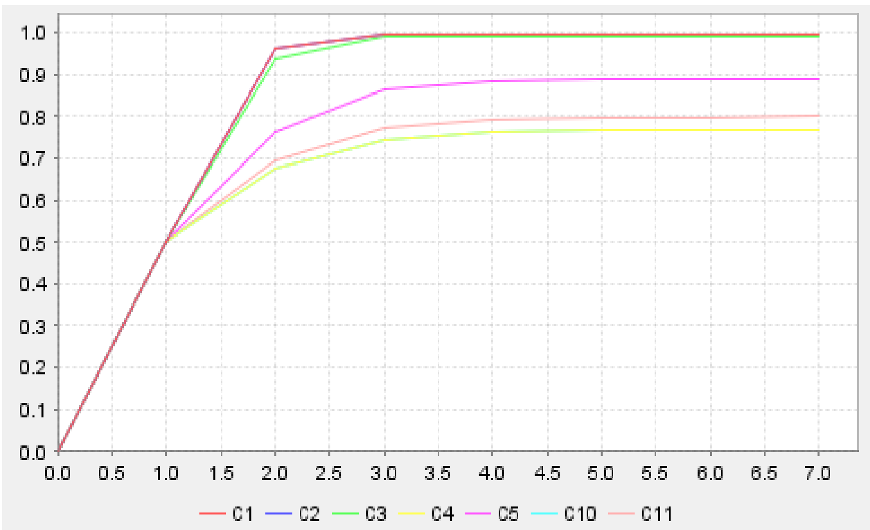


Fig. 6. FCM of P&E inference process.

The graphical representation of the interrelational impacts between Personal (P) and Health (H) factors and the respective inference process are presented in *Fig. 7* and *Fig. 8*, respectively.

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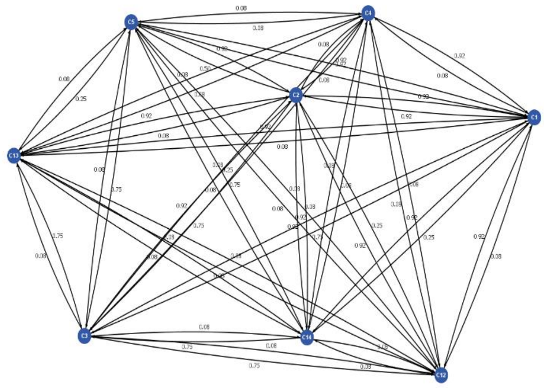

Fig. 7. Graphical representation of P&H factors of FCM.

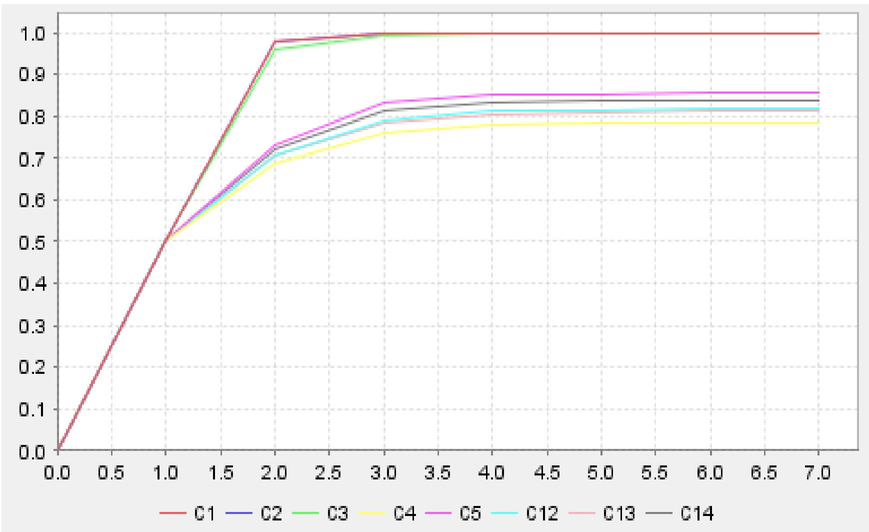

Fig. 8. FCM of P&H inference process.

The graphical representation of the interrelational impacts between Social (S) and Economic (E) factors and the respective inference process are presented in *Fig. 9* and *Fig. 10*, respectively.

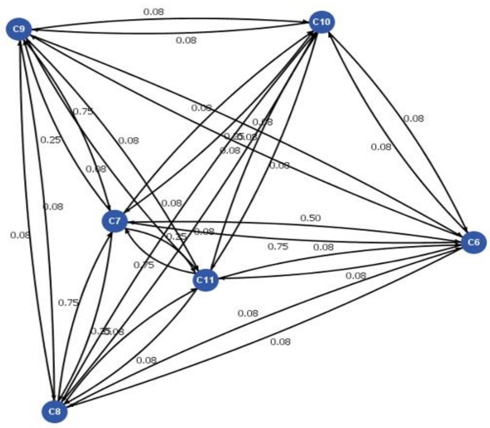

Fig. 9. Graphical representation of S&E factors of FCM.

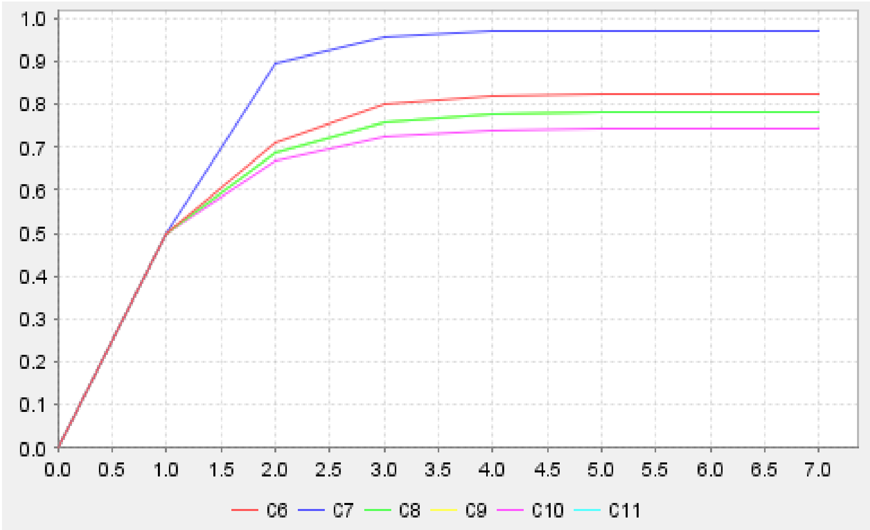

Fig. 10. FCM of S&E inference process.

The graphical representation of the interrelational impacts between Social (S) and Health (H) factors and the respective inference process are presented in *Fig. 11* and *Fig. 12*, respectively

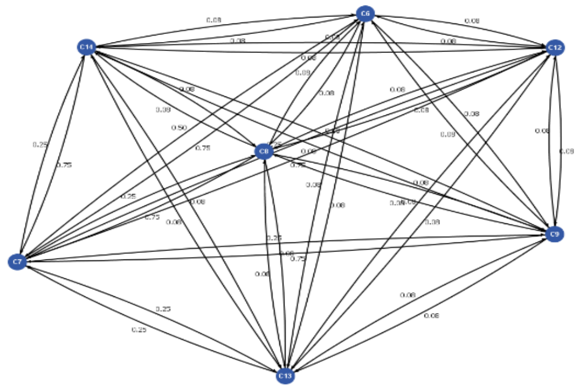

Fig. 11. Graphical representation of S&H factors of FCM.

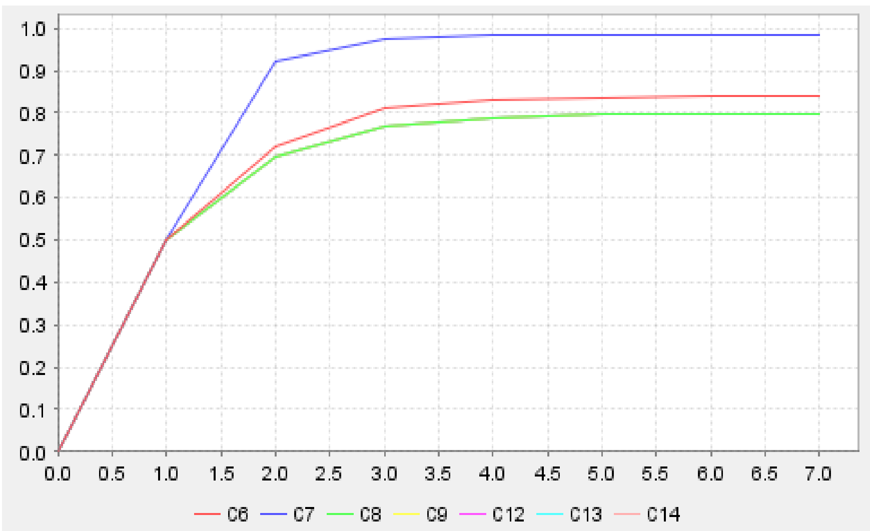

Fig. 12. FCM of S&E inference process.

The graphical representation of the interrelational impacts between Economic (E) and Health (H) factors and the respective inference process are presented in *Fig. 13* and *Fig. 14*, respectively.

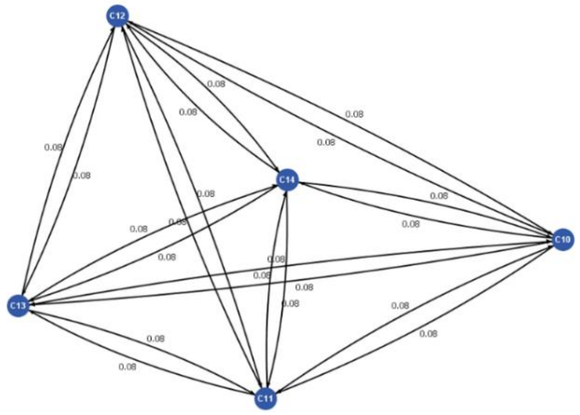

Fig. 13. Graphical representation of E & H factors of FCM.

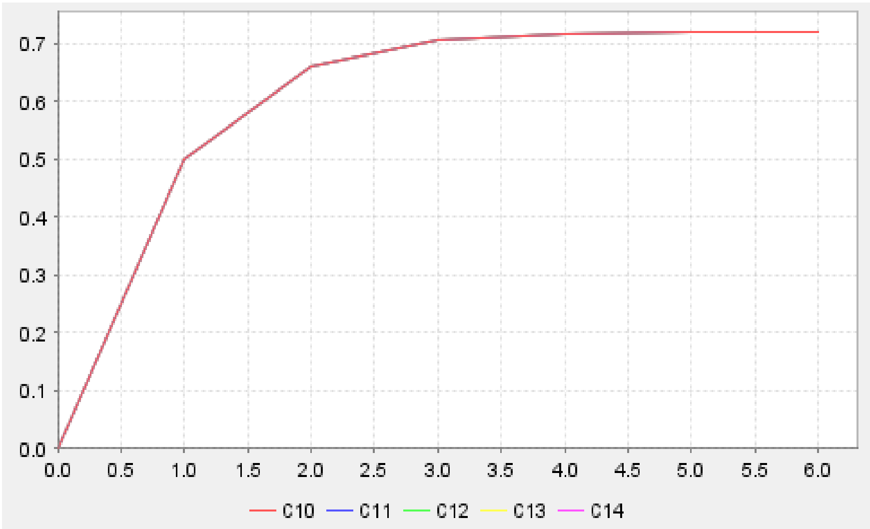

Fig. 14. FCM of E & H inference process.

4 | Discussion

Fig. 1 represents the graphical representation between all the sub-factors of the core factors and in *Fig. 1* the values taken by the concepts over a time of iterations and the convergence of the concept values are also represented. In the inference process, the Kosko [2] activation rule is used with sigmoid function and the concepts are assumed as the decision concept. It is also inferred that a steady state is arrived after a minimum number of iterations. In *Figs. 3, 5* and *7* the inter associational impacts between the sub factors of personal with social, economic and health are represented respectively and the respective *Figs. 4, 6* and *8* present the inference processes of FCM obtaining the steady state values over a period of time. Also *Figs. 9* and *11* present the graphical inter associational impacts between the sub factors of social with emotional and health sub factors. The respective FCM inference processes in *Figs. 10* and *12* presents the values of the concepts over a period of time. The inter associational impacts between economic and health were presented in *Fig. 13* and the respective inference process.

The sub-factors of the core-factors are assumed to be the concepts of FCM and in the above cases the concepts are not assigned to be decision concepts. In the later cases, on assuming one of the concepts in each of the sub-factors as decision concepts, it is inferred that the personal and social sub-factors have greater inter associational impacts between other factors and values in the below table substantiate the same. The factor C1, C2, C3 under personal core factor, C7 under social core factor, C11 under economic core factor and C14 under health core factor assumes higher values in comparison with other sub-factors of the core-factors.

Table 5. The personal and social sub-factors.

Step	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11	C12	C13	C14
0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
2	0.9985	0.9985	0.9954	0.7883	0.8334	0.8153	0.9924	0.7667	0.7667	0.735	0.7667	0.7667	0.7667	0.7816
3	1.0	1.0	0.9999	0.8964	0.9399	0.9279	0.9998	0.8793	0.8763	0.8345	0.8763	0.8793	0.8763	0.8948
4	1.0	1.0	1.0	0.9163	0.9527	0.9419	0.9999	0.8984	0.8972	0.8579	0.8972	0.8984	0.8972	0.9132
5	1.0	1.0	1.0	0.9195	0.9544	0.9437	0.9999	0.9013	0.9005	0.8623	0.9005	0.9013	0.9005	0.9159
6	1.0	1.0	1.0	0.92	0.9547	0.944	0.9999	0.9018	0.901	0.8631	0.901	0.9018	0.901	0.9163
7	1.0	1.0	1.0	0.9201	0.9547	0.944	0.9999	0.9019	0.9011	0.8632	0.9011	0.9019	0.9011	0.9163
8	1.0	1.0	1.0	0.9201	0.9547	0.944	0.9999	0.9019	0.9011	0.8633	0.9011	0.9019	0.9011	0.9163
9	1.0	1.0	1.0	0.9201	0.9547	0.944	0.9999	0.9019	0.9011	0.8633	0.9011	0.9019	0.9011	0.9163
10	1.0	1.0	1.0	0.9201	0.9547	0.944	0.9999	0.9019	0.9011	0.8633	0.9011	0.9019	0.9011	0.9163

4.1.1 | Other Findings

It is also found that around 62% of people wishes to practice their earlier occupation as they are not personally convinced by the occupational change also the social impacts hurdle them a lot in sticking on to their new occupations. The remaining percentage of the working rural population who have shifted their occupation adopted and adapted to new working environment by convincing themselves over the reasons of sustaining their family.

4.1.2 | Proposed Suggestions

I. The working rural populace who aspires to practice their earlier occupations due to personal dissatisfaction shall be supported both financially and emotionally, as their personal disassociations with the occupational shift have caused emotional chaos.

II. Counselling and rehabilitation programmes for promoting their emotional quotient shall be exclusively organized for the rural populace and the non-governmental workers together along with the volunteers shall be made involve in such remedial activities.

III. The rural populace who have adopted and adapted to the new working environment shall be set as models and be made to interact and motivate the peer workers so as to gain an exposure to the strategies of adaptation. The hosting of such interactive sessions moderated by experts will certainly facility the learning of new skill sets of getting accustomed to changes.

5 | Conclusion

The research work on the occupational shift on rural populace caused by pandemic shifts has been investigated on the grounds of the dimensions of personal, social, economic and health aspects of the rural populace. This study has deeply examined the inter associational impacts between the factors. This work shall be extended by making a comparative analysis on the impacts of occupational shifts between urban and rural populace of Madurai regions. The same impact study using FCM shall be applied to other social issues and other decision-making problems in the field of education, business, medicine and so many other fileds.

Acknowledgement

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Paper Type: Research Paper

Detection of Counterfeit Banknotes Using Genetic Fuzzy System

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Abstract

Due to developments in printing technology, the number of counterfeit banknotes is increasing every year. Finding an effective method to detect counterfeit banknotes is an important task in business. Finding a reliable method to detect counterfeit banknotes is a crucial challenge in the world of economic transactions. Due to technological development, counterfeit banknotes may pass through the counterfeit banknote detection system based on physical and chemical properties undetected. In this study, an intelligent counterfeit banknote detection system based on a Genetic Fuzzy System (GFS) is proposed to detect counterfeit banknotes efficiently. GFS is a hybrid system that uses a network architecture to fine-tune the membership functions of a fuzzy inference system. The learning algorithms Fuzzy Classification, Genetic Fuzzy Classification, ANFIS Classification, and Genetic ANFIS Classification were applied to the dataset in the UCI machine learning repository to detect the authenticity of banknotes. The developed model was evaluated based on Accuracy (ACC), Mean Absolute Error (MAE), Root Mean Square Error (RMSE), Error Mean, Error STD, and confusion matrix. The experimental results and statistical analysis showed that the classification performance of the proposed model was evaluated as follows: Fuzzy = 97.64%, GA_Fuzzy = 98.60%, ANFIS = 80.83%, GA_ANFIS = 97.72% accuracy (ACC). This shows the significant potential of the proposed GFS models for fraud detection.

Keywords: ANFIS, Counterfeit banknotes, Fuzzy inference system, Genetic fuzzy system, Genetic algorithm.

1 | Introduction

Paper money remains a common means of exchanging goods and services. With advances in digital imaging technology, color scanners, and laser printers, it is becoming easier to create high-resolution counterfeit banknotes. Counterfeit banknotes are becoming more common because they look very similar to real money and are difficult for the untrained eye to detect. Companies and organizations are losing money due to the widespread use of counterfeit banknotes. Therefore, it is important to develop an effective technique for detecting counterfeit banknotes. Counterfeit detection devices [1] exist, but they are sometimes prohibitively expensive, making counterfeit detection a major concern for financial and government institutions with little community involvement.

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The process of certifying banknotes also continues to improve as new strategies for producing counterfeit money are invented every day. The elimination of transaction problems is inextricably linked to the successful detection of counterfeit banknotes. Serious measures are needed to protect the economy from such immoral acts. Artificial intelligence approaches based on Machine Learning (ML) have recently become the de facto standard for banknote categorization difficulties [2]-[4]. The goal of ML must be to complement human decision making, but some approaches are superior at doing so. For applications that require explanation and are prone to unforeseen and unpredictable failures, ML techniques should be preferred over traditional approaches.

There are approaches to this problem based on both the latest technology and traditional computer vision methods, as well as alternative solutions. Nearest Neighbor Interpolation [5]-[7], evolutionary algorithms [8]-[10], and fuzzy systems [11]-[13] are examples of techniques that can be used. Due to its high accuracy and generalization capability for new data, it can beat both standard ML approaches and humans in classification tasks, which is compatible with learning-based methods. Various options were presented to detect counterfeit banknotes [14]-[16].

Fuzzy Inference System (FIS) is an intelligent system capable of explaining difficult facts [17]-[20]. Fuzzy systems are architectures capable of understanding language norms in decision scenarios and effectively ensuring membership in each category across a wide range of input values. The FIS parameters used in this work were optimized using the Genetic Algorithm (GA) [21]. The term "Genetic Fuzzy System" (GFS) refers to the application of a GA -optimized FIS (GFS) [22]-[27]. When it comes to detecting counterfeit banknotes, a False Positive (FP) is often more damaging than a False Negative (FN), as counterfeit banknotes can lead to greater financial losses if they are not detected.

The remainder of this paper is organized as follows: the materials and methods are described in Secion 2. The data set and the GFS are discussed in this section. Secion 3 presents the experimental results. Finally, Secion 4 contains the conclusion.

2 | Material and Methods

It is difficult to distinguish between counterfeit money and genuine banknotes. It should be possible to automate this process. Because of the accuracy with which counterfeit banknotes are produced, it is necessary to develop an algorithm that can predict whether a particular banknote is genuine or counterfeit. For this purpose, a model was created with the features obtained by analyzing the wavelet variance, wavelet skew, wavelet kurtosis and image entropy of an image sequence derived from real and imaginary banknote-like patterns. Since the variable to be estimated is a binary variable, this is a classification question (fake or legal). In this case, the objective is to simulate the possibility that a banknote is counterfeit while maintaining the functionality of its features.

2.1 | Data Set

The dataset [28] consists of 1372 samples (rows) and 5 variables (columns). Data was collected by digitizing photographs of genuine and counterfeit banknote-like samples using an industrial camera commonly used for inspecting printed products. Features were then extracted from the images using the Wavelet Transformation tool. The following variables are used as inputs to this problem: the Variance of the Wavelet Transformed Image (VWTI), the Skewness of the Wavelet Transformed Image (SWTI), the Kurtosis of the Wavelet Transformed Image (KWTI), and the Entropy of the Image (EI). The target was used as a counterfeit. It can have only two possible values: 0 (no counterfeit) or 1 (counterfeit). The proposed categorization model is shown in *Fig. 1*.

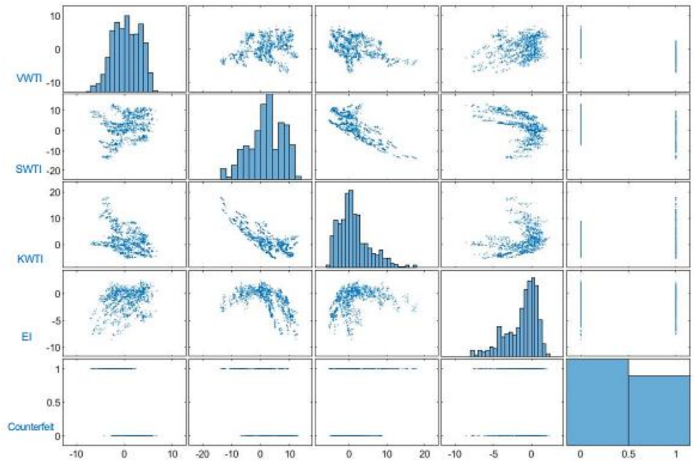

Fig. 1. The scatterplot matrix of the banknote authentication dataset [28].

2.2 | Genetic Algorithm

GA is a technique developed by Holland that is frequently used [29], [30]. To get the best performance from the FIS, its settings need to be adjusted. The procedure is to choose a random solution set for each parameter and update it until an optimal parameter set is reached. This first population is referred to as the "initial population" on GA. By far the most important component of GA is the chromosome. Each chromosome contains genes that serve as parameters for the respective task. To begin solving a problem, an initial population must be created. The responsible member then compares this response to the others based on the survival criteria. Finally, the requirements for optimization completion are set by the number of chromosomes created, and the work is typically done after a certain number of conditions is satisfied [31], [32].

2.2 | Adaptive Neural Fuzzy Inference System (ANFIS)

The FIS is an application of artificial intelligence developed by Jang [33] that mimics human reasoning. It is a simple approach to data learning that uses fuzzy principles (IF THEN) and given inputs and outputs to transform inputs and information links from strongly connected parts of the neural network into desired outputs. ANFIS uses both ANN and fuzzy inference methods to deal with non-linear and complex problems in a unified framework [34], [35]. ANFIS consists of nodes and routed paths, and all input-output values can be changed using the various parameter sets defined when designing the network. ANFIS systems can be used in conjunction with a variety of optimization techniques to minimize errors in the training phase. This goal was also achieved in the scenario used in this study [36]. ANFIS is classified into five levels. They consist of a network of neurons that communicate between the input and hidden layers and the hidden and output layers. Each layer consists of neurons constructed according to the principles of fuzzy control. *Fig. 2* illustrates the structure of the ANFIS algorithm.

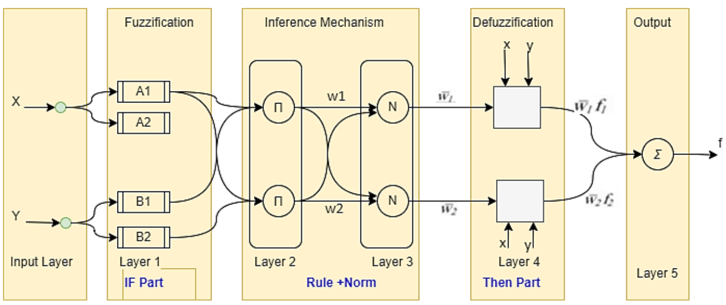

Fig. 2. Structure of ANFIS algorithm [37].

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The proposed GFS model is described with GA-optimized membership function parameters (MFs). These are updated with the release of each GA iteration. Each fuzzy set has a corresponding membership value for each variable, which is in the range [0,1].

2.3 | Genetic Fuzzy System Parameter Optimization

GAs are computer systems based on natural evolutionary processes that use operators that follow the heuristic search process in a search space containing the optimal answer to the optimization question [38]. GA is a stochastic optimization approach based on the principles of genetics and natural selection. GA [39]-[42] is a meta-heuristic optimization approach inspired by natural processes and well suited for optimizing membership function components in FIS [43]-[45]. GA is able to discover extremely large solution spaces due to probabilistic variations. GA is divided into three phases: population generation, GA operators (selection, crossover and mutation) and fitness function evaluation. GA selects participants in several ways, including tournaments and the roulette wheel. Two randomly selected individuals exchange their genes with the crossover operator to produce the next generation. Compared to crossover, the probability of a mutation occurring is low. Since it is easier to construct and debug than the round-based or tournament selection algorithms, a proportional roulette wheel selection algorithm is used in this study instead of the round-based or tournament selection algorithms. It also gives much faster results than the other two methods. One-point crossover algorithms have been developed as part of GA to transfer solution proposals or chromosomes between two different systems. The proposed GFS integrations are very useful in solving complex and nonlinear equations. *Fig. 3* shows the GFS architecture.

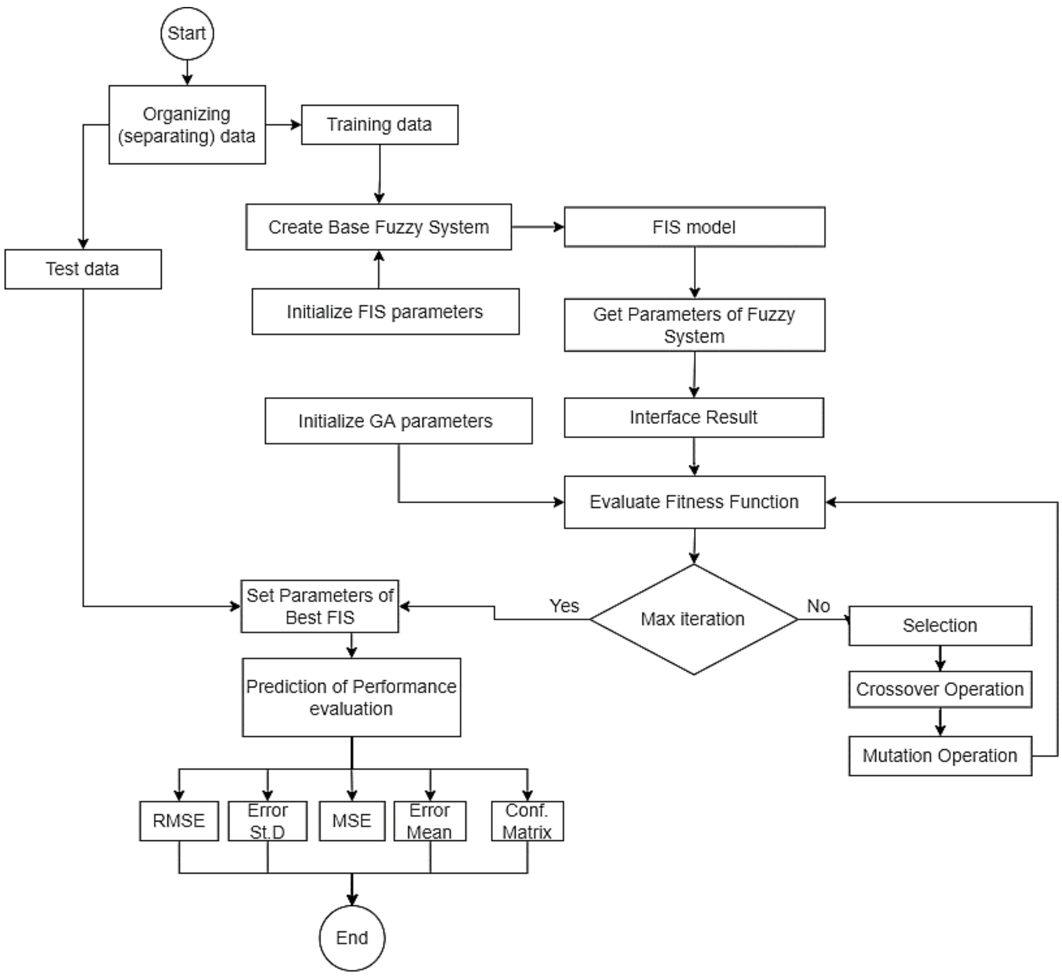

Fig. 3. Flowchart of GFS.

The initial parameters of the algorithm for the proposed model are given below.

Table 1. Initial parameters GFS.

Algorithm	Parameters	Values/types
	Epoch	80
	Error Goal	0
	Input membership shape	Gaussian
ANFIS	Output membership shape	Linear
	FIS generation	FCM
	Step Size Decrease Rate	0.9
	Step Size Increase	1.1
	Initial Step Size	1.1
GA	alpha	1
	VarMin	-(10^alpha)
	VarMax	10^alpha
	MaxIt	25
	nPop	7
	Crossover Percentage	0.7
	Mutation Percentage	0.5
	Mutation Rate	0.1
	gamma	0.2
	Selection Pressure	8
FIS	fcm_U	2
	MaxIter	100
	MinImp	1e-5

3 | Experiments

In this section, we test and compare the performance of the proposed GFS. The parameters of the FIS structure used here are optimized by a GA. The model to be built uses the decision mechanism to classify the banknotes in question as genuine or counterfeit. As a result, an expert system will increase the quality and efficiency of services while minimizing human error and the need for additional staff. As the following figures show, the technique based on training the FIS with the GA algorithm is more efficient. It has also been shown that the FIS network can be used for a wide range of problems as the GA algorithm has no limitations compared to inference-based techniques and is easy to implement.

3.1 | Evaluation Metrics

MSE, RMSE, error mean, error STD and the confusion matrix were used to assess the performance of the GFS system. Quantitative assessments of the models produced were carried out using a set of performance criteria (*Eqs. (1)-(4)*). The details of each equation can be found in the corresponding reference.

Mean squared error

The mean square error describes the closeness of a regression curve to a given collection of points. The MSE quantifies the performance of an estimator, a ML model. It is always positive, and it can be argued that estimators with an MSE close to zero perform better [46], [47].

$$\mathrm{MSE} = \frac{1}{\mathrm{N}} \sum_{\mathrm{i}=1}^{\mathrm{N}} (\mathrm{x_i} - \mathrm{y_i})^2. \tag{1}$$

Root mean square error

Root Mean Squared Error (RMSE) is a squared metric that evaluates the magnitude of an error in a ML model. It is often used to measure the difference between the expected values of the predictor and the actual values. The RMSE is the standard deviation of the estimation error. An RMSE value of 0 means that the model was error-free [48].

$$\text{RMSE} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - y_i)^2}. \tag{2}$$

Error mean

Mean error is the average error between the predicted values of a ML model and the actual values. In this context, the error is the measurement uncertainty or the difference between the estimated value and the actual value [47].

$$\text{Error Mean} = \frac{1}{N}\sum_{i=1}^{N}(x_i - y_i). \tag{3}$$

Error STD

As a method of calculation, it can be expressed as the square root of the mean of the sum of the squares of the deviations of the data from the mean, as shown in *Eq. (4)*. The variance is the square of the standard deviation [49], [50].

$$\text{Error St. D} = \sqrt{\sum_{i=1}^{N}\frac{(x_i - \bar{x}_i)^2}{N-1}}. \tag{4}$$

In the equations (*Eqs. (1)-(4)*), N is the number of data, $\bar{x}$ and $\bar{y}$ are the average of the predicted and actual values, x_i and y_i are the predicted and actual values, respectively.

The confusion matrix is divided into four groups, as shown in *Table 2*. "True Positive" (TP), "False Positive" (FP), "True Negative" (TN) and "False Negative" (FN). In a successful model, there are no false positives or negatives [51], [52].

Table 2. Confusion Matrix.

		Actual Value	
		Positive	Negative
Predicted Value	Positive	TP (true positive)	FN(false negative)
	Negative	FP(false positive)	TN (true negative)

The following equation is used to perform performance evaluation calculations based on the confusion matrix (*Eq. (5)*). For more information on this formula, see the relevant references [52].

$$\text{Accuracy (ACC)} = \frac{TP + TN}{TP + TN + FP + FN}. \tag{5}$$

3.2 | Experimental Results

In this section we discuss the results of the proposed GFS models for detecting counterfeit banknotes. GFS has been used in combination with ML techniques to develop and test categorization models. These strategies have proven successful in categorization and are used extensively. Each model was validated ten times through cross-validation. *Table 3* summarizes the accuracy of the developed GFS models by class. To compare the performance of the proposed approach, the fuzzy/ANFIS network is additionally trained with GA counterfeit banknote detection algorithms. *Table 3* compares the classification results of the developed fuzzy, GA fuzzy, ANFIS and GA ANFIS models. *Fig. 4* shows the development of the training error values (RMSE) over 50 iterations of the approaches.

Table 3. Performance indices for proposed GFS model.

Models	MSE	RMSE	Error Mean	Error STD	Accuracy (%)
Fuzzy	0.033372	0.18268	-2,07E-13	0.18275	97.64
GA_Fuzzy	0.02403	0.15502	0.012779	0.15455	98.60
ANFIS	0.65563	0.80971	0.31436	0.74647	80.83
GA_ANFIS	0.031879	0.17855	0.0078403	0.17844	97.72

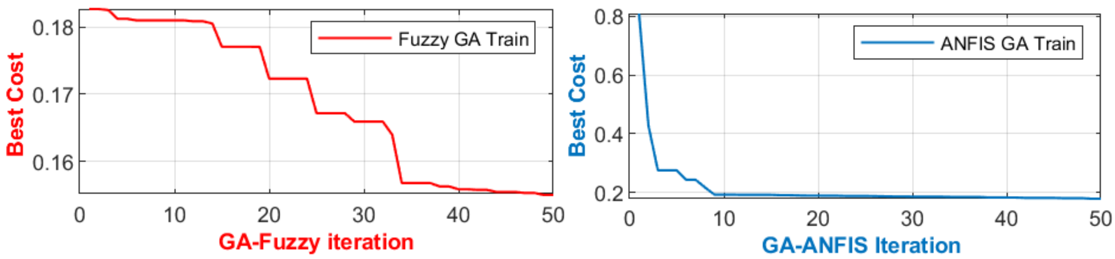

Fig. 4. Diagram of the best cost of the proposed GFS model.

Fig. 5 summarizes the classification performance obtained with the optimal parameter values derived from the simulation. The GA fuzzy model performed best here, with a classification rate of 98.6%. *Table 3* also shows the average performance of the categorization techniques and the percentage improvement compared to each other. When the GA method is used to train the ANFIS network, the classification performance increases by 20.9% compared to the regular ANFIS algorithm. It is found that the GA fuzzy classifier optimized using GA outperforms the classical fuzzy classifier by 0.98%. The improvements have shown that the GA increases the performance of the classifiers.

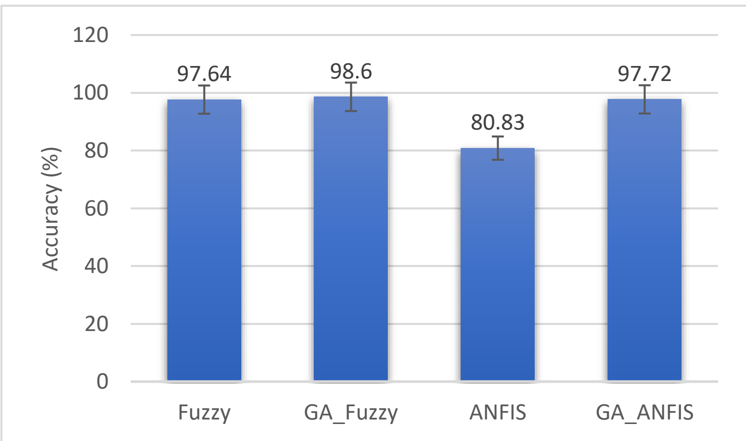

Fig. 5. Classification performance of the algorithms.

The performance metric used to evaluate the system in this case is the complexity matrix, which was discovered to be a measure of the correlation between predicted and observed values. The diagonal value of this matrix indicates the correct class, while the values outside the diagonal represent miscategorized elements. *Fig. 6* shows the confusion matrix of the proposed model.

The confusion matrix is used to analyze the results of a previously constructed classification model and to investigate errors in the mapping between real and predicted values during cross-validation. The positive and negative components in this matrix do not refer to accuracy or inaccuracy, but to the classes to be distinguished. Based on a dataset of counterfeit banknotes, this study created a model that attempts to predict whether the banknotes are counterfeit or not. When evaluating the results of the created classification model, TP, TN, FN, and FP are determined based on the matrix. TP and TN indicate the number of valid class predictions. FN and FP indicate how many inaccurate predictions the classes made in relation to each other. Here, the fuzzy classifier correctly predicted all counterfeit notes while it incorrectly predicted 32 non-counterfeit notes. The GA Fuzzy classifier has incorrectly predicted 19 non-counterfeit banknotes while it has correctly predicted all counterfeit banknotes. The GA_ANFIS classifier misclassified 30 non-counterfeit banknotes and misidentified 1 counterfeit banknote. The traditional ANFIS model, which has the lowest percentage of accuracy, incorrectly predicted 68 non-counterfeit banknotes while correctly identifying 186 counterfeit banknotes.

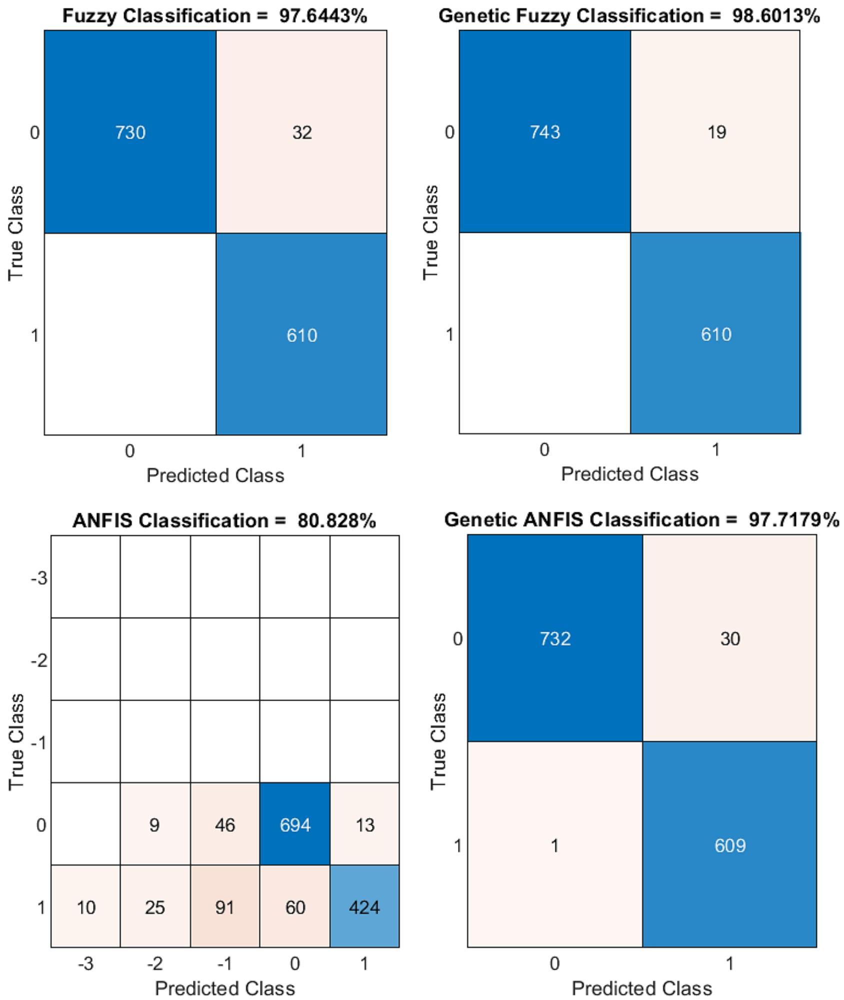

Fig. 6. Confusion matrix for detecting counterfeit banknotes.

4 | Conclusion

In this study, a method for detecting counterfeit banknotes based on a GFS is proposed. To classify the data of counterfeit banknotes, the fuzzy/ANFIS model was trained with the GA optimization algorithm and its performances were compared. From the results, it is found that the approach based on training Fuzzy and ANFIS with GA algorithm is more successful. It is shown that GFSs can be used to solve classification problems. GFS can be used in areas where ML algorithms need to be explainable due to the sensitivity of transactions. It was also found that the FIS network can be used in applications for various problems because the GA algorithm does not contain any constraints like derivative-based algorithms and can be easily applied to problems. According to the results of this study, the proposed GSF model was successfully applied in this theoretical study. Moreover, a practical application of this design seems to be possible. The method has a number of important advantages. It can distinguish genuine banknotes from counterfeit ones and thus prevent counterfeiting. The proposed model is fed with data from the counterfeit banknote dataset. Additional features that increase the discriminatory power of our system are currently being investigated. Furthermore, banknotes are susceptible to contamination due to their widespread distribution. It is certain that the degree of contamination varies from banknote to banknote. In addition, original banknotes may have defects and differ in appearance. Therefore, image-based categorization can provide more accurate results and can be applied in real time with real banknote photos and Deep Learning.

Data availability

The datasets presented in this study are freely available at [28].

Declaration of Competing Interest

The authors state that they have no known conflicting financial or personal interests that might seem to have influenced the work presented in this study.

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Paper Type: Research Paper

Soft Set Product Extended to HyperSoft Set and IndetermSoft Set Product Extended to IndetermHyperSoft Set

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Abstract

In this paper we define the Soft Set Product as a product of many soft sets and afterwards we extend it to the HyperSoft Set. Similarly, the IndetermSoft Product is extended to the IndetermHyperSoft Set. We also present several applications of the Soft Set Product to Fuzzy (and fuzzy-extensions) Soft Set Product and to IndetermSoft Set and IndetermHyperSoft Set.

Keywords: HyperSoft set, IndetermSoft set, IndetermHyperSoft set, product.

1 | Introduction

Using the Product of the Soft Set [1] we connect it with its generalization called HyperSoft Set [2], [3]. Then, similarly, through another Product the IndetermSoft Set [4] is connected to the IndetermHyperSoft Set [4] and we present several applications of them.

2 | Definition of Soft Set Product

Let's have $n \geq 2$ Soft Sets

$$F_1: A_1 \rightarrow P(H),$$
$$F_2: A_2 \rightarrow P(H),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$F_n: A_n \rightarrow P(H),$$

where $a_1, a_2, \ldots, a_n$ are n distinct attributes, and respectively $A_1, A_2, \ldots, A_n$ their corresponding sets of attributes' values, such that

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$A_i \cap A_j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \ldots, n\}$.

The Soft Set Product $F_1 \times F_2 \times \ldots \times F_n$ is defined as follows:

$$F_1 \times F_2 \times \ldots \times F_n : A_1 \times A_2 \times \ldots \times A_n \to \underbrace{P(H) \times P(x) \times \ldots \times P(H)}_{n \text{ times}}.$$

3 | The Soft Set Product transformed into a HyperSoft Set

Let's denote $F = F_1 \times F_2 \times \ldots \times F_n$.

Then we define the HyperSoft Set in the following way:

$$F : A_1 \times A_2 \times \ldots A_n \to P(H).$$

such that for each $(e_1, e_2, \ldots, e_n) \in A_1 \times A_2 \times \ldots A_n$,

$$F(e_1, e_2, \ldots, e_n) = F_1(e_1) \cap F_2(e_2) \cap \ldots \cap F_n(e_n).$$

4 | Fuzzy (and Any Fuzzy-Extension) Soft Set Product

Let's denote the fuzzy (and any fuzzy-extension degree in general) by d_0.

For example, if one uses the Fuzzy Soft Set, then $d_0 = T$ (degree of truth), $T \in [0, 1]$.

If one uses the Intuitionistic Fuzzy Set, the $d_0 = (T, F)$ or degree of truth/ falsehood, $T, F \in [0, 1]$, $0 \le T + F \le 1$;

If one uses the Neutrosophic Set, then $d_0 = (T, I, F)$, or degree of truth/ indeterminacy/ falsehood, one has:

$$0 \le T + I + F \le 3.$$

Let $H = \{h_1, h_2, h_3, h_4\}$, a set of houses.

Let the attributes a_1 = size, whose set of values is A_1 = {small, medium, big} and a_2 = location, whose set of values is A_2 = {central, peripherical}.

The soft sets

$$F_1 : A_1 \to P(H),$$

$$F_2 : A_2 \to P(H),$$

and their Product

$$F = F_1 \times F_2,$$

$$F : A_1 \times A_2 \to P(H).$$

F: {small, medium, big} $\times$ {central, peripherical} $\to P(\{h_1, h_2, h_3, h_4\})$.

F_1 (small) $= \{h_1, h_3, h_4\} \overset{def}{=} H_{11}$.

F_1 (medium) $= \{h_1, h_4\} \overset{def}{=} H_{12}$.

F_1 (big) $= \{h_2\} \overset{def}{=} H_{13}$.

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F_2 (central) $= \{h_1\} \overset{def}{=}$ H_{21}.

F_2 (peripheral) $= \{h_2, h_3, h_4\} \overset{def}{=}$ H_{22}.

Location / Size	Central	Peripheral
small	$\{h_1\}$	$\{h_3, h_4\}$
medium	$\{h_1\}$	$\{h_4\}$
big	$\varnothing$ (no house)	$\{h_2\}$

5 | HyperSoft Set Resulted from a Product of Two Soft Sets

How to read the above table, for example the intersection between the row 1 (small) and the column 2 (peripheral) gives

F [small, peripheral] = F_1 (small) $\cap$ F_2 (peripheral) = $H_{11} \cap H_{22} = \{h_1, h_3, h_4\} \cap \{h_2, h_3, h_4\} = \{h_3, h_4\}$.

6 | Fuzzy and Fuzzy-Extensions Soft Set and HyperSoft Set

F_1 (small) = $H_2(d_1^0) = \{h_1, h_3, h_4\}(d_{11}^0)$.

F_2 (peripheral) = $H_2(d_2^0) = \{h_2, h_3, h_4\}(d_{22}^0)$.

F (small, peripheral) = $H_{11}(d_{11}^0) \cap H_{22}(d_{22}^0) = \{h_3, h_4\}(d_{11}^0 \wedge d_{22}^0)$.

7 | Definition of the IndetermSoft Set

Let U be a universe of discourse, H a non-empty subset of U, and P(H) the powerset of H. Let a be an attribute, and A be a set of this attribute values.

A function F: A ⟶ (H) is called an IndetermSoft Set (Function) if

I. The set A has some indeterminacy with respect to one or more attribute's values.
II. Or P(H) has some indeterminacy.
III. Or there exist at least an attribute value v ∊ A, such that F(v) = indeterminate (unclear, uncertain, or not unique).
IV. Or any two or all three of the above situations.

In other words, an IndetermSoft Set, introduced by Smarandache in [4], is a soft set that has some degree of indeterminate (unclear, uncertain, alternative, conflicting) data or procedure.

8 | Example of IndetermSoft Set and IndetermHyperSoft Set

F_1 (small) = $\{h_1, or\ h_4\}$ = {either h_1, or h_2, or h_1 and h_2} $\overset{def}{=}$ H_{12}.

F_2 (peripheral) = $\{\ noth_2\ \}$ = {h_1, or h_3, or h_4} $\overset{def}{=}$ H_{134}.

F (small, peripheral) = $H_{12} \cap H_{134}$ = {h_1, or ϕ (no house)} $\overset{def}{=}$ H_{10}.

The first two equations above represent two IndetermSoft Set, only the third one gets the Product of them and obtains a IndetermHyperSoft Set.

As seeing, one has indeterminacy with respect to H_{12}, H_{123}, and the final result H_{10}.

Conclusion

We presented for the first time the connection between the Product of Soft Sets and the HyperSoft Set, and afterwards the Product of IndetermSoft Set that produced the IndetermHyperSoft Set, together with several applications.

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Supply Chain Management Problem Modelling in Hesitant Fuzzy Environment

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Abstract

Complex nature of the current market is often caused by uncertainties, data uncertainties, their manner of use, and differences in managers' viewpoints. To overcome these problems, Hesitant Fuzzy Sets (HFSs) can be useful as the extension of fuzzy set theory, in which the degree of membership of an element can be a set of possible values and provide greater flexibility in design and, thus, model performance. The power of this application becomes clear when different decision-makers tend to independently record their views. In most real-world situations, there are several goals for managers to achieve the desired performance. Therefore, in this study, a description of the solution of the Hesitant Fuzzy Linear Programming (HFLP) problem for solving hesitant fuzzy multi-objective problems is considered. In the following, the multi-objective and three-level supply chain management problem is modeled with the hesitant fuzzy approach. Then, with an example, the flexibility of the model responses is evaluated by the proposed method. The hesitant fuzzy model presented in this study can be extended to other supply chain management problems.

Keywords: Hesitant fuzzy, Supply chain, Multi-objective, Multi-level.

1 | Introduction

One of the key principles of various businesses to compete in the today's complex and turbulent markets is proper management of the Supply Chains (SCs) with the rapid changes in information and level of needs being met. Indeed, in the current wide market and in the presence of various levels of quality, price, service, and other factors affecting product delivery and satisfying customer satisfaction, if an SC fails to deliver superior customer service, products, and services to others, it will gradually be excluded from the competition market and lose its market share and, thus, its customers will be attracted by the competitors. Therefore, one of the best factors for staying profitable in these conditions is to be properly responsive to customer needs, have performance efficiency, and show greater adaptation to the environment. From Hughes's view, SC management is the coordination and

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transportation across SC units [1]. It is, therefore, necessary to consider two major points:

1. Improving all processes and actions in SC simultaneously.
2. Making models more compatible with the real world, due to the high level of uncertainty in the market.

It is clear that the decisions made in each sector can only lead to profitability and optimism in the same goals and do not provide the optimal global response for the whole chain. Therefore, the optimal problems in this area are modeled as multi-objective and multi-level to consider the optimal policy of all units in the overall structure.

The main performance of multi-level networks can be to supply, produce, and distribute goods to customers. One of the pioneers of multi-level models is Clark and Scarf [2], who examined the two-level inventory model in their research. In a review article, Gümüs and Güneri [3]extensively studied multi-level models.

Crisp numbers operate with limitations on their ability to perform mathematical modeling inefficiently. In the absence of comprehensive and accurate information, fuzzy execution is an effective tool for modeling complex systems. In fact, fuzzy set theory has the ability to represent many inaccurate and ambiguous concepts and systems in the mathematical form, thereby providing a basis for decision-making in an environment of uncertainty.

The complex structure of real-world problems is caused by uncertainty as well as some ambiguity in their meaning and definition. Nowadays, uncertainty has been the focus of many researchers on the way to better develop the models and adapt them to different domains, especially concerning the planning of SC management problems.

In 2006, Kumar et al. [5] used fuzzy goal programming to solve the problem of vendor selection in the SC with uncertain information. The hybrid problem of the three-objective fuzzy integers programming is used to solve the net costs of the network, number of network recurrences, and number of delayed sending and realistic constraints, in which the triangular fuzzy numbers are considered for objective function information [4]. Next, using a multi-objective fuzzy programming provided by Kumar et al. [5] solved a relatively similar problem for vendor selection in order to minimize cost and maximize quality and timely delivery of goods. This approach provides a decision-making tool, in which vendor selection and quota allocation under varying degrees of information uncertainty in the model decision parameters are facilitated. In their paper, Baykasoğlu and Goecken [6], while presenting a categorization of fuzzy mathematical programming problems, identified and presented methods for solving them including fuzzy ranking, fuzzy satisfaction criterion, meta-heuristic algorithms, and so on.

AmirKhan et al. [7] proposed a two-objective feasible linear programming model for solving the problem of multi-level, multi-commodity, and multi-period SC design considering uncertainties, time, and cost. They employed an interactive fuzzy approach.

Bashiri and Sherafati [8] introduced a two-objective model with the objective of minimizing cost and maximizing SC utility in order to design closed-loop SCs considering correlated indices under fuzzy conditions. They used the criterion as the principal component score to integrate and reduce the dimensions of the indices, eliminate the correlation between them in decision-making, and obtain the final answer using the metric LAP method. Pishvaee and Razmi [9] designed a two-objective model to minimize the total cost and environmental impact of an SC network with simultaneous inherent data uncertainties. Using the James' method, they applied a model of interactive approaches to solve the problem.

Bashiri et al. [10] employed a direct solution approach based on fuzzy ranking method and with a heuristic algorithm to balance the feasibility of constraints and optimality of the objective function in designing the three-level logistic network with fuzzy variables. In another study, a new mixed integer multi-objective linear programming model were applied for solving fully fuzzy multi-objective supplier selection problem as an important part in a SC by Nasseri et al. [11].

In 2022, the several sustainable objectives in the pharmaceutical SC optimization scheme under different uncertain constraints has extended by Ahmad et al. [12]. The trade-off between socio-economic and environmental objectives is investigated by ensuring the optimal assignment of various products among some levels and three robust techniques have presented to solve the main model.

Marzband [13], in order to obtain the performance of the SC in a manufacturing company applied the hierarchical analysis process for all suppliers were ordered and weighted based on each index in a fuzzy environment. Then, he evaluated all suppliers using the super efficiency data envelopment analysis. In 2020, Ghasempoor Anaraki et al. [14] determined reliable results for supplier selection model by combining three methods; simple multi-attribute rating technique, DEMATEL method and analytical network process in fuzzy state. Shafi Salimi and Edalatpanah [15], evaluate the suppliers by two methods of fuzzy hierarchical analysis with D-numbers. Then, as case study is different suppliers are ranked using two methods and then the findings are compared with each other.

The framework of a repurchase agreement related to the amount of good remaining in the two-echelon SC between the retailer and the manufacturer is evaluated by two (centralized and decentralized) scenarios in 2021, [16]. Recently, Nasiri et al. [17], by applying statistical methods of Kolmogorov-Smirnov, mean and Stepwise Weight Assessment Ratio Analysis (SWARA) approach, examined of effective factors of green SC management at famous Petrochemical Company.

In the past decade, various fuzzy researches and industrial fields have been observed and studied in some sciences by introducing hesitant fuzzy numbers [11], [12]. Ahmad et al. [20] constructed a multi-objective nonlinear programming problem in the manufacturing system. They gave a new approach based on single-valued neutrosophic hesitant fuzzy set to show the superiority of proposed method. To overcome the uncertainty and hesitation of the variables, Bharati [21], introduced two functions where called the hesitant fuzzy membership and non-membership functions and defined hesitant intuitionistic fuzzy pareto optimal solution. In another research, the definition of the neutrosophic hesitant fuzzy pareto optimal solution and two different optimization methods were given by Ahmad and John [22].

In this research, for the first time as far as the author's knowledge is concerned, a three-objective, three-level problem is modeled with the hesitant fuzzy approach. In this context, HFSs can be useful in modeling with ambiguity as an extension of fuzzy set theory where the element degree can be a set of possible values adopted by decision-makers. In this research, in addition to modeling, the hesitant fuzzy programming method for solving this model is developed and improved. To this end, the continuation of this paper is organized as follows:

Section 2 presents some of the prerequisites and concepts required for fuzzy sets and decision-making. In Section 3, with the overview of hesitant fuzzy programming problems, a model of multi-objective programming problems, in which objective functions and right values can be expressed as HFSs, is presented along with a method for its solving. In Section 4, the multi-objective and three-level SC management problem is presented under uncertain fuzzy conditions. Modeling with hesitant fuzzy approach is provided in Section 5. Via applying a practical example, the solution method outlined in Section 3 is evaluated in Section 6, and the findings and sensitive analysis with numerical results are proposed in Section 6. In Section 7, conclusions of the work are presented and suggestions are made for future research.

2 | Definitions and Concepts Related to Uncertain Fuzzy Sets (Hfss)

This article introduces the HFSs with respect to the issues that will be discussed in the next sections.

Definition 1. Consider the reference set X. An HFS is a set of values that, when apply on X, it returns a subset of [0,1]. Xia and Xu [23], described HFS using the following notation:

$$\mathrm{H} = \{\langle \mathrm{x}, \mathrm{h_H(x)} \rangle | \mathrm{x} \in \mathrm{X}\},$$

where, $h_H(x)$ is a set of multiple values within [0,1] and represents the degree of possible membership for element $x \in X$ relative to set H. It is easier to call $h_H(x)$ the Hesitant Fuzzy Element (HFE).

Some operators on HFEs are listed below:

$$\mathrm{h_1(x)} \cup \mathrm{h_2(x)} = \bigcup_{\gamma_1 \in \mathrm{h_1(x)}, \gamma_2 \in \mathrm{h_2(x)}} \max\{\gamma_1, \gamma_2\}.$$

$$\mathrm{h_1(x)} \cap \mathrm{h_2(x)} = \bigcap_{\gamma_1 \in \mathrm{h_1(x)}, \gamma_2 \in \mathrm{h_2(x)}} \min\{\gamma_1, \gamma_2\}.$$

$$(\mathrm{h_1(x)})^{\lambda} = \bigcup_{\gamma_1 \in \mathrm{h_1(x)}} \{\gamma_1{}^{\lambda}\}.$$

$$\lambda(\mathrm{h_1(x)}) = \bigcup_{\gamma_1 \in \mathrm{h_1(x)}} \left\{1 - (1 - \gamma_1)^{\lambda}\right\}.$$

We have a special case in HFS as the ordinary fuzzy sets, in which $h_H(x)$ is finite per $x \in X$. In this paper, HFS means that each member is a fuzzy number rather than a set of values within [0,1].

To solve the fuzzy programming problems from Bellman and Zadeh [24] view, G is assumed to be a fuzzy goal and C is a fuzzy constraint in the space of X. Then, C and G are combined to decide like D, which is the fuzzy decision of C and G. Symbolically, $D = G \cap C$ and, correspondingly, $\tau(h_G, h_c)$ where τ is used as the fuzzy operator in the fuzzy environment to compute the membership values of fuzzy elements sharing.

For the fuzzy multi-objective programming problem, we need to define a decision in the uncertain fuzzy environment. We employ this idea by extending the definition of decision-making in the fuzzy environment from Ranjbar and Effati [25] perspective:

Definition 2. Suppose $\tilde{\tilde{G}}$ is a hesitant fuzzy objective and $\tilde{\tilde{C}}$ is a hesitant fuzzy constraint in multiple choice space. In this case, decision $\tilde{\tilde{\tilde{D}}}$ from the combination of $\tilde{\tilde{C}}, \tilde{\tilde{G}}$ is called the fuzzy uncertain decision. Symbolically, we have $\tilde{\tilde{\tilde{D}}} = \tilde{\tilde{G}} \cap \tilde{\tilde{C}}$ and $h_{\tilde{\tilde{D}}} = \tau(h_{\tilde{\tilde{G}}}, h_{\tilde{\tilde{G}}})$ where τ as the T-norm in the environment hesitant fuzzy is used to compute membership values related to the HFEs subscription. We also have

$$\mathrm{h}_{\tilde{\tilde{C}}} = \left\{\mathrm{h}_{\tilde{\tilde{C}}}{}^{1}, \mathrm{h}_{\tilde{\tilde{C}}}{}^{2}, \dots, \mathrm{h}_{\tilde{\tilde{C}}}{}^{P_C}\right\}, \qquad \mathrm{h_G} = \left\{h_{\tilde{\tilde{G}}}{}^{1}, h_{\tilde{\tilde{G}}}{}^{2}, \dots, h_{\tilde{\tilde{G}}}{}^{P_G}\right\}.$$

P_C *and* P_G represent a number of decision-makers who select different levels of the objective function and constraints, respectively.

In multi-objective problems, one can consider n objectives $\tilde{\tilde{G}}_1, \tilde{\tilde{G}}_2, \dots, \tilde{\tilde{G}}_n$ and m constraints $\tilde{\tilde{C}}_1, \tilde{\tilde{C}}_2, \dots, \tilde{\tilde{C}}_n$. In that case, the decision will lead to:

$$\tilde{\tilde{\tilde{D}}} = (\tilde{\tilde{\tilde{G}}}_1 \cap \tilde{\tilde{\tilde{G}}}_2 \cap \dots \cap \tilde{\tilde{\tilde{G}}}_n) \cap (\tilde{\tilde{\tilde{C}}}_1 \cap \tilde{\tilde{\tilde{C}}}_2 \cap \dots \cap \tilde{\tilde{\tilde{C}}}_n) = \tilde{\tilde{\tilde{G}}} \cap \tilde{\tilde{\tilde{C}}}.$$

Since T-norms use HFE intersection to calculate membership values for decision-making in the hesitant fuzzy environment as a concurrent operator, we provide the following definition adopted by Santos et al. [26] for T-norms on HFSs:

Definition 3. Suppose $\tau: H^{(m)} \times H^{(m)} \to H^{(m)}$ where $H^{(m)}$ is an HFS of m members. In this case, τ is a common hesitant triangle (HT-norm). If for each $h_1, h_2, h_3 \epsilon H^{(m)}$ then, the following principles are satisfied:

$$\tau(h_1, h_2) = \tau(h_2, h_1); \text{Commutative},$$

$$\tau(h_1, \tau(h_2, h_3)) = \tau(\tau(h_1, h_2), h_3); \text{Associative},$$

$$\text{If } h_2 \leq_{H^{(m)}} h_3 \text{ then } \tau(h_1, h_2) \leq_{H^{(m)}} \tau(h_1, h_3); \text{monotony},$$

$$\tau\left(h_1, 1_{H^{(m)}}\right) = h_1; \text{Neutral member},$$

where $1_{H^{(m)}} = \{1,1, \dots ,1\}$ with m element is a complete HFE.

This definition depends on the comparison operator $\leq_{H^{(m)}}$. In this paper, we use the operator for HT-norm on HFE with fuzzy numerical members defined as follows:

Definition 4. Suppose $h \epsilon H^{(m)}$ is an HFE with m fuzzy member obtained using one of the ranking methods such as $\mathbb{R}$. Then, for every $h_1, h_2 \epsilon H^{(m)}$:

$$h_1{}^i <_{\sim\Re} h_2{}^i \quad \forall\ i = 1, \dots , m \quad \Leftrightarrow \quad h_1 <_{\approx\Re} h_2.$$

where $<_{\Re}$ with respect to the ranking function R indicates h_1 is less than h_2.

Remark 1. Suppose the number of values in HFEs can be different. The two HFEs must be of the same length in order to have the correct comparison. Then, for the two HFEs where $h_2 \epsilon H^{(n)}$ *and* $h_1 \epsilon H^{(m)}$, if $n < m$, then an expansion of h_1 by repeating the minimum value until being equal in length must be done. Choosing these values depends on the degree of risk in decision-makers' preferences. From the pessimistic view, expectation of undesirable results increases and, hence, can add minimal values, while optimistic prediction can give us more favorable results. Therefore, max values can be added.

A number of scoring functions for HFE are introduced as $S: [0,1]^n \longrightarrow [0,1]$, which establish the properties of boundary conditions and non-descending monotone. In this paper, in order to obtain the optimal solution for the hesitant fuzzy multi-objective fuzzy problem, we use a set of scoring functions defined as follows [27].

Definition 5. Suppose $h_H(x) = (h_H{}^1(x), \dots , h_H{}^m(x))$ be HFE. Then, we have following score functions:

$$S_{min}(h_H(x)) = \min\left\{h_H{}^1(x), \dots , h_H{}^m(x)\right\}; \text{Minimum scoring function},$$

$$S_{AM}(h_H(x)) = \frac{1}{m}\sum_{i=1}^{m} h_H{}^i(x); \text{Arithmetic mean},$$

$$S_{max}(h_H(x)) = \max\left\{h_H{}^1(x), \dots , h_H{}^m(x)\right\}; \text{Maximum scoring function}.$$

This definition introduces a suitable set of scoring functions appropriate to the decision-maker.

3 | Definitions Method for Solving Hesitant Fuzzy Multi-Objective

In this section, as an application of the HFSs, while introducing the HFMP, a method is presented for solving this kind of problem.

3.1 | HFMP

The HFLP can be expressed [25] as follows (HFLP):

$$
\begin{aligned}
&\text{Max } z = \tilde{\tilde{c}}^{T}\tilde{\tilde{x}}, \\
&\text{s.t.} \quad \tilde{\tilde{A}}\tilde{\tilde{x}} <_{\approx} \tilde{\tilde{b}}, \\
&\tilde{\tilde{x}} >_{\approx} 0,
\end{aligned} \tag{1}
$$

where, $\tilde{\tilde{A}}$ is a hesitant fuzzy matric and $\tilde{\tilde{c}}$, $\tilde{\tilde{b}}$ and $\tilde{\tilde{x}}$ are hesitant fuzzy vectors. In their work, they identified five categories of hesitant fuzzy programming:

1. Symmetric HFLP where the right-hand side values and objective function are fuzzy uncertain.
2. Asymmetric HFLP where only the right-hand side values are fuzzy uncertain.
3. The HFLP where the technological coefficients and right-hand side values are hesitant fuzzy.
4. The HFLP where the objective function coefficients are hesitant fuzzy.
5. The full HFLP where the objective function and the right-hand side values are hesitant fuzzy.

With the extension of models for multi-objective problems, we have

$$
\begin{aligned}
&(\text{HFMP}): \text{Max } z = \left(\tilde{\tilde{c}}_1{}^{T}\tilde{\tilde{x}}, \tilde{\tilde{c}}_2{}^{T}\tilde{\tilde{x}}, \dots, \tilde{\tilde{c}}_r{}^{T}\tilde{\tilde{x}}\right), \\
&\text{s.t.} \quad \tilde{\tilde{A}}\tilde{\tilde{x}} <_{\approx} \tilde{\tilde{b}}, \\
&\tilde{\tilde{x}} >_{\approx} 0.
\end{aligned} \tag{2}
$$

Since the five proposed for HFLP modes are extensible to HFMOLP problems and given that the methods for solving different modes are different, here, an extension of the symmetrical HFLP is considered. In this concept, the right-hand side values and the objective functions of the problem can be expressed as hesitant fuzzy numbers; so, we have:

$$
\begin{aligned}
&(\text{HFMP}): \text{Max } z = \left(\tilde{\tilde{c}}_1{}^{T}\tilde{\tilde{x}}, \tilde{\tilde{c}}_2{}^{T}\tilde{\tilde{x}}, \dots, \tilde{\tilde{c}}_r{}^{T}\tilde{\tilde{x}}\right) \geq \tilde{\tilde{z}}_o, \\
&\text{s.t.} \quad \tilde{\tilde{A}}\tilde{\tilde{x}} <_{\approx} \tilde{\tilde{b}}, \\
&x \geq 0,
\end{aligned} \tag{3}
$$

where $\tilde{\tilde{z}}_o = [\tilde{\tilde{z}}_1, \tilde{\tilde{z}}_2, \dots, \tilde{\tilde{z}}_r]^T$ is the hesitant fuzzy lower bound to maximize $(\tilde{\tilde{c}}_1{}^{T}\tilde{\tilde{x}}, \tilde{\tilde{c}}_2{}^{T}\tilde{\tilde{x}}, \dots, \tilde{\tilde{c}}_r{}^{T}\tilde{\tilde{x}})$ and $\tilde{\tilde{b}}$ is the HFEs components with fuzzy membership values. In this case, there is no distinction between goals and constraints. And several decision-makers can submit different views for the value of objective functions and constraints. The problem formulation can be transformed as follows:

$$
\begin{aligned}
&\text{Find } x \\
&\text{s.t.} \\
&c_1{}^{T}x \geq \tilde{\tilde{z}}_1, \\
&c_2{}^{T}x \geq \tilde{\tilde{z}}_2, \\
&c_r{}^{T}x \geq \tilde{\tilde{z}}_r, \\
&Ax \leq \tilde{\tilde{b}}, \\
&x \geq 0.
\end{aligned} \tag{4}
$$

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The current set of constraints includes the set of goals and hesitant fuzzy constraints.

If we have r goals and m constraints, then

$$B=\begin{bmatrix}-c_{11} & -c_{12} & \cdots & -c_{1n}\\ \vdots & & \ddots & \vdots\\ -c_{r1} & -c_{r2} & \cdots & -c_{rn}\\ a_{11} & a_{12} & \cdots & a_{rn}\\ \vdots & & \ddots & \vdots\\ a_{r1} & a_{r2} & \cdots & a_{mn}\end{bmatrix},\ \tilde{\tilde{d}}=\begin{bmatrix}-\tilde{\tilde{z}}_1\\ \vdots\\ -\tilde{\tilde{z}}_r\\ \tilde{\tilde{b}}_1\\ \vdots\\ \tilde{\tilde{b}}_m\end{bmatrix}.$$

All $m + r$ on row $\tilde{\tilde{d}}$ are specified below by HF elements:

$$\tilde{\tilde{d}}_i=\{h_i^{\,1},h_i^{\,2},\dots,h_i^{\,p_i}\},\qquad i=1,2,\dots,m+r,$$

where $\tilde{\tilde{d}}_i$ are fuzzy numbers and p_i are the number of decision-makers, satisfaction levels of which represent the values of the objective functions, and each constraint is based on the i^{th} line according to knowledge and experience. We consider $h_i^{\,k}$ for $k_i = 1,2,\dots,p_i$ with decreasing membership function as follows:

$$h_i^{\,k_i}(x)=\begin{cases}1 & B_i x\le d_i^{\,k_i}\\ 1-\dfrac{B_i x-d_i^{\,k_i}}{q_i^{\,k_i}}, & d_i^{\,k_i}<B_i x\le d_i^{\,k_i}+q_i^{\,k_i}\\ 0 & B_i x\ge d_i^{\,k_i}+q_i^{\,k_i}\end{cases}. \tag{5}$$

That is, where B_i represents the i^{th} row of B ($i = 1,2,\dots,m + r$), $d_i^{\,k_i}$is the constant value i^{th} of the selected row, and $q_i^{\,k_i}$ is an acceptable error corresponding to i^{th} row which is selected by the $k_i^{\,th}$ decision-maker.

3.2 | HFMP Solving Method

First, in terms of the hesitant fuzzy decision definition of the model, we state:

$$h_D=\tau_M(h_1,h_2,\cdots,h_{m+r})=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_{m+r}\in h_{m+r}}\min\{\gamma_1,\gamma_2,\cdots,\gamma_{m+r}\}.$$

In this case, $h_D = \{\ h_D^{\,1}, h_D^{\,2},\dots,h_D^{\,p_1p_2\dots p_{m+r}}\}$ is a set of fuzzy numbers. Now, for the optimal solution to this problem, we can recommend the maximum of each member of h_D as follows:

$$\begin{aligned}&\max\ h_D^{\,s}(x^s),\\ &\text{s.t.}\quad x^s\ge 0,\qquad s=1,2,\cdots,(p_1p_2\cdots p_{m+r}).\end{aligned} \tag{6}$$

By introducing the variable λ^s that corresponds to $h_D^{\,s}(x^s)$ in the model, we have

$$\begin{aligned}&LP_s:\ \ \max\ \lambda^s\\ &\text{s.t}\quad \lambda^s q_i^{\,k_i}+B_i x^s\le q_i^{\,k_i},\qquad i=1,2,\dots,m+r\\ &0\le\lambda^s\le 1,\\ &x^s\ge 0.\end{aligned} \tag{7}$$

Then, after solving this model, λ^{*s} is the maximum degree corresponding to the level of satisfaction of the goals and constraints that can establish i^{th}. The $x^{*s} = (x_1^{\,*s}, x_2^{\,*s},\cdots,x_n^{\,*s})$ is an HFMOLP problem solution. So, by solving $(p_1p_2\cdots p_{m+r})$, we have the LP problem as the following model:

$$h_D(x^*)=\{\lambda^{*1},\lambda^{*2},\cdots,\lambda^{*(p_1p_2\cdots p_{m+r})}\},$$

where $x^* = (x_1^{\,*}, x_2^{\,*},\cdots,x_n^{\,*})$, such that

$$x_1^* = \{x_1^{*1}, x_1^{*2}, \cdots, x_1^{*(p_1p_2\cdots p_{m+r})}\}.$$

$$\vdots$$

$$x_n^* = \{x_n^{*1}, x_n^{*2}, \cdots, x_n^{*(p_1p_2\cdots p_{m+r})}\}.$$

Remark 2. If decision-makers state only some goals as hesitant fuzzy, that is to say, there exist crisp goal or goals in the model. Here, sub-problems are as MOLP where weighted average method can be used for their solving. In that case, only a weighty goal based on the decision-maker's priorities plays a role in the importance of the goals [28].

Remark 3. It is possible to examine responses at different levels of decision-makers' views with alpha levels in mind. In this case, in addition to the constraints presented in LP_s, we will have a constraint as ≥.

Remark 4. If the decision-makers are not interested in the hesitant fuzzy solution, then, the optimal solution of the problem can be found by using the scoring functions from different points of view, similar to those presented in *Table 1*, where l is the minimum membership index of $h_D(x^*)$ and u is the maximum index of $h_D(x^*)$.

Table 1. Optimal solutions to the MOHFLP problem from different perspectives.

λ^*	x^*	View
$\lambda^{*l} = S_{min}(h_D((x^*))$	$(x_1^{*l}, x_2^{*l}, \ldots, x_n^{*l})$	pessimistic
$\lambda^* = S_{AM}(h_D((x^*))$	$\left(\frac{\sum_{r=1}^{(p_1\cdots p_{m+r})} \lambda^* x_1^{*r}}{\sum_{r=1}^{(p_1\cdots p_{m+r})} \lambda^*}, \frac{\sum_{r=1}^{(p_1\cdots p_{m+r})} \lambda^* x_2^{*r}}{\sum_{r=1}^{(p_1\cdots p_{m+r})} \lambda^*}, \ldots, \frac{\sum_{r=1}^{(p_1\cdots p_{m+r})} \lambda^* x_n^{*r}}{\sum_{r=1}^{(p_1\cdots p_{m+r})} \lambda^*}\right)$	Normal
$\lambda^{*u} = S_{max}(h_D((x^*))$	$(x_1^{*u}, x_2^{*u}, \ldots, x_n^{*u})$	Optimistic

4 | Multi-Objective SC Problem with Hesitant Fuzzy Approach

In this section, with some limitations, we consider the multi-objective, three-level, single-product chain management model as *Fig. 1* in a form that should be considered by decision-makers for various purposes, some of which are conflicting. The following are the indices, parameters, decision variables, constraints, and goals.

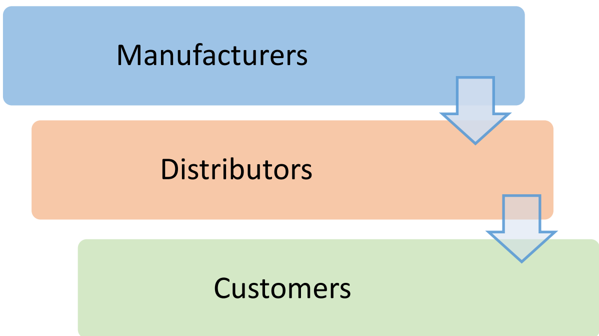

Fig. 1. Three-level supply chain structure.

Indices

Manufactures $(i\epsilon I)$, $i = 1,2, \dots, m$.

Distributors $(j \in J)$, $j = 1,2, \dots, n$.

Customers $(k \in O)$, $k = 1,2, \dots, o$.

Parameters

Q_i: Product quality produced by the i^{th} manufacturer.

${P_{ij}}^T$: Cost of shipping the product from i^{th} manufacturer to j^{th} distributor.

${C_i}^V$: Product shipping capacity from i^{th} manufacturer warehouse to warehouse distribution centers.

${P_j}^H$: Cost of maintaining each unit of goods in the j^{th} distributor warehouse.

${P_{jk}}^R$: Cost of payment for each unit of fine returned by the distributor j^{th} to the customer i^{th}.

${B_{jk}}^R$: Return percentage of goods sold by distributor j^{th} to customer k^{th}.

${T_{jk}}^S$: Delivery time from distributor j^{th} to customer k^{th}.

${C_j}^V$: Freight forwarding capacity of distributor j^{th}.

S_{jk}: Sales price per unit of product from distributor j^{th} to customer k^{th}.

${U_i}^P$: Maximum amount of product manufactured by i^{th} manufacturer to send to distribution center.

${L_j}^D$: Minimum customer required demand for distributor.

Decision variables

x_{ij}: Quantity of product sent by manufacturer i^{th} to distributor j^{th}.

y_{jk}: Amount of customer demand k^{th} from distributor j^{th}.

Constraints

Product lack constraints: Obviously, one of the main reasons for developing and validating systems is to meet customer demand at the right time. Therefore, we need constraints that ensure that the amount of production is sufficient to meet the needs of the customers and does not increase warehousing costs. To this end, the following constraints may apply

$$\sum_{i=1}^{m} x_{ij} = \sum_{k=1}^{o} y_{jk}, \quad (j = 1,2, \dots, n). \tag{8}$$

Maximum production capacity constraints: This type of constraint ensures that the amount of product produced by the i^{th} manufacturer to deliver to distributors has a certain maximum value. For this purpose, we have:

$$\sum_{i=1}^{m} x_{ij} \leq U_i^P, \qquad (i = 1,2, \dots, m). \tag{9}$$

Customer demand minimum constraints: This type of constraint ensures that the quantity of product requested by the distributor i^{th} is minimal. For this purpose, we have:

$$\sum_{k=1}^{n} y_{jk} \geq L_j^D, \qquad (j = 1,2, \dots, n). \tag{10}$$

In addition to the three types of *Constraints (8), (9)* and *(10)* mentioned above, we present the non-negative constraints of decision variables:

$$x_{ij}, \qquad y_{jk} \geq 0, \quad (i = 1,2, \dots, m), \qquad (j = 1,2, \dots, n), (k = 1,2, \dots, o). \tag{11}$$

Objective functions

Quality objective function: this objective function aims to maximize the quality of products sent by the manufacturer i^{th} to the distributor j^{th} in order to deliver more quality goods to distributors and, thus, to customers. For this purpose, we have the following objective function:

$$F_q = \sum_{i=1}^{m} \sum_{j=1}^{n} Q_i x_{ij}. \tag{12}$$

Total cost objective function: To minimize total system costs, including shipping, maintenance, and penalties for returning goods, it is formulated as follows:

$$\begin{aligned} F_p = {} & \sum_{i=1}^{m} \sum_{j=1}^{n} P_{ij}^T (x_{ij} / C_i^V) + \sum_{i=1}^{m} \sum_{j=1}^{n} H_j (x_{ij}/2) \\ & + \sum_{j=1}^{n} \sum_{k=1}^{o} P_{jk} (B_{ij}^R y_{jk}), \end{aligned} \tag{13}$$

which includes, respectively, the total shipping costs from the manufacturer i^{th} to the distributor j^{th}, the maintenance cost of the product shipped by the manufacturer i^{th} to the distributor j^{th}, and the return fine.

Delivery time objective function: it aims to minimize product delivery time by the distributor, as follows:

$$F_t = \sum_{j=1}^{n} \sum_{k=1}^{i} T_{jk}^S (y_{ij} / C_j^V). \tag{14}$$

Income objective function: to maximize revenue from product sales from the distributor i^{th} to the customer k^{th}, it will generate more revenue from selling the product to customers.

$$F_s = \sum_{j=1}^{n} \sum_{i=1}^{o} S_{jk} y_{jk}. \tag{15}$$

The objective functions presented in *Eqs. (12)-(15)* along with the deterministic model *Constraints (8)-(11)* form multi-objective SC management problem.

5 | Modeling with Hesitant Fuzzy Approach

Product quality, total cost, delivery time, and optimal revenue, which are considered definite goals in the model presented in the previous section, may be influenced by various factors such as management, competitor's status, inflation, and so on. Therefore, these goals may be desirable from the point of view of different decision-makers at a particular level and may allow a certain level of violation. For modeling the problem, the goals can be considered fuzzy by considering the decision-makers with the help of hesitant fuzzy numbers. This idea can be limited by constraints such as the amount of production capacity due to changes in the amount of raw materials available and overtime human force hours, limitation in the minimum amount of customer demand by product quality, relative satisfaction with after-sales service, manner of advertising develops status of competitors in the market, and so on. Hence, the model presented in the previous section can be modeled by the hesitant fuzzy approach:

$$F_q = \sum_{i=1}^{m} \sum_{j=1}^{n} Q_i x_{ij} \geq \tilde{\tilde{z}}_q.$$

$$F_p = \sum_{i=1}^{m} \sum_{j=1}^{n} P_{ij}^{\ T}(x_{ij} / C_i^{\ V}) + \sum_{i=1}^{m} \sum_{j=1}^{n} H_j(x_{ij}/2) + \sum_{j=1}^{n} \sum_{k=1}^{o} P_{jk}(B_{ij}^{\ R}\, y_{jk}) \leq \tilde{\tilde{z}}_p.$$

$$F_s = \sum_{j=1}^{n} \sum_{i=1}^{o} S_{jk} y_{jk} \leq \tilde{\tilde{z}}_s.$$

$$F_t = \sum_{j=1}^{n} \sum_{k=1}^{i} T_{jk}^{\ S}(y_{ij} / C_j^{\ V}) \geq \tilde{\tilde{z}}_t.$$

s. t.

$$\sum_{i=1}^{m} x_{ij} = \sum_{k=1}^{o} y_{jk} \quad (j = 1,2, \dots, n).$$

$$\sum_{i=1}^{m} x_{ij} \leq \tilde{\tilde{U}}_i^{\ P} \qquad (i = 1,2, \dots, m).$$

$$\sum_{k=1}^{n} y_{jk} \geq \tilde{\tilde{L}}_j^{\ D} \qquad (k = 1,2, \dots, o).$$

$$x_{ij}, y_{jk} \geq 0 \quad (i = 1,2, \dots, m), (j = 1,2, \dots, n), (k = 1,2, \dots, o).$$

Where fuzzy numbers are uncertain. A summary of the solution is given as flowchart in *Fig. 2* in accordance with the material presented in Section 3. The following section provides a numerical example to analyze the model and discuss and evaluate its results.

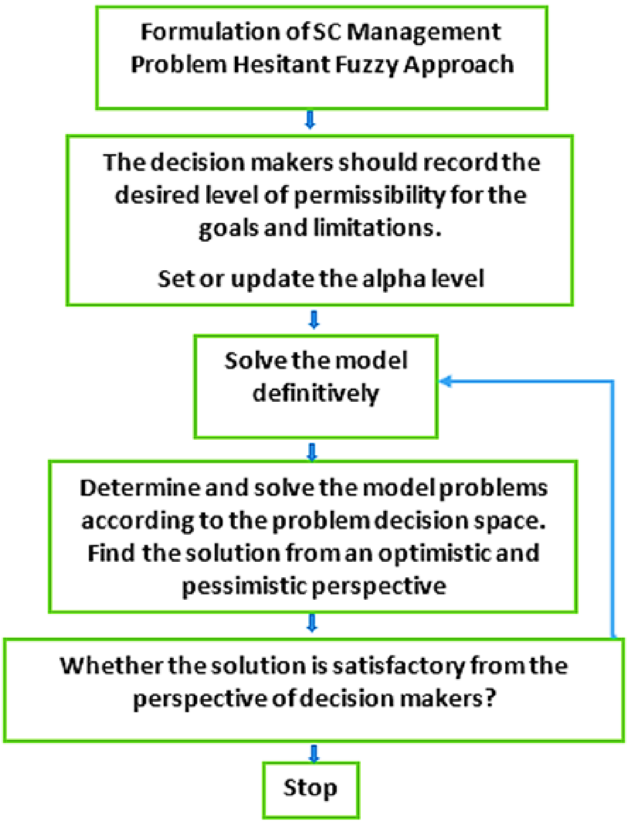

Fig. 2. Flowchart for fuzzy SC problem solving with hesitant approach.

i \ j		Distributor 1	Distributor 2
Distributer	1	5	3
	2	4	5

a.

i \ j		Distributor 1	Distributor 2
Distributer	1	3	4
	2	3	4

b.

j \ k		Customer 1	2	3	4
Distributer	1	2	3	3	2
	2	3	2	2	3

c.

j \ k		Customer 1	2	3	4
Distributer	1	40	48	48	40
	2	48	40	40	48

d.

j \ k		Customer 1	2	3	4
Distributer	1	20	24	24	20
	2	24	20	20	24

e.

j \ k		Customer 1	2	3	4
Distributer	1	0.03	0.02	0.02	0.03
	2	0.02	0.03	0.03	0.02

f.

j	1	2
C_j^V	50	45

g.

i	1	2
C_i^V	45	50

h.

j	1	2
L_j^D	800	700

i.

i	1	2
U_i^P	1200	1800

j.

Fig 3. SC problem data: a. shipping cost from manufacturer to distributor warehouse (in Currency); b. cost of keeping the manufacturer's goods in the distributor's warehouse (in currency); c. delivery time from the distributor's warehouse to the customer (in units of time); d. sales price per unit of distribution to customer (in units of time); e. amount of the fine paid by the distributor to the customer; f. distributor return percentage rate; g. capacity of carriers used by distribution center (in commodity units); h. capacity of carriers used by production center (in units of goods); i. minimum customer demand from distribution centers (in units); j. maximum production capacity (in units of commodity).

6 | Empirical Numerical Analysis

Consider the multi-objective, three-level problem of 2 manufacturers, 2 distributors, and 4 customers as in *Fig. 4*. Supplementary information is provided in *Figs. 3.a-3.j*. The return penalty per unit of commodity is half of its sales price. In addition, the quality percentages per unit of product produced by manufacturers 1 and 2 are 0.86 and 0.9, respectively. The two decision-makers record the desired values and the virtual violations for the second and third objective functions and the first and second constraints, whose views are presented in the *Table 2*.

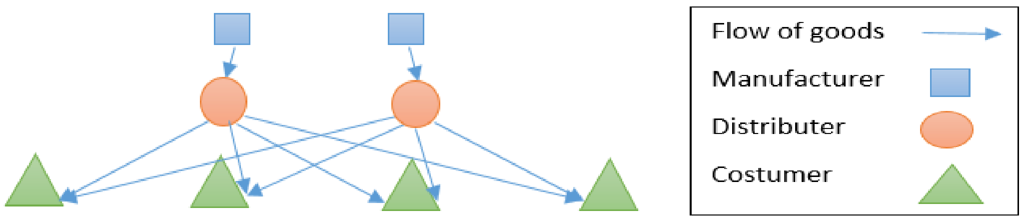

Fig. 4. Three-level diagram: with 2 manufacturers, 2 distributors, 4 customers.

Table 2. Desired values and permitted violations from the point of view of decision-makers for some objective and constraints.

DM	The desired value range and the permissible violation from the decision-maker's point of view					
	Obj-1	Obj-2	Obj-3	Obj-4	Cons-5	Cons-6
DM 1	(2800,1000)	(4000,900)	(70,8)	(150000,15000)	(700,140)	(650,250)
DM 2	(3200,800)	(4400,400)	(95,10)	(160000,20000)	(1250,50)	(1340,40)

According to the problem information, the following formulation is provided:

$$\widetilde{\widetilde{\max}}\ F_q = 0.86\,x_{11} + 0.86x_{12} + 0.9x_{21} + 0.9x_{22},$$

$$\begin{aligned}\widetilde{\widetilde{\min}}\ F_p = ((\tfrac{5}{45})\,x_{11} + (\tfrac{3}{45})\,x_{12} \\ &+ (\tfrac{4}{50})\,x_{21} + \left(\tfrac{5}{50}\right)x_{22} + 1.5(x_{11} + x_{21}) + 2(x_{12} + x_{22}) + 0.6\,y_{11} \\ &+ 0.48\,y_{12} + 0.48\,y_{13} + 0.6\,y_{14} + 0.48\,y_{21} + 0.6\,y_{22} + 0.6\,y_{23} \\ &+ 0.48\,y_{24},\end{aligned}$$

$$\begin{aligned}\widetilde{\widetilde{\min}}\ F_t = (\tfrac{2}{50})\,y_{11} + (\tfrac{3}{50})\,y_{12} + \left(\tfrac{3}{50}\right)y_{13} + \left(\tfrac{2}{50}\right)y_{14} + \left(\tfrac{3}{50}\right)y_{21} + \left(\tfrac{2}{45}\right)y_{22} + \left(\tfrac{2}{45}\right)y_{23} \\ &+ \left(\tfrac{3}{45}\right)y_{24},\end{aligned}$$

$$\widetilde{\widetilde{max}}\ F_s = 40\,y_{11} + 48\,y_{12} + 48\,y_{13} + 40\,y_{14} + 48\,y_{21} + 40\,y_{22} + 40\,y_{23} + 48\,y_{24},$$

s. t.

$$x_{11} + x_{21} = y_{11} + y_{12} + y_{13} + y_{14},$$

$$x_{12} + x_{22} = y_{21} + y_{22} + y_{23} + y_{24},$$

$$x_{11} + x_{12} \leq 1200,$$

$$x_{21} + x_{22} \leq 1800,$$

$$y_{11} + y_{12} + y_{13} + y_{14} \geq \widetilde{\widetilde{800}},$$

$$y_{21} + y_{22} + y_{23} + y_{24} \geq \widetilde{\widetilde{700}},$$

$$x_{ij}, y_{jk} \geq 0 \quad (i = 1,2, \quad j = 1,2, \quad k = 1,2,3,4).$$

Results of the deterministic modeling with respect to the objectives are presented separately and together in *Table 3*.

Table 3. Example results considering objectives separately and multi-objectively.

Objective Variables	F_s 34984	F_q 2652	F_p 3433	F_t 60	F_s 144000	$f = 0.25 * (F_q + F_p + F_t + F_s)$ 34984
x_{11}	0	0	0	500	500	500
x_{12}	1200	700	700	700	700	700
x_{21}	1800	800	800	1800	1800	1800
x_{22}	0	0	0	0	0	0
y_{11}	0	0	0	0	0	0
y_{12}	0	0	0	0	0	0
y_{13}	0	800	0	2300	2300	2300
y_{14}	1800	0	800	0	0	0
y_{21}	0	0	0	0	700	700
y_{22}	0	0	0	0	0	0
y_{23}	0	0	700	0	0	0
y_{24}	12000	700	0	700	0	0

In this case, using the method presented in Section 1, we have 64 sub-problems; the results of solving each are given by selecting zero for alpha in *Table 4*.

Table 4. Example results considering objectives separately and multi-objectively.

$MOLP_r$	λ^*	x^*	f^*
$MOLP_1$	0.6	(1200,0,0,1800,0,0,1200,0,1800,0,0,0)	34830
$MOLP_2$	0.315	(1200,0,0,1800,0,0,665.6,534.4,1800,0,0,0)	33748
$MOLP_3$	0.6	(1200,0,0,1800,0,0,1200,0,1800,0,0,0)	34830
$MOLP_4$	0.6	(1200,0,0,1800,0,0,1200,0,1800,0,0,0)	34830
$MOLP_5$	0.2	(1200,0,0,1800,0,0,1200,0,1800,0,0,0)	34830
$MOLP_6$	0.6	(1200,0,30,1770,0,0,1230,0,1770,0,0,0)	34834
$MOLP_7$	0.6	(1200,0,476,1324,0,0,1676,0,1324,0,0,0)	34892
$MOLP_8$	0.315	(1200,0,0,1800,0,0,665.6,534.4,1800,0,0,0)	337.8
$MOLP_9$	0.315	(1200,0,0,1800,0,0,665.6,534.4,1800,0,0,0)	33748
$MOLP_{10}$	0.6	(1200,0,0,1800,0,0,1200,0,1800,0,0,0)	34830
$MOLP_{11}$	0.2	(1200,0,10,1790,0,0,1210,0,1790,0,0,0)	34831
$MOLP_{12}$	0.315	(1200,0,0,1800,0,0,665.6,534.4,1800,0,0,0)	33748
$MOLP_{13}$	0.2	(1200,0,0,1800,0,0,1200,0,1800,0,0,0)	34830
$MOLP_{14}$	0.315	(1200,0,15.8,1784.2,0,0,681.4,534.4,1784.2,0,0,0)	33750
$MOLP_{15}$	0.315	(1200,0,0,1800,0,0,665.6,534.4,1800,0,0,0)	33748
$MOLP_{16}$	0.6	(1200,0,0,1800,0,0,1200,0,1800,0,0,0)	34830
$MOLP_{17}$	0.2	(1200,0,0,1800,0,0,1200,0,1800,0,0,0)	34830
$MOLP_{18}$	0.6	(1200,0,30,1770,0,0,1230,0,1770,0,0,0)	34834
$MOLP_{19}$	0.6	(1200,0,184,1616,0,0,1384,0,1616,0,0,0)	34854
$MOLP_{20}$	0.2	(1200,0,0,1800,0,0,1200,0,1800,0,0,0)	34830
$MOLP_{21}$	0.6	(1200,0,30,1770,0,0,1230,0,1770,0,0,0)	34834
$MOLP_{22}$	0.6	(1200,0,476,1324,0,0,1676,0,1324,0,0,0)	34892
$MOLP_{23}$	0.2	(1200,0,10,1790,0,0,1210,0,1790,0,0,0)	34831
$MOLP_{24}$	0.2	(1200,0,492,1308,0,0,1692,0,1308,0,0,0)	34894
$MOLP_{25}$	0.2	(1200,0,0,1800,0,0,1200,0,1800,0,0,0)	34830
$MOLP_{26}$	0.315	(1200,0,15.8,1784.2,0,0,681.4,534.4,1784.2,0,0,0)	33750
$MOLP_{27}$	0.315	(1200,0,0,1800,0,0,665.6,534.4,1800,0,0,0)	33748
$MOLP_{28}$	0.2	(1200,0,0,1800,0,0,1200,0,1800,0,0,0)	34830
$MOLP_{29}$	0.315	(1200,0,15.7,1784.2,0,0,681.4,534.4,1784.2,0,0,0)	33750
$MOLP_{30}$	0.315	(1200,0,0,1800,0,0,665.6,534.4,1800,0,0,0)	33748
$MOLP_{31}$	0.2	(1200,0,10,1790,0,0,1210,0,1790,0,0,0)	34831
$MOLP_{32}$	0.2	(1200,0,0,1800,0,0,1200,0,1800,0,0,0)	34830
$MOLP_{33}$	0.2	(1200,0,10,1790,0,0,1210,0,1790,0,0,0)	34831
$MOLP_{34}$	0.2	(1200,0,10,1790,0,0,1210,0,1790,0,0,0)	34831
$MOLP_{35}$	0.2	(1200,0,10,1790,0,0,1210,0,1790,0,0,0)	34831
$MOLP_{36}$	0.2	(1200,0,10,1790,0,0,1210,0,1790,0,0,0)	34831

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Table 4. Continued.

$MOLP_r$	λ^*	x^*	f^*
$MOLP_{37}$	0.315	(1200,0,15.7,1784.3,0,0,681.4,534.4,1784.3,0,0,0)	33751
$MOLP_{38}$	0.2	(1200,0,0,1800,0,0,1200,0,1800,0,0,0)	34830
$MOLP_{39}$	0.2	(1200,0,10,1790,0,0,1210,0,1790,0,0,0)	34831
$MOLP_{40}$	0.2	(1200,0,10,1790,0,0,1210,0,1790,0,0,0)	34831
$MOLP_{41}$	0.2	(1200,0,10,1790,0,0,1210,0,1790,0,0,0)	34831
$MOLP_{42}$	0.6	(1200,0,30,1770,0,1230,0,1770,0,0,0)	34834
$MOLP_{43}$	0.6	(1200,0,476,1324,0,0,1676,0,1324,0,0,0)	34892
$MOLP_{44}$	0.2	(1200,0,10,1790,0,0,1210,0,1790,0,0,0)	34831
$MOLP_{45}$	0.6	(1200,0,30,1770,0,0,1230,0,1770,0,0,0)	34834
$MOLP_{46}$	0.2	(1200,0,0,1800,0,0,1200,0,1800,0,0,0)	34830
$MOLP_{47}$	0.2	(1200,0,10,1790,0,0,1210,0,1790,0,0,0)	34831
$MOLP_{48}$	0.2	(1200,0,10,1790,0,0,1210,0,1790,0,0,0)	34831
$MOLP_{49}$	0.315	(1200,0,15.8,1784.2,0,0,681.4,534.4,1784.2,0,0,0)	33750
$MOLP_{50}$	0.2	(1200,0,0,1800,0,0,1200,0,1800,0,0,0)	34830
$MOLP_{51}$	0.2	(1200,0,10,1790,0,0,1210,0,1790,0,0,0)	34831
$MOLP_{52}$	0.315	(1200,0,15.8,1784.2,0,0,681.4,534.4,1784.2,0,0,0)	33750
$MOLP_{53}$	0.2	(1200,0,0,1800,0,0,1200,0,1800,0,0,0)	34830
$MOLP_{54}$	0.2	(1200,0,10,1790,0,0,1210,0,1790,0,0,0)	34831
$MOLP_{55}$	0.315	(1200,0,0,1800,0,0,665.6,534.4,1800,0,0,0)	33748
$MOLP_{56}$	0.315	(1200,0,15.8,1784.2,0,0,681.4,534.4,1784.2,0,0,0)	33750
$MOLP_{57}$	0.6	(1200,0,30,1770,0,0,1230,0,1770,0,0,0)	34834
$MOLP_{58}$	0.2	(1200,0,492,1308,0,0,1692,0,1308,0,0,0)	34894
$MOLP_{59}$	0.2	(1200,0,10,1790,0,0,1210,0,1790,0,0,0)	34831
$MOLP_{60}$	0.6	(1200,0,30,1770,0,0,1230,0,1770,0,0,0)	34834
$MOLP_{61}$	0.2	(1200,0,0,1800,0,0,1200,0,1800,0,0,0)	34830
$MOLP_{62}$	0.2	(1200,0,10,1790,0,0,1210,0,1790,0,0,0)	34831
$MOLP_{63}$	0.2	(1200,0,0,1800,0,0,1200,0,1800,0,0,0)	34830
$MOLP_{64}$	0.6	(1200,0,30,1770,0,0,1230,0,1770,0,0,0)	34834

If the decision-maker intends to obtain definite results, from the optimistic and pessimistic points of view, we obtain the values presented in *Table 5*.

Table 5. Results from the optimistic and pessimistic perspectives.

Views	λ^*	x^*	f^*
Pessimistic	0.2	(1200,0,10,1790,0,0,1210,0,1790,0,0,0)	34831
Optimist	0.6	(1200,0,476,1324,0,0,1676,0,1324,0,0,0)	34892

7 | Findings and Sensitive Analysis

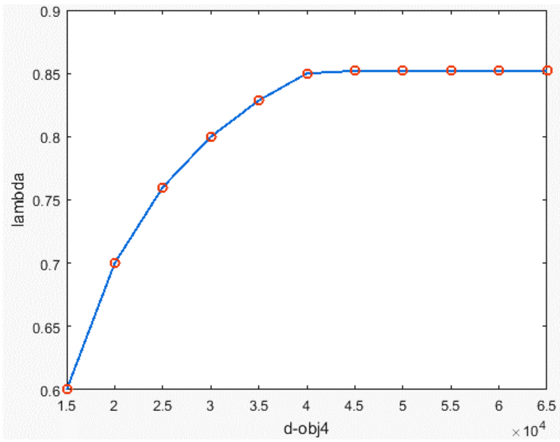
In the previous section, a multi-objective problem of the SC was solved extensively by presenting a practical example. According to the recording of the expected values and the acceptable violation from the point of view of two decision makers about objective *Function (4)* and *Constraints (5)* and *(6)*, 64 sub-problems were extracted. The results presented in *Table 4* have provided the optimal expectation level and the average objective function value in the λ^* and f^* columns for these 64 sub-problems. From the point of view of the decision maker with a higher level of expectation, $\lambda^* = 0.6$ the value of $f^* = 34892$ is obtained and from the point of view of a decision maker with a lower level of expectation, $\lambda^* = 0.2$ the value of $f^* = 34831$ is obtained (*Table 5*). Of course, considering the variety of solutions and changes caused by real world conditions, decision makers can finally use the results of one of the sub-problems in *Table 4* as the optimal solution.

For sensitivity analysis, since the expected value and the deviation depend on the opinion of the decision makers, it is clear that any increase or decrease in these values may cause a change in the final response (values of λ^* and f^*) for the related sub-problems. For example, if we keep the expected value for the fourth objective function constant and increase the acceptable deviation value, then the results in the sub-problems related to this value may change. For example, for sub-problem $MOLP_1$, results similar to those seen in *Table 6* and *Fig. 5* are obtained. As can be seen in *Table 6*, with the increase of the acceptable violation from 15000 to 65000, the value of λ^* increased from 0.6 to 0.852 and the value of f^* decreased from 34830 to 33914.

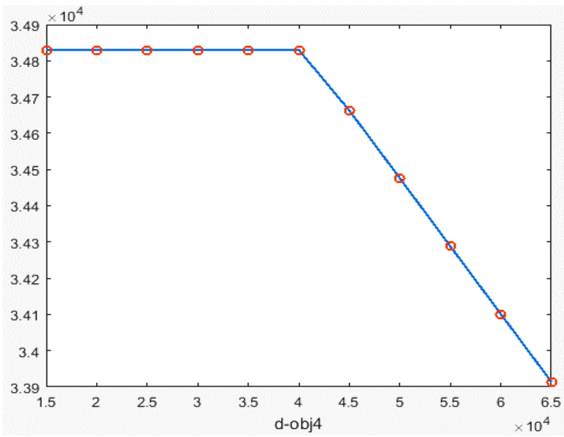
Table 6. Results of the changes in $\tilde{\tilde{d}}_{\tilde{\tilde{z}}_4}$ for the fourth objective of $MOLP_1$.

$(\tilde{\tilde{z}}_4, \tilde{\tilde{d}}_{\tilde{\tilde{z}}_4})$	λ^*	x^*	f^*
(150000,15000)	0.6000	(1200,0,0,1800,0,0,1200,0,1800,0,0,0)	34830
(150000,20000)	0.7000	(1200,0,0,1800,0,0,1200,0,1800,0,0,0)	34830
(150000,25000)	0.7600	(1200,0,0,1800,0,0,1200,0,1800,0,0,0)	34830
(150000,30000)	0.8000	(1200,0,0,1800,0,0,1200,0,1800,0,0,0)	34830
(150000,35000)	0.8286	(1200,0,0,1800,0,0,1200,0,1800,0,0,0)	34830
(150000,40000)	0.8500	(1200,0,0,1800,0,0,1200,0,1800,0,0,0)	34830
(150000,45000)	0.8520	(1200,0,0,1800,0,0,1117.5,82.5,1800,0,0,0)	34663
(150000,50000)	0.8520	(1200,0,0,1800,0,0,1025,175,1800,0,0,0)	34476
(150000,55000)	0.8520	(1200,0,0,1800,0,0,932.5,267.5,1800,0,0,0)	34288
(150000,60000)	0.8520	(1200,0,0,1800,0,0,840,360,1800,0,0,0)	34101
(150000,65000)	0.8520	(1200,0,0,1800,0,0,747.5,452.5,1800,0,0,0)	33914

In *Fig. 5*, the results obtained due to the increase of the acceptable violation for the fourth objective function and related to sub-problem $MOLP_1$ are depicted. The horizontal axis and the vertical axes show the acceptable violation and the amount of λ^* and f^*, respectively.



a.



b.

Fig. 5. a. Variations of λ^* by the changes in $\tilde{\tilde{d}}_{\tilde{\tilde{z}}_4}$ for fourth objective of $MOLP_1$; b. Variations of f^* by the changes in $\tilde{\tilde{d}}_{\tilde{\tilde{z}}_4}$ for fourth objective of $MOLP_1$.

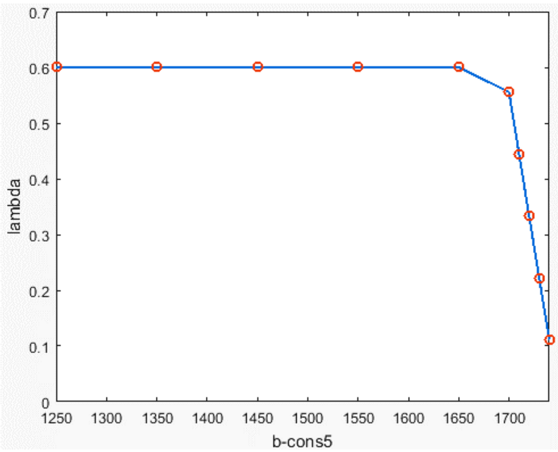
As an example, for sensitivity analysis in constraints, if we increase the expected value for the 5th constraint and keep the acceptable violation value constant, then the results for the sub-problems related to this value may change. For example, for sub-problem $MOLP_{52}$, results similar to those seen in *Table*

7 and *Fig. 6* are obtained. As can be seen in *Table 7*, by increasing the value on the right side from 1250 to 1740, the value of λ^* decreased from 0.6 to 0.1111 and the value of f^* increased from 34834 to 34886 and then decreased to 33038.

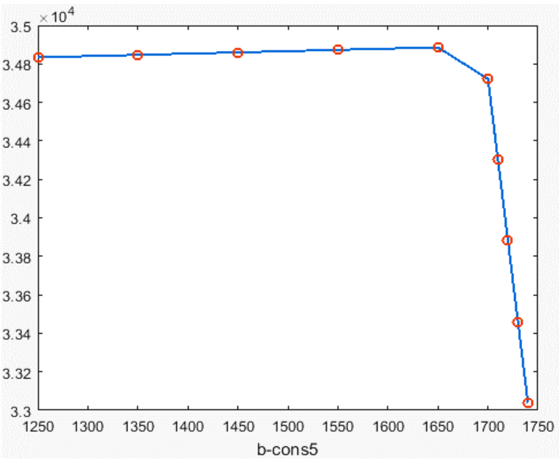
Table 7. Results of the changes in $\tilde{\tilde{b}}_5$ for the fifth constraint of MOLP$_{52}$.

$(\tilde{\tilde{b}}_5, \tilde{\tilde{d}}_{\tilde{\tilde{b}}_5})$	λ^*	x^*	f^*
(1250,50)	0.6000	(1200,0,30,1770,0,0,1230,0,1770,0,0,0)	34834
(1350,50)	0.6000	(1200,0,130,1670,0,0,1330,0,1670,0,0,0)	34847
(1450,50)	0.6000	(1200,0,230,1570,0,0,1430,0,1570,0,0,0)	34860
(1550,50)	0.6000	(1200,0,330,1470,0,0,1530,0,1470,0,0,0)	34873
(1650,50)	0.6000	(1200,0,430,1370,0,0,1630,0,1370,0,0,0)	34886
(1700,50)	0.5556	(1200,0,477.8,1322.2,0,0,1594.4,83.3,1322.2,0,0,0)	34722
(1710,50)	0.4444	(1200,0,482.2,1317.8,0,0,1390.6,291.7,1317.8,0,0,0)	34303
(1720,50)	0.3333	(1200,0,486.7,1313.3,0,0,1186.7,500,1313.3,0,0,0)	33881
(1730,50)	0.2222	(1200,0,491.1,1308.9,0,0,982.8,708.3,1308.9,0,0,0)	33460
(1740,50)	0.1111	(1200,0,495.6,1304.4,0,0,778.9,916.7,1304.4,0,0,0)	33038

In *Fig. 6*, the results obtained due to increasing the value on the right side for the fifth constraint and related to sub-problem *MOLP*$_{52}$ are depicted.



a.



b.

Fig. 6. a. Variations of λ^* by the changes in $\tilde{\tilde{b}}_5$ for the fifth constraint of MOLP$_{52}$; b. Variations of f^* by the changes in $\tilde{\tilde{b}}_5$ for the fifth constraint of MOLP$_{52}$.

8 | Conclusion

Uncertainties and their investigation manner in applied models are common research topics. In this study, an initial step was taken to apply the hesitant fuzzy programming to the uncertainties caused by these numbers in SC management problems. To this end, we extended and applied the method proposed by Ranjbar and Effati for symmetric and asymmetric HFLP problems for multi-objective fuzzy programming problems. Afterwards, we modeled a three-level four-objective SC problem in the hesitant fuzzy environment and provided an example to evaluate the efficiency of the proposed method. For 64 sub-problems, the results established in *Table 4* and shows the optimal expectation level, optimal point and the average objective function value. Due to the variety of responses, decision makers have a wide range of choices as the optimal response. The results presented in *Table 6* and *Fig. 5* showed that by keeping the expected value for the fourth objective function constant and increasing the acceptable violation value, the optimal response for sub-problem $MOLP_1$ has increasing or decreasing changes in the values of $\lambda^*$and f^* will be. Also, the results of increasing the expected value and keeping the acceptable violation value of the fifth constant illustrated differing changes in the values of $\lambda^*$and f^*. The exponential increase in the number of sub-problems with the increase in the number of opinions of decision makers and the increase in the number of goals and constraints are among the basic limitations in the application of this type of problems. Therefore, for problems with a higher volume, it is better to use more efficient algorithms such as heuristic and hybrid algorithms.

Among the benefits of this research were the opening up of a new view of applied research into SC management, group decision-making capability and, in addition, weight allocation for decision-makers.

Some of the innovations in this article are as follows:

- *Formulating the fuzzy symmetric multi-objective programming problem and its solution.*
- *Using weighted average objective function to solve sub-problems.*
- *Having ability to consider alpha cuts in each of the sub-problems and, hence, examine responses at different levels.*
- *Expressing a model of SC management with hesitant fuzzy approach and solving it using the proposed method.*
- *Providing strategies to improve model performance.*

To continue with the hesitant fuzzy number approach, the following points may be of interest for researchers:

1. Developing and interpreting uncertainties arising from uncertain fuzzy data for other SC-related areas.
2. Considering the uncertainties arising from fuzzy data over other model parameters.
3. Improving the solution methods presented in this paper to deal with uncertainties caused by fuzzy data.
4. Considering more goals or levels to solve the problem.
5. Implementing the method on higher-dimensional models, in particular solving them by combinatorial, heuristic, and meta-heuristic methods, and comparing responses with deterministic methods.
6. With the information obtained from the solutions presented, enabling managers to observe and identify the full range of outcomes, from the worst to best, for final decision-making.

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Basic Fuzzy Arithmetic Operations Using α–Cut for the Gaussian Membership Function

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Abstract

Currently fuzzy set theory has a wide range to model real life problems with incomplete or vague information which perfectly suits the reality and its application is theatrically increasing. This work explored the basic fuzzy operations with the Gaussian Membership using the α-cut method. As it is known that, the Gaussian membership function has a great role in modelling the fuzzy problems this is what impelled to explore its operation which can further be used in analysis of fuzzy problems. Primarily the basic operations which has been discussed here are addition, subtraction, multiplication, division, reciprocal, exponential, logarithmic and nth power.

Keywords: Fuzzy arithmetic, α–cut and Gaussian membership function.

1 | Introduction

Over the recent years, since its foundation by Zadeh [13] fuzzy sets and logic has been usefully in solving the real life problems which has partial or vague information which has wider scope of application and has dramatically increased in 1990's in area of decision making problem till this time for instance in the field like of pattern classification and information processing [7]. In the future by studying effectually and reconnoitering fuzzy implications can be the step towards the simulation of human thinking [3].

Furthermore, as fuzzy can be applied in different other areas like that of optimization which were specifically done by Shirin [9] in their work they proposed optimization solution to the problem by computing using three methods, which were the Bellman-Zadeh's method, Zimmerman's method, and fuzzy version of Simplex method, are compared to each other. Others are Tang et al. [12] and Sahayasudha and Vijayalakshmi [11] who did in fuzzy optimization and transportation problem, respectively.

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Largely, fuzzy arithmetic operation has a great role in the analysis of fuzzy problems a number of works has been done involving fuzzy arithmetic operations like that of Bobillo and Straccia [1], Dutta et al. [4], Mazeika et al. [7], Stefanini et al. [10], Sahayasudha and Vijayalakshmi [11], Raju and Jayagopal [8] and Iliadis et al. [6] but almost all of them did not use the approach of operations using α–cut for the Gaussian membership function in their works.

Other work is that of Hassanzadeh et al. [5] in their paper titled "an α-cut approach for fuzzy product and its use in computing solutions of fully fuzzy linear systems" whereby they used a regression model to obtain the membership function of the product.

The gaussian membership function has now been used in solving the fuzzy problems, for instance Mazeika et al. [7] in their work they used trapezoidal and Gaussian membership function and introduced a new general set approach to compute fuzzy sets based on interval analysis. They protracted their work techniques to handle multidimensional continuous membership functions, also Bundy and Wallen [2], Iliadis et al. [6] used gaussian distribution in their papers, whereby in this work we showed the Gaussian membership function using α–cut which can operated to both the continuous and discrete fuzzy set.

The core principle of the fuzzy set is the construction of the membership function and its operations in the fuzzy environment. Therefore, for this work focused on the basic arithmetic operation using alpha cut for the Gaussian function as this approach will be usefully for other scientists in handling and analyzing the real-life problem in the fuzzy environment.

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2 | Basic Concepts and Notations

2.1 | Fuzzy Set and Membership Function

Let X be a non-empty crisp set, a fuzzy set A is define as set of pairs $A = \{(x, \mu_A(x))\}$, where $x \in X$ and $\mu_A(x)$ is membership value for the corresponding crisp value $x \in X$ which is defined by a membership function as $\mu_A: X \to [0,1]$ [13].

There are many membership functions so far which has been commonly used are triangular function, trapezoidal function, singleton function, L–function, gamma function, S–function, Gaussian function, Sigmoidal and Pseudo–Exponential function [14].

2.2 | Normal Fuzzy Set

A fuzzy set A of the universe of discourse X is called a normal fuzzy set implying that there exists at least one $x \in X$ such that $\mu_A(x) = 1$.

2.3 | Support

The support of a fuzzy set A defined on X is a crisp set defined as

$$\text{Support(A)} = x\epsilon X: \mu_A(x) > 0.$$

2.4 | Fuzzy Number

A fuzzy set A defined on the set of real numbers $\mathbb{R}$ is said to be a fuzzy number if its membership function: $\mu_A: \mathbb{R} \to [0,1]$ has the following properties

I. A must be a normal fuzzy set.
II. A_α must be a closed interval for every a $\alpha \in (0,1]$.
III. The support of A, must be bounded.

2.5 | Gaussian Function with an Example

The Gaussian membership function of a crisp set A of a non-empty universal set X is defined by

$$\mu_A(x) = \exp\left[\frac{-(x-m)^2}{2\sigma^2}\right]. \tag{1}$$

For all $x \in X$ or generally $x \in \mathbb{R}$ as the crisp set, m can be taken as the mid value or mean and $\sigma > 0$ can be taken as the standard deviation of the crisp set. This Gaussian function will take a bell-shaped curve and the smaller σ the narrower the bell [14].

For example, to express a specific Gaussian membership function for the following discrete crisp set B which is chosen arbitrary as, $B = 2,3,5,9,8,14$, using the data for set B here we can find $m = 6.8$ as mean and $\sigma = 4.1$ as standard deviation therefore the membership function will be

$$\mu_B(x) = \exp\left[\frac{-(x-6.8)^2}{2 \times 4.1^2}\right]. \tag{2}$$

Solving using the given Gaussian membership function then we can have the fuzzy set which will be as $\beta = \{(2, 0.50\ ), (3, 0.65\ ), (5, 0.91), (8, 0.96), (9, 0.87), (14, 0.21)\}$.

Generally, the above Gaussian membership geometrically as continuous can be seen as shown:

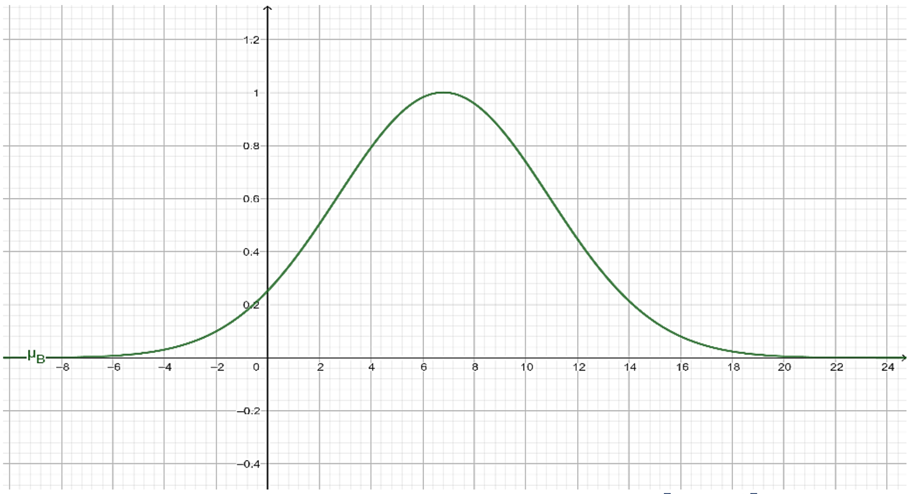

Fig. 1. Gaussian membership for $\mu_B(x) = \exp\left[\frac{-(x-6.8)^2}{2\times4.1^2}\right]$.

2.6 | Alpha (α)-Cut

Let X be a non- empty crisp set, an α–cut for a given fuzzy set A denoted by A_α is defined as the crisp set of all elements of A whose membership grades are greater than α,

$$\forall\ \alpha \in (0,1], \quad \text{that is } A_{\alpha} = \{x \in X | \mu_A(x) \geq \alpha\}. \tag{3}$$

Now from the Gaussian membership function given above we can find the alpha-cut as follows:

$$\mu_A(x) = \exp\left[\frac{-(x-m)^2}{2\sigma^2}\right] \geq \alpha. \tag{4}$$

To simplify the calculations, we have taken the alpha – cut for equality, that is

$$\exp\left[\frac{-(x-m)^2}{2\sigma^2}\right] = \alpha. \tag{5}$$

Then we solve for x to obtain the α–cut for the corresponding fuzzy set, apply logarithm throughout we have

$$\left[\frac{-(x-m)^2}{2\sigma^2}\right] = \ln\alpha. \tag{6}$$

$$x = m + \sigma\sqrt{-2\ln\alpha}.$$

Simplifying and taking the term with plus we have,

Therefore, the α – cut will be given by

$$A_{\alpha} = m + \sigma\sqrt{-2\ln\alpha}. \tag{7}$$

3 | Results and Discussions

Here we have explored all the basic operations and showing in detail how to find the membership of all the operations with α–cut using the Gaussian Membership function, the operations discussed here are addition, subtraction, multiplication, division, reciprocal, exponential, logarithmic and nth power. The graphs to help the analysis were drawn using GeoGebra Calculator suite for graphing software.

3.1 | Basic Operations

3.1.1 | Addition

Let the fuzzy sets A and B with their corresponding membership as $\mu_A(x) = \exp\left[\frac{-(x-m_A)^2}{2\sigma_A{}^2}\right]$, $\mu_B(x) = exp\left[\frac{-(x-m_B)^2}{2\sigma_B{}^2}\right]$ respectively Solving the α–cut for the two fuzzy set we have, $A_\alpha = m_A + \sigma_A\sqrt{-2\,ln\,\alpha}$ and $B_\alpha = m_B + \sigma_B\sqrt{-2\,ln\,\alpha}$ respectively for fuzzy set A and B.

By adding we have, $A_\alpha + B_\alpha = m_A + \sigma_A\sqrt{-2\,ln\,\alpha} + m_B + \sigma_B\sqrt{-2\,ln\,\alpha}.$

Upon simplifying we have, $A_\alpha + B_\alpha = (m_A + m_B) + (\sigma_A + \sigma_B)\sqrt{-2\ln\alpha}.$

Now to get the membership for addition we have to let $x = A_\alpha + B_\alpha$ and solve for α, that is $x = (m_A + m_B) + (\sigma_A + \sigma_B)\sqrt{-2\ln\alpha}$, then $\ln\alpha = \left(\frac{-(x-(m_A+m_B))^2}{2(\sigma_A+\sigma_B)^2}\right)$ henceforth, we have $\alpha = exp\left(\frac{-(x-(m_A+m_B))^2}{2(\sigma_A+\sigma_B)^2}\right)$, lastly the membership will be

$$\mu_{(A+B)}(x) = \exp\left(\frac{-(x-(m_A+m_B))^2}{2(\sigma_A+\sigma_B)^2}\right)\ \forall x \in \mathbb{R}. \tag{8}$$

For example, if we choose arbitrary the specific values for $\sigma_A = 1$, $\sigma_B = 2$, $m_A = 10$ and $m_B = 20$, therefore we will have the membership as, $\mu_A(x) = exp\left(\frac{-(x-10)^2}{2(1)^2}\right)$, $\mu_B(x) = exp\left(\frac{-(x-20)^2}{2(2)^2}\right)$ and $\mu_{(A+B)}(x) = exp\left(\frac{-\left(x-(10+20)\right)^2}{2(1+2)^2}\right)$. On the same axes their graphs will be see as in the *Fig. 2* below ($\mu_A(x)$ in green, $\mu_B(x)$ in Blue and $\mu_{(A+B)}(x)$ in Red).

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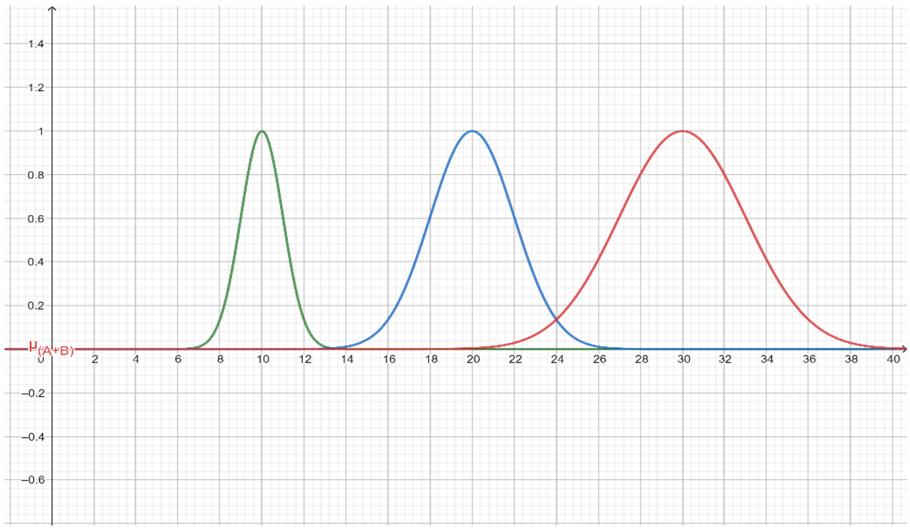

Fig. 2. Graphs of $\mu_A(x) = \exp\left(\frac{-(x-10)^2}{2(1)^2}\right)$, $\mu_B(x) = \exp\left(\frac{-(x-20)^2}{2(2)^2}\right)$ **and** $\mu_{(A+B)}(x) = \exp\left(\frac{-\left(x-(10+20)\right)^2}{2(1+2)^2}\right)$.

3.2 | Subtraction

Let the fuzzy sets A and B with their corresponding membership as $\mu_A(x) = exp\left[\frac{-(x-m_A)^2}{2{\sigma_A}^2}\right]$, $\mu_B(x) = exp\left[\frac{-(x-m_B)^2}{2{\sigma_B}^2}\right]$, respectively.

Solving the α–cut for the two fuzzy set we have, $A_\alpha = m_A + \sigma_A\sqrt{-2\ln\alpha}$ and $B_\alpha = m_B + \sigma_B\sqrt{-2\ln\alpha}$ respectively for fuzzy set A and B.

By subtracting we have, $A_\alpha - B_\alpha = m_A + \sigma_A\sqrt{-2\ln\alpha} - (m_B + \sigma_B\sqrt{-2\ln\alpha})$.

Upon simplifying we have, $A_\alpha - B_\alpha = (m_A - m_B) + (\sigma_A - \sigma_B)\sqrt{-2\ln\alpha}$.

Now to get the membership for subtraction we have to let $x = A_\alpha - B_\alpha$ and solve for α, that is $x = (m_A - m_B) + (\sigma_A - \sigma_B)\sqrt{-2\ln\alpha}$, then $\ln\alpha = \left(\frac{-(x-(m_A-m_B))^2}{2(\sigma_A-\sigma_B)^2}\right)$ hence, we have $\alpha = exp\left(\frac{-(x-(m_A-m_B))^2}{2(\sigma_A-\sigma_B)^2}\right)$, lastly the membership will be

$$\mu_{(A+B)}(x) = \exp\left(\frac{-(x-(m_A-m_B))^2}{2(\sigma_A-\sigma_B)^2}\right)\forall x \in \mathbb{R}. \tag{9}$$

For example, if we choose arbitrary the specific values for $\sigma_A = 1$, $\sigma_B = 2$, $m_A = 10$ and $m_B = 20$, therefore we will have the membership as, $\mu_A(x) = exp\left(\frac{-(x-10)^2}{2(1)^2}\right)$, $\mu_B(x) = exp\left(\frac{-(x-20)^2}{2(2)^2}\right)$ and $\mu_{(A-B)}(x) = exp\left(\frac{-\left(x-(10-20)\right)^2}{2(1-2)^2}\right)$.

On the same axes their graphs will be see as in the *Fig. 3* below $\mu_A(x)$ (in green, $\mu_B(x)$ in Blue and $\mu_{(A-B)}(x)$ in Red).

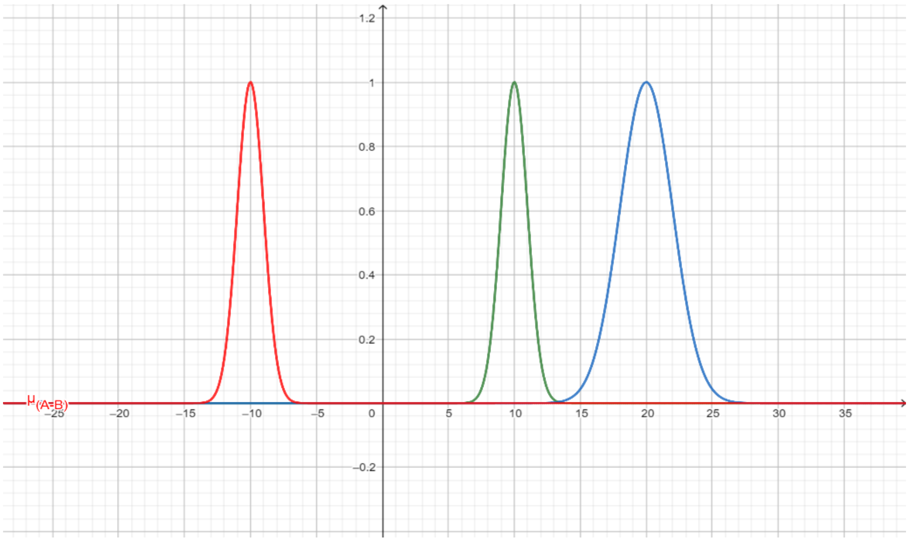

Fig. 3. Graphs of $\mu_A(x) = \exp\left(\frac{-(x-10)^2}{2(1)^2}\right)$, $\mu_B(x) = \exp\left(\frac{-(x-20)^2}{2(2)^2}\right)$ and $\mu_{(A-B)}(x) = \exp\left(\frac{-(x-(10-20))^2}{2(1-2)^2}\right)$.

3.3 | Multiplication

For the fuzzy sets A and B with their corresponding membership as $\mu_A(x) = exp\left[\frac{-(x-m_A)^2}{2\sigma_A{}^2}\right]$, $\mu_B(x) = exp\left[\frac{-(x-m_B)^2}{2\sigma_B{}^2}\right]$, respectively.

Solving the α – cut for the two fuzzy set we have $A_\alpha = m_A + \sigma_A\sqrt{-2\,ln\,\alpha}$ and $B_\alpha = m_B + \sigma_B\sqrt{-2\,ln\,\alpha}$ respectively for fuzzy set A and B.

Now, multiplication $A_\alpha * B_\alpha = (m_A + \sigma_A\sqrt{-2\,ln\,\alpha}) * (m_B + \sigma_B\sqrt{-2\,ln\,\alpha})$

$$\text{Let } x = (m_A + \sigma_A\sqrt{-2\ln\alpha}) * \left(m_B + \sigma_B\sqrt{-2\ln\alpha}\right).$$

$$x = (m_A m_B) + (m_B\sigma_A + m_A\sigma_B) * \left(\sqrt{-2\ln\alpha}\right) + (\sigma_A\sigma_B) * \left(\sqrt{-2\ln\alpha}\right)^2. \quad (10)$$

Upon simplifying and making α the subject we have

$$\alpha = \exp\left(-\frac{1}{2}\left(\frac{(m_B\sigma_A+m_A\sigma_B)\pm\sqrt{(m_B\sigma_A+m_A\sigma_B)^2+4(\sigma_A\sigma_B)*(x-(m_A m_B))}}{(2\sigma_A\sigma_B)}\right)^2\right). \quad (11)$$

Therefore,

$$\mu_{(A\times B)}(x) = \exp\left(-\frac{1}{2}\left(\frac{(m_B\sigma_A+m_A\sigma_B)+\sqrt{(m_B\sigma_A+m_A\sigma_B)^2+4(\sigma_A\sigma_B)*(x-(m_A m_B))}}{(2\sigma_A\sigma_B)}\right)^2\right). \quad (12)$$

or

$$\mu_{(A\times B)}(x) = \exp\left(-\frac{1}{2}\left(\frac{(m_B\sigma_A+m_A\sigma_B)-\sqrt{(m_B\sigma_A+m_A\sigma_B)^2+4(\sigma_A\sigma_B)*(x-(m_A m_B))}}{(2\sigma_A\sigma_B)}\right)^2\right). \quad (13)$$

For example, if we choose arbitrary the specific values for $\sigma_A = 1$, $\sigma_B = 2$, $m_A = 10$ and $m_B = 20$, therefore we will have the membership as, $\mu_A(x) = exp\left(\frac{-(x-10)^2}{2(1)^2}\right)$ and

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$$\mu_{(A\times B)}(x) = exp\left(-\frac{1}{2}\left(\frac{(20\times1+10\times2)+\sqrt{(20\times1+10\times2)^2+4(2\times1)(x-(10\times20))}}{(2\times1\times2)}\right)^2\right)$$ Simplifying, $\mu_{(A\times B)}(x) = exp\left(-\frac{1}{2}\left(\frac{40+\sqrt{8x}}{4}\right)^2\right)$.

On the same axes their graphs will be see as in the *Fig. 4* below $\mu_A(x)$ in green $\mu_{(A\times B)}(x)$ in Blue.

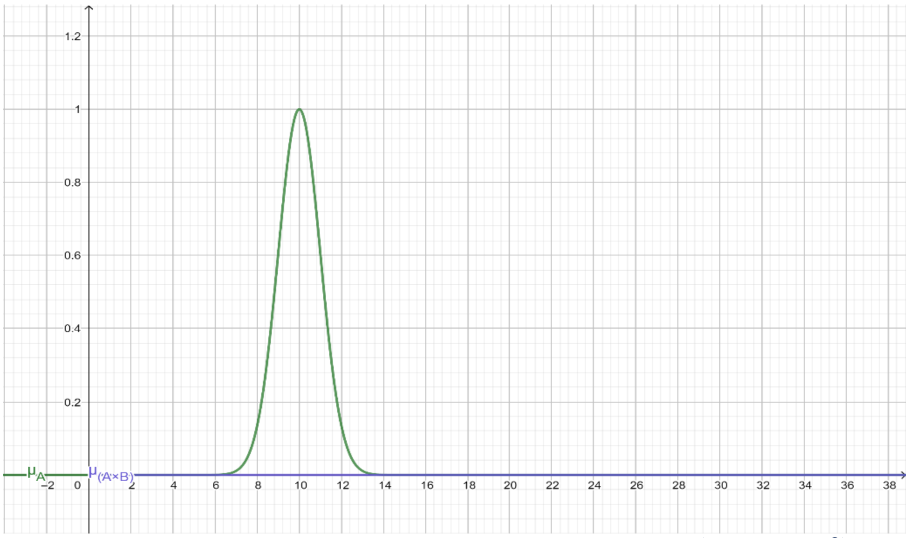

Fig. 4. Graphs $\mu_A(x) = \exp\left(\frac{-(x-10)^2}{2(1)^2}\right)$ and $\mu_{(A\times B)}(x) = \exp\left(-\frac{1}{2}\left(\frac{40+\sqrt{8x}}{4}\right)^2\right)$.

3.4 | Division

For the fuzzy sets A and B with their corresponding membership as $\mu_A(x) = exp\left[\frac{-(x-m_A)^2}{2{\sigma_A}^2}\right]$, $\mu_B(x) = exp\left[\frac{-(x-m_B)^2}{2{\sigma_B}^2}\right]$, respectively.

Solving the α–cut for the two fuzzy set we have, $A_\alpha = m_A + \sigma_A\sqrt{-2\ln\alpha}$ and $B_\alpha = m_B + \sigma_B\sqrt{-2\,ln\,\alpha}$ respectively for fuzzy set A and B. Dividing we have

$$\frac{A_\alpha}{B_\alpha} = \frac{m_A+\sigma_A\sqrt{-2\ln\alpha}}{m_B+\sigma_B\sqrt{-2\ln\alpha}}. \quad (14)$$

$$\text{Let } x = \frac{m_A+\sigma_A\sqrt{-2\ln\alpha}}{m_B+\sigma_B\sqrt{-2\ln\alpha}}.$$

Then, $x\left(m_B + \sigma_B\sqrt{-2\,ln\,\alpha}\right) = m_A + \sigma_A\sqrt{-2\,ln\,\alpha}$

Simplifying and making ∝ the subject we have,

$$\mu_{\frac{A}{B}}(x) = \exp\left(\frac{-(xm_B - m_A)^2}{(\sigma_A - \sigma_B)^2}\right). \quad (15)$$

For example, if we choose arbitrary the specific values for $\sigma_A = 1$, $\sigma_B = 2$, $m_A = 10$ and $m_B = 20$, therefore we will have the membership as, $\mu_A(x) = exp\left(\frac{-(x-10)^2}{2(1)^2}\right)$, $\mu_B(x) = exp\left(\frac{-(x-20)^2}{2(2)^2}\right)$ and $\mu_{\frac{A}{B}}(x) = exp\left(\frac{-(20x-10)^2}{(1)^2}\right)$. On the same axes their graphs will be see as in the *Fig. 5* below $\mu_A(x)$ in blue, $\mu_B(x)$ in Green and $\mu_{\left(\frac{A}{B}\right)}(x)$ in Red).

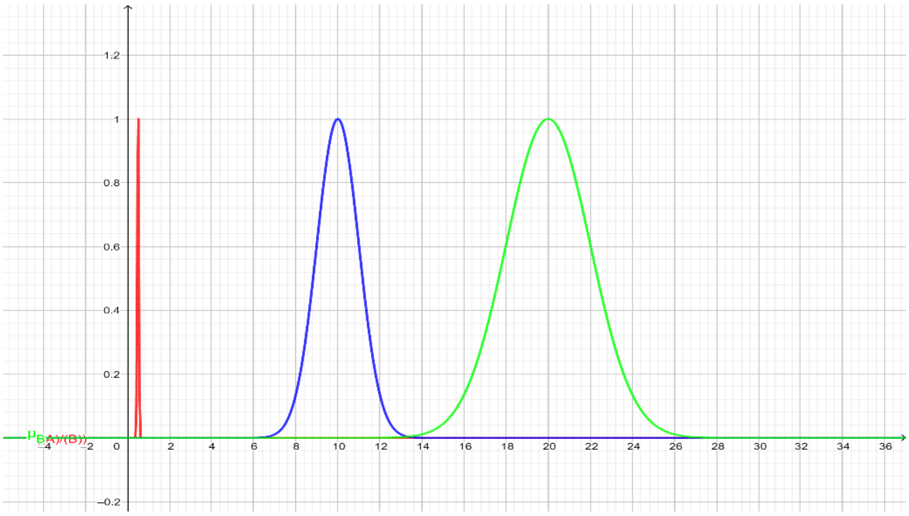

Fig. 5. Graphs of $\mu_A(x)$, $\mu_B(x)$ and $\mu_{(A/B)}(x)$.

3.5 | Reciprocal

For the fuzzy sets A with membership as $\mu_A(x) = exp\left[\frac{-(x-m_A)^2}{2\sigma_A{}^2}\right]$, by solving the α – cut for the given fuzzy set we have, $A_\alpha = m_A + \sigma_A\sqrt{-2\ln\alpha}$.

The reciprocal membership can be found as $\frac{1}{A_\alpha} = \frac{1}{m_A+\sigma_A\sqrt{-2\ln\alpha}}$. Then let $x = \frac{1}{m_A+\sigma_A\sqrt{-2\ln\alpha}}$,

$$xm_A + x\sigma_A\sqrt{-2\ln\alpha} = 1. \tag{16}$$

Making α the subject we have

$$\alpha = \exp\left(-\frac{1}{2}\left(\frac{xm_A - 1}{x\sigma_A}\right)^2\right). \tag{17}$$

Therefore, the reciprocal membership will be written as

$$\mu_{A^{-1}} = \exp\left(-\frac{1}{2}\left(\frac{xm_A - 1}{x\sigma_A}\right)^2\right). \tag{18}$$

For example, if we choose arbitrary the specific values for $\sigma_A = 1$ and $m_A = 10$ therefore we will have the membership as$\mu_A(x) = exp\left(\frac{-(x-10)^2}{2(1)^2}\right)$, and $\mu_{A^{-1}} = exp\left(-\frac{1}{2}\left(\frac{10x-1}{x}\right)^2\right)$.

On the same axes their graphs will be see as in the *Fig. 6* below $\mu_A(x)$ in Green and $\mu_{A^{-1}}$ in Blue.

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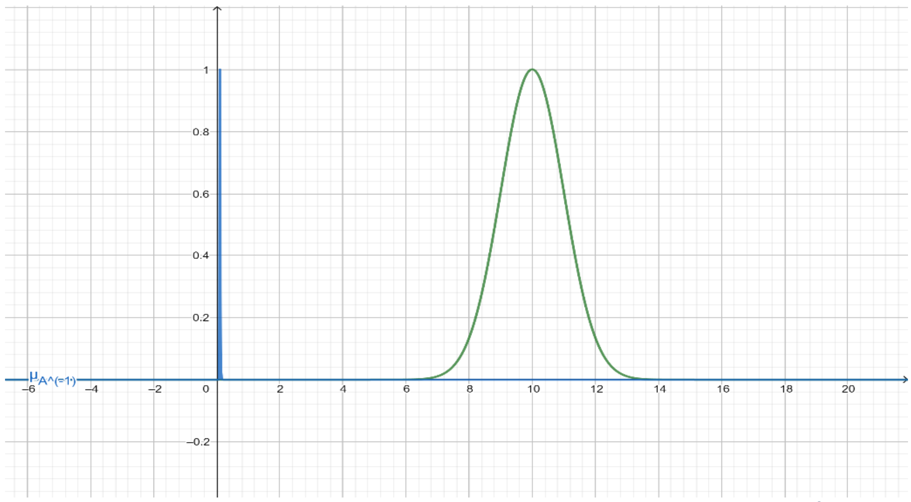

Fig. 6. Graphs of , $\mu_A(x) = \exp\left(\frac{-(x-10)^2}{2(1)^2}\right)$ and $\mu_{A^{-1}} = \exp\left(-\frac{1}{2}\left(\frac{10x-1}{x}\right)^2\right)$.

3.6 | Exponential

For the fuzzy sets A with membership as $\mu_A(x) = \exp\left[\frac{-(x-m_A)^2}{2\sigma_A{}^2}\right]$, by solving the α – cut for the given fuzzy set we have, $A_\alpha = m_A + \sigma_A\sqrt{-2\ln\alpha}$. Then apply exponential throughout, that is $exp\,(A_\alpha) = exp\,(m_A + \sigma_A\sqrt{-2\,ln\,\alpha})$.

$$\text{Let } x = \exp\left(m_A + \sigma_A\sqrt{-2\ln\alpha}\right).$$

Solve for α, that is $ln\,x = m_A + \sigma_A\sqrt{-2\,ln\,\alpha}$, then we have the member ship as

$$\mu_{e^A}(x) = \exp\left(-\frac{1}{2}\left(\frac{\ln x - m_A}{\sigma_A}\right)^2\right). \tag{19}$$

For example, if we choose arbitrary the specific values for $\sigma_A = 1$ and $m_A = 5$, therefore we will have the membership as, $\mu_A(x) = exp\left(\frac{-(x-5)^2}{2(1)^2}\right)$ and $\mu_{e^A}(x) = exp\left(-\frac{1}{2}\left(\frac{(ln\,x) - 5}{1}\right)^2\right)$.

On the same axes their graphs will be see as in the *Fig. 7*, $\mu_A(x)$ in Green and $\mu_{e^A}(x)$ in Red color.

3.7 | Logarithmic

For the fuzzy sets A with membership as $\mu_A(x) = exp\left[\frac{-(x-m_A)^2}{2\sigma_A{}^2}\right]$, by solving the α–cut for the given fuzzy set we have, $A_\alpha = m_A + \sigma_A\sqrt{-2\ln\alpha}$. Apply logarithm throughout, we have $ln\,A_\alpha = ln\left(m_A + \sigma_A\sqrt{-2\ln\alpha}\right)$.

Then, let $x = ln\left(m_A + \sigma_A\sqrt{-2\ln\alpha}\right)$, from this make α the subject.

Apply exponential throughout we have

$$e^x = \left(m_A + \sigma_A\sqrt{-2\ln\alpha}\right). \tag{20}$$

$$-2\ln\alpha = \left(\frac{e^x - m_A}{\sigma_A}\right)^2. \tag{21}$$

Then,

$$\alpha = \exp\left(-\frac{1}{2}\left(\frac{e^x - m_A}{\sigma_A}\right)^2\right). \tag{22}$$

Therefore, the membership function for logarithm is given by

$$\mu_{\ln(A)}(x) = \exp\left(-\frac{1}{2}\left(\frac{e^x - m_A}{\sigma_A}\right)^2\right). \tag{23}$$

For example, if we choose arbitrary the specific values for $\sigma_A = 1$ and $m_A = 10$, therefore we will have the membership as, $\mu_A(x) = exp\left(\frac{-(x-10)^2}{2(1)^2}\right)$ and $\mu_{ln(A)}(x) = exp\left(-\frac{1}{2}\left(\frac{e^x-10}{1}\right)^2\right)$.

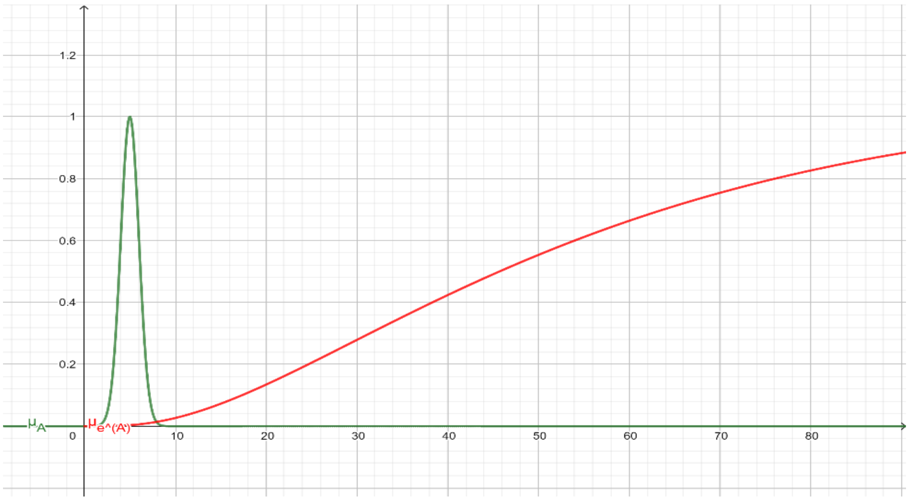

Fig. 7. Graphs of $\mu_A(x) = \exp\left(\frac{-(x-5)^2}{2(1)^2}\right)$ and $\mu_{e^A}(x) = \exp\left(-\frac{1}{2}\left(\frac{\ln x - 5}{1}\right)^2\right)$.

On the same axes their graphs will be see as in the *Fig. 8* below $\mu_A(x)$ in Green and $\mu_{ln\,A}(x)$ in Red.

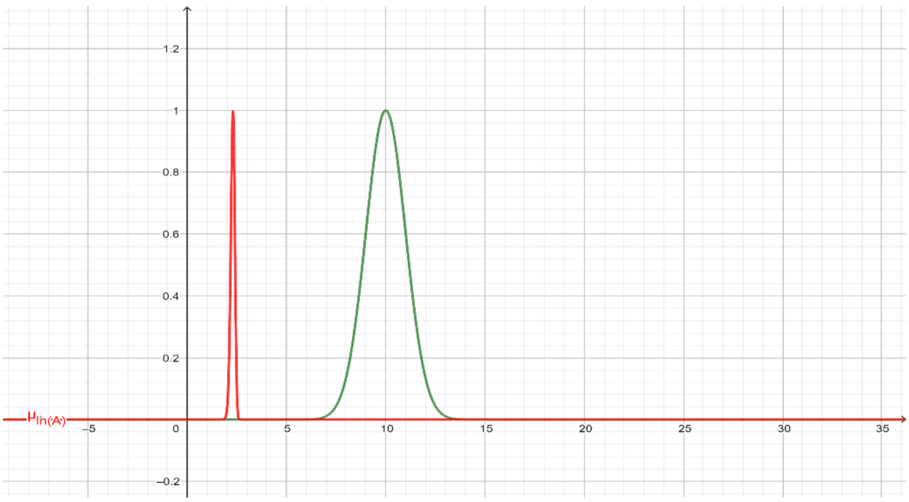

Fig. 8. Graphs of $\mu_A(x) = \exp\left(\frac{-(x-10)^2}{2(1)^2}\right)$ and $\mu_{\ln(A)}(x) = \exp\left(-\frac{1}{2}\left(\frac{e^x - 10}{1}\right)^2\right)$.

3.8 | The nth power

The n[th] power can be used to solve the fuzzy sets A with membership as $\mu_A(x) = \exp\left[\frac{-(x-m_A)^2}{2{\sigma_A}^2}\right]$,

Solving the α–cut for the given fuzzy set we have, $A_\alpha = m_A + \sigma_A\sqrt{-2\,ln\,\alpha}$.

For nth power we have, $A_\alpha{}^n = \left(m_A + \sigma_A\sqrt{-2\ln\alpha}\right)^n \;\; \forall n\epsilon\mathbb{R}$.

Letting $x = A_\alpha{}^n$, that is

$$x = \left(m_A + \sigma_A\sqrt{-2\ln\alpha}\right)^n \;\; \forall n\epsilon\mathbb{R}. \tag{24}$$

Simplifying we have, $\sqrt[n]{x} = \left(m_A + \sigma_A\sqrt{-2\ln\alpha}\right) \; \forall n\epsilon\mathbb{R}$.

Then making α the subject we get

$$\mu_{(A^n)}(x) = \exp\left(-\frac{1}{2}\left(\frac{\sqrt[n]{x} - m_A}{\sigma_A}\right)^2\right) \;\; \forall n\epsilon\mathbb{R}. \tag{25}$$

For example, if we choose arbitrary the specific values for $\sigma_A = 1$, $m_A = 10$ and $n = 1.3$ therefore we will have the membership as, $\mu_A(x) = exp\left(\frac{-(x-10)^2}{2(1)^2}\right)$ and $\mu_{(A^n)}(x) = exp\left(-\frac{1}{2}\left(\frac{\sqrt[1.3]{x} - 10}{1}\right)^2\right)$.

On the same axes their graphs will be see as in the *Fig. 9* below $\mu_A(x)$ in Green and $\mu_{(A^n)}(x)$ in Red.

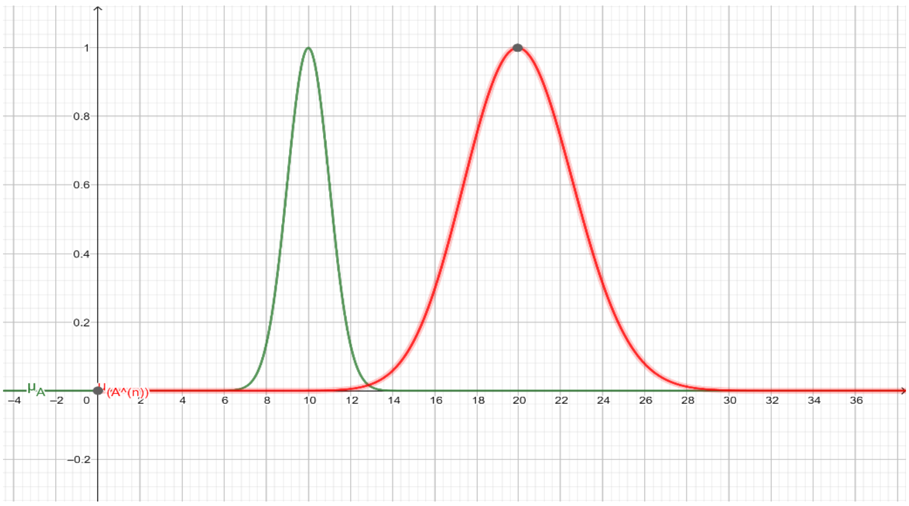

Fig. 9. Graphs of $\mu_A(x) = \exp\left(\frac{-(x-10)^2}{2(1)^2}\right)$and $\mu_{(A^{1.3})}(x) = \exp\left(-\frac{1}{2}\left(\frac{\sqrt[1.3]{x} - 10}{1}\right)^2\right)$.

4 | Conclusions

We have explored the basic operations which are, addition, subtraction, multiplication, division, logarithm, reciprocal, exponential and n^{th} power for the fuzzy set with Gaussian membership function by using the alpha–cut. The sightseen operations can further be used in analysis of fuzzy sets with Gaussian Membership and using the alpha-cut method make things easier for the calculations of all the basic operations. Therefore, we propose this approach to be used when analyzing the fuzzy problem with Gaussian Membership.

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Some Picture Fuzzy Mean Operators and Their Applications in Decision-Making

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Abstract

Picture fuzzy set is the generalization of fuzzy set and intuitionistic fuzzy set. It is useful for handling uncertainty by considering the positive membership, neutral membership and negative membership degrees independently for each element of a universal set. The main objective of this article is to develop some picture fuzzy mean operators, including Picture Fuzzy Harmonic Mean (PFHM), Picture Fuzzy Weighted Harmonic Mean (PFWHM), Picture Fuzzy Arithmetic Mean (PFAM), Picture Fuzzy Weighted Arithmetic Mean (PFWAM), Picture Fuzzy Geometric Mean (PFGM) and Picture Fuzzy Weighted Geometric Mean (PFWGM), to aggregate the picture fuzzy sets. Moreover, we discuss some relevant properties of these operators. Furthermore, we apply these mean operators to make decisions with practical examples. Finally, to show the efficiency and the validity of the proposed operators, we compare our results with the results of existing methods and concluded from the comparison that our proposed methods are more effective and reliable.

Keywords: Picture fuzzy set, Harmonic mean operator, Arithmetic mean operator, Geometric mean operator.

1 | Introduction

Many branches of science and engineering and also in the field of medical science, management science, economics, environmental science and so on, we face various problems where data are more ambiguous than precise. To describe such ambiguous data Zadeh [31] introduced the concept of fuzzy set in 1965. Then this was successfully applied in different branches of science and engineering by a host of researchers. There are some generalizations of fuzzy set according to the application in different fields of our real life problems. One of the generalization of fuzzy set is intuitionistic fuzzy fuzzy [1] which is capable to describe uncertainty more precisely than fuzzy set by taking positive membership and negative membership of an element of a universal set. But in many cases of our real life, we face some problems, where the term neutrality becomes essential to describe uncertainty. Voting is an example of such situation, where the human voters may be divided into four groups of those who: vote for, abstain, vote against, the refusal of the voting.

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To describe such situation Cuong and Kreinovich [3] developed the latest generalization of fuzzy set as picture fuzzy set which is the direct extension of intuitionistic fuzzy set. Later a huge amount of works have been emerged on diverse aspects of picture fuzzy sets and their applications [see ([2], [4]-[6], [7], [10], [11], [13]-[15], [17], [19]-[24], [27], [30], [32])]. The averaging operators on picture fuzzy sets along their applications are also becoming a great attention by numerous researchers. In 2017, Wei [28] discussed the arithmetic and geometric operations for picture fuzzy sets and applied them in multiple attribute decision-making problems. Khan et al. [11], [12] investigated the information aggregation operators' method under the picture fuzzy environment with the help of Einstein norms operations and applied a group decision-making problem in 2019 [12]. In 2018, Wei [30] discussed the multiple attribute decision-making problem based on the arithmetic, geometric aggregation operators and Hamacher operations of picture fuzzy [30]. Luo and Long [16] studied picture fuzzy geometric aggregation operators based on a trapezoidal fuzzy number and applied it to Multi-Attribute Decision-Making (MADM) and pattern recognition in 2021. Picture fuzzy aggregation operators are also discussed some researchers (see, [8], [9], [12], [18], [25], [26], [28]). In the above aggregation operators related to picture fuzzy set, the authors described the score function in such a way, where the properties of neutrality coincided with negative membership degree. But the properties neutrality should coincide of the term of positive membership degree. So in this article, we redefine the score function, where the the properties of neutrality coincide with the positive membership degree. On the other hand the existing aggregation operators are more complicated, because the aggregated value cannot find from the direct definition of the aggregation. In this article, to overcome these difficulties, we have defined some mean operators, where the aggregated value can be found from direct definition. Some related properties of the operators are also explored. The practical application of these methods is also described with comparison in existing methods.

The article is organized as follows: In Section 2, some basic definitions are given which are essential to rest of the paper. In Section 3, Picture Fuzzy Harmonic Mean (PFHM) operator and Picture Fuzzy Weighted Harmonic Mean (PFWHM) operator are discussed. In Section 4, picture fuzzy arithmetic operator and Picture Fuzzy Weighted Arithmetic Mean (PFWAM) operator are discussed. In Section 5, Picture Fuzzy Geometric Mean (PFGM) operator and picture Fuzzy Weighted Geometric Mean (PFWGM) operator are deliberated. In Section 6, the application of the proposed methods is illustrated. In Section 7, the comparison studies are showed.

2 | Preliminaries

In this section, we recall some basic definitions which are used in later sections.

Definition 1 ([31]). Let X be non-empty set. A fuzzy set A in X is given by

$$A = \{(x, \mu_A(x)) : x \in X\}, \text{ where } \mu_A : X \to [0,1].$$

Definition 2 ([1]). Let X be non-empty set. An intuitionistic fuzzy set A in X is given by

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}, \text{ where } \mu_A : X \to [0,1] \text{ and } \nu_A : X \to [0,1].$$

The values $\mu_A(x)$ and $\nu_A(x)$ represent the membership degree and non-membership degree of the element x to the set A respectively. The pair $(\mu_A(x),\ \nu_A(x))$ is called intuitionistic fuzzy value satisfying the condition,

$$0 \le \mu_A(x) + \nu_A(x) \le 1 \ \forall x \in X.$$

For any intuitionistic fuzzy set A on the universal set X, for $x \in X$,

$$\pi_A(x) = 1 - (\mu_A(x) + \nu_A(x)).$$

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is called the hesitancy degree (or intuitionistic fuzzy index) of an element x in A. It is the degree of indeterminacy membership of the element x whether belonging to A or not.

Obviously, $0 \le \pi_A(x) \le 1$ for any $x \in X$.

Definition 3 ([3]). A picture fuzzy set A on a universal set $X \neq \emptyset$ is of the form

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) : x \in X\},$$

where $\mu_A(x) \in [0,1]$ is the degree of positive membership, $\eta_A(x) \in [0,1]$ is the degree of neutral membership and $\nu_A(x) \in [0,1]$ is the degree of negative membership of x in A, where $\mu_A(x), \eta_A(x)$ and $\nu_A(x)$ satisfy the following condition,

$$0 \le \mu_A(x) + \eta_A(x) + \nu_A(x) \le 1 \ \forall x \in X.$$

Here $1 - (\mu_A(x) + \eta_A(x) + \nu_A(x)); \forall x \in X$ is called the degree of refusal membership of x in A. The pair (μ_A, η_A, ν_A) is called picture fuzzy value.

Definition 4 ([3]). Let $A = (\mu_A, \eta_A, \nu_A)$ and $B = (\mu_B, \eta_B, \nu_B)$ be two picture fuzzy values of X. Then

I. $A \le B$ iff $\mu_A \le \mu_B, \eta_A \le \eta_B$ and $\nu_A \ge \nu_B$.
II. $A = B$ iff $\mu_A = \mu_B, \eta_A = \eta_B$ and $\nu_A = \nu_B$.

Definition 5. Let $A = (\mu_A, \eta_A, \nu_A)$ be a picture fuzzy value. Then the score function $S(A)$ and the accuracy function $H(A)$ are defined as

$$S(A) = \mu_A + \eta_A - \nu_A.$$

and

$$H(A) = \mu_A + \eta_A + \nu_A,$$

where $S(A) \in [-1,1]$ and $H(A) \in [0,1]$.

Definition 6. Let $A = (\mu_A, \eta_A, \nu_A)$ and $A = (\mu_A, \eta_A, \nu_A)$ be two picture fuzzy values. Then the following comparison rules can be used:

I. If $S(A) > S(B)$, then A is greater than B, denoted by $A \succ B$.
II. If $S(A) = S(B)$, then
III. $H(A) = H(B)$, implies that A is equivalent to B, denoted by $A \sim B$.
IV. $H(A) > H(B)$, implies that A is greater than B, denoted by $A \succ B$.

3 | PFHM Operators

Definition 7. Let $A_i = (\mu_{A_i}, \eta_{A_i}, \nu_{A_i})$ $(i = 1, 2, \cdots, n)$ be collection of picture fuzzy values. Then the PFHM operator is mapping $PFHM: A^n \to A$ such that

$$PFHM(A_1, A_2, \cdots, A_n) = \begin{pmatrix} n\left(\sum_{i=1}^{n}(\mu_{A_i})^{-1}\right)^{-1}, n\left(\sum_{i=1}^{n}(\eta_{A_i})^{-1}\right)^{-1}, \\ n\left(\sum_{i=1}^{n}(\nu_{A_i})^{-1}\right)^{-1} \end{pmatrix}.$$

Definition 8. Let $A_i = (\mu_{A_i}, \eta_{A_i}, \nu_{A_i})$ $(i = 1,2,\cdots,n)$ be collection of picture fuzzy values and $w = (w_1, w_2, \cdots, w_n)^T$ be the weighting vector of A_i $(i = 1,2,\cdots,n)$ such that $w_i \in [0,1]$, $(i = 1,2,\cdots,n)$ and $\sum_{i=1}^{n} w_i = 1$. Then the PFWHM operator is a mapping $PFWHM: A^n \to A$ such that

$$\mathrm{PFWHM}(A_1, A_2, \cdots, A_n) = \left(\left(\sum_{i=1}^{n} \frac{w_i}{\mu_{A_i}}\right)^{-1}, \left(\sum_{i=1}^{n} \frac{w_i}{\eta_{A_i}}\right)^{-1}, \left(\sum_{i=1}^{n} \frac{w_i}{\nu_{A_i}}\right)^{-1}\right),$$

where $\mu_{A_i}, \eta_{A_i}, \nu_{A_i} \neq 0$.

The following axioms are satisfied for *PFHM* and *PFWHM*:

Theorem 1 (Idempotency). Let $A_i = (\mu_{A_i}, \eta_{A_i}, \nu_{A_i})$ $(i = 1,2,\cdots,n)$ be collection of picture fuzzy values.

If $A_i = A$, $(i = 1,2,\cdots,n)$, then

$$\mathrm{PFHM}(A_1, A_2, \cdots, A_n) = A,$$

And

$$\mathrm{PFWHM}(A_1, A_2, \cdots, A_n) = A.$$

Proof. For $A_i = A$ and $\sum_{i=1}^{n} w_i = 1$, we have

$$\mathrm{PFHM}(A_1, A_2, \cdots, A_n) = \begin{pmatrix} n\left(\sum_{i=1}^{n}(\mu_{A_i})^{-1}\right)^{-1}, n\left(\sum_{i=1}^{n}(\eta_{A_i})^{-1}\right)^{-1}, \\ n\left(\sum_{i=1}^{n}(\nu_{A_i})^{-1}\right)^{-1} \end{pmatrix} =$$

$$\begin{pmatrix} n\left(\sum_{i=1}^{n}(\mu_{A})^{-1}\right)^{-1}, n\left(\sum_{i=1}^{n}(\eta_{A})^{-1}\right)^{-1}, \\ n\left(\sum_{i=1}^{n}(\nu_{A})^{-1}\right)^{-1} \end{pmatrix} = \left(\frac{n}{n(\mu_A)^{-1}}, \frac{n}{n(\eta_A)^{-1}}, \frac{n}{n(\nu_A)^{-1}}\right) = (\mu_A, \eta_A, \nu_A) = A.$$

And where

$$\mathrm{PFWHM}(A_1, A_2, \cdots, A_n) = \left(\left(\sum_{i=1}^{n} \frac{w_i}{\mu_{A_i}}\right)^{-1}, \left(\sum_{i=1}^{n} \frac{w_i}{\eta_{A_i}}\right)^{-1}, \left(\sum_{i=1}^{n} \frac{w_i}{\nu_{A_i}}\right)^{-1}\right) \mu_{A_i}, \eta_{A_i}, \nu_{A_i} \neq 0$$

$$= \left(\left(\sum_{i=1}^{n} \frac{w_i}{\mu_A}\right)^{-1}, \left(\sum_{i=1}^{n} \frac{w_i}{\eta_A}\right)^{-1}, \left(\sum_{i=1}^{n} \frac{w_i}{\nu_A}\right)^{-1}\right) =$$

$$\begin{pmatrix} \left(\sum_{i=1}^{n} w_i\right)^{-1}\left((\mu_A)^{-1}\right)^{-1}, \left(\sum_{i=1}^{n} w_i\right)^{-1}\left((\eta_A)^{-1}\right)^{-1}, \\ \left(\sum_{i=1}^{n} w_i\right)^{-1}\left((\nu_A)^{-1}\right)^{-1} \end{pmatrix}; \ \sum_{i=1}^{n} w_i = 1 = (\mu_A, \eta_A, \nu_A) = A.$$

Theorem 2 (Monotonicity). If $A_i \leq A_i^*$, then

$$\mathrm{PFHM}(A_1, A_2, \cdots, A_n) \leq \mathrm{PFHM}(A^*_1, A^*_2, \cdots, A^*_n).$$

And

$$\mathrm{PFWHM}(A_1, A_2, \cdots, A_n) \leq \mathrm{PFWHM}(A^*_1, A^*_2, \cdots, A^*_n).$$

Proof.

$$\mathrm{PFHM}(\mathrm{A}_1, \mathrm{A}_2, \cdots, \mathrm{A}_n) - \mathrm{PFHM}(\mathrm{A}^*_1, \mathrm{A}^*_2, \cdots, \mathrm{A}^*_n)$$

$$= \begin{pmatrix} \dfrac{n}{\frac{1}{\mu_{A_1}} + \frac{1}{\mu_{A_2}} + \cdots + \frac{1}{\mu_{A_n}}} - \dfrac{n}{\frac{1}{\mu_{A_1^*}} + \frac{1}{\mu_{A_2^*}} + \cdots + \frac{1}{\mu_{A_n^*}}}, \\ \dfrac{n}{\frac{1}{\eta_{A_1}} + \frac{1}{\eta_{A_2}} + \cdots + \frac{1}{\eta_{A_n}}} - \dfrac{n}{\frac{1}{\eta_{A_1^*}} + \frac{1}{\eta_{A_2^*}} + \cdots + \frac{1}{\eta_{A_n^*}}}, \\ \dfrac{n}{\frac{1}{\nu_{A_1}} + \frac{1}{\nu_{A_2}} + \cdots + \frac{1}{\nu_{A_n}}} - \dfrac{n}{\frac{1}{\nu_{A_1^*}} + \frac{1}{\nu_{A_2^*}} + \cdots\cdots + \frac{1}{\nu_{A_n^*}}} \end{pmatrix} \leq 0.$$

Since $A_i \leq A_i^*$ or $\frac{1}{A_i} \geq \frac{1}{A_i^*}$, for $i = 1, 2, \cdots, n$.

Similarly, we can prove that

$$\mathrm{PFWHM}(\mathrm{A}_1, \mathrm{A}_2, \cdots, \mathrm{A}_n) - \mathrm{PFWHM}(\mathrm{A}_1^*, \mathrm{A}_2^*, \cdots, \mathrm{A}_n^*) \leq 0.$$

Theorem 4 (Boundedness). Let $A_{min} = min(A_1, A_2, \cdots, A_n)$ and $A_{max} = max(A_1, A_2, \cdots A_n)$, for $i = 1, 2, \cdots, n$. Then $A_{min} \leq PFHM(A_1, A_2, \cdots, A_n) \leq A_{max}$ and $A_{min} \leq PFWHM(A_1, A_2, \cdots A_n) \leq A_{max}$.

Proof. Boundedness is the consequence of monotonicity and idempotency.

Theorem 5 (Commutatively). *If* $\left(A_1^0, A_2^0, \cdots, A_n^0\right)$ be any permutation of $(A_1, A_2, \cdots, A_n)$, then

$$\mathrm{PFHM}(\mathrm{A}_1, \mathrm{A}_2, \cdots, \mathrm{A}_n) = \mathrm{PFHMO}\left(\mathrm{A}^0_1, \mathrm{A}^0_2, \cdots, \mathrm{A}^0_n\right).$$

And

$$\mathrm{PFWHM}(\mathrm{A}_1, \mathrm{A}_2, \cdots, \mathrm{A}_n) = \mathrm{PFWHM}\left(\mathrm{A}^0_1, \mathrm{A}^0_2, \cdots, \mathrm{A}^0_n\right).$$

Proof.

$$\mathrm{PFHM}(\mathrm{A}_1, \mathrm{A}_2, \cdots, \mathrm{A}_n) - \mathrm{PFHMO}\left(\mathrm{A}^0_1, \mathrm{A}^0_2, \cdots, \mathrm{A}^0_n\right)$$

$$= \begin{pmatrix} n\left(\sum_{i=1}^{n}\left(\mu_{A_i}\right)^{-1}\right)^{-1} - n\left(\sum_{i=1}^{n}\left(\mu_{A_i^0}\right)^{-1}\right)^{-1}, \\ n\left(\sum_{i=1}^{n}\left(\eta_{A_i}\right)^{-1}\right)^{-1} - n\left(\sum_{i=1}^{n}\left(\eta_{A_i^0}\right)^{-1}\right)^{-1}, \\ n\left(\sum_{i=1}^{n}\left(\nu_{A_i}\right)^{-1}\right)^{-1} - n\left(\sum_{i=1}^{n}\left(\nu_{A_i^0}\right)^{-1}\right)^{-1} \end{pmatrix} = 0,$$

because $\left(A_1^0, A_2^0, \cdots\cdots, A_n^0\right)$ be any permutation of $(A_1, A_2, \cdots\cdots, A_n)$.

Hence, we have

$$\mathrm{PFHM}(\mathrm{A}_1, \mathrm{A}_2, \cdots, \mathrm{A}_n) = \mathrm{PFHM}\left(\mathrm{A}^0_1, \mathrm{A}^0_2, \cdots, \mathrm{A}^0_n\right)$$

Again,

$$\mathrm{PFWHM}(A_1, A_2, \cdots\cdots, A_n) - \mathrm{PFWHM}(A^0_1, A^0_2, \cdots\cdots, A^0_n)$$
$$= \begin{pmatrix} \left(\sum_{i=1}^{n} w_i (\mu_{A_i})^{-1}\right)^{-1} - \left(\sum_{i=1}^{n} w_i (\mu_{A_i^0})^{-1}\right)^{-1}, \\ \left(\sum_{i=1}^{n} w_i (\eta_{A_i})^{-1}\right)^{-1} - \left(\sum_{i=1}^{n} w_i (\eta_{A_i^0})^{-1}\right)^{-1}, \\ \left(\sum_{i=1}^{n} w_i (\nu_{A_i})^{-1}\right)^{-1} - \left(\sum_{i=1}^{n} w_i (\nu_{A_i^0})^{-1}\right)^{-1} \end{pmatrix}.$$

Because $(A_1^0, A_2^0, \cdots, A_n^0)$ be any permutation of $(A_1, A_2, \cdots, A_n)$.

Hence, we have

$$\mathrm{WHM}(A_1, A_2, \cdots, A_n) = \mathrm{PFWHM}(A^0_1, A^0_2, \cdots, A^0_n).$$

4 | Picture Fuzzy Arithmetic Mean (PFAM) Operators

Definition 9. Let $A_i = (\mu_{A_i}, \eta_{A_i}, \nu_{A_i})$ $(i = 1, 2, \cdots, n)$ be collection of picture fuzzy values. Then the PFAM operator is mapping $PFAM: A^n \to A$ such that

$$\mathrm{PFAM}(A_1, A_2, \cdots, A_n) = \left(\frac{1}{n}\sum_{i=1}^{n} \mu_{A_i}, \frac{1}{n}\sum_{i=1}^{n} \eta_{A_i}, \frac{1}{n}\sum_{i=1}^{n} \nu_{A_i}\right).$$

Definition 10. Let $A_i = (\mu_{A_i}, \eta_{A_i}, \nu_{A_i})$ $(i = 1, 2, \cdots, n)$ be collection of picture fuzzy values and $w = (w_1, w_2, \cdots, w_n)^T$ be the weighting vector of A_i $(i = 1, 2, \cdots, n)$ such that $w_i \in [0,1]$, $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} w_i = 1$. Then the PFWAM operator is a mapping $PFWAM: A^n \to A$ such that

$$\mathrm{PFWAM}(A_1, A_2, \cdots, A_n) = \left(\frac{1}{n}\sum_{i=1}^{n} w_i\mu_{A_i}, \frac{1}{n}\sum_{i=1}^{n} w_i\eta_{A_i}, \frac{1}{n}\sum_{i=1}^{n} w_i\nu_{A_i}\right).$$

The following axioms are satisfied for $PFAM$ and $PFWAM$.

Theorem 6 (Idempotency). Let $A_i = (\mu_{A_i}, \eta_{A_i}, \nu_{A_i})$ $(i = 1, 2, \cdots, n)$ be collection of picture fuzzy values.

If $A_i = A$, $(i = 1, 2, \cdots, n)$, then $PFAM(A_1, A_2, \cdots, A_n) = A$ and $PFWAM(A_1, A_2, \cdots, A_n) = A$.

Proof. For $A_i = A$ and $\sum_{i=1}^{n} w_i = 1$, we have

$$\mathrm{PFAM}(A_1, A_2, \cdots, A_n) = \left(\frac{1}{n}\sum_{i=1}^{n} \mu_{A_i}, \frac{1}{n}\sum_{i=1}^{n} \eta_{A_i}, \frac{1}{n}\sum_{i=1}^{n} \nu_{A_i}\right)$$
$$= \left(\frac{1}{n}\sum_{i=1}^{n} \mu_{A}, \frac{1}{n}\sum_{i=1}^{n} \eta_{A}, \frac{1}{n}\sum_{i=1}^{n} \nu_{A}\right)$$
$$= \left(\frac{1}{n} n\mu_A, \frac{1}{n}. n\eta_A, \frac{1}{n}. n\nu_A\right) = (\mu_A, \eta_A, \nu_A) = A.$$

Theorem 7 (Monotonicity). If $A_i \le A_i^*$, then

$$\mathrm{PFAM}(A_1, A_2, \cdots, A_n) \le \mathrm{PFAM}(A^*_1, A^*_2, \cdots, A^*_n).$$

And

$$\mathrm{PFWAM}(A_1, A_2, \cdots, A_n) \le \mathrm{PFWAM}(A^*_1, A^*_2, \cdots, A^*_n).$$

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Proof.

$$\mathrm{PFAM}(A_1, A_2, \cdots, A_n) - \mathrm{PFAM}(A^*_1, A^*_2, \cdots, A^*_n) = \begin{pmatrix} \frac{\mu_{A_1}+\mu_{A_2}+\cdots\cdots+\mu_{A_n}}{n} - \frac{\mu_{A_1^*}+\mu_{A_2^*}+\cdots\cdots+\mu_{A_n^*}}{n}, \\ \frac{\eta_{A_1}+\eta_{A_2}+\cdots\cdots+\eta_{A_n}}{n} - \frac{\eta_{A_1^*}+\eta_{A_2^*}+\cdots\cdots+\eta_{A_n^*}}{n}, \\ \frac{\nu_{A_1}+\nu_{A_2}+\cdots\cdots+\nu_{A_n}}{n} - \frac{\nu_{A_1^*}+\nu_{A_2^*}+\cdots\cdots+\nu_{A_n^*}}{n} \end{pmatrix} \leq 0.$$

Since $A_i \leq A_i^*$ or $\frac{1}{A_i} \geq \frac{1}{A_i^*}$, for $i = 1, 2, \cdots\cdots\cdots, n$.

Similarly, we can prove that

$$\mathrm{PFWAM}(A_1, A_2, \cdots, A_n) - \mathrm{PFWAM}(A^*_1, A^*_2, \cdots, A^*_n) \leq 0.$$

This proves the monotonicity of *PFAM* and *PFWAM*.

Theorem 8 (Boundedness). Let $A_{min} = min(A_1, A_2, \cdots, A_n)$ and $A_{max} = max(A_1, A_2, \cdots A_n)$, for $i = 1, 2, \cdots, n$, then $A_{min} \leq PFAM(A_1, A_2, \cdots, A_n) \leq A_{max}$ and $A_{min} \leq PFWAM(A_1, A_2, \cdots A_n) \leq A_{max}$.

Proof. Boundedness is the consequence of monotonicity and idempotency.

Theorem 9 (Commutatively). If $\left(A_1^0, A_2^0, \cdots, A_n^0\right)$ be any permutation of $(A_1, A_2, \cdots, A_n)$, then

$$\mathrm{PFAM}(A_1, A_2, \cdots, A_n) = \mathrm{PFAM}\left(A^0_1, A^0_2, \cdots, A^0_n\right),$$

and

$$\mathrm{PFWAM}(A_1, A_2, \cdots, A_n) = \mathrm{PFWAM}\left(A^0_1, A^0_2, \cdots, A^0_n\right).$$

Proof.

$$\mathrm{PFAM}(A_1, A_2, \cdots, A_n) - \mathrm{PFAM}\left(A^0_1, A^0_2, \cdots, A^0_n\right) = \begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n}\mu_{A_i} - \frac{1}{n}\sum_{i=1}^{n}\mu_{A_i^0}, \\ \frac{1}{n}\sum_{i=1}^{n}\eta_{A_i} - \frac{1}{n}\sum_{i=1}^{n}\eta_{A_i^0}, \\ \frac{1}{n}\sum_{i=1}^{n}\nu_{A_i} - \frac{1}{n}\sum_{i=1}^{n}\nu_{A_i^0} \end{pmatrix} = 0$$

Hence, we have

$$\mathrm{PFAM}(A_1, A_2, \cdots, A_n) = \mathrm{PFAM}\left(A^0_1, A^0_2, \cdots, A^0_n\right).$$

Again,

$$\mathrm{PFWAM}(A_1, A_2, \cdots\cdots, A_n) - \mathrm{PFWAM}\left(A^0_1, A^0_2, \cdots\cdots, A^0_n\right) = \begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n} w_i\mu_{A_i} - \frac{1}{n}\sum_{i=1}^{n} w_i\mu_{A_i^0}, \\ \frac{1}{n}\sum_{i=1}^{n} w_i\eta_{A_i} - \frac{1}{n}\sum_{i=1}^{n} w_i\eta_{A_i^0}, \\ \frac{1}{n}\sum_{i=1}^{n} w_i\nu_{A_i} - \frac{1}{n}\sum_{i=1}^{n} w_i\nu_{A_i^0} \end{pmatrix} = 0$$

because $\left(A_1^0, A_2^0, \cdots, A_n^0\right)$ be any permutation of $(A_1, A_2, \cdots, A_n)$.

Hence, we have

$$\mathrm{PFWAM}(A_1, A_2, \cdots, A_n) = \mathrm{PFWAM}\left(A^0_1, A^0_2, \cdots, A^0_n\right).$$

5 | PFGM Operators

Definition 11. Let $A_i = \left(\mu_{A_i}, \eta_{A_i}, \nu_{A_i}\right)$ $(i = 1, 2, \cdots, n)$ be collection of picture fuzzy values. Then the PFGM operator is mapping $PFGM: A^n \to A$ such that

$$\mathrm{PFGM}(A_1, A_2, \cdots, A_n) = \left(\sqrt[n]{\prod_{i=1}^{n} \mu_{A_i}}, \sqrt[n]{\prod_{i=1}^{n} \eta_{A_i}}, \sqrt[n]{\prod_{i=1}^{n} \nu_{A_i}}\right).$$

Definition 12. Let $A_i = \left(\mu_{A_i}, \eta_{A_i}, \nu_{A_i}\right)$ $(i = 1, 2, \cdots, n)$ be collection of picture fuzzy values and $w = (w_1, w_2, \cdots, w_n)^T$ be the weighting vector of A_i $(i = 1, 2, \cdots, n)$ such that $w_i \in [0, 1]$, $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} w_i = 1$. Then the picture PFWGM operator is a mapping $PFWGM: A^n \to A$ such that

$$\mathrm{PFWGM}(A_1, A_2, \cdots, A_n) = \left(\sqrt[n]{\prod_{i=1}^{n} w_i\mu_{A_i}}, \sqrt[n]{\prod_{i=1}^{n} w_i\eta_{A_i}}, \sqrt[n]{\prod_{i=1}^{n} w_i\nu_{A_i}}\right).$$

The following axioms are satisfied for *PFGM* and *PFWGM*.

Theorem 10 (Idempotency). Let $A_i = \left(\mu_{A_i}, \eta_{A_i}, \nu_{A_i}\right)$ $(i = 1, 2, \cdots, n)$ be collection of picture fuzzy values. If $A_i = A$, $(i = 1, 2, \cdots, n)$, then

$$PFGM(A_1, A_2, \cdots, A_n) = A.$$

And

$$\mathrm{PFWGM}(A_1, A_2, \cdots, A_n) = A.$$

Proof. For $A_i = A$ and $\sum_{i=1}^{n} w_i = 1$, we have

$$\mathrm{PFGM}(A_1, A_2, \cdots, A_n) = \left(\sqrt[n]{\prod_{i=1}^{n} \mu_{A_i}}, \sqrt[n]{\prod_{i=1}^{n} \eta_{A_i}}, \sqrt[n]{\prod_{i=1}^{n} \nu_{A_i}}\right)$$
$$= \begin{pmatrix} \sqrt[n]{\prod_{i=1}^{n} \mu_{A}}, \sqrt[n]{\prod_{i=1}^{n} \eta_{A}}, \\ \sqrt[n]{\prod_{i=1}^{n} \nu_{A}} \end{pmatrix} = \left(\sqrt[n]{(\mu_A)^n}, \sqrt[n]{(\eta_A)^n}, \sqrt[n]{(\nu_A)^n}\right)$$
$$= (\mu_A, \eta_A, \nu_A) = A.$$

Theorem 11 (Monotonicity). If $A_i \leq A_i^*$, then

$$\mathrm{PFGM}(A_1, A_2, \cdots, A_n) \leq \mathrm{PFGM}(A^*_1, A^*_2, \cdots, A^*_n).$$

$$\mathrm{PFWGM}(A_1, A_2, \cdots, A_n) - \mathrm{PFWGM}(A^*_1, A^*_2, \cdots, A^*_n) \leq 0.$$

And

$$\mathrm{PFWGM}(A_1, A_2, \cdots, A_n) \leq \mathrm{PFWGM}(A^*_1, A^*_2, \cdots, A^*_n).$$

Proof.

$$\mathrm{PFGM}(A_1, A_2, \cdots, A_n) - \mathrm{PFGM}(A^*_1, A^*_2, \cdots, A^*_n) =$$
$$\begin{pmatrix} \left(\mu_{A_1}.\mu_{A_2}.\cdots\cdots.\mu_{A_n}\right)^{\frac{1}{n}} - \left(\mu_{A_1^*}.\mu_{A_2^*}.\cdots\cdots.\mu_{A_n^*}\right)^{\frac{1}{n}}, \\ \left(\eta_{A_1}.\eta_{A_2}.\cdots\cdots.\eta_{A_n}\right)^{\frac{1}{n}} - \left(\eta_{A_1^*}.\eta_{A_2^*}.\cdots\cdots.\eta_{A_n^*}\right)^{\frac{1}{n}}, \\ \left(\nu_{A_1}.\nu_{A_2}.\cdots\cdots.\nu_{A_n}\right)^{\frac{1}{n}} - \left(\nu_{A_1^*}.\nu_{A_2^*}.\cdots\cdots.\nu_{A_n^*}\right)^{\frac{1}{n}} \end{pmatrix} \leq 0.$$

Similarly, we can prove that

This proves the monotonicity of *PFGM* and *PFWGM*.

Theorem 12 (**Boundedness**). Let $A_{min} = min(A_1, A_2, \cdots, A_n)$ and $A_{max} = max(A_1, A_2, \cdots A_n)$, for $i = 1, 2, \cdots, n$, Then $A_{min} \leq PFGM(A_1, A_2, \cdots, A_n) \leq A_{max}$ and $A_{min} \leq PFWGM(A_1, A_2, \cdots A_n) \leq A_{max}$.

Proof. Boundedness is the consequence of monotonicity and idempotency.

Theorem 13 (Commutatively). If $\left(A_1^0, A_2^0, \cdots, A_n^0\right)$ be any permutation of $\left(A_1, A_2, \cdots, A_n\right)$, the

$$\mathrm{PFGM}(\mathrm{A}_1, \mathrm{A}_2, \cdots, \mathrm{A}_n) = \mathrm{PFGM}\left(\mathrm{A^0}_1, \mathrm{A^0}_2, \cdots, \mathrm{A^0}_n\right).$$

And

$$\mathrm{PFWGM}(\mathrm{A}_1, \mathrm{A}_2, \cdots, \mathrm{A}_n) = \mathrm{PFWGM}\left(\mathrm{A^0}_1, \mathrm{A^0}_2, \cdots, \mathrm{A^0}_n\right).$$

Proof.

$$\mathrm{PFGM}(\mathrm{A}_1, \mathrm{A}_2, \cdots, \mathrm{A}_n) - \mathrm{PFGM}\left(\mathrm{A^0}_1, \mathrm{A^0}_2, \cdots, \mathrm{A^0}_n\right) = \begin{pmatrix} \sqrt[n]{\prod_{i=1}^n \mu_{A_i}} - \sqrt[n]{\prod_{i=1}^n \mu_{A_i^0}}, \\ \sqrt[n]{\prod_{i=1}^n \eta_{A_i}} - \sqrt[n]{\prod_{i=1}^n \eta_{A_i^0}}, \\ \sqrt[n]{\prod_{i=1}^n \nu_{A_i}} - \sqrt[n]{\prod_{i=1}^n \nu_{A_i^0}} \end{pmatrix} = 0$$

because $\left(A_1^0, A_2^0, \cdots \cdots, A_n^0\right)$ be any permutation of $\left(A_1, A_2, \cdots \cdots, A_n\right)$.

Hence, we have

$$\mathrm{PFGM}(\mathrm{A}_1, \mathrm{A}_2, \cdots, \mathrm{A}_n) = \mathrm{PFGM}\left(\mathrm{A^0}_1, \mathrm{A^0}_2, \cdots, \mathrm{A^0}_n\right).$$

Again,

$$\mathrm{PFWGM}(\mathrm{A}_1, \mathrm{A}_2, \cdots \cdots, \mathrm{A}_n) - \mathrm{PFWGM}\left(\mathrm{A^0}_1, \mathrm{A^0}_2, \cdots \cdots, \mathrm{A^0}_n\right) = \begin{pmatrix} \sqrt[n]{\prod_{i=1}^n w_i \mu_{A_i}} - \sqrt[n]{\prod_{i=1}^n w_i \mu_{A_i^0}}, \\ \sqrt[n]{\prod_{i=1}^n w_i \eta_{A_i}} - \sqrt[n]{\prod_{i=1}^n w_i \eta_{A_i^0}}, \\ \sqrt[n]{\prod_{i=1}^n w_i \nu_{A_i}} - \sqrt[n]{\prod_{i=1}^n w_i \nu_{A_i^0}} \end{pmatrix} = 0.,$$

Because $\left(A_1^0, A_2^0, \cdots, A_n^0\right)$ be any permutation of $(A_1, A_2, \cdots, A_n)$.

Hence, we have

$$\mathrm{PFWGM}(\mathrm{A}_1, \mathrm{A}_2, \cdots, \mathrm{A}_n) = \mathrm{PFWGM}\left(\mathrm{A^0}_1, \mathrm{A^0}_2, \cdots, \mathrm{A^0}_n\right).$$

6 | Application of the Picture Fuzzy Weighted Mean Operators to Multiple Attribute Decision-Making

MADM problems are common in everyday decision environments. An MADM problem is to find a great concession solution from all possible alternatives measured on multiple attributes.

Let the discrete set of alternatives and attributes are $A = \{A_1, A_2, \cdots A_n\}$ and $C = \{C_1, C_2, \cdots C_m\}$ respectively. Let $w = (w_1, w_2, \cdots, w_m)^T$ be the weighting vector of attributes C_j $(j = 1, 2, \cdots, m)$ such that $w_j \in [0,1]$, $(j = 1, 2, \cdots, m)$ and $\sum_{j=1}^{m} w_j = 1$. Suppose decision maker gives the picture fuzzy values for the alternatives A_i $(i = 1, 2, \cdots, n)$ on attributes C_j $(j = 1, 2, \cdots, m)$ are $k_{ij} = \left(\mu_{k_{ij}}, \eta_{k_{ij}}, \nu_{k_{ij}}\right)$, where $\mu_{k_{ij}}, \eta_{k_{ij}}$ and $\nu_{k_{ij}}$are positive, neutral and negative membership values of A_i under C_j respectively. Here $\mu_{k_{ij}}, \eta_{k_{ij}}, \nu_{k_{ij}} \in [0,1]$ and $0 \leq \mu_{k_{ij}} + \eta_{k_{ij}} + \nu_{k_{ij}} \leq 1$. Hence, an MADM problem can be briefly stated in a picture fuzzy decision matrix

$$\mathrm{K} = \left(\mathrm{k_{ij}}\right)_{\mathrm{n \times m}}.$$

Step 1. Utilize the decision information given in matrix K, and the $PFWHM$, $PFWAM$ and $PFWGM$ operators to derive the overall preference values d_i $(i = 1, 2, \cdots, n)$of the alternative A_i $(i = 1, 2, \cdots, n)$.

Step 2. Calculate the scores $S(d_i)$ $(i = 1, 2, \cdots, n)$ of the overall picture fuzzy values d_i $(i = 1, 2, \cdots, n)$.

Step 3. Rank all the alternatives A_i $(i = 1, 2, \cdots, n)$ in accordance with the values of $S(d_i)$ $(i = 1, 2, \cdots, n)$ and select the best one(s). If there is no difference between two scores $S(d_i)$ and $S(d_j)$, then we need to calculate the accuracy degrees $H(d_i)$ and $H(d_j)$ of the overall picture fuzzy values d_i and d_j, respectively, and then rank the alternatives A_i and A_j in accordance with the accuracy degrees $H(d_i)$ and $H(d_j)$.

Step 4. End.

6.1 | Numerical Example

A ceramic factory is looking for a general manager. There are five applicants $A = \{A_1, A_2, A_3, A_4, A_5\}$ for this position. The company is also looking for four attributes $C = \{C_1, C_2, C_3, C_4\}$ from these applicants. These attributes are leadership, problem-solving skill, communication skill, and experimentation. An expert will be graded for the four attributes. The decision matrix $K = \left(k_{ij}\right)_{5\times 4}$ is presented in *Table 1*, where k_{ij} $(i = 1, 2, \cdots, 5,\ j = 1, 2, \cdots, 4)$ are in the form of picture fuzzy values.

Table 1. Picture fuzzy decision matrix.

	C_1	C_2	C_3	C_4
A_1	(0.5,0.1,0.3)	(0.4,0.2,0.4)	(0.7,0.1,0.1)	(0.2,0.4,0.1)
A_2	(0.4,0.3,0.3)	(0.2,0.5,0.2)	(0.4,0.2,0.4)	(0.5,0.1,0.3)
A_3	(0.2,0.3,0.4)	(0.5,0.2,0.3)	(0.5,0.2,0.1)	(0.5,0.4,0.1)
A_4	(0.8,0.1,0.1)	(0.7,0.2,0.1)	(0.4,0.2,0.4)	(0.3,0.2,0.4)
A_5	(0.3,0.2,0.4)	(0.6,0.1,0.1)	(0.4,0.2,0.2)	(0.5,0.2,0.3)

The information about the attribute weights is known as: $w = (0.30, 0.35, 0.15, 0.20)$.

Step 1. Utilize the decision information given in matrix K and $PFWHM$, $PFWAM$ and $PFWGM$ operators, we have overall preference values d_i as following *Table 2*.

Table 2. Preference values $\mathbf{d_i (i = 1, 2, \cdots, 5)}$ for the operators PFWHM, PFWAM and PFWGM.

	d_1	d_2	d_3	d_4	d_5
PFWHM	(0.37, 0.15, 0.19)	(0.31, 0.22, 0.26)	(0.34, 0.25, 0.18)	(0.52, 0.15, 0.14)	(0.42, 0.15, 0.18)
PFWAM	(0.11, 0.05, 0.07)	(0.09, 0.08, 0.07)	(0.10, 0.07, 0.07)	(0.15, 0.04, 0.05)	(0.12, 0.04, 0.06)
PFWGM	(0.10, 0.04, 0.04)	(0.08, 0.06, 0.07)	(0.09, 0.06, 0.04)	(0.12, 0.04, 0.05)	(0.10, 0.04, 0.05)

Step 2. The scores $S(d_i)$ $(i = 1, 2, \cdots, 5)$ of the overall picture fuzzy values d_i $(i = 1, 2, \cdots, 5)$ are as following *Table 3*.

Table 4. Ranking all the alternatives A_i $(i = 1, 2, \cdots, 5)$ in accordance with the values of $S(d_i)$ $(i = 1, 2, \cdots, 5)$.

Operators	Ranking	Best Alternatives
PFWHM	$A_4 \succ A_3 \succ A_5 \succ A_1 \succ A_2$	A_4
PFWAM	$A_4 \succ A_3 \succ A_2 \succ A_5 \succ A_1$	A_4
PFWGM	$A_4 \succ A_3 \succ A_1 \succ A_5 \succ A_2$	A_4

7 | Comparison Studies

Comparing our results with the method using Picture fuzzy aggregation operator Wei [29] we get following score values of weighted picture fuzzy aggregation operator

$$S(d_1) = 0.23.$$

$$S(d_2) = 0.09.$$

$$S(d_3) = 0.20.$$

$$S(d_4) = 0.49.$$

$$S(d_5) = 0.26.$$

Rank all the alternatives A_i $(i = 1, 2, \cdots, 5)$ in accordance with the values of $S(d_i)$ $(i = 1, 2, \cdots, 5)$,

$$A_4 \succ A_5 \succ A_1 \succ A_3 \succ A_2.$$

Hence the best alternative is A_4, which is same as our result.

We compare our result with method of some geometric aggregation operators given by Wang et al. [26] we have following score values of weighted geometric aggregation operator

$$S(d_1) = 0.18.$$

$$S(d_2) = 0.11.$$

$$S(d_3) = 0.14.$$

$$S(d_4) = 0.37.$$

$$S(d_5) = 0.20.$$

Rank all the alternatives A_i $(i = 1, 2, \cdots, 5)$ in accordance with the values of $S(d_i)$ $(i = 1, 2, \cdots, 5)$,

$$A_4 \succ A_5 \succ A_1 \succ A_3 \succ A_2.$$

Hence the best alternative is A_4, which is same as our result.

8 | Conclusions

Mean operators are very useful tools to aggregate some picture fuzzy sets. It also helps us to make a decision in many problems of our real life. In literature, a host of researchers studied on different kind of aggregation operators of picture fuzzy sets and applied them to solve many problems in practical life. In this article, we have introduced some picture fuzzy mean operators and explored some related properties of them. A practical example is illustrated by using our proposed operators. Comparison studied are also discussed to show the effectiveness of our proposed operators.

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