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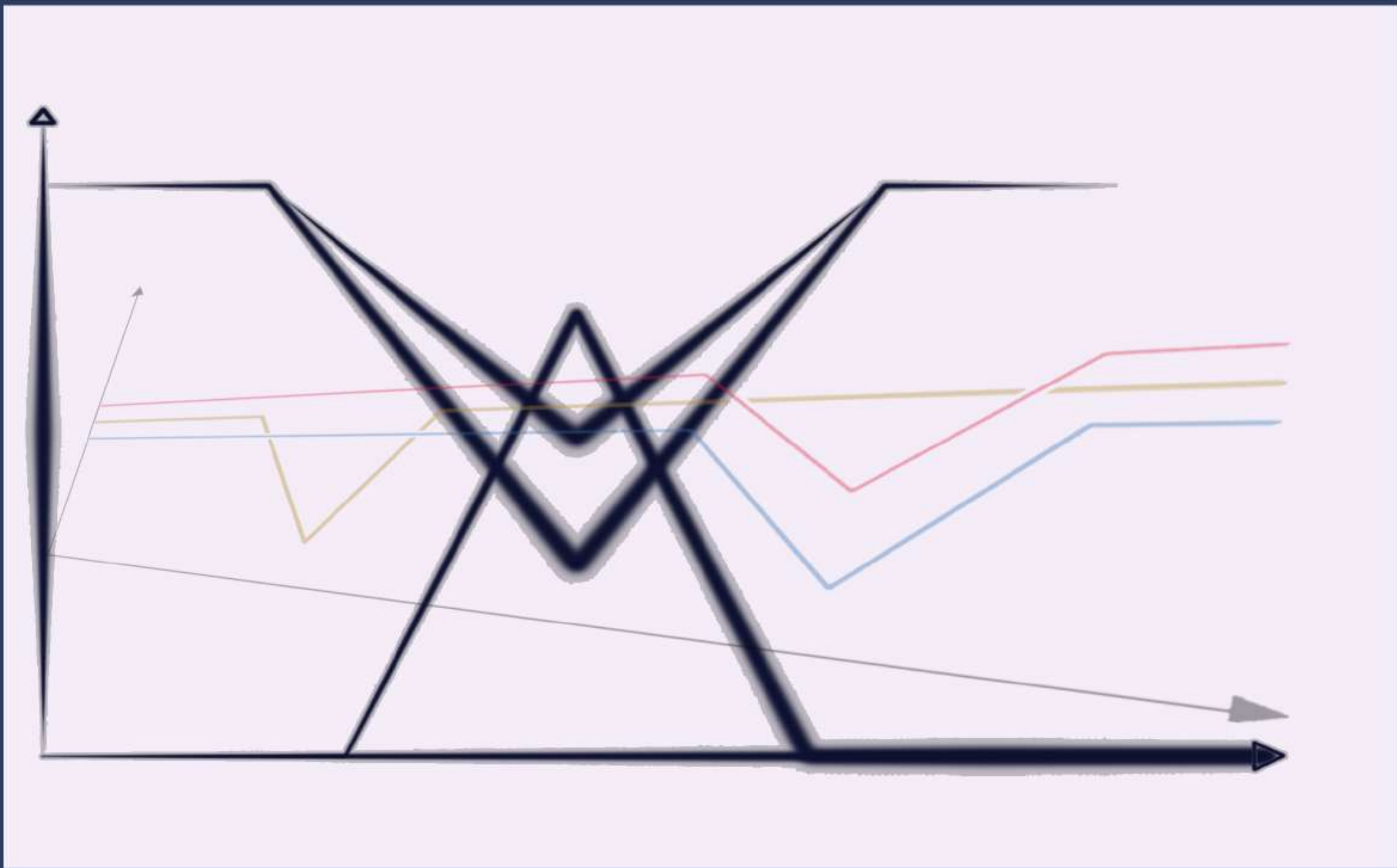

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The Nilpotent (p-Group) of $(D_{2^5} \times C_{2^n})$ for $m \geq 5$

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Abstract

Every finite p-group is nilpotent. The nilpotence property is an hereditary one. Thus, every finite p-group possesses certain remarkable characteristics. Efforts are carefully being intensified to calculate, in this paper, the explicit formulae for the number of distinct fuzzy subgroups of the cartesian product of the dihedral group of order 2^5 with a cyclic group of order of an m power of two for, which $n \geq 5$.

Keywords: Finite p-groups, Nilpotent group, Fuzzy subgroups, Dihedral group, Inclusion-Exclusion principle, Maximal subgroups.

1 | Introduction

The aspect of pure Mathematics has undergone a lot of dynamic developments over the years. Since inception, the study has been extended to some other important classes of finite abelian and nonabelian groups such as the dihedral, quaternion, semidihedral, and hamiltonian groups. Other different approaches have been so far, applied for the classification. The fuzzy sets were introduced by Zadeh [10]. Even though, the story of fuzzy logic started much more earlier, it was specially designed mathematically to represent uncertainty and vagueness. It was also, to provide formalized tools for dealing with the imprecision intrinsic to many problems. The term fuzzy logic is generic as it can be used to describe the likes of fuzzy arithmetic, fuzzy mathematical programming, fuzzy topology, fuzzy graph theory ad fuzzy data analysis which are customarily called fuzzy set theory. This theory of fuzzy sets has a wide range of applications, one of which is that of fuzzy groups de-veloped by Rosenfeld [11]. This by far, plays a pioneering role for the study of fuzzy algebraic structures. Other notions have been developed based on this theory. These, amongst others, include the notion of level subgroups by Das [16] used to characterize fuzzy subgroups of finite groups and that of equivalence of fuzzy subgroups introduced by Murali and Makamba [3] which we use in this work [1]-[9]. Essentially, this work is actually one of the follow up of our paper [15].

By the way, a group is nilpotent if it has a normal series of a finite length n:

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$$G = G_0 \geq G_1 \geq G_2 \dots \geq G_n = \{e\},$$

where

$$G_i/G_{i+1} \leq Z\ G/G_{i+1}).$$

By this notion, every finite p-group is nilpotent. The nilpotence property is an hereditary one. Thus,

I. Any finite product of nilpotent group is nilpotent.
II. If G is nilpotent of a class c, then, every subgroup and quotient group of G is nilpotent and of class ≤ c.

The problem of classifying the fuzzy subgroups of a finite group has so far experienced a very rapid progress. One particular case or the other have been treated by several papers such as the finite abelian as well as the non-abelian groups. The number of distinct fuzzy subgroups of a finite cyclic group of square-free order has been determined. Moreover, a recurrence relation is indicated which can successfully be used to count the number of distinct fuzzy subgroups for two classes of finite abelian groups. They are the arbitrary finite cyclic groups and finite elementary abelian p-groups. For the first class, the explicit formula obtained gave rise to an expression of a well-known central Delannoy numbers. Some forms of propositions for classifying fuzzy subgroups for a class of finite p-groups have been made by Tarnauceaus [5], [6]. It was from there, the study was extended to some important classes of finite non-abelian groups such as the dihedral and hamiltonian groups. And thus, a method of determining the number and nature of fuzzy sub-groups was developed with respect to the equivalence relation. There are other different approaches for the classification. The corresponding equivalence classes of fuzzy subgroups are closely connected to the chains of subgroups, and an essential role in solving counting problem is again played by the inclusion-exclusion principle. This hereby leads to some recurrence relations, whose solutions have been easily found. For the purpose of using the Inclusion-Exclusion principle for generating the number of fuzzy subgroups, the finite p-groups has to be explored up to the maximal subgroups. The responsibility of describing the fuzzy subgroup structure of the finite nilpotent groups is the desired objective of this work. Suppose that (G, ., e) is a group with identity e. Let S(G) denote the collection of all fuzzy subsets of G. An element $\lambda \in S(G)$ is called a fuzzy subgroup of G whenever it satisfies some certain given conditions. Such conditions are as follows:

I. $\lambda\ ab) \geq \in \{\lambda\ a), \lambda\ b)\}\ \ \forall\ a, b \in G$.

II. $\lambda(a^{-1} \geq \lambda\ a)\lambda$ for any $a \in G$.

And, since $(a^{-1})^{-1} = a$, we have that $\lambda(a^{-1}) = \lambda(a)$, for any $a \in G$.

Also, by this notation and definition, $\lambda(e) = \sup \lambda(G)$ [6].

Theorem 1. The set FL(G) possessing all fuzzy subgroups of G forms a lattice under the usual ordering of fuzzy set inclusion. This is called the fuzzy subgroup lattice of G.

We define the level subset:

$$\lambda G_\beta = \{a \in G/\lambda\ a) \geq \beta\} \text{ for each } \beta \in [0,1].$$

The fuzzy subgroups of a finite p-group G are thus, characterized, based on these subsets. In the sequel, λ is a fuzzy subgroup of G if and only if its level subsets are subgroups in G. This theorem gives a link between FL(G) and L(G), the classical subgroup lattice of G.F.

Moreover, some natural relations on S(G) can also be used in the process of classifying the fuzzy subgroups of a finite q-group G. One of them is defined by: $\lambda \sim \gamma$ if $(\lambda\ a) > \lambda\ b) \Longleftrightarrow v\ a) > v\ b)$, $\forall a, b \in G$. Alos, two fuzzy subgroups λ, γ of G and said to be distinct if $\lambda \times v$.

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As a result of this development, let G be a finite p-group and suppose that: G→[0,1] is a fuzzy subgroup of G. Put λ G) = $\{\beta_1, \beta_2, \dots, \beta_k\}$ with the assumption that $\beta_1 < \beta_2 > \cdots > \beta_k$. Then, ends in G is determined by λ.

$$\lambda G_{\beta_1} \subset \lambda G_{\beta_2} \subset \cdots \subset \lambda G_{\beta_k} = G. \tag{a}$$

Also, we have that:

$$\lambda\ a) = \beta_t \Longleftrightarrow t = \max\left\{\frac{r}{a} \in \lambda G_{\beta_r}\right\} \Longleftrightarrow a \in \lambda G_{\beta_t}/\lambda G_{\beta_{t-1}}.$$

For any a ∈ G and t = 1, …, k, where by convention, set $\lambda G_{\beta_0} = \phi$.

2 | Methodology

We are going to adopt a method that will be used in counting the chains of fuzzy subgroups of an arbitrary finite p-group G is described. Suppose that $M_1, M_2, \dots, M_t$ are the maximal subgroups of G, and denote by h(G) the number of chains of subgroups of G which ends in G. By simply applying the technique of computing h(G), using the application of the Inclusion-Exclusion Principle, we have that:

$$h\ G) = 2(\sum_{r=1}^{t} h(M_r - \sum_{1\le r_1 \le r_2 \le t} h\left(M_{r_1} \cap M_{r_1}\right) + \cdots + \ -1)^{t-1} h(\bigcap_{r=1}^{t} M_r)). \tag{1}$$

In [6], *Eq. (1)* was used to obtain the explicit formulas for some positive integers n.

Theorem 2 ([5], [6]). The number of distinct fuzzy subgroups of a finite p-group of order p^n which have a cyclic maximal subgroup is:

I. $h\left(\mathbb{Z}_{p^n}\right) = 2^n$.

II. $h\left(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}\right) = 2^{n-1}[2 + \ n-1)p]$.

3 | The District Number of the Fuzzy Subgroups of the Nilpotent Group of (D_{2^3} × C_{2^m}) for m ≥ 3

Proposition 1 ([13]). Suppose that $G = \mathbb{Z}_4 \times \mathbb{Z}_{2^n}$, $n \ge 2$. Then, $h(G) = 2^n[n^2 + 5n - 2]$.

Proof: G has three maximal subgroups of which two are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^n}$ and the third is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}$.

Hence,

$$\begin{aligned} h\ \mathbb{Z}_4 \times \mathbb{Z}_{2^n}) &= 2h\ \mathbb{Z}_2 \times \mathbb{Z}_{2^n}) + 2^1 h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}) + 2^2 h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}) + \\ &\quad 2^3 h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-3}}) + 2^4 h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-4}}) + \cdots + 2^{n-2} h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) \\ &= 2^{n+1}[2\ n+1) + \sum_{j=1}^{n-1}[\ n+1) - j] = 2^{n+1}\left[2\ n+1) + \frac{1}{2}\ n-2)\ n-3)\right] \\ &= 2^n\left[n^2 + 5n - 2\right], n \ge 2. \end{aligned}$$

We have that:

$$h\left(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}\right) = 2^{n-1}\left[(n-1)^2 + 5\ n-1) - 2\right] = 2^{n-1}\left[n^2 + 3n - 6\right], n > 2.$$

Corrolary 1. Following the last proposition, $h\ \mathbb{Z}_4 \times \mathbb{Z}_{2^5}), h\ \mathbb{Z}_4 \times \mathbb{Z}_{2^6}), h\ \mathbb{Z}_4 \times \mathbb{Z}_{2^7})$ and $h\ \mathbb{Z}_4 \times \mathbb{Z}_{2^8})$= 1536, 4096, 10496 and 26112 respectively.

Theorem 3 ([14]). Let $G = D_{2^n} \times \mathbb{C}_2$, the nilpotent group formed by the cartesian product of the dihedral group of order 2^n and a cyclic group of order 2. Then, the number of distinct fuzzy subgroups of G is given by : $h\ G) = 2^{2n}\ 2n + 1) - 2^{n+1}$, n > 3.

Proof: The group $D_{2^n} \times C_2$, has one maximal subgroup which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}$, two maximal subgroups which are isomorphic to $D_{2^{n-1}} \times C_2$, and 2^2 which are isomorphic to D_{2^n}.

It thus, follows from the Inclusion-Exclusion Principle using equation,

$$\frac{1}{2}h\ D_{2^n} \times C_2) = h\left(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}\right) + 4h\ D_{2^n}) - 8h\left(D_{2^{n-1}}\right) - 2h\left(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}\right) + 2h\left(D_{2^{n-1}} \times C_2\right).$$

By recurrence relation principle we have:

$$h\ D_{2^n} \times C_2) = 2^{2n}\ 2n + 1) - 2^{n+1}, n > 3.$$

By the fundermental principle of mathematical induction, set $F\ n) = h\ D_{2^n} \times C_2)$, assuming the truth of $F\ k) = h\left(D_{2^k} \times C_2\right) = 2h\left(\mathbb{Z}_2 \times \mathbb{Z}_{\ k-1}\right) + 8h(D_{2^k} - 16hD_{2^{k-1}} - 4h\left(\mathbb{Z}_2 \times \mathbb{Z}_{\ k-2}\right) + 4h\left(D_{2^{k-1}} \times C_2\right) = 2^{2k}\ 2k + 1) - 2^{k+1}$,

$F\ k + 1) = h\left(D_{2^{k+1}} \times C_2\right) = 2h\left(\mathbb{Z}_2 \times \mathbb{Z}_{\ 2^k}\right) + 8h(D_{2^{k+1}} - 16h(D_{2^k} - 4h\left(\mathbb{Z}_2 \times \mathbb{Z}_{\ k-1}\right) + 4h\left(D_{2^k} \times C_2\right) = 2^2[2^{2k}\ 2k - 3) - 2^k]$, which is true.

Proposition 2 ([12]). Suppose that $G = D_{2^n} \times \mathbb{C}_4$.Then, the number of distinct fuzzy subgroups of G is given by:

$$2^{2\ n-2)}\ 64n + 173) + 3\sum_{j=1}^{n-3} 2^{\ n-1+j)}(2n + 1 - 2j).$$

Proof:

$$\frac{1}{2}h\ D_{2^n} \times C_4) = h\ D_{2^n} \times C_2) + 2h\left(D_{2^{n-1}} \times C_4\right) - 4h\left(D_{2^{n-1}} \times C_2\right) + h\left(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}\right) - 2h\left(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}\right) - 2h\left(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-2}}\right) + 8h\left(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}\right) + h\left(\mathbb{Z}_{2^{n-1}}\right) - 4h\left(\mathbb{Z}_{2^{n-2}}\right).$$

$$h\ D_{2^n} \times C_4) = \ n - 3).2^{2n+2} + 2^{2\ n-3)}\ 1460) + 3\left[2^n\ 2n - 1) + 2^{n+1}\ 2n - 3) + 2^{n+2}\ 2n - 5) + \cdots + 7\left(2^{2\ n-2)}\right)\right] = \ n - 3).2^{2n+2} + 2^{2\ n-3)}\ 1460) + 3\sum_{j=1}^{n-3} 2^{n-1+j}\ 2n + 1 - 2j) = 2^{2\ n-2)}\ 64n + 173) + 3\sum_{j=1}^{n-3} 2^{n-1+j}\ 2n + 1 - 2j).$$

Proposition 3. Let G be an abelian p-group of type $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^n}$, where p is a prime and n ≥ 1; the number of distinct fuzzy subgroups of G is $h(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^n}) = 2^n p\ p + 1)\ n - 1)\ 3 + np + 2p) + 2^n - 2)p^3 - 2^{n+1}\ n - 1)p^3 + 2^n\left[p^3 + 4\ 1 + p + p^2)\right]$.

Proof: there exist exactly $1 + p + p^2$ maximal subgroups for the abelian $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^n}$ [17]. One of them is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}$ while each of the remaining $p + p^2$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$. Thus, by the application of the Inclusion-Exclusion principle, we have as follows: $h(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^n}) = 2^n p\ p + 1)\ n - 1)\ 3 + np + 2p) + 2^n - 2)p^3 - 2^{n+1}\ n - 1)p^3 + 2^n\left[p^3 + 4\ 1 + p + p^2)\right]$ and thus,

$$h(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}) = 2^{n-2}\left[43\ 3n - 5)p + \left(n^2 - 5\right)p^2 + \left(n^2 - 5n + 8\right)p^3\right] - 2p^2.$$

Corrolary 2. From Proposition 3, obsreve that, we are going to have that:

$$h(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^n}) = 2^{n+1}\left[\left(18n^2 + 9n + 26\right)\right] - 54.$$

Similarly, for p = 5, using the same analogy, we have

$$h(\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_{5^n}) = 2\left[30h\ \mathbb{Z}_5 \times \mathbb{Z}_{5^n}) + h(\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_{5^{n-1}}) - p^3 h\ \mathbb{Z}_{5^n}) - 30h(\mathbb{Z}_{5^{n-1}})\right] + 125.$$

And for p = 7,

$$h(\mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_{7^n}) = 2\left[56h\ \mathbb{Z}_7 \times \mathbb{Z}_{7^n}) + h(\mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_{7^{n-1}}) - 343h\ \mathbb{Z}_{7^n}) - 56h(\mathbb{Z}_{7^{n-1}})\right] + 343.$$

We have, in general, $h(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}) = 2^{n-2}\left[4 +\ 3n - 5)p +\ n^2 - 5)p^2 +\ n^2 - 5n + 8)p^3\right] - 2p^2$.

Proposition 4 ([15]). Let $G = (D_{2^3} \times C_{2^m})$ for $m \geq 3$. Then, $h\ G) = m\ 89 - 23m) +\ 85)2^{m+3} - 124$.

Proof: there exist seven maximal subgroups, of which one is isomorphic to $D_{2^3} \times C_{2^{m-1}}$, two being isomorphic to $C_{2^m} \times C_2 \times C_2$, two isomorphic to $C_{2^m} \times C_2$, and one each isomorphic to $C_{2^m} \times C_4$, and C_{2^m} respectively. Hence, by the inclusion - exclusion principle, using the propositions 1 to 3, and Theorem 2 we have that:

$$\frac{1}{2}h\ G) = h\left(D_{2^3} \times C_{2^{m-1}}\right) + 2h(C_{2^m} \times C_2) \times C_2) + 2h(C_{2^m} \times C_2) + 2h(C_{2^m} \times C_4) + h(C_{2^m}) - 12h(C_{2^m} \times C_2) - 6h(C_{2^{m-1}} \times C_2) \times C_2) - 3h(C_{2^{m-1}}) \times C_4) + 28h(C_{2^{m-1}} \times C_2) + 2h(C_{2^{m-1}} \times C_2) \times C_2) + 4h(C_{2^m} \times C_2) \times h(C_{2^{m-1}} \times C_4) - 35h(C_{2^{m-1}} \times C_2) - 7h(C_{2^{m-1}} \times C_2) + h(C_{2^{m-1}} \times C_2) = h\left(D_{2^3} \times C_2{}^{m-1}\right) + 2h(C_{2^m} \times C_2) \times C_2) - 6h(C_{2^m} \times C_2) + h(C_{2^m} \times C_4) + h(C_{2^m}) - 4h(C_{2^{m-1}} \times C_2) \times C_2) - 2h(C_{2^{m-1}}) \times C_4) + 8h(C_{2^{m-1}} \times C_2) = h\left(D_{2^3} \times C_2{}^{m-1}\right) + 2^{m+2}(6m^2 + 7m + 9) - 32 - (6)2^m(2m + 2) + 8m(2^m) - 2^m 6m^2 - 5m + 8 + 2^6 + 2^m(m^2 + 5m - 2) - 2^m(3m + m^2 - 6) + 2^m = h(D_{2^3} \times C_2{}^{m-1}) + 2^m(46m - 4) + 2^m + 32 = h(D_{2^3} \times C_2{}^{m-1}) + 2^m(46m - 3) + 32.$$

Hence

$$h\ G) = 2h\left(D_{2^3} \times C_2{}^{m-1}\right) + 2^{m+1}\ 46m - 3) + 64 = 2^{m+1}\ 46m - 3) + 64 + 2\left[2^m\ 46m - 49) + 64 + 2h\left(D_{2^3} \times C_2{}^{m-2}\right)\right] = 2^{m+1}\ 46m - 3) + 64 + 2^{m+1}\ 46m - 49) + 2^7 + 2^2 h\left(D_{2^3} \times C_2{}^{m-2}\right) = 2^{m+1}\ 46m - 3) + 2^6 + 2^{m+1}\ 46m - 49) + 2^7 + 2^2[2^{m-1}\ 46m - 95) + 64 + 2h\left(D_{2^3} \times C_2{}^{m-3}\right).$$

$$\begin{aligned} h\ D_{23n} \times C_{2^m}) &= 2^{m+1}\ 46m - 3) + 2^6 + 2^{m+1}\ 46m - 49) + 2^7 + 2^{m-1}\ 46m - 95) + 2^8 \\ &\quad + 2^3 h\left(D_{2^3} \times\ C_{2^{m-3}}\right) \\ &= 2^{m+1}\ 46m - 3) +\ 46m - 49) + (46m - 95)] + 2^6 + 2^7 + 2^8 \\ &\quad + 2^3 h(D_{2^3} \times C_2{}^{m-3}) = \underbrace{2^6 + 2^7 + 2^8 + \cdots + 2^{5+k}}_{\text{Series (1)}} \end{aligned}$$

$$+2^{m+1}\left[46mk + \underbrace{\left(-3 - 49 - 95 \ldots \left(-3 - 46\ k - 1)\right)\right)}_{\text{Series (2)}}\right]$$

$$+2^k h\left(D_{2^3} \times\ C_{2^{m-k}}\right), k \in \{1, 2, 3, \ldots, n\ \in \mathbb{N}\}.$$

For the Series (1), we have that, $U_m = 2^6.2^{m-1} = |2^{5+k}, m+5 = k+5, \Rightarrow m = k$. We have that $S_{m=k} = 2^6\left|\left[\frac{2^k-1}{2-1}\right]\right| = 2^6(2^k.1)$. And for the second Series (2), we have that, $T_m = -3 + \ m-1) -46) = -3 - 46\ k-1) \Longrightarrow m-1 = k-1, n = k$. Hence ,$S_m = k = \frac{k}{2}[2\ -3) + \ k-1) -46)] = \frac{k}{2}\ -6 - 46k + 46) = \frac{k}{2}(40 - 46k)$, We have that $h\ D_{23n} \times C_{2^m}) = \frac{k}{2}\ 40 - 46k) + 2^6\left(2^k.1\right) + 2^k h(D_3 \times C_{2^{m}-k}$. By setting m = k we have that k = m-3. Hence, $h\ D_{23n} \times C_{2^m}) = \ m-3)\ 20 - 23m) + 2^6\ 2^{m-3} - 1) + 2^m - 3h\ D_3 \times C_{2^3})$.

$$h\ G) = \ m-3)\ 20 - 23m) + 2^6\left(2^{m-3} - 1\right) + 2^{m-3}\ 5376) = \ m-3)\ 20 - 23m) + 2^{m-3} - 2^6 + 2^{m+5}\ 21) = 20m - 23m^2 - 60 + 69m + 2^{m+3} - 2^6 + 2^{m+5}\ 21) = \left(89m - 23m^2 - 60\right) + 2^{m+3} - 2^6 + 2^{m+5}\ 21) = m\ 89 - 23m) - 124 + \ 85)2^{m+3}.$$

4 | Applications

From the calculations, the Inclusion-Exclusion principle can be certified here as being very useful in the computations of the district number of fuzzy subgroups for the finite nilpotent p-groups.

5 | Examples

Now, since the stipulated condition that $m \geq 3$ must definitely be fulfilled then the readers may consider the examples below in tabular format.

Table 1. Table summarizing the some number of distinct fuzzy subgroups of ($D_{2^3} \times C_{2^m}$) for m ≥ 3.

S/N for theNumber of m	3	4	5	6	7	8	9	10
$h(G) = (D_{2^3} \times C_{2^m})$	5376	10728	21506	43347	86536	173320	347098	694910

Computation for $G = D_{2^5} \times \mathbb{C}_{2^n}, n \geq 5$.

Suppose that $G = D_{2^5} \times C_{2^n}$ for $n \geq 5$. Then,

$$\frac{1}{2}h\ G) = h\left(D_{2^5} \times C_{2^{n-1}}\right) + 2h\left(D_{2^4} \times C_{2^n}\right) + 2h\left(D_{2^n} \times C_{2^3}\right) + h\left(D_{2^n} \times C_{2^4}\right) - 4h\left(D_{2^4} \times C_{2^{n-1}}\right) - 4h\left(C_{2^3} \times C_{2^n}\right) - 2h\left(C_{2^4} \times C_{2^{n-1}}\right) + 8h\left(C_{2^3} \times C_{2^{n-1}}\right) - 3h\ C_{2^n}).$$

So, $h\ G) = 2h\ D_{2^5} \times C_{2^{n-1}}) + 4h\ D_{2^4} \times C_{2^n}) + 4h\ D_{2^n} \times C_{2^3}) + 2h\ D_{2^n} \times C_{2^4}) - 8h\ D_{2^4} \times C_{2^{n-1}}) - 8h\ C_{2^3} \times C_{2^n}) - 4h\ C_{2^4} \times C_{2^{n-1}}) + 16h\ C_{2^3} \times C_{2^{n-1}}) - 6h(C_{2^n})$.

6 | Conclusion

So far from our studies and discoveries it has been observed that any finite product of nilpotent group is nilpotent. Also, the problem of classifying the fuzzy subgroups of a finite group has experienced a very rapid progress. Finally, the method can be used in further computations up to the generalizations of similar and other given structures. Acknowledgments the authors are grateful to the anonymous reviewers for their helpful comments and suggestions which has improved the overall quality and the earlier version of the work.

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Competing of Interests Statement

The authors declare that in this paper, there is no competing of interests.

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Global Domination in Fuzzy Graphs Using Strong Arcs

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Abstract

Sampathkumar [7] introduces the notion of global domination in graphs. Nagoorgani and Hussain [24] introduced the concept of global domination in fuzzy graphs using effective arcs. This paper presents global domination in fuzzy graphs using strong arcs. The strong global domination number of different classes of fuzzy graphs is obtained. An upper bound for the strong global domination number of fuzzy graphs is obtained. Strong global domination in fuzzy trees is studied. It is established that every node of a strong global dominating set of a fuzzy tree is either a fuzzy cut node or a fuzzy end node. It is proved that in a fuzzy tree, each node of a strong global dominating set is incident on a fuzzy bridge. Also, the characteristic properties of the existence of a strong global dominating set for a fuzzy graph and its complement are established.

Keywords: Fuzzy graph, Strong arcs, Weight of arcs, Strong domination, Strong global domination.

1 | Introduction

Fuzzy graphs were introduced by Rosenfeld [26]. Rosenfeld [26] has described the fuzzy analog of several graph theoretic concepts like paths, cycles, trees, and connectedness and established some of their properties. Bhutani and Rosenfeld [9] have introduced the concept of strong arcs. Akram [1] also does several works on fuzzy graphs, Akram and Dudek [2], Akram et al. [3], Manjusha and Sunitha [16], Narayan and Sunitha [28], Mathew and Sunitha [14], and Rashmanlou et al. [5]. Global domination in graphs was discussed by Sampathkumar [7]. Somasundaram and Somasundaram [25] discussed domination in fuzzy graphs. They defined domination using effective edges in fuzzy graphs [19], [25]. Nagoorgani and Chandrasekharan [21] explained domination in fuzzy graphs using strong arcs. Manjusha and Sunitha [15], [16] discussed some concepts of domination and total domination in fuzzy graphs using strong arcs. Akram [1] did related works on bipolar fuzzy graphs, Akram et al. [3], Akram and Waseem [4], and Nagoorgani et al. [22]. This paper discusses global domination in fuzzy graphs using strong arcs.

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2 | Preliminaries

It is pretty well-known that graphs are simply models of relations. A graph is a convenient way of representing information involving the relationship between objects. Vertices and relations by edges represent the objects. When there is vagueness in the description of the objects, their relationships, or both, it is natural that we must design a 'fuzzy graph model.' It briefly summarises some basic definitions in fuzzy graphs presented in [8], [9], [12], [15], [17], [20], [21], [23], [25], [26].

A fuzzy graph is denoted by $G:(V,\sigma,\mu)$, where V is a node-set, σ and μ are mappings defined as $\sigma: V \to [0,1]$ and $\mu: V\times V \to [0,1]$, where σ and μ represent the membership values of a node and an arc respectively. For any fuzzy graph, $\mu(x,y) \le \sigma(x)\wedge\sigma(y)$. We consider fuzzy graph G with no loops and assume that V is finite and nonempty, μ is reflexive (i.e. $\mu(x,x)\le\sigma(x)$, for all x) and symmetric (i.e. $\mu(x,y)=\mu(y,x)$, for all (x, y)). In all the examples, σ is chosen suitably. Also, we denote the underlying crisp graph by G*: (σ*, μ*) where σ* = $\{u\in V:\sigma(u)>0\}$ and $\{(u,v)\in V\times V:\mu(u,v)>0\}$. Throughout, we assume that $\sigma^*=V$. The fuzzy graph $H:(\tau,\nu)$ is said to be a partial fuzzy subgraph of $G:(V,\sigma,\mu)$ if $\nu\subseteq\mu$ and $\tau\subseteq\sigma$. In particular, we call $H:(\tau,\nu)$ a fuzzy subgraph of $G:(V,\sigma,\mu)$ if $\tau(u)=\sigma(u)$ for all $u\in\tau^*$ and $\nu(u,v)=\mu(u,v)$ for all $(u,v)\in\nu^*$. A fuzzy graph $G:(V,\sigma,\mu)$ is called trivial if $\sigma^*=1$. Two nodes u and v in a fuzzy graph G are said to be adjacent (neighbors) if $\mu(u,v)>0$. The set of all neighbors of u is denoted by $N(u)$. An arc (u,v) of a fuzzy graph $G:(V,\sigma,\mu)$ with $\mu(u,v)>0$ is called a the weakest arc of G if (u,v) is an arc with a minimum $\mu(u,v)$. A path P of length n is a sequence of distinct nodes $u_o,u_1,\ldots\ldots,u_n$ such that $\mu(u_{i-1},u_i)>0, i=1,2,\ldots\ldots n$ and the degree of membership of the weakest arc is defined as its strength. If $u_0=u_n$, and $n\ge 3$, then P is called a cycle, and P is called a fuzzy cycle if it contains more than one weakest arc. A cycle's strength is the strength of its weakest arc. The strength of connectedness between two nodes, x and y, is defined as the maximum of the strengths of all paths between x and y and is denoted by $CONN_G(x,y)$. A fuzzy graph $G:(V,\sigma,\mu)$ is connected if for every x, y in σ^*, $CONN_G(x,y)>0$. An arc (u,v) of a fuzzy graph is called an effective arc if $\mu(u,v)=\sigma(u)\wedge\sigma(v)$. neighbors of u are called the effective neighborhood of u and are denoted by $EN(u)$.

A fuzzy graph $G:(V,\sigma,\mu)$ is said to be complete if μ $\mu(u,v)=\sigma(u)\wedge\sigma(v)$, for all *u, v* ∈ σ*.

The order p and size q of a fuzzy graph $G:(V,\sigma,\mu)$ is defined to be $p=\sum_{x\in V}\sigma(x)$ and $q=\sum_{(x,y)\in V\times V}\mu(x,y)$.

Let $G:(V,\sigma,\mu)$ be a fuzzy graph and $S\subseteq V$. Then the scalar cardinality of S is defined to be $\sum_{v\in S}\sigma(v)$, and it is denoted by $|S|_s$. Let p denotes the scalar cardinality of V, also called the order of G.

The complement of a fuzzy graph $G:(V,\sigma,\mu)$, denoted by $\overline{G}$, is defined to be G = (V, σ, μ) where $\overline{\mu(x,y)}=\sigma(x)\wedge\sigma(y)-\mu(x,y)$ for all x, y ∈ V [13]. An arc of a fuzzy graph G: (V, σ, μ) is called strong if its weight is at least as great as the strength of connectedness of its end nodes when it is deleted. A fuzzy graph G is called a strong fuzzy graph if each arc in G is a strong arc. Depending on $CONN_G(x, y)$ of an arc (x, y) in a fuzzy graph G, Mathew and Sunitha [17] defined three different types of arcs. Note that

$CONN_{G-(x,y)}(x, y)$ is the strength of connectedness between x and y in the fuzzy graph obtained from G by deleting the arc (x, y). An arc (x, y) in G is α−strong if $\mu(x, y) > CONN_{G-(x,y)}(x, y)$.

An arc (x, y) in G is β−strong if $\mu(x, y) = CONN_{G-(x,y)}(x, y)$. An arc (x, y) in G is δ−arc if $\mu(x, y) > CONN_{G-(x,y)}(x, y)$.

Thus, an arc (x, y) is strong if it is either α strong or β strong. Also, y is called a strong neighbor of x if the arc (x, y) is strong. The set of all strong neighbors of x is called the strong neighborhood of x and is denoted by $N_s(x)$. The closed strong neighborhood $N_s[x]$ is defined as $N_s[x] = N_s(x) \cup \{x\}$ A path P is called a strong path if P contains only strong arcs.

A fuzzy graph G: (V, σ, μ) is said to be bipartite [25] if the vertex set V can be partitioned into two non-empty sets V_1 and V_2 such that $\mu(v_1, v_2) = 0$ if $V_1, V_2 \in V_1$ or $V_1, V_2 \in V_2$. Further if $\mu(x, y) = \sigma(x) \wedge \sigma(y)$ for all u ∈ V_1 and v ∈ V_2, then G is called a complete bipartite graph and is denoted by $K_{\sigma 1, \sigma 2}$, where σ_1 and σ_2 are respectively the restrictions of σ to V_1 and V_2.

A node u is said to be isolated if μ(u, v) = 0 for all $v \neq u$.

3 | Strong Global Domination in Fuzzy Graphs

This section introduces the concept of global domination in fuzzy graphs using strong arcs. Recall the notion of global domination in graphs introduced by Sampathkumar [7]. According to him, a dominating set S of G is a global dominating set of G if S is also a dominating set of the complement of G. The minimum number of vertices in a global dominating set of G is the global domination number $\gamma_g(G)$ of G.

Nagoorgani and Hussain [24] introduced the concept of global domination in fuzzy graphs using effective arcs. According to then a set $D \subseteq V$ is a dominating set of a fuzzy graph G if every vertex in V\ D is effective adjacent to some vertex in D. A fuzzy dominating set D of an effective fuzzy graph G is a global fuzzy dominating set if D is also a fuzzy dominating set of the complement of a fuzzy graph G. The global fuzzy domination number is the minimum fuzzy cardinality of a fuzzy global dominating set. These concepts have motivated researchers to reformulate the concept of global domination more effectively. The studies in [24] are the main motivation of this article, and it is introduced the definition of global domination of a fuzzy graph using strong arcs. This introduction is essential because the parameter 'global domination number' defined by Nagoorgani and Hussain [24] is inclined more towards graphs than fuzzy graphs. Using the new definition of global domination number, it is possible to reduce the value of the old global domination number and extract classic results in a fuzzy graph.

Definition 1 ([21]). A node v in a fuzzy graph G : (V, σ, μ) is said to strongly dominate itself and each of its strong neighbors; that is, v strongly dominates the nodes in $N_s[v]$. A set D of nodes of G is a strong dominating G set if every node of V(G) −D is a strong neighbor of some node in D.

Definition 2. A strong dominating set D of a fuzzy graph G : (V, σ, μ) is called a strong global dominating set of G if D is also a strong dominating set of the complement of the fuzzy graph G.

Definition 3. The weight of a strong global dominating set D is defined as W (D) = u∈D μ(u, v), where μ(u, v) is the minimum of the membership values (weights) of the strong arcs incident on u. The strong global domination number of a fuzzy graph G is defined as the minimum weight of strong global dominating sets of G, and it is denoted by $\gamma_{sg}(G)$ or simply γ_{sg}. A minimum strong global dominating set in a fuzzy graph G is a strong global dominating set of minimum weight.

Let $\gamma_{sg}(G)$ or γ_{sg} denote the strong global domination number of the complement of a fuzzy graph G.

Remark 1. Note that in any undirected fuzzy graph $G:(V, \sigma, \mu)$, for any $x, y \in V$, if (x, y) is a strong arc in G, then (x, y) need not be a strong arc in $\overline{G}$. That is, if x strongly dominates y in G, then x need not strongly dominate y in $\overline{G}$.

Remark 2. If all the nodes are isolated, then V is the only strong global dominating set of G of order p and $\gamma_{sg} = 0$. That is, $N_s(u) = \phi$ for each $u \in V$.

Example 1. Consider the fuzzy graph given in *Fig. 1*.

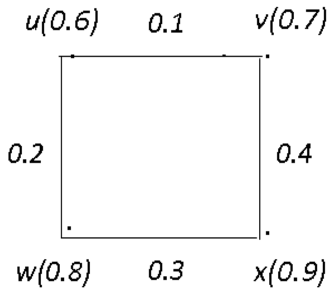

Fig. 1. Illustration of strong global domination in fuzzy graphs.

In this fuzzy graph, strong arcs are (u, w), (w, x), and (x, v). The strong global dominating sets are $D_1 = \{u, x\}$, $D_2 = \{u, v\}$, $D_3 = \{w, x\}$, and $D_4 = \{v, w\}$. Among these, the minimum strong global dominating sets are D_1 and D_3 where

$$W(D_1) = 0.2 + 0.3 = 0.5 \text{ and } W(D_3) = 0.2 + 0.3 = 0.5.$$

Hence $\gamma_{sg} = 0.5$.

4 | Strong Global Domination Number for Classes of Fuzzy Graphs

This section determines the strong global domination number of a complete fuzzy graph, complete bipartite fuzzy graphs, and fuzzy cycles.

Proposition 1. If $G:(V, \sigma, \mu)$ is a complete fuzzy graph, then

$$\gamma_{sg}\ G) = n\mu\ u, v).$$

Where n is the number of nodes in G and $\mu(u, v)$ is the weight of the weakest arc in G.

Proof. Since G is a complete fuzzy graph, all arcs are strong [25] and each node is adjacent to all other nodes. Also, all nodes in $\overline{G}$ are isolated nodes hence the set of all nodes of G is the strong global dominating set of G. Hence the result follows.

Proposition 2. In any fuzzy graph $G:(V, \sigma, \mu)$, the number of elements in any minimum strong global dominating set of both G and $\overline{G}$ are the same.

Proof. By definition, a strong global dominating set is a dominating set of both G and $\overline{G}$. Hence the proposition.

Proposition 3. $\gamma_{sg}(K_{\sigma1,\sigma2}) = 2\mu(u, v)$, where $\mu(u, v)$ is the weight of the weakest arc in $K_{\sigma1,\sigma2}$ and $u \in V_1$, $v \in V_2$.

Proof. In $K_{\sigma1,\sigma2}$, all arcs are strong. Also, each node in V_1 is adjacent to all nodes in V_2. Hence in $K_{\sigma1,\sigma2}$, the strong global dominating set is any set containing precisely two nodes, one in V_1 and the other in V_2. The end nodes say {u, v} of any weakest arc (u, v) in $K_{\sigma1,\sigma2}$ forms the minimum strong global dominating set of G. Hence $\gamma_{sg}(K_{\sigma1,\sigma2}) = \mu(u, v) + \mu(u, v) = 2\mu(u, v)$. So the proposition is proved.

Theorem 1. Let $G: (V, \sigma, \mu)$ be a fuzzy cycle where G^* is a cycle. Then, $\gamma_{sg}(G) = \vee\{W(D): D$ is a strong global dominating set in G with $|D| \geq \left\lfloor \frac{n}{3} \right\rfloor$, where n is the number of nodes in G.

Proof. In a fuzzy cycle, every arc is strong. Also, the number of nodes in a strong global dominating set of both G and G^* are the same because each arc in both graphs is strong. In graph G^*, the strong global domination number of G^* is obtained as $\lfloor \frac{n}{3} \rfloor$ [27]. Hence the minimum number of nodes in a strong global dominating set of G is $\lfloor \frac{n}{3} \rfloor$. Therefore the result follows.

Proposition 4. Let $G: (V, \sigma, \mu)$ be a non trivial fuzzy graph of size q. Then $\gamma_{sg}(G) = q$ if and only if all arcs are strong and each node is either isolated or has a unique, strong neighbor.

Proof. If all arcs are strong and each node is either an isolated node or has a unique, strong neighbor, then the minimum strong global dominating set of G is a set D containing nodes, each of which is either an isolated node or an end node of each unique, strong arc. Hence the weight of D is exactly

$$W(D) = \sum_{u \in D} \mu(u, v) = q.$$

Hence $\gamma_{sg} = q$.

Conversely, suppose that $\gamma_{sg} = q$. To prove that all arcs are strong and each node is either isolated or has a unique, strong neighbor. If possible, let (u, v) be an arc of G, which is not strong. Then the weight of this arc is not counted for getting γ_{sg}. Hence $\gamma_{sg} < q$, a contradiction. Hence all arcs are strong.

Let x be any node of G. If x is an isolated node, then clearly, x is contained in the minimum strong global dominating set. If possible, suppose x has two strong neighbors say v and w. Then exactly one of the weights of the arcs (x, v) and (x, w) contribute to the weight of the minimum strong global dominating set. Hence $\gamma_{sg} < q$, a contradiction. Hence each node has a unique, strong neighbor.

Remark 3. In proposition 4, G is exactly $S \cup N$ where S is a set of isolated nodes, may be empty, and N is a union of K_2 s.

Remark 4. In any fuzzy graph $G: (V, \sigma, \mu)$, $\gamma_{sg} < p$ always holds since $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in \sigma^*$, [p is the scalar cardinality of G, which is obtained by using the node weights, and γ_{sg} is the weight of the minimum strong global dominating set, which is obtained by using the arc weights].

For the strong global domination number γ_{sg}, the following theorem gives a Nordhaus-Gaddum type result.

Theorem 2. For any fuzzy graph $G: (V, \sigma, \mu), \gamma_{sg} + \gamma_{sg} < 2p$.

Proof. Since $\gamma_{sg} < p$ and $\gamma_{sg} < p$ by remark 4, we have $\gamma_{sg} + \gamma_{sg} < p + p = 2p$.

Definition 4. A strong global dominating set D of a fuzzy graph $G:(V, \sigma, \mu)$ is called a minimal strong global dominating set if no proper subset of D is a strong global dominating set.

Example 2. In *Fig. 1* of example 1, $D=\{u,v\}$ is a minimal strong global dominating set.

Definition 5 ([18]). A strong dominating set D of a fuzzy graph $G:(V, \sigma, \mu)$ is a strongly connected dominating set of G if the induced fuzzy sub graph $< D >$ is connected.

Remark 5 ([18]). Note that a fuzzy graph $G:(V, \sigma, \mu)$ contains a strong, connected dominating set if and only if G is connected.

Definition 6 ([18]). The weight of a strong, connected dominating set D is defined as $W D) = \sum_{u\in D} \mu(u,v)$, where $\mu(u, v)$ is the minimum of the membership values(weights) of strong arcs incident on u. The strong connected domination number of a fuzzy graph G is defined as the minimum weight of strong, connected dominating sets of G, and it is denoted by $\gamma_{sc}(G)$ or simply γ_{sc}. A minimum strong, connected dominating set in a fuzzy graph G is a strong, connected dominating set of minimum weight.

Let $\gamma_{sc}(\overline{G})$ or $\overline{\gamma_{sc}}$ denote the strong connected domination number of the complement of a fuzzy graph G.

Remark 6. Let D be a minimum strong global dominating set of a fuzzy graph $G:(V, \sigma, \mu)$. Then D induces a connected subgraph in G or G^c. Hence D is a strongly connected dominating set of G or G^c. Thus the following proposition is established.

Proposition 5. For any fuzzy graph $G:(V, \sigma, \mu)$, at least one of the following holds;

- $\gamma_{sc} \le \gamma_{sg}$.
- $\overline{\gamma_{sc}} \le \overline{\gamma_{sg}}$.

5 | Strong Global Domination in Complement of Fuzzy Graphs

Sunitha and Vijayakumar [13] have defined the present notion of the complement of a fuzzy graph. Sandeep and Sunitha [28] have studied the connectivity concepts in a fuzzy graph and its complement. The complement of a fuzzy graph G, denoted by $\overline{G}$ or G^c is defined to be $\overline{G}=(V,\sigma,\overline{\mu})$ where $\overline{\mu}(x,y)=\sigma(x)\wedge\sigma(y)-\mu(x,y)$ for all $x,y\in V$ [13]. Bhutani [8] has defined the isomorphism between fuzzy graphs.

Consider the fuzzy graphs $G_1:(V_1, \sigma_1, \mu_1)$ and $G_2:(V_2, \sigma_2, \mu_2)$ with $\sigma_1^*=V_1$ and $\sigma_2^*=V_2$. An isomorphism [8] between two fuzzy graphs G_1 and G_2 is a bijective map h: $V_1 \to V_2$ that satisfies;

- $\sigma_1(u)=\sigma_2(h(u))$ *for all* $u \in V_1$.
- $\mu_1(u, v) = \mu_2(h(u), h(v))$ *for all* $u, v \in V_1$ *and we write* $G_1 \approx G_2$.

A fuzzy graph G is self-complementary [13] if $G\approx\overline{G}$.

Theorem 3 ([13]). If G is an M-strong fuzzy graph, then G^c is also an M-strong fuzzy graph.

Theorem 4. If G is an M-strong fuzzy graph, then G and G^c have the same strong global dominating set.

Proof. By theorem 3, if G is an M-strong fuzzy graph, then G^c is also an M-strong fuzzy graph. Then the end nodes of the M-strong arcs in G are isolated nodes in $G^{c,}$ and isolated nodes in G are the end nodes of M-strong arcs in G^c. Hence every strong global dominating set of G is a strong global dominating set of G^c and vice-versa. Thus the theorem follows.

Theorem 5. Every non-trivial self-complementary connected fuzzy graph G has a strong global dominating set D whose complement $V \setminus D$ is also a strong global dominating set.

Proof. Every non-trivial connected fuzzy graph G has a strong dominating set D whose complement V − D is also a strong dominating set [16]. Since G is self-complementary, G ≅ G^c. Hence G and G^c are connected. Hence the theorem follows by using the result in [16].

Theorem 6. For any self-complementary connected fuzzy graph $G:(V, \sigma, \mu)$, $\gamma_{sg} \le p/2$.

Proof. Let D be a minimal strong global dominating set of G. Then, by theorem 3, V − D is a strong global dominating set of G. Then $\gamma_{sg} \le W(D)$ and $\gamma_{sg} \le W(V-D)$.

Therefore $2\gamma_{sg} \le W(D) + W(V-D) \le p$ implies $\gamma_{sg} \le p/2$. Hence the proof.

Corollary 1. Let G be a self complimentary connected fuzzy graph. Then $\gamma_{sg} + \overline{\gamma_{sg}} \le p$ further equality holds if and only if $\gamma_{sg} = \overline{\gamma_{sg}} = p/2$.

Proof. By theorem 6, $\gamma_{sg} \le p/2, \overline{\gamma_{sg}} \le p/2 \Rightarrow \gamma_{sg} + \overline{\gamma_{sg}} \le p/2 + p/2 = p$, that is $\gamma_{sg} + \overline{\gamma_{sg}} \le p$. If $\gamma_{sg} = \overline{\gamma}_{sg} = p/2$, then obviously $\gamma_{sg} + \overline{\gamma}_{sg} = p$. Conversely, suppose $\gamma_{sg} + \overline{\gamma}_{sg} = p$. Then, by theorem 6, we have $\gamma_{sg} \le p/2$, $\overline{\gamma_{sg}} \le p/2$. If either $\gamma_{sg} < p/2$ or $\overline{\gamma_{sg}} < p/2$, then $\gamma_{sg} + \overline{\gamma}_{sg} < p$, which is a contradiction. Hence the only possibility is that $\gamma_{sg} = \overline{\gamma}_{sg} = p/2$.

Theorem 7. In any fuzzy graph, $G:(V, \sigma, \mu)$, $\gamma_{sg} = p/2$ if and only if the following conditions hold.

I. The graph is a self-complementary fuzzy graph.
II. All nodes have the same weight.
III. All arcs are M-strong arcs.
IV. For every minimum strong global dominating set D of G, $|D| = n/2$, where n is the number of nodes of G, and n is even.

Proof. If all the above conditions hold, then obviously $\gamma_{sg} = p/2$.

Conversely, suppose $\gamma_{sg} = p/2$. If the graph is not self-complementary, then clearly $\gamma_{sg} < p/2$. If some nodes say u and v have different weights, then the arc weight corresponding to these nodes is $\mu(x, y) \le \sigma(x) \wedge \sigma(y)$.

If $\mu(x, y) < \sigma(x) \wedge \sigma(y)$, then obviously $\gamma_{sg} < p/2$, a contradiction.

If $\mu(x, y) = \sigma(x) \wedge \sigma(y)$, then (x,y) is a M-strong arc.

If $|D| < n/2$, then clearly $\gamma_{sg} < p/2$, a contradiction.

Hence all the conditions are sufficient.

6 | Strong Global Domination in Fuzzy Trees

A fuzzy subgraph $H:(\tau, \nu)$ spans the fuzzy graph $G:(V, \sigma, \mu)$ if $\tau = \sigma$ [20]. A connected fuzzy graph $G = (V, \sigma, \mu)$ is called a fuzzy tree (f-tree) if it has a fuzzy spanning subgraph $F:(\sigma, \nu)$, which is a tree, where for all arcs (x, y) not in F there exists a path from x to y in F whose strength is more than $\mu(x, y)$ [26]. Note that here F is a tree that contains all nodes of G and hence is a spanning tree of G. Also, note that F is the unique Maximum Spanning Tree (MST) of G [11], where a MST of a connected fuzzy graph $G:(V, \sigma, \mu)$ is a fuzzy spanning subgraph $T:(\sigma, \nu)$, such that T^* is a tree, and for which $\sum_{u=v} \nu(u, v)$ is maximum [20].

An arc is called a fuzzy bridge (f-bridge) of a fuzzy graph $G: (V, \sigma, \mu)$ if its removal reduces the strength of connectedness between some pair of nodes in G [26].

Similarly, a fuzzy cut node (f-cut node) w is a node in G whose removal from G reduces the strength of connectedness between some pair of nodes other than w [26].

A node z is called a fuzzy end node (f-end node) if it has precisely one strong neighbor in G [10].

A non-trivial fuzzy tree G contains at least two fuzzy end nodes, and every node in *G* is either a fuzzy cut node or a fuzzy end node [10].

In an f-tree G, an arc is strong if and only if it is an arc of F, where F is the associated unique MST of G [9], [11]. Note that these strong arcs are α-strong, and there are no β-strong arcs in an f-tree [17]. Also, note that in an f-tree G, an arc (x, y) is α-strong if and only if (x, y) is an f-bridge of G [17].

Theorem 8 ([12]). The strong arc incident with a fuzzy end node is a fuzzy bridge in any non-trivial fuzzy graph $G:(V, \sigma, \mu)$.

Corollary 2 ([12]). In a non-trivial fuzzy tree $G:(V, \sigma, \mu)$ except K_2, the strong neighbor of a fuzzy end node is a fuzzy cut node of G.

Theorem 9. In a non-trivial fuzzy tree $G:(V, \sigma, \mu)$, every node of a strong global dominating set is either a fuzzy cut node or a fuzzy end node.

Proof. A non-trivial fuzzy tree G contains at least two fuzzy end nodes, and every node in G is either a fuzzy cut node or a fuzzy end node [10]. Hence the theorem.

Theorem 10. In a non-trivial fuzzy tree $G:(V, \sigma, \mu)$, each node of a strong global dominating set is incident on a fuzzy bridge of G.

Proof. Let D be a strong global dominating set of G. Let $u \in D$. Since D is a strong global dominating set, there exists $v \in V \setminus D$ such that (u, v) is a strong arc. Also, D is a strong global dominating set of $\overline{G}$. Then (u, v) is an arc of the unique MST F of G [9], [11]. Hence (u, v) is an f-bridge of G [26]. Since u is arbitrary, this is true for every node of the strong global dominating set of G. This completes the proof.

7 | Practical Application

Let G be a graph that represents the road network connecting various hospitals. Let the vertices denote the hospitals, and the edges denote the roads connecting the hospitals. Suppose we need to navigate patients in between hospitals during busy hours. The membership functions σ and μ on the vertex set and the edge set of G's can be constructed from the statistical data that represents the number of ambulances going to various hospitals and the number of ambulances passing through multiple roads during a busy hour. Now the term' busy' is vague in nature. It depends on the availability of ambulances, time of journey, hospital's demands, special requirements of patients, etc. Thus, we get a fuzzy graph model. Some of the roads may be too traffic during the busy hour. So, we must think of taking patients to various hospitals through secret roads. In this fuzzy graph, a strong global dominating set D can be interpreted as a set of busy hospitals in the sense that every hospital not in D is connected to a hospital in D by a road or a secret road in which the traffic flow is full.

8 | Conclusion

Global fuzzy domination yields specific, adaptable, and conformable results compared to classical domination and fuzzy domination. Hence it introduced global domination in fuzzy graphs using strong arcs and found some results using the newly defined parameter' global domination number.' It is established that every node of a strong global dominating set of a fuzzy tree is either a fuzzy cut node or a fuzzy end node. It is proved that in a fuzzy tree, each node of a strong global dominating set is incident on a fuzzy bridge. Also, the characteristic properties of the existence of a strong global dominating set for a fuzzy graph and its complement are established.

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Fuzzy Metric Topology Space and Manifold

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Abstract

This paper, considers the fuzzy topological subsets, fuzzy topological spaces and introduces a novel concept of fuzzy Hausdorff space and fuzzy manifold space in this regards. Based on these concepts, we present a concept of fuzzy metric Hausdorff spaces and fuzzy metric manifold spaces via the notations of KM-fuzzy metric spaces. This study, generalises the concept of fuzzy metric space to union and product of fuzzy metric spaces in classical logic and in this regard investigates the some product of fuzzy metric fuzzy manifold spaces. We apply the notation of valued-level subsets and make a relation between of topological space, Husdorff space, manifold space and fuzzy topological space, fuzzy Husdorff space and fuzzy manifold space. In final, we extended the fuzzy topological space, fuzzy Husdorff space and fuzzy manifold space to fuzzy metric topological space, fuzzy metric Husdorff space and fuzzy metric manifold space Indeed, this study analyses the notation of fuzzy metric manifold based on valued-level subset.

Keywords: Fuzzy metric space, T-norm, Fuzzy (metric) Hausdorff space, Fuzzy (metric) manifold space.

1 | Introduction

As a generalization of the classical set theory, fuzzy set theory was introduced by Zadeh [17] to deal with uncertainties. Fuzzy set theory is playing an important role in modeling and controlling unsure systems in nature, society and industry. Fuzzy set theory also plays a vital role in complex phenomena which is not easily characterized by classical set theory. After the pioneering work of Zadeh [17], there has been a great effort to obtain fuzzy analogues of classical theories. Among other fields, a progressive developments is made in the field of fuzzy topology. Fuzzy topology is a fundamental branch of fuzzy theory which has become an area of active research in the last arse because of its wide range of applications. One of the most important problems in fuzzy topology is to obtain an proprieties concept of fuzzy metric space. This problem has been investigated by many authors from different points view. In particular, George and Veeramani [5] have introduced and studied a notion of fuzzy metric space with respect to the concept of t-norms. Furthermore, the class of topological spaces that are fuzzy metrizable agrees with the class of metrizable-topological spaces. This result permits Gregori et al. [4] to restate some classical theorems on metric completeness and metric (pre) compactness in the realm of fuzzy metric. Kaleva and Manjusha [11] generalized the notion of the

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metric space by setting the distance between two points to be a non-negative fuzzy number. They by defining an ordering and an addition in the set of fuzzy numbers obtained a triangle inequality which is analogous to the ordinary triangle inequality. Historically, the notion of a differentiable manifold, that is, a set that looks locally like Euclidean Space, has been an integral part of various fields of mathematics. One may note their applications in the fields of Differential Topology [7], Algebraic Geometry, Algebraic Topology, and Lie Groups and their associated algebras. They will base our work upon the already well-established fuzzy structures viz. fuzzy topological spaces, fuzzy topological vector spaces, fuzzy derivatives. However, the definition of a fuzzy derivative provided in Foster [7], is not easily generalized to a general k derivatives. Consequently, the existence of a fuzzy differentiable manifold of class greater than one has not yet been established. They shall give topological separation axioms that have not been given previously, for sake of completeness. Our principal approach is quite similar to the methods used in [7]. Namely, they shall take definitions in [3] of fuzzy continuity and fuzzy topological vector space, and use these notions to give a fuzzy topological vector space differential structure by constructing a fuzzy homemorphism and, naturally, a fuzzy diffeomorphism of class k. To do so, they provide a new definition of a fuzzy object, known as fuzzy vectors. They then define proper fuzzy directional derivatives along these abstract fuzzy vectors, and allude to their applications in manifold learning. For completeness, they shall define tangent vector spaces to these fuzzy manifolds. Further materials regarding fuzzy topological spaces are available in the literature too [1], [2], [8]-[13], [15], [16]. Regarding these points, we apply the concept of fuzzy metric spaces and make a connection between of manifolds and fuzzy subsets. The fuzzy metric spaces are not necessarily finite space, so one of our motivation of this work is a construction of finite fuzzy metric space. This study presents a concept of fuzzy (metric) Hausdorff space and fuzzy (metric) manifold space. The main our motivation of this work is the concept of fuzzy (metric) Hausdorff space and fuzzy (metric) manifold space based on t-norm such as Domby t-norm, Godel t-norm and etc.

Motivation fuzzy subsets are mail tools in construction of classic structures via valued cuts. We try to construct topology spaces, Hausdorff spaces and manifolds in this regard. Thus this probleem is a main motivation to extension of topology spaces, Hausdorff spaces and manifolds to fuzzy (metric) Hausdorff space and fuzzy (metric) manifold space.

2 | Preliminaries

In this section, we recall some definitions and results, which we need in what follows.

Theorem 1 ([6]) (Inverse Function Theorem). Let U be an open subset of $\mathbb{R}^n$, and let $p \in U$. Let $g: U \to \mathbb{R}^n$ be a smooth map. If $d_{g_p} \in$ Hom $(\mathbb{R}^n, \mathbb{R}^n)$ is a linear isomorphism, then there exist open neighbourhoods U_0, v_0 of p, g(p) such that $g|U_0 : U_0 \to V_0$ is a diffeomorphism.

Definition 1 ([6]). Let M be a Hausdorff topological space. We say that M is an n-dimensional topological manifold if it satisfies the following condition: for any $p \in M$, there exists:

- *an open subset U with $p \in U \subseteq M$.*
- *an open subset $E \subseteq \mathbb{R}^n$.*
- *a homeomorphism $\psi: U \to E$.*

Such a U is called a (local) coordinate neighbourhood, and ψ is called a (local) coordinate function. We write $x = \psi(p)$ and regard $(x_1,..., x_n)$ as local coordinates for the manifold M.

Definition 2 ([6]). Let M be a topological manifold. Let A be a set. We say that S is a C^0-atlas (or coordinate neighbourhood system) for M if $S = \{(U_\alpha, \psi_\alpha) | \alpha \in A\}$ where

- U_α *is an open subset of M, for all* $\alpha \in A$.
- $\psi_\alpha : U_\alpha \to E_\alpha$ *is a homeomorphism to an open subset* E_α *of* $\mathbb{R}^n$, *for all* $\alpha \in A$.
- $\bigcup_{\alpha \in A} U_\alpha = M$.

Definition 3 ([6]). Let S be a C^0-atlas for M. If $\psi_\alpha \circ \psi_\beta^{-1}$ is a C^∞-map for all $\alpha, \beta \in A$, we say that M is a C^∞-atlas for M. We call $\psi_\alpha \circ \psi_\beta^{-1}$ a coordinate transformation or transition function. The domain of the map $\psi_\alpha \circ \psi_\beta^{-1}$ is assumed to be $\psi_\beta(U_\alpha \cap U_\beta)$ (which could be the empty set). Thus, $\psi_\alpha \circ \psi_\beta^{-1}$ is a homeomorphism from $\psi_\beta(U_\alpha \cap U_\beta)$ to $\psi_\alpha(U_\alpha \cap U_\beta)$.

Definition 4 ([6]). Let M be a topological manifold. Let S be a C^∞-atlas for M. We say that M is a C^∞-manifold (or smooth manifold, or differentiable manifold). The concept of C^r-manifold can be defined in a similar way. However, from now on, manifold will always mean C^∞-manifold. The concept of complex manifold can be defined in a similar way, using coordinate charts $\psi : U \to C^n$. However, the term complex manifold will always mean complex manifold with holomorphic (complex analytic) transition functions.

Definition 5 ([6]). Let M, M' be smooth manifolds with dim M = n, dim M' = n'. Let $\phi : M \to M'$ be a map. Let $p \in M$.

- *We say that* ϕ *is smooth (or* C^∞*) at p if* $\psi' \circ \phi \circ \psi{-1}$ *is smooth at* $\psi(p)$ *for some local coordinate functions* $\psi : U \to E \subseteq Rn$, $\psi' : U' \to E' \subseteq R^{n'}$ *with* $p \in U$, $\phi(p) \in U'$.
- *We say that* ϕ *is a smooth map (or* C^∞*-map) if* ϕ *is smooth at all points of M.*

If $\phi : M \to M'$ satisfies the conditions; 1) ϕ is bijective, 2) ϕ is smooth and 3) $\phi{-1}$ is smooth. Then we say that ϕ is a diffeomorphism and M, M' are diffeomorphic.

Definition 6 ([17]). Let X be a non-empty set. A fuzzy subset A of X is defined as, $A = \{(x,\mu_A) \mid x \in A\} = \mu_A$ where $\mu_A : A \to [0, 1]$.

Definition 7 ([14]). A map $T : [01] \times [01] \to [01]$ ois called a t-norm, if for all x, y, z ∈ X;

- $T(x, y) = T(y, x)$.
- $T(x, 1) = T(1, x) = x$.
- $T(x, T(y, z)) = T(T(x, y), z)$.
- *if* $x \leq y$, *then* $T(x, z) \leq T(y, z)$.

Definition 8 ([14]). A triplet $(X, \rho, T)$is called a KM-fuzzy metric space, if X is an arbitrary non-empty set, T is a left-continuous t-norm and $\rho : X^2 \times \mathbb{R}^{\geq 0} \to [0, 1]$ is a fuzzy set, such that for each x, y, z, ∈ X and t, s ≥ 0, we have:

I. $\rho(x, y, 0) = 0$.
II. $\rho(x, x, t) = 1$ for all $t > 0$.
III. $\rho(x, y, t) = \rho(y, x, t)$(commutative property).
IV. $T(\rho(x, y, t), \rho(y, z, s)) \leq \rho(x, z, t + s)$(triangular inequality).
V. $\rho(x, y, -) : \mathbb{R} \geq 0 \to [0, 1]$ is a left-continuous map.
VI. $\rho(x, y, t) \to 1$, when $t \to \infty$.
VII. $\rho(x, y, t) = 1, \forall\, t > 0$ implies that $x = y$.

If (X, ρ, T)satisfies in conditions (I)–(VII), then it is called KM-fuzzy pseudometric space and ρ is called a KM -fuzzy pseudometric

3 | Fuzzy Topological Space

In this section, we recall some definitions and results, which we need in what follows.

Definition 9. Let M be an arbitrary set and $F(M) = \{\mu : M \to [0, 1]\}$. A family $\mathscr{F}_\tau$ of subset of F (M) is called a fuzzy topological if satisfied in the following conditions:

I. $\mu_0 \in \mathscr{F}_\tau$ and $\mu_1 \in \mathscr{F}_\tau$.

II. every $i \in I$, $\mu_i \in \mathscr{F}_\tau$, implies that $(\bigcup_{i \in I} \mu_i) \in \mathscr{F}_\tau$.

III. each $1 \leq i \leq n$, $\mu_i \in \mathscr{F}_\tau$, implies that $(\bigcap_{i=1}^{n} \mu i) \in \mathscr{F}_\tau$.

So $(M, \mathscr{F}_\tau)$ is called a fuzzy topological space and the members of $\mathscr{F}_\tau$ are called open fuzzy subsets, where $\mu_0 \equiv 0$ and $\mu_1 \equiv 1$.

Example 1. Consider $M = \mathbb{R}$ and $\mathscr{F}_\tau = \{\mu_1, \mu_i \mid$ where $\frac{i}{\mu_i = i + x_2}$ and $i \in N^*$ }. One can see that $(M, \mathscr{F}_\tau)$ is a fuzzy topological space.

Theorem 2. Let $\alpha \in [0, 1]$. Then $(M, \mathscr{F}_\tau)$ is a fuzzy topological space if and only if $(M, \mathscr{F}_\tau{}^{\alpha^+})$ is a topological space.

Proof. Since $\mu_0 \in \mathscr{F}_\tau$,we get that $\mu_0{}^{\alpha^+} = \{x \mid \mu_0(x) > \alpha\} = \emptyset$ and so $\emptyset \in \mathscr{F}_\tau{}^{+\alpha}$. In additon, $\mu_1 \in \mathscr{F}_\tau$, implies that $\mu_0{}^{\alpha^+} = \{x \mid \mu_0(x) > \alpha\} = X$, then $X \in \mathscr{F}_\tau{}^{+\alpha}$. Let $\{\mu_i{}^{\alpha^+}\}_{i \in} \in \mathscr{F}_\tau{}^{\alpha^+}$. Since $\bigvee_{i \in I} \mu_i\ x) = (\bigcup_{i \in I} \mu_i)(x)$ and for all $i \in I$, $\mu_i(x) > \alpha$, we have $\bigvee_{i \in I} \mu_i\ x) > \alpha$ and so $\bigcup_{i \in I} \mu_i{}^{\alpha^+} \in \mathscr{F}_\tau{}^{\alpha^+}$.

If $\{\mu_i{}^{\alpha^+}\}_{i=1}^{n} \in \mathscr{F}_\tau{}^{+\alpha}$, then $(\bigcap_{i=1}^{n} \mu_i)\ x) = (\bigwedge_{i=1}^{n} \mu_i)\ x) = \bigwedge_{i=1}^{n}(\mu_i(x)) > \alpha$. It follows that $(X, \mathscr{F}_\tau{}^{+\alpha})$ is a topological space. The converse is similar to.

Example 2. Consider the fuzzy topological space $(M, \mathscr{F}_\tau)$ in Example 1 and $\alpha = \frac{1}{2}$. Then $\mu_0{}^{\frac{1}{2}^+} = \{x \in| 0 > \frac{1}{2}\} = \varnothing$, $\mu_1{}^{\frac{1}{2}^+} = \{x \in M \mid 1 > \frac{1}{2}\} = \mathbb{R}$, and for $i \geq 2$, $\mu_1{}^{\frac{1}{2}^+} = \{x \in M \mid \frac{i}{i+x^2} > \frac{1}{2}\} = \{x \in M \mid - \cdots < x < \cdots\}$. So $(\mathbb{R}, \mathscr{F}_\tau{}^{\frac{1}{2}^+})$ is a topological space, where $\mathscr{F}_\tau{}^{\frac{1}{2}^+} = \{\varnothing, \mathbb{R}, (-\sqrt{i}, \sqrt{i}) \mid i \geq 2\}$.

Theorem 3. Let M ' be a set where $|M| = |M'|$ and $(M, \mathscr{F}_\tau)$ be a fuzzy topological space. Then there exists a fuzzy topology $\mathscr{F}_\tau'$ on M ' in such a way that $(M', \mathscr{F}_\tau')$ is a fuzzy topological space.

Proof. Since $|M| = |M'|$, there is a bijection $\varphi : M' \to M$. Consider $\mathscr{F}_\tau' = \{\mu_i \circ \varphi \mid \mu_i \in \mathscr{F}_\tau'\}$, clearly $(M', \mathscr{F}_\tau')$ is a fuzzy topological space.

Example 3. Consider the fuzzy topological space $(\mathbb{R}, \mathscr{F}_\tau)$ in Example 2 and $M = (0, 1)$. Define a map $F : R \to M$ by $f\ x) = \frac{1}{1+e^x}$. Then $((0, 1), \mathscr{F}_\tau')$ is a fuzzy topological space.

Corollary 1. Let M be a set where $|M| = |\mathbb{R}|$. Then there exists a topology $\mathscr{F}_\tau$ on M in such a way that $(M, \mathscr{F}_\tau)$ is a fuzzy topological space.

Definition 10. Let (M, $\mathscr{F}_\tau$) be a fuzzy topological space. A subfamily $\mathscr{F}B_\tau$ of $\mathscr{F}_\tau$ is called a base if

I. for all x ∈ M, we have $\bigvee_{\mu\in\mathscr{F}B\tau}\mu\ x) = 1$.

II. μ_1, $\mu_2 \in \mathcal{F}B\tau$ implies that $\mu_1 \cap \mu_2 \in \mathscr{F}B\tau$.

Theorem 4. Let (M, $\mathscr{F}_\tau$) be a fuzzy topological space and $\mathscr{F}B_\tau$ be a base for $\mathscr{F}_\tau$. Then every elementof $\mathscr{F}_\tau$ is inclused in union of elements of $\mathscr{F}B_\tau$.

Proof. Let $\mu \in \mathscr{F}_\tau$. Then for all x ∈ M, we have $\mu(x) = \mu(x)\vee 1 = \mu(x) \vee (\bigvee_{\mu_i\in\mathscr{F}B\tau}\mu_i\ x)) \leq \bigvee_{\mu_i\in\mathscr{F}B\tau}\mu_i\ x)$. It follows that $\mu \subseteq \bigcup_{\mu_i\in\mathscr{F}B\tau}\mu_i$.

Definition 11. Let (M, $\mathscr{F}_\tau$) be a fuzzy topological space and $\mathscr{F}B_\tau$ be a base for $\mathscr{F}_\tau$. Define $\langle \mathscr{F}B_\tau \rangle = \{\nu \in \mathscr{F}_\tau \mid \exists\, \mu \in \mathscr{F}_\tau , \mu \subseteq \nu\}$ and it called by generated fuzzy topology by $\mathscr{F}B_\tau$.

Theorem 5. Let (M, $\mathscr{F}_\tau$) be a fuzzy topological space. Then $\langle \mathscr{F}B_\tau \rangle$ is a fuzzy topology on M.

Proof. Since for all $\mu \in \mathscr{F}_\tau$, $\mu \subseteq \mu_1$, we get that $\mu_1 \in \langle \mathscr{F}B_\tau \rangle$. If $\mu_0 \in \langle \mathscr{F}B_\tau \rangle$, then $\mu \subseteq \mu_0$ implies that $\mu = \mu_0$. Let $\{\nu i\}_{i\in I}$ a family of elements $\langle \mathcal{F}B_\tau \rangle$ and $\bigcup_{i\in I} v_i = v$. Then there exist $\mu_i \in \mathscr{F}_\tau$ in such a way that $\mu_i \subseteq \nu_i$. So $\mu = \bigcup_{i\in I}\mu_i \subseteq \bigcup_i v_i = v$. Because $\mu \in \mathscr{F}_\tau$ and $\mu \subseteq \nu$, we get that $\nu \in \langle \mathscr{F}B_\tau \rangle$. Now, if for n ∈ N, $\{\nu_i\}_{i=1}^n$ is a family of elements of $\langle \mathscr{F}B_\tau \rangle$ in a similar way we have $\bigcap_{i=1}^n v_i \in \langle \mathscr{F}B_\tau \rangle$.

Proposition 1. Let (M, $\mathscr{F}_\tau$) be a fuzzy topological space and $\mathscr{F}B_\tau$ be a base for (M, $\mathscr{F}_\tau$). Then $\mathscr{F}B_\tau^{\alpha^+}$ is a base for topological space (M, $\mathscr{F}_\tau^{\alpha^+}$).

Proof. Let $B = \{\mu^{\alpha^+} \mid \mu \in \mathscr{F}B_\tau , \alpha \in [0, 1]\}$, x ∈ M and $\mu(x) = \alpha$. Then $x \in \mu^{\alpha^+} + \subseteq \bigcup_{\alpha\in\mathscr{F}B\tau}\mu^{\alpha^+}$ and so $\bigcup_{\alpha\in\mathscr{F}B\tau}\mu^{\alpha^+} = M$. Suppose $x \in \mu_1^{\alpha^+} \cap \mu_2^{\alpha^+}$. Since $\mathscr{F}B_\tau$ is a base for fuzzy topological space (*M*, $\mathscr{F}_\tau$), we get $\mu_1 \cap \mu_2 \in \mathscr{F}B_\tau$. Now, $x \in (\mu_1 \cap \mu_2)^{\alpha^+} + \subseteq \mu_1^{\alpha^+} \cap \mu_2^{\alpha^+}$ and so *B* is a base for $\mathscr{F}B_\tau$ .In the following, we will construct fuzzy topological space via topological spaces.

Definition 12. Let (M, τ_M) be a topological space. For all $M_i \in \tau_M$, define $\tau_\emptyset = \mu_0$, $\tau_M = \mu_1$ and for any i ∈ I, τ_{M_i} : M ⟶ [0, 1] by $\tau_{M_i}(x) = \mu_i(x) = \alpha_i \in [0, 1]$.

So we have the following lemma.

Theorem 6. Let (M, τ_M) be a topological space. Then (M, $\mathscr{F}_\tau = \{\tau_{M_i}\}_{i\in I}$) is a fuzzy topological space.

Proof. Since (M, τ_M) is a topological space, we have ∅, M ∈ τ_M so by definition $\mu_0 = \tau_\emptyset$, $\mu_0 = \tau_M \in \tau$.

Let for any i ∈ I, we have $\mu_i \in \mathscr{F}_\tau$. Since for all x ∈ M, $(\bigcup_{i\in I}\mu_i)\ x) = \bigvee_{i\in I}\mu_i\ x) = \bigvee_{M_i\in\tau_i}\tau_{M_i}\ x) = (\bigcup_{M_i\in\tau_i}\tau_{M_i})(x)$ and (M, τ_M) is a topological space, we get that $\bigcup_{M_i\in\tau_i}\tau_{M_i} \in \tau_M$ and so $\bigcup_{i\in I}\mu_i \in \mathscr{F}_\tau$. Let n ∈ℕ and $\{\mu_i\}_{i=1}^n$ be a set of elements $\mathscr{F}_\tau$. In a similar way, it is shoed that $\bigcap_{i=1}^n\mu_i \in \mathscr{F}_\tau$. therefore, (*M*, $\mathscr{F}_\tau$ is a fuzzy topological space.

Corollary 2. Let M be a non-empty set. Then there exists a fuzzy topology $\mathscr{F}_\tau$ on M such that (M, $\mathscr{F}_\tau$) is a fuzzy topological space.

Definition 13. Let (M, $\mathscr{F}_\tau$) and (M′ ,$\mathscr{F}_\tau'$) be fuzzy topological spaces and f : (M, $\mathscr{F}_\tau$) ⟶ (M′ ,$\mathscr{F}_\tau'$)be a homeomorphism, define f^{α^+}: (M, $\mathscr{F}_\tau^{\alpha^+}$) ⟶ (M′ , $(\mathscr{F}_{\tau'})^{\alpha^+}$) by $f^{\alpha^+}(x) = f(x)$, where x ∈ M.

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Theorem 7. Let $(M, \mathscr{F}_\tau)$ and $(M', \mathscr{F}_\tau')$ be fuzzy topological spaces. If $f : (M, \mathscr{F}_\tau) \to (M', \mathscr{F}_\tau')$ isa bijection and a fuzzy continuous map, then $f^{\alpha^+}: (M, \mathscr{F}_\tau^{\alpha^+}) \to (M', (\mathscr{F}_{\tau'})^{\alpha^+})$ is a continuous map.

Proof. Let $(\mu')^{\alpha^+} \in ((\mathscr{F}_{\tau'})^{\alpha^+}$. Then

$$(f^{\alpha^+})^{-1}\ \mu')^{\alpha^+} = \{x \in M \,|f^{\alpha^+}\ x) \in \ \mu')^{\alpha^+}\} = \{x \in M \,|\, f(x) \in \ \mu')^{\alpha^+}\} = \{x \in$$

$$M \,|\, \mu'(f(x)) > \alpha\} = \{x \in M \,\big|\, \exists \mu \in \mathscr{F}_\tau \text{s. t}\ \mu\ x) > \alpha\big\} = \mu^{\alpha^+} \in \mathscr{F}_\tau.$$

Since $f^{\alpha^+}$is a bijection, we get $f^{\alpha^+}$is a homeomorphism.

3.1 | Fuzzy Manifold Space

In this subsection, we introduce a concept of fuzzy Hausdorff space based on valued-cuts and in this regards, the concept of fuzzy manifold space is presented.

Definition 14. Let $(M, \mathscr{F}_\tau)$ be a fuzzy topological space. Then $(M, \mathscr{F}_\tau)$ is called a fuzzy Hausdorff space if, for all $x, y \in M$ there exist $\mu_1, \mu_2 \in \mathscr{F}_\tau$ and $0 \le \alpha < \beta \le 1$ in such a way that $x \in \mu^{(\alpha,\beta)^+} = \{x \in M \mid \alpha < \mu(x) < \beta\}$.

Example 4. Consider the fuzzy topological space, which is defined in Example 1. A simple computations show that for $i \neq i' \in N^*$.

$$J_i = (-\sqrt{i(\frac{1}{\alpha}-1)}, -\sqrt{i(\frac{1}{\beta}-1)}), \cup (\sqrt{i(\frac{1}{\beta}-1)}, \sqrt{i(\frac{1}{\alpha}-1)}).$$

$$J_{i'} = (-\sqrt{i'(\frac{1}{\alpha}-1)}, -\sqrt{i'(\frac{1}{\beta}-1)}), \cup (\sqrt{i'(\frac{1}{\beta}-1)}, \sqrt{i'(\frac{1}{\alpha}-1)}).$$

Where $J_i = \mu_i^{(\alpha,\beta)^+}$ and $J_{i'} = \mu_{i'}^{(\alpha,\beta)^+}$. If $x, y \in \mathbb{R}$, since $\mathbb{R}$ is a Hausdorff space, there exists $i, i' \in N^*$ such that $x \in J_i$, $y \in J_{i'}$, and $J_i \cap J_{i'} = \emptyset$ (For all $x \in \mathbb{R}$, consider $0 \le \alpha < \beta \le 1$, then $\frac{\alpha x^2}{1-\alpha} < i < \frac{\beta x^2}{1-\beta}$.)

Theorem 8. Let $(M, \mathscr{F}_\tau)$ be a fuzzy Hausdorff space and $\alpha \in [0, 1]$. Then $(M, \mathscr{F}_\tau^\alpha)$ is a Hausdorff space.

Proof. Since $(M, \mathscr{F}_\tau)$ is a fuzzy topological space, $(M, \mathscr{F}_\tau)$ is a topological space. Let $x, y \in M$ and $\alpha, \beta \in [0, 1]$, because of $(M, \mathscr{F}_\tau)$ is a fuzzy Hausdorff space, there exist, $\mu_1, \mu_2 \in \mathscr{F}_\tau$ and $0 \le \alpha' < \beta' < 1$ such that $x \in \mu_1^{(\alpha',\beta')^+}$ and $y \in \mu_2^{(\alpha',\beta')^+}$. Now consider $\alpha \le T_{\min}\{\alpha', \beta'\}$. It follows that, $x \in \mu^\alpha$ and $y \in \mu^\beta$ and $\mu^\alpha \cap \mu^\beta = \emptyset$. So $(M, \mathscr{F}_\tau^\alpha)$ is a Haussdorff space.

Definition 15. Let $(M, \mathscr{F}_\tau)$ be a fuzzy Hausdorff space. Then $(M, \mathscr{F}_\tau)$ is called a fuzzy manifold if, for all $x \in M$, there exists $\mu \in \mathscr{F}_\tau$ and homeomorphism $\phi : \text{supp}(\mu) \to \mathbb{R}^n$ such that $x \in \text{Supp}(\mu)$. Each (μ, ϕ) is called a fuzzy chart and $A = \{(\mu, \phi) \mid \mu \in \mathscr{F}_\tau, \phi : \text{supp}(\mu) \to \mathbb{R}^n\}$ is called a fuzzy atlas. Let (μ, ϕ), (υ, ψ) be two fuzzy charts of fuzzy atlas A. Then (μ, ϕ), (υ, ψ) are called C^∞-compatible charts if $\phi : \text{supp}(\mu) \to \mathbb{R}^n$, $\psi : \text{supp}(\upsilon) \to \mathbb{R}^n$ and $\phi \circ \psi{-1} : \psi(\text{Supp}(\mu) \cap \text{Supp}(\upsilon)) \to \phi(\text{Supp}(\mu) \cap \text{Supp}(\upsilon))$ are a C^1-fuzzy diffeomorphism.

Example 5. Consider the fuzzy Hausdorff space, which is defined in Example 6. It is clear that for all x ∈ ℝ, and for all i ∈ ℕ, we have x ∈ Supp(μ_i) = ℝ, we get that x ∈ Supp(μ_i). Define $(Ln)_i$: Supp(μ_i) ⟶ℝ, so A = {(μ_i, $(Ln)_i$) | i ∈ ℕ } is a fuzzy atlas.

Theorem 9. Let (M, $\mathscr{F}_\tau$) be a fuzzy manifold and α ∈ [0, 1]. Then (M, $\mathscr{F}_\tau^\alpha$)) is a manifold.

Proof. Since (M, $\mathscr{F}_\tau$) is a fuzzy manifold, by theorem 8, (M, $\mathscr{F}_\tau^\alpha$) is a Hausdorff topological space. Inaddition, for all x ∈ M, there exists μ ∈ $\mathscr{F}_\tau$ and homeomorphism ϕ: supp(μ) →$\mathbb{R}^n$ such that x ∈ Supp(μ). Now consider $A^\alpha = \{(\mu^\alpha, \phi) \mid \mu \in \mathscr{F}_\tau, \phi : supp(\mu) \to \mathbb{R}^n\}$. One can see that A^α is an atlas. Thus (M, $\mathscr{F}_\tau^\alpha$) is a manifold.

In the following, we wnant to extend two fuzzy topological space to a larger class of fuzzy topological spaces.

Definition 16. Let $\mathscr{F}_\tau$ and $\mathscr{F}_{\tau'}$be fuzzy topology on M and M′ , respectively. Define $\mathscr{F}_\tau \times \mathscr{F}_{\tau'} = \{\mu \times \mu' \mid \mu \in \mathscr{F}_\tau, \mu' \in \mathscr{F}_{\tau'}\}$, where for all x, y ∈ M ×M′ and ($\mu \times \mu'$)(x, y) = T_{min}(μ (x), μ (y)).

Theorem 10. Let (M, $\mathscr{F}_\tau$) and (M′ ,$\mathscr{F}_{\tau'}$) be fuzzy topological spaces. Then (M ×M′ ,$\mathscr{F}_\tau \times \mathscr{F}_{\tau'}$) is a fuzzy topological space.

Proof. Since $\mu_0 \in \mathscr{F}_\tau \cap \mathscr{F}_{\tau'}$ for all (x, y) ∈ M×M′ we have ($\mu_0 \times \mu_0$)(x, y) = $T_{min}\{\mu_0(x), \mu_0(y)\}$ = 0, and so ($\mu_0 \times \mu_0$) ∈ $\mathscr{F}_\tau \times \mathscr{F}_{\tau'}$. In addition, $\mu_1 \in \mathscr{F}_\tau \cap \mathscr{F}_{\tau'}$, implies that ($\mu_1 \times \mu_1$)(x, y) = $T_{pr}\{\mu_1(x), \mu_1(y)\}$ = 1 and so $\mu_1 \times \mu_1 \in \mathscr{F}_\tau \times \mathscr{F}_{\tau'}$. Let $\{\mu_i\}_{i\in I}$ and $\{\mu'_j\}_{j\in J}$ be two families of fuzzy topologies on M and M′ , respectively. Then $(\bigsqcup_{i\in I\, j\in J}(\mu_i \times \mu'_j))\, x, y) = (\bigvee_{i\in I\, j\in J} T_{min}(\mu_i \times \mu'_j))$ implies that $\bigsqcup_{i\in I\, j\in J}(\mu_i \times \mu'_j) \in \mathscr{F}_\tau \times \mathscr{F}_{\tau'}$. Let $\{\mu_i\}_{i=1}^{n}$ and $\{\mu'_j\}_{j=1}^{m}$ be two families of fuzzy topologies on M and M′ ,respectively. Then $(\bigcap_{i=1 j=1}^{n,m}(\mu_i \times \mu'_j))(x, y) = (\bigvee_{i=1 j=1}^{n,m}(\mu_i \times \mu'_j))(x, y) = \bigvee_{i=1 j=1}^{n,m}(T_{min}(\mu_i \times \mu'_j))$ implies that $\bigcap_{i=1 j=1}^{n,m}(\mu_i \times \mu'_j) \in \mathscr{F}_\tau \times \mathscr{F}_{\tau'}$. Thus (M ×M′ ,$\mathscr{F}_\tau \times \mathscr{F}_{\tau'}$) is a fuzzy topological space.

3.2 | Fuzzy Metric Manifolds

In this subsection, we introduce the concept of fuzzy metric topological space and investigate its properties.

Definition 17. Let (M, ρ, T) be a fuzzy metric space and F(M) = {μ : M → [0, 1]}. Then (M, ρ, T, $\mathscr{F}_\tau$) is called a fuzzy metric topological space, if

- *$\mathscr{F}_\tau$ is a fuzzy topology on M.*
- *for all x, y ∈M, t ∈R^+ and $\mu_1 \neq \mu \in \mathscr{F}_\tau$, we have T(μ(x), μ(y)) ≤ ρ(x, y, t).*

Example 6. Let M = ℝ. Then (M, T_{pr}, ρ) is a fuzzy metric space, where for all x, y ∈ ℝ and t ∈ $\mathbb{R}^+$, ρ (x, y, t) =$\left|\frac{\min\{x,y\}+t}{\max\{x,y\}+t}\right|$. It is easy to see that (M, ρ, T, $\mathscr{F}_\tau$) is a fuzzy metric topological space, where $\mathscr{F}_\tau$ = { μ_1, μ_i | where $\frac{i}{1+x^2}$ and i ∈ $\mathbb{N}^*$}.

Theorem 11. Let (M, τ_M) be a topological space and T be a t-norm on M. Then there exists a fuzzy subset ρ : $M_2 \times \mathbb{R}^+$ ⟶ [0, 1] such that (M, ρ, T, $\mathscr{F}_\tau$) is a fuzzy metric topological space. Proof. Let μ ∈ $\mathscr{F}_\tau$, t ∈ $\mathbb{R}^+$ and x, y ∈ M. Define

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$$\rho(x,y,t) = \begin{cases} T(\mu(x),\mu(y)), & \text{if } x \neq y, \\ 1, & \text{otherwise.} \end{cases}$$

Now, we show that $H = (M, \rho, T, \mathcal{F}_\tau)$ is a fuzzy metric topological space. By definition, for all $x \in M$, $\rho(x, x, t) = 1$ and for all $y \in M$, $\rho(x, y, t) > 0$. Let $x, y, z \in M$. Then for all $t, s \in R^+$,

$$T(\rho(x,y,t),\rho(x,y,s)) = T(T(\mu(x),\mu(y),T(\mu(y),\mu(z))) \leq T(\mu(x),\mu(z)) = \rho(x,y,t+s).$$

So (M, ρ, T) is a fuzzy metric space. Let $x, y \in M$, $t \in \mathbb{R}^+$ and $\mu_1 \neq \mu_2 \in \mathcal{F}_\tau$. Hence $T(\mu(x), \mu(y)) = \rho(x, y, t) \leq \rho(x, y, t)$. Therefore, $(M, \rho, T, \mathcal{F}_\tau$ is a fuzzy metric topological space.

Theorem 12. Let $(M, \rho, T, \mathcal{F}_\tau)$ be a fuzzy metric topological space, $x, y \in M$ and $\alpha \in [0, 1]$. Then

- *If $\mu = \mu_1$, then $T(\mu(x), \mu(y)) \leq \rho(x, y, t)$ implies that $x = y$.*
- *For any $i \in I$, $\mu_i \in \mathcal{F}_\tau$, we have $T((\bigsqcup_{i\in I} \mu_i)(x), (\bigsqcup_{i\in I} (\mu_i)(y)) \leq \rho(x,y,t)$.*
- *For any $i \in \mathbb{N}$, $\mu_i \in \mathcal{F}_\tau$, we get $T((\bigcap_{i=1}^{n} \mu_i)(x), \bigcap_{i=1}^{n}(\mu_i)(y)) \leq \rho(x,y,t)$.*

Proof.

I. Since for all $x \in M$, $\mu(x) = 1$, we get $T(\mu(x), \mu(x)) = T(1, 1) = 1 \leq \rho(x, y, t)$. Thus $\rho(x, y, t) = 1$ and so $x = y$.

II. Let $x, y \in M$. Then $T((\bigsqcup_{i\in I} \mu_i)(x), (\bigsqcup_{i\in I} (\mu_i)(y)) = T(\bigwedge_{i=1}^{n} \mu_i(x), \bigwedge_{i=1}^{n} \mu_i(y)) \leq T(\mu_i(x), \mu_i(y)) \leq \rho(x,y,t)$.

III. Since $(M, \mathcal{F}_\tau)$ is a fuzzy topological space, for all $i \in I$ and $\mu_i \in \mathcal{F}_\tau$, $\mu = \bigcup_{i\in I} \mu_i \in \mathcal{F}_\tau$.

So,

$$T((\bigsqcup_{i\in I} \mu_i)(x), (\bigsqcup_{i\in I} (\mu_i)(y)) = T(\mu(x),\mu(y)) \leq \rho(x,y,t).$$

Theorem 13. Let $(M, \rho, T, \mathcal{F}_\tau)$ be a fuzzy metric topological space and $\alpha \in [0, 1]$. Then $(M, \rho, T, \mathcal{F}_\tau^{+\alpha})$ is a topological space.

Proof. It is similar to Theorem 5.

Theorem 14. Let M be a non-empty set. Then there exists a t-norm T, a fuzzy metric subset $\rho : M_2 \times \mathbb{R}^+ \to [0, 1]$, and fuzzy topology $\mathcal{F}_\tau$ on M such that $(M, \rho, T, \mathcal{F}_\tau)$ is a fuzzy metric topological space.

Proof. Let M be a non-empty set. Then there exists a topology τ on M such that (M, τ) is a topological space. Using Corollary 2, there exists a fuzzy topology $\mathcal{F}_\tau$ on M such that $(M, \mathcal{F}_\tau)$ is a fuzzy topological space. Hence there exists a fuzzy subset $\rho : M_2 \times \mathbb{R}^+ \to [0, 1]$, such that (M, ρ, T_{pr}) is a fuzzy metric space. Thus there exists a t-norm T, a fuzzy metric subset $\rho : M_2 \times \mathbb{R}^+ \to [0, 1]$, and fuzzy topology $\mathcal{F}_\tau$ such that $(M, \rho, T, \mathcal{F}_\tau)$ is a fuzzy metric topological space.

Theorem 15. Let M' be a set where $|M| = |M'|$ and $(M, \rho, T, \mathcal{F}_\tau)$ be a fuzzy metric topological space. Then there exists a topology $\mathcal{F}_\tau'$ on M' and fuzzy subset $\rho : M_2 \times \mathbb{R}^+ \to [0, 1]$ in such a way that $(M', \rho, T, \mathcal{F}_\tau')$ is a fuzzy metric topological space.

Proof. Since $|M| = |M'|$, there is a bijection $\phi : M' \to M$. Consider $\mathcal{F}_\tau' = \{\mu_i \circ \phi \mid \mu_i \in \mathcal{F}_\tau\}$, clearly $(M', \mathcal{F}_\tau')$ is a fuzzy topological space. Based on theorem 11, there exists a fuzzy subset $\rho : M_2 \times \mathbb{R}^+ \to [0, 1]$ and a t-norm T such that $(M', \rho, T, \mathcal{F}_\tau')$ is a fuzzy metric topological space.

Corollary 3. Let M be a set where $|M| = |\mathbb{R}|$. Then there exists a fuzzy topology $\mathscr{F}_\tau$ on M, a fuzzy subset $\rho : M_2 \times \mathbb{R}^+ \to [0, 1]$ and a t-norm T such that $(M, \rho, T, \mathscr{F}_\tau)$ is a fuzzy metric topological space.

In the following, results are similar to perevious section.

Definition 18. Let $(M, \rho, T, \mathscr{F}_\tau)$ be a fuzzy metric topological space. A subfamily $\mathscr{F}B_\tau$ of $\mathscr{F}_\tau$ is called a base if

- *for all* $x \in M$, *we have* $\bigvee_{\mu_i \in \mathscr{F}B\tau} \mu\ x) = 1$ *and*
- $\mu_1, \mu_2 \in \mathscr{F}B_\tau$ *implies that* $\mu_1 \cap \mu_2 \in \mathscr{F}B_\tau$.

Theorem 16. Let $(M, \rho, T, \mathscr{F}_\tau)$ be a fuzzy metric topology space and $\mathscr{F}B_\tau$ be a base for $\mathscr{F}_\tau$. Then every element of $\mathscr{F}_\tau$ is inclosed in union of elements of $\mathscr{F}B_\tau$.

Proof. The proof is similar to Theorem 4.

Definition 19. Let $(M, \rho, T, \mathscr{F}_\tau)$ be a fuzzy metric topological space and $\mathscr{F}B_\tau$ be a base for $\mathscr{F}_\tau$. Define $\langle \mathscr{F}B_\tau \rangle = \{\nu \in \mathscr{F}_\tau \mid \exists\ \mu \in \mathscr{F}_\tau, \mu \subseteq \nu\}$ and it called by generated topology by $\mathscr{F}B_\tau$.

Theorem 17. Let $(M, \rho, T, \mathscr{F}_\tau)$ be a fuzzy metric topological space. Then $\langle \mathscr{F}B_\tau \rangle$ is a fuzzy topology on M.

Proof. The proof is similar to theorem 5.

Proposition 2. Let $(M, \rho, T, \mathscr{F}_\tau)$ be a fuzzy metric topological space and $\mathscr{F}B_\tau$ be a base for $(M, \rho, T, \mathscr{F}_\tau)$. Then $\mathscr{F}B_\tau^{\alpha^+}$ is a base for topological space $(M, \mathscr{F}_\tau{}^{\alpha^+})$.

Definition 20. Let $(M, \rho, T, \mathscr{F}_\tau)$ and $(M', \rho, T, \mathscr{F}_{\tau'})$ be fuzzy metric topological spaces and f $:(M, \mathscr{F}_\tau) \to (M', \mathscr{F}_{\tau'})$ be a homeomorphism, define $f^{\alpha^+}: (M, \mathscr{F}_\tau^{\alpha^+}) \to (M', (\mathscr{F}_{\tau'})^{\alpha^+})$ by $f^{\alpha^+}(x) = f(x)$, where $x \in M$.

Theorem 17. Let $(M, \rho, T, \mathscr{F}_\tau)$ and $(M', \rho, T, \mathscr{F}_{\tau'})$ be fuzzy metric topological spaces. If $f : (M, \rho, T, \mathscr{F}_\tau) \to (M', \rho, T, \mathscr{F}_{\tau'})$ be a fuzzy continuous map, then $f^{\alpha^+}: (M, \mathscr{F}_\tau^{\alpha^+}) \to (M', ((\mathscr{F}_{\tau'})^{\alpha^+})$ is a continuous map.

Proof. The proof is similar to Theorem 7.

Definition 21. Let $(M, \rho, T, \mathscr{F}_\tau)$ be a fuzzy metric topological space. Then $(M, \rho, T, \mathscr{F}_\tau)$ is called a fuzzy metric Hausdorff space if, for all $x, y \in M$ there exist $\mu_1, \mu_2 \in \mathscr{F}_\tau$ in such a way that $x \in \text{supp}(\mu_1)$, $y \in \text{supp}(\mu_2)$ and $\mu_1 \cap \mu_2 = \emptyset$ where $\text{Supp}(\mu) = \{x \mid \mu(x) \neq 0\}$ and $\mu_1 \cap \mu_2 = \emptyset$ means that $(\mu_1 \cap \mu_2)(x) = 0$.

Definition 22. Let $(M, \rho, T, \mathscr{F}_\tau)$ be a fuzzy metric Hausdorff space. Then $(M, \rho, T, \mathscr{F}_\tau)$ is called a fuzzy metric manifold if, for all $x \in M$, there exists $\mu \in \mathscr{F}_\tau$ and homeomorphism $\phi : \text{supp}(\mu) \to \mathbb{R}^n$ such that $x \in \text{Supp}(\mu)$. Each (μ, ϕ) is called a fuzzy chart and $A = \{(\mu, \phi) \mid \mu \in \mathscr{F}_\tau, \phi : \text{supp}(\mu) \to \mathbb{R}^n\}$ is called a fuzzy atlas. Let (μ, ϕ), (υ, ψ) be two fuzzy chart of fuzzy atlas A. Then (μ, ϕ), (υ, ψ) are called C^∞-compatible charts if phi : $\text{supp}(\mu) \to \mathbb{R}^n$, $\psi : \text{supp}(\upsilon) \to \mathbb{R}^n$ and $\phi \circ \psi{-1} : \psi(\text{Supp}(\mu) \cap \text{Supp}(\upsilon)) \to \phi(\text{Supp}(\mu) \cap \text{Supp}(\upsilon))$ are a C^1-fuzzy diffeomorphism.

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Example 7. Consider the fuzzy Hausdorff space, which is defined in Example 6. It is clear that for all x ∈ ℝ, and for all i ∈ N, we have x ∈ Supp(μ i) = ℝ, we get that x ∈ Supp(μ_i). Define $(Ln)_i$: Supp(μ_i) ⟶ ℝ, so A = {(μ_i, $(Ln)_i$) | i ∈ N} is a fuzzy atlas. Now, define for all x, y ∈ ℝ and $t \in \mathbb{R}^+$, ρ (x, y, t) $=\left|\frac{\min\{x,y\}+t}{\max\{x,y\}+t}\right|$ It is easy to see that (M, ρ , T, $\mathscr{F}_\tau$) is a fuzzy metric topological space, where $\mathscr{F}_\tau$ = { μ 1, μ i | where $\mu_i = \frac{i}{1+x^2}$ and i ∈ N*} and consequently (M, ρ , T, $\mathscr{F}_\tau$) is a fuzzy.

4 | Conclusion

The current paper has introduced a novel concept of fuzzy Hausdorff space, fuzzy manifold space. Also:

I. Based on fuzzy toplology, every non empty set converted to a fuzzy Hausdorff space.
II. It is showed that the product and union of fuzzy Hausdorff spaces is a fuzzy Hausdorff space.
III. The extended fuzzy metric spaces are constructed using the some algebraic operations on fuzzy metric spaces.
IV. The concept of fuzzy Hausdorff space and fuzzy manifold space is defined and investigated some its properties.

We hope that these results are helpful for further studies in theory of fuzzy metric Hausdorff space fuzzy metric manifold space. In our future studies, we hope to obtain more results regarding instuitic metric Hausdorff spaces, neutrosophic metric manifold spaces and their applications.

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Spherical Distance Measurement Method for Solving MCDM Problembs under Pythagorean Fuzzy Environment

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Abstract

Multiple Criteria Decision Analysis (MCDA) has been widely investigated and successfully applied to many fields,owing to its great capability of modeling the process of actual decision-making problems and establishing proper evaluation and assessment mechanisms. With the development of management and economics, real-world decision-making problem are becoming diversified and complicated to an increasing extent, especially within a changeable and unpredictable enviroment. Multi-criteria is a decision-making technique that explicitly evaluates numerous contradictory criteria. TOPSIS is a well-known multi-criteria decision-making process. The goal of this research is to use TOPSIS to solve MCDM problems in a Pythagorean fuzzy environment. The distance between two Pythagorean fuzzy numbers is utilized to create the model using the spherical distance measure. To construct a ranking order of alternatives and determine the best one,the revised index approach is utilized. Finally, we look at a set of MCDM problems to show how the proposed method and approach work in practice. In addition, it shows comparative data from the relative closeness and updated index methods.

Keywords: Multiple attribute decision making, TOPSIS, Pythagorean fuzzy sets, Score function, Spherical distance measurement, Revised index method.

1 | Introduction

Zadeh [1] introduced fuzzy set theory that provides a convenient and efficient tool for characterizing by mambership functionin [0,1] and managing Multiple Criteria Decision Analysis (MCDA) problems with vagueness and uncertainty. Nonetheless, in real decision situations, sometimes the membership function of an ordinary fuzzy set is not enough to depict the characters of assessment information because of the complexity of evaluation values and the ambiguity of human subjective judgments. Adak et al. [2], [3] had been extentd the MCDA problems in generalized form.

However in reality, it may not always be true that the degree of non-membership of an element in a fuzzy set is equal to 1 minus the membership degree because there may be some hesitation degree. To overcome this situation, Atanassov [4], [5] introduced the concept of Intuitionistic Fuzzy Sets (IFSs), which is a generalization of fuzzy sets and incorporate with the membership degree(α), non-

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membership degree (β) and hesitation degree(γ) (defined as 1 minus the sum of membership and non-membership). The notion of intutionistic fuzzy set is quite interesting and useful in many application areas. The knowledge and semantic representation of IFS become more meaningful, resourceful and applicable since it includes the degree of belongingness, degree of non-belongingness and the hesitation margin. Several research work done in the field of IFSs [6]-[10].

In IFSs, the pair of membership grades and non-membership grades are denoted by (α, β) satisfying the condition $0 \leq \alpha + \beta \leq 1$. Recently, Yager and Abbasov [11] and Yager [12], [13] extended the condition $\alpha + \beta \leq 1$ to $\alpha^2 + \beta^2 \leq 1$ and then introduced a class of Pythagorean Fuzzy Sets (PFSs) whose membership values are ordered pairs (α, β) that fulfill the required condition $\alpha^2 + \beta^2 \leq 1$. The space of all intuitionistic fuzzy values is also Pythagorean Fuzzy Values (PMVs), but converse is not necessary true. For instance, consider the situation when $\alpha = 0.7$ and $\beta = 0.4$, we can use PFSs, but IFSs cannot be used since $\alpha + \beta \leq 1$, but $\alpha^2 + \beta^2 \leq 1$ PFSs are wider than IFSs so that they can tackle more daily life problems under imprecision and uncertainty cases. Due to this relaxed condition, PF sets are more general than other nonstandard sets, such as IFSs, making the PF theory more powerful and useful than other nonstandard fuzzy models. Furthermore, Zhang and Xu [14] presented the detailed mathematical expression for PF sets and put forward the concept of PF numbers. In recent years, various results have been introduced in PFSs [15]-[17].

How to measure the distance between two PFSs is still an open issue. Different methods had been proposed to present the question in former researches. However, not all existing methods can accurately manifest differences among PFSs and satisfy the property of similarity. And some other kinds of methods neglect the relationshiop among three variables of PFS.

Some researchers have extended the distance measure of IF sets. Zhang and Xu [14] considered three parameters of PFSs, namely, the membership degree, the non-memership degree, and the hesitation degree, while ignoring the direction of commitment, the strength of commitment, and the radian. Li and Zeng [18] considered four basic parameters (the membership degree, the non-memership degree, and the hesitation degree, the strength of commitment, the direction of commitment) of PF sets in the distance measure equation. Wang et al. [20] introduced a distance measure that is based on the long distance and angular distance and angular distance in a bidirectional projection model under the PF environment. Yu et al. [21] proposed a new distance formula that employs Induced Ordered Weighted Averaging (IOWA) with PF information; however, this basic distance formula considers only three parameters, which are the same as the parameters that are considered in the method of Zhang and Xu [14]. Peng and Li [22] proposed a new distance measure for IVPF sets that has two parameters (the membership degree and the non-membership degree) for resolving the counter-intuitive situation.

To address the problem, a new method of measuring distance is proposed that meets the requirements of all axioms of distance measurements and is able to indicate the degree of distinction of PFSs well. For a Pythagorean fuzzy number, membership (α) and non-membership (β) degree satisfying the condition $0 \leq \alpha + \beta \leq 1$ and hesitation degree is $\gamma = \sqrt{1 - \alpha^2 - \beta^2}$, i.e, $\alpha^2 + \beta^2 + \gamma^2 = 1.$ Utilizing the relation $\alpha^2 + \beta^2 + \gamma^2 = 1$, we may assume that the triplet (α, β, γ) lies on the spherical surface of unit radius and centre at the origin. This interpretation encourages defining the spherical distance between two Pythagorean fuzzy numbers on restricted spherical surface.

The purpose of this paper is to propose a spherical distance measurement and use to find the distance between two Pythagorean fuzzy numbers and applied in Pythagorean fuzzy TOPSIS method. This measurement is essential to determine distances for both the Pythagorean fuzzy positive ideal solution and Pythagorean fuzzy negative ideal solution. The score function of Pythagorean fuzzy number is used to determine PFPIS and PFNIS in this approch. Revised index and relative closeness are used to rank the alternatives.

The remainder of this paper is organized as follows. Section 2 briefly introduces some basic concepts of PF sets. Section 3 formulates spherical distance measurement method for Pythagorean fuzzy numbers.

Moreover, some comparative discussions with other measurement method are conducted to demonstrate the effectiveness and advantages of the devloped method. Section 4 develops TOPSIS for solving MCDM problems in Pythagorean fuzzy environment. Section 5 applies the proposed methodology to a real-life problem to demonstrate its feasibility and practically. Finally, section 6 presents the conclusions and scope for the future work.

2 | Preliminaries and Definitions

In this section, we recall some basic notions such as the IFSs and the PFSs. Also,we include some elementary aspects that are necessary for this paper.

Definition 1 ([4]). IFS let X be a set of finite universal sets. An IFS I in X is an expression having the form

$$\mathrm{I}=\left\{\left\langle x,\alpha_{\mathrm{I}}(x),\beta_{\mathrm{I}}(x)\right\rangle : x\in X\right\}.$$

Where $\alpha_I(x)$ and $\beta_I(x)$ are the degree of membership and the degree of non-membership of element of $x\in X$ respectively. Also $\alpha_I : X\to[0,1]$, $\beta_I : X\to[0,1]$ and $0\le\alpha_I(x)+\beta_I(x)\le 1$, for all $x\in X$. The degree of indeterminacy $\pi_I(x)=1-\alpha_I(x)-\beta_I(x)$.

In practice for some reson, the condition $0\le\alpha+\beta\le 1$, is not always true. For instance, $0.6+0.7=1.3>1$ but $0.6^2+0.7^2<1$,or $0.7+0.7=1.4>1$ but $0.7^2+0.7^2<1$. To overcome this situations, in 2013, Yager [12] introduced the concept of PFS.

Definition 2 ([12]). PFS A Pythagorean fuzzy set P in a finite universe of discourse X is given by:

$$\mathrm{P}=\left\{\left\langle x,\alpha_p(x),\beta_p(x)\right\rangle \mid x\in X\right\},$$

Where $\alpha_p : X\to[0,1]$ denotes the degree of membership and $\beta_p : X\to[0,1]$ denotes the degree of non-membership of the element of $x\in X$ to the set A respectively with the condition that $0\le(\alpha_p(x))^2+(\beta_p(x))^2\le 1$.The degree of indeterminacy, $\gamma_p(x)=\sqrt{1-(\alpha_p(x))^2-(\beta_p(x))^2}$.

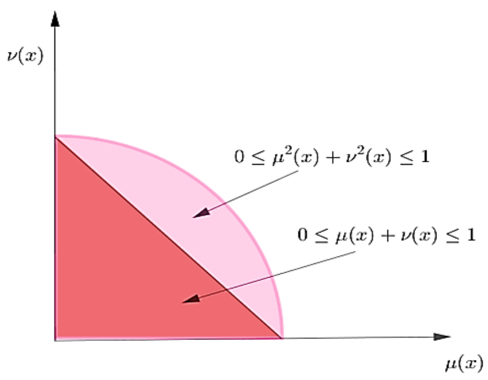

Fig. 1. Comparison space for IFSs and PFSs.

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Zhang and Xu [14] introduced the concept of Pythagorean fuzzy numbers. A PFS $P=\langle \alpha_p(x),\beta_p(x)\rangle$ can be expressed as Pythagorean fuzzy numbers $\langle \alpha_p,\beta_p\rangle$, where $\alpha_p,\beta_p \in [0,1], \gamma_p=(1-(\alpha_p)^2-(\beta_p)^2)^{\frac{1}{2}}$ and $0 \le (\alpha_p)^2+(\beta_p)^2 \le 1$.

Yager [12] proposed another PFN formulation, namely, $p=\langle r,d\rangle$,where r denotes the strength of commitment and $r\in[0,1]$. The larger value of r is the stronger commitment and the lower the uncertainty of the commitment. D denote the direction of commitment. r and d associates with the membership grade and non-membership grade; $\alpha = r\cos\theta$ and $\beta = r\sin\theta$. Therefore, $d=1-\frac{2}{\pi}\theta$.

Example 1. Let us consider a Pythagorean fuzzy number $p=\langle 0.7,0.4\rangle$, then we have:

$$\alpha_p=0.7,\beta_p=0.4,\gamma_p=\sqrt{1-(0.7)^2-(0.4)^2}=0.592, r=\sqrt{(0.7)^2+(0.4)^2}=0.806,$$

$$\theta=\arctan\left(\frac{0.4}{0.7}\right)=0.519 \text{ and } d=1-2\frac{0.519}{\pi}=0669.$$

2.1 | Some Operations on Pythagorean Fuzzy Numbers

Here we discussed some operations on Pthagorean fuzzy numbers and PFSs and operations are used in the rest of the paper.

Given three PFNs $p=\langle\alpha,\beta\rangle$, $p_1=\langle\alpha_1,\beta_1\rangle$ and $p_2=\langle\alpha_2,\beta_2\rangle$. The basic operations can be defined as follows:

- $\overline{p}=\langle\beta,\alpha\rangle$.
- $p_1\cup p_2=\langle max\{\alpha_1,\alpha_2\}, min\{\beta_1,\beta_2\}\rangle$.
- $p_1\cap p_2=\langle min\{\alpha_1,\alpha_2\}, max\{\beta_1,\beta_2\}\rangle$.

Example 2. Let $p_1=\langle 0.5,0.2\rangle$ and $p_2=\langle 0.6,0.4\rangle$,then $p_1\cup p_2=\langle 0.6,0.2\rangle$, $p_1\cap p_2=\langle 0.5,0.4\rangle$ and $\overline{p_1}=\langle 0.2,0.5\rangle$.

Definition 3 ([23]). Let $p=\langle\alpha,\beta\rangle$ be a Pythagorean fuzzy number. The score function of p is defined as $s(p)=(\alpha)^2-(\beta)^2$, where $s(p)\in[-1,1]$.

Example 3. Let $p_1=\langle 0.7,0.3\rangle$ and $p_2=\langle 0.4,0.6\rangle$, then $s(p_1)=0.40$ and $s(p_1)=-0.20$.

There are some situations, the score function is not sufficient for magnitude comparison of Pythagorean fuzzy numbers.Using the concept of score function Peng and Yang [22] developed accuracy function for magnitude comparison of Pythagorean fuzzy numbers.

Definition 4. Let $p=\langle\alpha,\beta\rangle$ be a Pythagorean fuzzy number. The accuracy function of p is defined as $h(p)=(\alpha)^2+(\beta)^2$, where $h(p)\in[0,1]$.

Let $p_1=\langle\alpha_1,\beta_1\rangle$ and $p_2=\langle\alpha_2,\beta_2\rangle$ be two PFNs; $s(p_1)=\alpha_1^{\,2}-\beta_1^{\,2}$ and $s(p_2)=\alpha_2^{\,2}-\beta_2^{\,2}$ be their score functions; $h(p_1)=\alpha_1^{\,2}+\beta_1^{\,2}$ and $h(p_2)=\alpha_2^{\,2}+\beta_2^{\,2}$ be the accuracy functions of p_1 and p_2, then yager and abbasov [11] defined the following:

I. If $s(p_1)<s(p_2)$, then p_1 is smaller than p_2, that is $p_1<p_2$.

II. If $s(p_1)>s(p_2)$, then $p_1>p_2$.

III. If $s(p_1)=s(p_2)$, then:

- *If* $h(p_1)<h(p_2)$, *then* $p_1<p_2$.
- *If* $h(p_1)>h(p_2)$, *then* $p_1>p_2$.
- *If* $h(p_1)=h(p_2)$, *then* p_1 *and* p_2 *represent the same information, that is* $p_1=p_2$.

3 | Spherical Distance Measurement Method for Pythagorean Fuzzy Numbers

In this section, we analyze Hamming distance measurement method and Euclidean distance measurement method. Then, we propose a spherical distance measurement method of PFNs.

3.1 | Distance Measurement Method for PFNs

Zhang and Xu [14] presented the Hamming distance measurement method for Pythagorean fuzzy numbers with help of membership,non-membership and indeterminacy degree.

Definition 5. Let $p_i=\langle\alpha_i,\beta_i\rangle, i=1,2$ be two Pythagorean fuzzy numbers. Then the Hamming distance between p_1 and p_2 can be denied as follows:

$$D_{ZH}(p_1,p_2)=\frac{1}{2}(|(\alpha_1)^2-(\alpha_2)^2|+|(\beta_1)^2-(\beta_2)^2|+|(\gamma_1)^2-(\gamma_2)^2|). \quad (1)$$

Li and Zeng [18] proposed a new distance measurement method containing four parameters α,β,γ and d of Pythagorean fuzzy numbers.

Definition 6. Let $p_i=\left\langle\alpha_i,\beta_i\right\rangle, i=1,2$ be two Pythgorean fuzzy numbers. Then the normalized Hamming distance and the normalized Euclidean distance measure between p_1 and p_2 can be denied as follow:

$$D_{LH}(p_1,p_2)=\frac{1}{4}(|\alpha_1-\alpha_2|+|\beta_1-\beta_2|+|r_1-r_2|+|d_1-d_2|). \quad (2)$$

$$D_{LE}(p_1,p_2)=\left[\frac{1}{4}(\alpha_1-\alpha_2)^2+(\beta_1-\beta_2)^2+(r_1-r_2)^2+(d_1-d_2)^2\right]^{\frac{1}{2}}. \quad (3)$$

New distance measurement method using five the parameters membership (α), non-membership (β), hesitation (γ), strength of commitment (r) and direction of commitment (d) of Pythagorean fuzzy numbers.

Definition 7. Let $p_i=\left\langle\alpha_i,\beta_i\right\rangle, i=1,2$ be two Pythgorean fuzzy numbers. Then the normalized Hamming distance and the normalized Euclidean distance measure between p_1 and p_2 can be denied as follows:

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$$D_{LH}(p_1,p_2)=\frac{1}{5}(|\alpha_1-\alpha_2|+|\beta_1-\beta_2|+|\gamma_1-\gamma_2|+|r_1-r_2|+|d_1-d_2|). \quad (4)$$

$$D_{LE}(p_1,p_2)=\left[\frac{1}{4}(\alpha_1-\alpha_2)^2+(\beta_1-\beta_2)^2+(\gamma_1-\gamma_2)^2+(r_1-r_2)^2+(d_1-d_2)^2\right]^{\frac{1}{2}}. \quad (5)$$

Example 4. Let $p_1=\langle 0.6,0.5\rangle$ and $p_2=\langle 0.8,0.2\rangle$ be two Pythagorean fuzzy numbers. Then utilize the *Eq.* (*1*) to (*5*), the different types of distance between p_1 and p_2 and are as follows:

$D_{ZH}(p_1,p_2)=0.280, D_{LH}(p_1,p_2)=0.206, D_{LE}(p_1,p_2)=0.232, D_{LH}(p_1,p_2)=0.178,$
$D_{LE}(p_1,p_2)=0.209.$

3.2 | Spherical Distance Measurement Method for PFNs

Let $p=\langle\alpha,\beta\rangle$ be a Pythagorean fuzzy number satisfying the condition $0\le\alpha^2+\beta^2\le 1$ and hesitation function is $\gamma=\sqrt{1-\alpha^2-\beta^2}$, i.e., $\alpha^2+\beta^2+\gamma^2=1$.

From this relation we may assume that the triplet (α,β,γ) lies on the spherical surface of unit radius and centre at origin. This interpretation encourage defining the spherical distance between two Pythagorean fuzzy numbers on restricted spherical surface.

On spherical surface the shortest distance is the length arc of the great circle passing through both points.

Definition 8. Let A and C be two points on the spherical surface with co-ordinate (x_1,y_1,z_1) and (x_2,y_2,z_2), then the spherical distance between these two points is defined as:

$$D_{SP}(A,C)=\arccos\{1-\frac{1}{2}[(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2]\}. \quad (6)$$

Incorporated this expression, the spherical distance between two Pythagorean fuzzy numbers defined as follows:

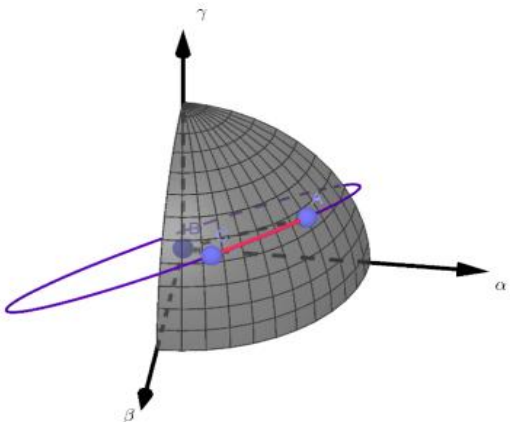

Fig. 2. Spherical distance between points A and B.

Definition 9. Let $p_1=\langle\alpha_1,\beta_1\rangle$ and $p_2=\langle\alpha_2,\beta_2\rangle$ be two Pythagorean fuzzy numbers with hesitation function γ_1 and γ_2 respectively. Then the spherical distance between these two Pythagorean fuzzy numbers is:

$$D_s(p_1,p_2)=\frac{2}{\pi}\arccos\{1-\frac{1}{2}[(\alpha_1-\alpha_2)^2+(\beta_1-\beta_2)^2+(\gamma_1-\gamma_2)^2]\}. \quad (7)$$

To get the distance value in between $[0,1]$ the factor $\frac{2}{\pi}$ is introduced.

Since, $\alpha_1^2+\beta_1^2+\gamma_1^2=1$ and $\alpha_2^2+\beta_2^2+\gamma_2^2=1$, so after simplifying the *Eq. (7)*, we have:

$$D_s(p_1,p_2)=\frac{2}{\pi}\arccos[\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2]. \quad (8)$$

Now, we define the spherical and normalized distances between two PFSs.

Definition 10. Let $P=\{x_i,\langle\alpha_p(x_i),\beta_p(x_i)\rangle : x_i\in X\}$ and $Q=\{x_i,\langle\alpha_Q(x_i),\beta_Q(x_i)\rangle : x_i\in X\}$ of the universe of discourse $X=\{x_1,x_2,.....x_n\}$,then their spherical and normalized spherical distances are;

I. Spherical distance:

$$D_s(P,Q)=\frac{2}{\pi}\sum_{i=1}^{n}\arccos[\alpha_P(x_i)\alpha_Q(x_i)+\beta_P(x_i)\beta_Q(x_i)+\gamma_P(x_i)\gamma_Q(x_i)]. \quad (9)$$

Where $0\le D_{NSP}(P,Q)\le n.$

II. Normalized spherical distance:

$$D_{NS}(P,Q)=\frac{2}{n\pi}\sum_{i=1}^{n}\arccos[\alpha_P(x_i)\alpha_Q(x_i)+\beta_P(x_i)\beta_Q(x_i)+\gamma_P(x_i)\gamma_Q(x_i)]. \quad (10)$$

Where $0\le D_{NSP}(P,Q)\le 1.$

Example 5. Let $p_1=\langle 0.9,0.2\rangle$ and $p_2=\langle 0.7,0.3\rangle$ be two Pythagorean fuzzy numbers. Then the spherical distance between p_1 and p_2 is:

$$D_s(p_1,p_2)=\frac{2}{\pi}\arccos[(0.9\times 0.7)+(0.2\times 0.3)\times(\sqrt{1-0.9^2-0.2^2}\times\sqrt{1-0.7^2-0.3^2})]=0.2198.$$

Definition 11. Let $p_1=(\alpha_{1j},\beta_{1j}),\ p_2=(\alpha_{2j},\beta_{2j}), j=1,2,3,.....,n,$ be two Pythagorean fuzzy numbers. w_j is the weight of j, i.e., $w=(w_1,w_2,...,w_n)^T$, where $0\le w_i\le 1$ and $\sum_{i=1}^{n}w_i=1$.

Then the weighted normalized spherical distance between p_1 and p_2 is defined as;

$$D_{NS}(p_1,p_2)=\frac{2}{n\pi}\sum_{i=1}^{n}\arccos[\alpha_{1j}\alpha_{2j}+\beta_{1j}\beta_{2j}+\gamma_{1j}\gamma_{2j}]. \quad (11)$$

Example 6. Let $p_1=\{\langle 0.6,0.3\rangle,\langle 0.8,0.2\rangle,\langle 0.5,0.4\rangle\}$ and $p_2=\{\langle 0.7,0.2\rangle,\langle 0.7,0.3\rangle,\langle 0.9,0.1\rangle\}$ be two PFSs with weights $w=\{0.2,0.5,0.3\}$. Then, the weighted spherical distance between P_1 and P_2 is calculated as:

$$\begin{aligned}D_{NS}=&\frac{2}{3\pi}[0.2\times\arccos((0.6\times 0.7)+(0.3\times 0.2)+(\sqrt{1-0.6^2-0.3^2}\times\sqrt{1-0.7^2-0.2^2}))\\&+0.5\times\arccos((0.8\times 0.7)+(0.2\times 0.3)+(\sqrt{1-0.8^2-0.2^2}\times\sqrt{1-0.7^2-0.3^2}))\\&+0.5\times\arccos((0.5\times 0.9)+(0.4\times 0.1)+(\sqrt{1-0.5^2-0.4^2}\times\sqrt{1-0.9^2-0.1^2}))]=0.0499.\end{aligned}$$

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4 | Proposed Pythagorean Fuzzy TOPSIS Method for MCDM Problems

In this section, we introduce multi-criteria Decision-making problem where the information has been taken in the form of the fuzzy numbers and apply spherical distance measurement method to solve this problems.

Let $A=\{A_1,A_2,...,A_n\},(m\geq 2)$ be a set alternatives and $C=\{C_1,C_2,...,C_n\},(n\geq 2)$ be a set of criterion, $w=(w_1,w_2,...w_n)^T$ where $0\leq w_i\leq 1$and $\sum_{j=1}^{n} w_i=1,$ be the weight vector for each criteria.

Let the Pythagorean fuzzy numbers $\langle\alpha_{ij},\beta_{ij}\rangle$denotes the assessment value of the i-th alternative for the j-th criteria, viz, $C_j(v_i)=\langle\alpha_{ij},\beta_{ij}\rangle$ and $R=(C_j(v_i))_{m\times n}$ denotes Pythagorean fuzzy decision matrix, where

$$\begin{bmatrix} \langle\alpha_{11},\beta_{11}\rangle & \langle\alpha_{12},\beta_{12}\rangle & \cdots & \langle\alpha_{1n},\beta_{1n}\rangle \\ \langle\alpha_{21},\beta_{21}\rangle & \langle\alpha_{22},\beta_{22}\rangle & \cdots & \langle\alpha_{2n},\beta_{2n}\rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle\alpha_{m1},\beta_{m1}\rangle & \langle\alpha_{m2},\beta_{m2}\rangle & \cdots & \langle\alpha_{mn},\beta_{mn}\rangle \end{bmatrix}.$$

4.1 | Process of the Proposed Method

To solve MCDM problems in Pythagorean fuzzy environment, we present Pythagorean fuzzy TOPSIS system. The primery concept of the TOPSIS approach is that the most preferred alternative should not only have shortest distance from the positive ideal solution but also have the furthest distance from the negative ideal solution.

We start this method by computing PFPIS and PFNIS. Let j_i be the set of benefit criteria and j_2 be the set of cost cretria. PFPIS and PFNIS were determined by score function. Let v^+ and v^- denote PFPIS and PFNIS respectively. These values are calculated using the following formula:

$$v^+=\{C_j,\max(S(C_j(v_i)))\,|\,j=1,2...,n\}. \tag{12}$$

$$v^-=\{C_j,\min(S(C_j(v_i)))\,|\,j=1,2...,n\}. \tag{13}$$

Next, we calculate normalized spherical distance from each alternative to the PFPIS $D_{NS}(v_i,v^-)$. Now, we obtain weighted normalized spherical distance of alternative v_i from PFPIS v^+ based on (11) which can be defined as follows:

$$D_{NS}(v_i,v^+)=\sum_{j=1}^{n} D_{NS}(C_j(v_i),C_j(v_i)^+)=\frac{2}{n\pi}\sum_{j=1}^{n} w_j \arccos((\alpha_{ij}\alpha_j^{\,+}+\beta_{ij}\beta_j^{\,+}+\gamma_{ij}\gamma^{+}). \tag{14}$$

Where $i=1,2...,n$.

According to the principle of TOPSIS, the smaller $D_{NS}(v_i,v^+)$ is the better alternative x_i.

Let

$$D_{\min}(x_i,x^+)=\min_i D_{NS}(v_i,v^+), i=1,2...,n.$$

Similarly, the weighted normalized spherical distance of alternative v_i from PFNIS v^- calculated as follows:

$$D_{NS}(v_i,v^-)=\sum_{j=1}^{n}D_{NS}(C_j(v_i),C_j(v^-))=\frac{2}{n\pi}\sum_{j=1}^{n}w_j\arccos((\alpha_{ij}\alpha_j^-+\beta_{ij}\beta_j^-+\gamma_{ij}\gamma^-). \tag{15}$$

Where $i=1,2...,n$.

According to the principle of TOPSIS, the greater $D_{NS}(v_i,v^-)$ is the better alternative v_i. Let $D_{max}(v_i,v^+)=\max_i D_{NS}(v_i,v^+), i=1,2,...,n$.

Now, we calculate relative closeness co-efficient of the alternative x_i with respect to PFPIS (x^+) and PFNIS (x^-) with the help of basic principle of classical TOPSIS method.

The formula for $RC(x_i)$ is as follows:

$$RC(v_i)=\frac{D_{NS}(v_i,v^-)}{D_{NS}(v_i,v^+)+D_{NS}(v_i,v^-)}. \tag{16}$$

According to the Hadi Venecheh [12], the optimal solution is the shortest distance from positive ideal solution and farthest distance from negative ideal solution. Consequently, Zhang and Xu [14] utilized revised index, which is denoted by $\xi(v_i)$ to determine the ranking order. The index formula is expressed as follow:

$$\xi(v_i)=\frac{D_{NS}(v_i,v^-)}{D_{max}(v_i,v^-)}-\frac{D_{NS}(v_i,v^+)}{D_{min}(v_i,v^-)}. \tag{17}$$

According to $RC(v_i)$ or $\xi(v_i)$, we obtain the rank of the alternatives x_i, which is used to determine the optimal solution according to the maximum value of $RC(v_i)$ or $\xi(v_i)$.

4.2 | Algorithm for Proposed Method

The traditional TOPSIS introduced by Yoon and Hwang [23] is a classic and useful method to solve the MCDM problems with crisp numbers. Zhang and Xu [14] developed a revised TOPSIS method to deal effectively with Pythagorean fuzzy information. The algorithm involves the following steps:

Step 1. For MCDM problem with PFNs, we construct the decision matrix $R=(C_j(v_i))_{m\times n}$, where the elements $C_j(v_i), i=1,2,...,m,\ \ j=1,2,...,n$ are the assessments of alternative v_i with respect to the criterion C_j.

Step 2. Utilize the score function to determine the Pythagorean fuzzy positive ideal solution (v^-).

Step 3. Use *Eq. (14)* and *(15)* to calculate the weighted spherical distances of each alternative v_i from the Pythagorean fuzzy PIS (v^+) and Pythagorean fuzzy NIS (v^-).

Step 4. Utlize *Eq. (16)* and *(17)* to calculate relative closeness $RC(v_i)$ and revised closeness $\xi(v_i)$ of the alternative v_i.

Step 5. Rank the alternative and select the best one(s) according to the decreasing relative closeness $RC(v_i)$ and revised closeness $\xi(v_i)$ obtained from Step 4.

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The bigger the $RC(v_i)$ the more desirable the v_i, $(i=1,2,...,m)$ will be.

5 | Illustrative Example

In this section, we consider a decision-making problem that concerns daily life problems to illustrate the proposed approach.

A decision maker want to buy a car. There are more than one branded cars with their criterion. Decision-maker considers only five banded cars v_1, v_2, v_3, v_4 and v_5 among these he/she want to buy a particular with his/her availability. In order to buy the cars four criterion viz., cost (C_1), fuel consumption (C_2), comfort (C_3) and attractiveness (C_4) are considered as avaluation factor. According to the assessment of attributes and criterion, Pythagorean fuzzy decision matrix are considered as follows:

$$\begin{array}{ccccc} & C_1 & C_2 & C_4 & C_5 \\ v_1 & \langle 0.7,0.3\rangle & \langle 0.5,0.4\rangle & \langle 0.7,0.6\rangle & \langle 0.9,0.2\rangle \\ v_2 & \langle 0.6,0.4\rangle & \langle 0.6,0.3\rangle & \langle 0.5,0.3\rangle & \langle 0.7,0.4\rangle \\ v_3 & \langle 0.5,0.6\rangle & \langle 0.7,0.4\rangle & \langle 0.5,0.4\rangle & \langle 0.6,0.4\rangle \\ v_4 & \langle 0.4,0.7\rangle & \langle 0.5,0.8\rangle & \langle 0.6,0.2\rangle & \langle 0.5,0.3\rangle \\ v_5 & \langle 0.5,0.8\rangle & \langle 0.7,0.2\rangle & \langle 0.6,0.5\rangle & \langle 0.6,0.2\rangle \end{array}.$$

Where $C_1(v_1)=\langle 0.7,0.3\rangle$ represents the degree to which alternative v_1 satisfies criteria C_1 is 0.7 and degree to which satisfies alternative v_1 dissatisfies criterion C_1 is 0.3.

Considering that fuel consumption, comfort and attractiveness of the cars as benefit criteria, $j_1=\{C_2,C_3,C_4\}$ and cost of the car is the cost criterion $j_2=\{C_1\}$.

To calculate score type Pythagorean fuzzy positive ideal solution (v^+) and Pythagorean fuzzy negative ideal solution (v^-), we utilize the *Eq. (12)* and *(13)*. We get the result as follows:

$$v^+=\{\langle 0.5,0.8\rangle,\langle 0.7,0.2\rangle,\langle 0.6,0.2\rangle,\langle 0.9,0.2\rangle\}.$$

$$v^-=\{\langle 0.7,0.3\rangle,\langle 0.5,0.8\rangle,\langle 0.5,0.4\rangle,\langle 0.5,0.3\rangle\}.$$

Next, utilize *Eq. (14)* and *(15)* to calculate the weighted spherical distances of each alternatives v_i from Pythagorean fuzzy positive ideal solution and Pythagorean fuzzy negative ideal solutions;

	$D_{NS}(v_i,v^+)$	$D_{NS}(v_i,v^-)$
v_1	0.0590	0.0554
v_2	0.0593	0.0475
v_3	0.0561	0.0475
v_4	0.0712	0.0359
v_5	0.0290	0.0742

We utilize *Eq. (16)* and *(17)* to compute the $RC(v_i)$ and $\xi(v_i)$ for each alternative v_i and results are listed below:

	$RC(v_i)$(Rank)	$\xi(v_i)$(Rank)
v_1	0.4842(2)	–1.2878(3)
v_2	0.4447(4)	–1.4046(4)
v_3	0.4642(3)	–1.2794(2)
v_4	0.3352(5)	–1.9713(5)
v_5	0.7189(1)	0(1)

According to $RC(v_i)$ rank of the alternatives are $v_5 \succ v_1 \succ v_3 \succ v_2 \succ v_4$ among which v_5 v_5 is the best alternative. However, according to the revised index $\xi(v_i)$ the ranking of the alternatives are $v_5 \succ v_3 \succ v_1 \succ v_2 \succ v_4$. Here also, the best alternative is v_5.

6 | Conclusion and Future Work

In this paper, the spherical distance measurements method has been introduced and are applied in TOPSIS method for MCDM approach with with Pythagorean fuzzy is developed. The main advantage of this method is that it is able to reflect the importance of the degree of membership, non-membership and hesitancy of decision-maker. Moreover, it provides a more complete representation of the decision process because the decision makers can consider many different scenarios depending on his interest by dealing with Pythagorean fuzzy environment. The spherical distance measurement method combined with the TOPSIS method by Pythagorean fuzzy data has enormous chance of success for MCDM problems. Ordering of the alternative by utilizing relative closeness and revised index method.

In future research, we expect to devlop further developments by using spherical distance measurement methods in the environment of intuitiontic fuzzy,picture fuzzy,fermatean fuzzy etc. This approach will be considered, especially in supply chain management and logistic, engineering, manufacturing system, business and marketing, human resources, and water resource management.

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Conflicts of Interest

No potential conflict of interest was reported by the authors.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authers.

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On Some Related Concepts of n-Cylindrical Fuzzy Neutrosophic Topological Spaces

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Abstract

This work is the continuation of our recent work entitled n-Cylindrical Fuzzy Neutrosophic Topological Spaces (n-CyFNTS). In this paper, we present n-Cylindrical Fuzzy Neutrosophic (n-CyFN) continuous functions and related results. A characterization on n-CyFN continuous functions is proposed as well. Along with familiarising the n-CyFN interior and n-CyFN closure of sets in n-Cylindrical Fuzzy Neutrosophic Topology (n-CyFNT) and its basic theorems, we define n-CyFN open function, n-CyFN closed function and n-CyFN homeomorphism.

Keywords: n-CyFN continuous functions, CyFN interior, n-CyFN closure, n-CyFN open function, n-CyFN closed function, n-CyFN homeomorphism.

1 | Introduction

Following Zadeh's invention of fuzzy sets [14], Chang [6] introduced the idea of fuzzy topological space, and numerous researchers adapted general topological notions in the context of fuzzy topology, among them. Atanassov's Intuitionistic Fuzzy Sets (IFSs) [5] are one of the fuzzy set's generalisations. Later, Coker [7] introduced the useful concept of intuitionistic fuzzy topological space by utilising the concept of the IFS. Jeon et al. [11] defined and investigated the concepts of intuitionistic fuzzy-continuity and pre-continuity. After Smarandache [8], [9] introduced the concepts of neutrosophy and neutrosophic set, Salama and Alblowi [1] introduced the concepts of neutrosophic crisp set and neutrosophic crisp topological spaces. Neutrosophy has laid the groundwork for a new class of mathematical theories that generalise both their crisp and fuzzy counterparts. The neutrosophic set is a generalisation of the IFS. Salama and Alblowi [1] introduced the concept of neutrosophic topological spaces in 2012. They defined neutrosophic topological space as a generalisation of intuitionistic fuzzy topological space and a neutrosophic set in addition to each element's degree of membership, degree of indeterminacy, and degree of non-membership. Smarandache [7], [8] introduced the dependence degree of (also, the independence degree of) the fuzzy and neutrosophic components for the first time. Arokiarani et al. [4] pioneered the concept of

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fuzzy neutrosophic set, defined as the sum of all three membership functions not exceeding 3. In 2017, Veereswari [16] proposed a Fuzzy Neutrosophic topological space and basic operations on it. Sarannya et al. [13] introduced n-Cylindrical Fuzzy Neutrosophic Sets (n-CyFNS), which have T and F as dependent components and I as independent components. Except for fuzzy neutrosophic sets, the n-CyFNS is the largest extension of fuzzy sets. In this case, the degree to which positive, neutral, and negative membership functions satisfy the condition, $0 \leq \beta A(x) \leq 1$ and $0 \leq \alpha A\ n(x) + \gamma An(x) \leq 1$, $n > 1$, is an integer. They also defined the distance between two n-CyFNS, as well as their properties and basic operations.

In this work, the membership functions of an image and its pre image in n-CyFNSs are defined at first. We then introduced the n-CyFN continuity of a function defined between two n-Cylindrical Fuzzy Neutrosophic Topological Spaces (n-CyFNTS) using this concept. In addition, we characterise the n-CyFN continuity and present some fundamental results associated with this idea. Additionally defined are n-CyFN interior and closure of n-CyFN subsets of n-CyFNTS. Based on these concepts, we investigate some properties. As well as this, the ideas of n-CyFN open function, n-CyFN closed function and n-CyFN homeomorphism are presented.

2 | Preliminaries

Throughout this paper, $\mho$ denotes the universe of discourse.

Definition 1 ([17]). A fuzzy set A in $\mho$ is defined by membership function μA: A→ [0, 1] whose membership value $\mu A(x)$ shows the degree to which $x \in \mho$ includes in the fuzzy set A, for all $x \in \mho$.

Definition 2 ([6]). A fuzzy topological space is a pair (X, T), where X is any set and T is a family of fuzzy sets in X satisfying following axioms:

I. $\Phi, X \in T$.
II. If $A, B \in T$, then $A \cap B \in T$.
III. If $A_i \in T$ for each $i \in I$, then $\cup_i A_i \in T$.

Definition 3 ([5]). An IFS A on $\mho$ is an object of the form $A = \{(x, \alpha_A(x), \gamma_A(x) \mid x \in \mho)\}$ where $\alpha_A(x) \in [0,1]$ is called the degree of membership of x in A, $\gamma_A(x) \in [0,1]$ is called the degree of non-membership of x in A, and where α_A and γ_A satisfy (for all $x \in \mho$) $(\alpha_A(x) + \gamma_A(x) \leq 1)$ IFS($\mho$) denote the set of all the IFSs on a universe $\mho$.

Definition 4 ([8]). A neutrosophic set A on $\mho$ is $A = < x, T_A(x), I_A(x), F_A(x) >$; $x \in \mho$, where T_A, I_A, F_A: $A \to\ ]^-0,1^+[$ and $^-0 < T_A(x) + I_A(x) + F_A(x) < 3^+$.

Definition 5 ([16]). A fuzzy neutrosophic set A on $\mho$ is $A = < x, T_A(x), I_A(x), F_A(x) >$; $x \in \mho$, where T_A, I_A, F_A: $A \to [0,1]$ and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 6 ([9]). A neutrosophic set A on $\mho$ is an object of the form $A = \{(x, u_A(x), \zeta_A(x), v_A(x)) : x \in \mho\}$, where $u_A(x), \zeta_A(x), v_A(x) \in [0,1]$, $0 \leq u_A(x) + \zeta_A(x) + v_A(x) \leq 3$, for all $x \in \mho$. $u_A(x)$ is the degree of truth membership, $\zeta_A(x)$ is the degree of indeterminacy and $v_A(x)$ is the degree of non-membership. Here (x) and (x) are dependent components and (x) is an independent component.

Definition 7 ([1]). A Neutrosophic Topology (NT) on a non-empty set X is a family τ of neutrosophic subsets in X satisfying the following axioms:

I. (NT1) $0_N, 1_N \in \tau$.
II. (NT2) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$.
III. (NT3) $\cup G_i \in \tau$, for all $\{G_i : i \in J\} \subseteq \tau$.

In this case the pair (X) is called a Neutrosophic Topological Space (NTS) and any neutrosophic set in τ is known as Neutrosophic Open Set (NOS) in X. The elements of τ are called open neutrosophic sets, A neutrosophic set F is closed if and only if it C(F) is neutrosophic open.

Definition 8 ([16]). A Fuzzy Neutrosophic Topology (FNT) a non-empty set X is a family τ of fuzzy neutrosophic subsets in satisfying the following axioms:

I. (FNT1) $0_N, 1_N \in \tau$.
II. (FNT2) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$.
III. (FNT3) $\cup G_i \in \tau$, for all $\{G_i: i \in J\} \subseteq \tau$.

In this case the pair (X) is called a Fuzzy Neutrosophic Topological Space (FNTS) and any fuzzy neutrosophic set in τ is known as Fuzzy Neutrosophic Open Set (FNOS) in. The elements of τ are called open fuzzy neutrosophic sets.

Definition 9 ([13]). A n-CyFNS A on U is an object of the form $A = \{\langle x, \alpha_A(x), \beta_A(x), \gamma_A(x)\rangle \mid x \in U\}$ where $\alpha_A(x) \in [0, 1]$, called the degree of positive membership of x in A, $\beta_A(x) \in [0,1]$, called the degree of neutral membership of x in A and $\gamma_A(x) \in [0, 1]$, called the degree of negative membership of x in A, which satisfies the condition, (for all $x \in U$) ($0 \leq \beta_A(x) \leq 1$ and $0 \leq \alpha_A{}^n(x) + \gamma_A{}^n(x) \leq 1$, $n > 1$, is an integer. Here T and F are dependent neutrosophic components and I is independent.

For the convenience, $\langle \alpha_A(x), \beta_A(x), \gamma_A(x)\rangle$ is called as n-Cylindrical Fuzzy Neutrosophic Number (n-CyFNN) and is denoted as $A = \langle \alpha_A, \beta_A, \gamma_A \rangle$.

Definition 10 ([13]). Let $\{A_i: i \in I\}$ be an arbitrary family of n-CyFNS in U. Then

- $\cap A_i = \{\langle x, inf(\alpha_{Ai}(x)), inf(\beta_{Ai}(x)), sup((\gamma_{Ai}(x))\rangle \mid x \in U\}$.
- $\bigcup A_i = \{\langle x, sup(\alpha_{Ai}(x)), sup(\beta_{Ai}(x)), inf((\gamma_{Ai}(x))\rangle \mid x \in U\}$.

Definition 11 ([13]). Let $\tau_N(U)$ denote the family of all n-CyFNSs on U.

1. Inclusion: for every two $A, B \in \tau_N(U)$, $A \subseteq B$ iff (for all $x \in U$, $\alpha_A(x) \leq \alpha_B(x)$ and $\beta_A(x) \leq \beta_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$) and $A = B$ iff ($A \subseteq B$ and $B \subseteq A$).
2. Union: for every two $A, B \in \tau_N(U)$, the union of two n-CyFNSs A and B is:

$$A \cup B(x) = \{\langle x, \max(\alpha_A(x), \alpha_B(x)), \max(\beta_A(x), \beta_B(x)), \min(\gamma_A(x), \gamma_B(x)) \rangle \mid x \in U\}.$$

3. Intersection: for every two $A, B \in \tau_N(U)$, the intersection of two n-CyFNSs A and B is:

$$A \cap B(x) = \{\langle x, \min(\alpha_A(x), \alpha_B(x)), \min(\beta_A(x), \beta_B(x)), \max(\gamma_A(x), \gamma_B(x)) \rangle \mid x \in U\}.$$

4. Complementation: for every $A \in \tau_N(U)$, the complement of an n-CyFNS A is $A^c = \{\langle x, \gamma_A(x), \beta_A(x), \alpha_A(x)\rangle \mid x \in U\}$.

Definition 12 ([13]).

I. If $A \subseteq B$ & $B \subseteq C$ then $A \subseteq C$.
II. $A \cup B = B \cup A$ & $A \cap B = B \cap A$.
III. $(A \cup B) \cup C = A \cup (B \cup C)$ & $(A \cap B) \cap C = A \cap (B \cap C)$.
IV. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ & $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.
V. $A \cap A = A$ & $A \cup A = A$.
VI. De Morgan's Law for A & B ie, $(A \cup B) = A^c \cap B^c$ & $(A \cap B) = A^c \cup B$.

Definition 13.

- $0_{CyN1}=\{\langle x, 0, 1, 1\rangle \mid x \in U\}$ and $1_{CyN1}=\{\langle x, 1, 1, 0\rangle \mid x \in U\}$ or,
- $0_{CyN2}=\{\langle x, 0, 0, 1\rangle \mid x \in U\}$ and $1_{CyN2}=\{\langle x, 1, 0, 0\rangle \mid x \in U\}$.

Commonly it can be denoted as 0_{CyN} and 1_{CyN}.

Definition 14. An n-Cylindrical Fuzzy Neutrosophic Topology (n-CyFNT) on a non-empty set X is a family, τ_X, of n-cylindrical fuzzy neutrosophic sets in X which satisfies the following conditions:

I. $0_{cyN}, 1_{cyN} \in \tau_X$.
II. $A_1 \cap A_2 \in \tau_X$.
III. $\cup A_i \in \tau_X$, for any arbitrary family $A_i \in \tau_X$, $i \in I$.

The pair (X, τ_X) is called an n-CyFNTS and any n-CyFNS belongs to τ_X is called an n-Cylindrical Fuzzy Neutrosophic Open Set (n-CyFNOS) and the complement of n-CyFNOS is called n-Cylindrical Fuzzy Neutrosophic Closed Set (n-CyFNCS) in X. Like classical topological spaces and fuzzy topological spaces, the family $\{0_{cyN}, 1_{cyN}\}$ is called indiscrete n-CyFNTS and the topology containing all the n-CyFN subsets is called Discrete n-CyFNTs.

Definition 15. Let (X, τ_X) be a CyFNTS on X. Then, $\mathscr{B} \subseteq \tau_X$, a sub family of τ_X is called an n-CyFN base for (X, τ_X), if each member of τ_X may be expressed as the union of members in $\mathscr{B}$.

$\mathscr{S} \subseteq \tau_X$, a sub family of τ_X is called a n-CyFN sub-base for (X, τ_X), if the family of all finite intersections of $\mathcal{S}$ forms a base for (X, τ_X).

3 | n-Cylindrical Fuzzy Neutrosophic Continuous Functions

In this section we will introduce n-CyFN continuous functions and related results.

For that, we define the positive, neutral and negative membership functions of image and pre image of a function in n-CyFNSs.

Definition 16. Let X and Y be two non-empty sets and let f: X→Y be a function. If $A=\{\langle x, \alpha_A(x), \beta_A(x), \gamma_A(x)\rangle \mid x \in X\}$ is an n-CyFNS in X, then the membership functions of image of A under f, defined as:

$f_{\alpha A}(y) = \sup\{\alpha_A(x)): x \in f^{-1}\ y)\}$, $if\ f^{-1}\ y)$ is non empty

$= 0$ if $f^{-1}\ y) = 0$.

$f_{\beta A}(y) = \sup\{\beta_A(x)): x \in f^{-1}\ y)\ \}$, $if\ f^{-1}\ y)$ is non empty

$= 0$ if $f^{-1}\ y) = 0$.

$f_{\gamma A}(y) = \inf\ \{\gamma_A(x)): x \in f^{-1}\ y)\}$, $if\ f^{-1}(y)$ is non empty

$= 1$ if $f^{-1}\ y) = 0$.

Definition 17. Let X and Y be two non-empty sets and let f: X→Y be a function. If $B=\{\langle y, \alpha_B(y), \beta_B(y)), \gamma_B(y)\rangle \mid y \in Y\}$, is an n-CyFNS in Y, then the membership functions of pre image of B under f are defined as:

I. $f^{-1}\alpha_B(x) = \alpha_B(f(x))$,
II. $f^{-1}\beta_B(x) = \beta_B(f(x))$,
III. $f^{-1}\gamma_B(x) = \gamma_B(f(x))$, respectively.

Now we define the image and pre image of n-CyFNSs.

Definition 18. Let X and Y be two non-empty sets and let f: X→Y be a function. If B = {⟨ y, $\alpha_B(y)$, $\beta_B(y)$), $\gamma_B(y)$ ⟩ | y∈Y}, is an n-CyFNS in Y, then the pre image of Y under f is denoted by f^{-1} B), is the n-CyFNS in X defined by:

$$f^{(-1)}(B) = \{\langle x, f^{(-1)}\alpha_B(x), f^{(-1)}\beta_B(x), f^{(-1)}\gamma_B(x)\rangle \mid x \in X\}.$$

Similarly, if A = {⟨ x, $\alpha_A(x)$, $\beta_A(x)$, $\gamma_A(x)$ ⟩ | x∈X} is an n-CyFNS in X, then the image of A under f, denoted by f(A), is the n-CyFNS in Y defined by f(A) = {⟨ y, $f\alpha_A(y)$, $f\beta_A(y)$, $f\gamma_A(y)$ ⟩ | y∈Y}.

It is evident that f(A) and f^{-1} B) are n-CyFNSs and f is called n-CyFN function.

Preposition 1. Let X and Y be two non-empty sets and let f: X→Y be an n-CyFN function.

Then

1. $f^{-1}\left(B^C\right) = f^{-1}[B]^C$ for any n-CyFN subset B of Y.
2. $f[A]^C \subset f[A^C]$ for any n-CyFN subset A of X.
3. If B_1 & B_2 are two n-CyFN subsets of Y & $B_1 \subset B_2$, then $f^{-1}(B_1) \subset f^{-1}(B_2)$.
4. If A_1 & A_2, are two n-CyFN subsets of X & $A_1 \subset A_2$, then $f(A_1) \subset f(A_2)$.
5. For any n-CyFN subset B of Y, $f[f^{-1}B)] \subset B$.
6. For any n-CyFN subset A of X, $A \subset f^{-1}[f(A)]$.
7. $f^{-1}\left(\cup B_j\right) = \cup f^{-1}(B_j)$ & $f^{-1}\left(\cap B_j\right) = \cap f^{-1}(B_j)$.
8. $f\left(\cup B_j\right) = \cup f(B_j)$ & $f\left(\cap B_j\right) \subseteq \cap f(B_j)$, if f is one-one, then $f\left(\cap B_j\right) = \cap f(B_j)$.
9. $f^{-1}(0_{CyN}) = 0_{CyN}$ & $f^{-1}(1_{CyN}) = 1_{CyN}$.
10. $f(0_{CyN}) = 0_{CyN}$ & $f(1_{CyN}) = 1_{CyN}$, if f is onto.

Proof:

For (1): we have f^{-1} B) = {⟨ x, $f^{-1}\alpha_B(x)$, $f^{-1}\beta_B(x)$, $f^{-1}\gamma_B(x)$ ⟩ | x ∈ X}. Then $f^{-1}\left(B^C\right)$ = {⟨ x, $f^{-1}\gamma_B(x)$, $f^{-1}\beta_B(x)$, $f^{-1}\alpha_B(x)$ ⟩ | x ∈ X} = {⟨ x, $\gamma_B(f(x))$, $\beta_B(f(x))$, $\alpha_B(f(x))$ ⟩ | x∈X}.

$$= \{f^{-1}[B]\}^C.$$

For (2): we have f(A) = {⟨ y, $f\alpha_A(y)$, $f_{\beta A}(y)$, $f\gamma_A(y)$ ⟩ | y∈Y}. Then

$f[A]^C$ = {⟨ y, $f\gamma_A(y)$, $f\beta_A(y)$, $f\alpha_A(y)$ ⟩ | y∈Y} = {⟨ y, inf $\gamma_A(x)$, sup $\beta_A(x)$, sup $\alpha_A(x)$ ⟩ | y∈Y}

Now we have, $f_{\alpha_{A^C}}(y) = \sup \alpha_{A^C}(z), z \in f^{-1}(y)$

$$\geq \alpha_{A^C}(y)$$

$$= \gamma_A(y)$$

$$\geq \inf \gamma_A(z) = f\gamma_A(y)$$

$= f_{\alpha_A}{}^{C}(y).$

Similarly, it follows for the other membership functions. Thus $f[A]^C \subset f[A^C]$.

For (3): we have $B_1 \subset B_2$, for any $x \in X$, $f^{-1}\alpha_{B1}(x) = \alpha_{B1}(f(x)) \leq \alpha_{B2}(f(x)) = f^{-1}\alpha_{B2}(x)$,

Similarly, we can prove that $f^{-1}\ \beta_{B1}(x) \leq f^{-1}\ \beta_{B2}(x)$ and $f^{-1}\ \gamma_{B1}(x) \geq f^{-1}\ \gamma_{B2}(x)$.

Hence $f^{-1}(B_1) \subset f^{-1}(B_2)$.

For (4): this is similar to 3.

For (5): for any $y \in Y$, such that $f(y) \neq \varnothing$, $f(f^{-1}\alpha_B(y)) = \sup\ (f^{-1}\alpha_B(z)), z \in f^{-1}(y)$

$= \sup \alpha_B\big(f\ z)\big), z \in f^{-1}\ y)$

$= \alpha_B\ y).$

If $f(y) = \varnothing$, then $f(f^{-1}\alpha_B(y)) = 0 \leq \alpha_B\ y)$, also $f(f^{-1}\beta_B(y)) \leq \beta_B\ y)$. Similarly, we have $f(f^{-1}\gamma_B(y)) \geq \gamma_B\ y)$.

For (6): for any $x \in X$ $f^{-1}(f\alpha_A\ x)) = f\alpha_A\ f(x))$

$= \sup \alpha_A(z), z \in f^{-1}(f(x))$

$\geq \alpha_A(x).$

Similarly, we have $f^{-1}(f\beta_A\ x)) \geq \beta_A(x)$ and $f^{-1}(f\gamma_A\ x)) \leq \gamma_A(x)$.

For (7): it follows immediately.

For (8): it is evident since 7 holds.

For (9): we have $f^{-1}(0_{CyN}) = f^{-1}(\langle y, 0, 0, 1 \rangle) = \langle x, f^{-1}(0), f^{-1}(0), f^{-1}(1) \rangle = \langle x, 0, 0, 1 \rangle = 0_{CyN}$ $f^{-1}(1_{CyN}) = f^{-1}(\langle y, 1,0, 0 \rangle) = \langle x, f^{-1}(1), f^{-1}(0), f^{-1}(0) \rangle = \langle x, 1, 0, 0 \rangle = 1_{CyN}$.

For (10): this is similar to 7.

Now we'll look at how to define n-CyFN continuity of a function.

Definition 19. Let (X, τ_X) and (Y, τ_Y) be two n-cylindrical fuzzy neutrosophic topological spaces and let f: X→Y be an n-CyFN function. Then f is said to be n-CyFN continuous if for any n-Cylindrical fuzzy neutrosophic subset A of X and for any neighborhood V of f[A] there exists a neighborhood U of A such that $f[U] \subset V$.

The following theorem gives the characterization of n-cylindrical fuzzy neutrosophic continuity.

Theorem 1. Let (X, τ_X) and (Y, τ_Y) be two n-cylindrical fuzzy neutrosophic topological spaces and let f: X→Y be a function. Then the following are equivalent:

I. f is n-cylindrical fuzzy neutrosophic continuous.
II. For any n-cylindrical fuzzy neutrosophic subset (n-CyFNS) A of X and for any neighborhood V of f[A], there exists a neighborhood U of A such that for any B ⊂ U implies, f[B] ⊂ V.
III. For any n-CyFNS A of X and for any neighborhood V of f [A], there exists a neighborhood U of A such that U ⊂ $f^{-1}[V]$.
IV. For any n-CyFNS A of X and for any neighborhood V of f[A], $f^{-1}[V]$ is a neighborhood of A.

Proof: for equivelency of I) and II),

Suppose f is n-cylindrical fuzzy neutrosophic continuous and A ⊂ X is n-CyFNS and V be the neighborhood of f[A]. Then there exist a neighborhood U of A such that f[U] ⊂ V, by the definition of continuity. If B ⊂ U, then f[B] ⊂ f[U] ⊂ V.

For equivelency of II) and III),

Suppose II) holds. Let A ⊂ X is n-CyFNS and V be the neighborhood of f[A], then there exist a neighborhood U of A such that for any B ⊂ U, then f[B] ⊂ V, then we can write B ⊂ f^{-1}f[B] ⊂ f^{-1}[V]. Since B is arbitrary, III) follows.

For equivelency of III) and IV),

Suppose III) holds ie, for any neighborhood U of A ⊂ X, A is n-CyFNS in X, U ⊂ f^{-1}[V]. By the definition of neighborhood, A⊂ C ⊂ U, C is n-CyFNOS of X.

But A⊂ C ⊂ U ⊂ f^{-1}[V]. Thus A⊂ C ⊂ f^{-1}[V]. This implies IV)

For equivelency of IV) and I),

Suppose IV) holds. ie, A ⊂ C ⊂ f^{-1}[V]; C is n-CyFNOS of X.

Now f[C] ⊂ f(f^{-1}[V]) this shows that f[C] ⊂ f(f^{-1}[V]) ⊂ V. Since C is open, it is a neighborhood of A. Thus f is n-CyFN continuous.

Example 1. Let X = {a, b, c} and family of n-CyFN subsets of X, $\tau_X = \{1cyN, 0cyN, A, B, C, D\}$. Also Y = { p, q, r } with τ_y = { 1_{cyN} , 0_{cyN}, P, Q, R, S} and let f: (X, τ_X) → (Y, τ_y) be a function. Also,

A = {<a; 0.4, 0.5, 0.6>, <b; 0.6, 0.5, 0.3>, <c; 0.7, 0.5, 0.5>},

B = {<a; 0.7, 0.5, 0.6>, <b; 0.7, 0.6, 0.4>, <c; 0.7, 0.6, 0.6>},

C = {<a; 0.7, 0.5, 0.6>, <b; 0.7, 0.6, 0.3>, <c; 0.7, 0.6, 0.5>},

D = {<a; 0.4, 0.5, 0.6>, <b; 0.6, 0.5, 0.4>, <c; 0.7, 0.5, 0.6>},

P = {<p; 0.6, 0.5, 0.3>, <q; 0.4, 0.5, 0.6>, <r; 0.7, 0.5, 0.5>},

Q = {<p; 0.7, 0.6, 0.4>, <q; 0.7, 0.5, 0.6>, <r; 0.7, 0.6, 0.6>},

R = {<p; 0.7, 0.6, 0.3>, <q; 0.7, 0.5, 0.6>, <r; 0.7, 0.6, 0.5>},

S = {<p; 0.6, 0.5, 0.4>, <q; 0.4, 0.5, 0.6>, <r; 0.7, 0.5, 0.6>}.

Clearly (X, τ_X) and (Y, τ_Y) are n-CyFNTS.

If f: $(X, \tau_X) \rightarrow (Y, \tau_y)$ is defined by f(a) = q, f(b) = p and f(c) = f(r), then f is n-CyFN continuous.

Theorem 2. Let (X, τ_X) and (Y, τ_Y) be two n-CyFNTS and let f: X→Y is n-CyFN continuous if and only if for each n-CyFN open subset B of Y, we have $f^{-1}[B]$ is an n-CyFN open subset of X.

Proof: let f: X→Y is n-CyFN continuous, then for any n-CyFNS A of X and V is a neighborhood of f[A], there exists a neighborhood U of A such that f[U] ⊂ V. Let B ⊂ Y be n-CyFN open in Y, then there is a neighborhood V of f[A] such that V ⊂ B, by iv) of *Theorem 1*, $f^{-1}[V]$ is a neighborhood of A. Since V ⊂ B implies $f^{-1}[V] \subset f^{-1}[B]$. Thus $f^{-1}[B]$ is a neighborhood and A is arbitrary, it follows the definition of n-CyFN open subset.

Conversely suppose, A ⊂ X is n-CyFNS and V is a neighborhood of f[A]. Then, there exists an n-CyFN open subset C such that f[A] ⊂ C ⊂ V and $f^{-1}[C]$ is open. Also we can write $A \subset f^{-1}f[A] \subset f^{-1}[C] \subset f^{-1}[V]$. Thus $f^{-1}[V]$ is a neighborhood of A. hence f is n-CyFN continuous.

Definition 20. Let (X, τ_X) and (Y, τ_Y) be two n-CyFNTS and let f: X→Y is an n-CyFN function. Then

I. f is called an n-CyFN open function if f(A) is n-CyFN open in Y for every n-CyFN open set A in X.
II. f is called n-CyFN closed function if f(B) is n-CyFN closed in Y for every n-CyFN closed set B in X.

Example 2. In *Example 1*, f is clearly an open function but not closed.

Theorem 3. Let X be a non-empty set and let (Y, τ_Y) be an n-CyFNTS and f: X→Y be a function. Then, there exists a coarsest n-cylindrical fuzzy neutrosophic topology τ_x on X such that f is n-CyFN continuous.

Proof: clearly,

I) $0_{cyN}, 1_{cyN} \in \tau_X$.
II) and III) are evident. Also from *Theorem 2*, f is n-CyFN continuous.

To prove that τ_X is the coarsest n-CyFNT on X such that the f is n-CyFN continuous.

Suppose $\tau_c \subset \tau_X$ is the coarsest n-CyFNT on X such that f is n-CyFN continuous. If $B \in \tau_X$, then there is V in τ_Y such that $f^{-1}[V] = B$. But f is n-CyFN continuous with respect to τ_c.

Then $B = f^{-1}[V] \in \tau_c$.

Thus $\tau_X \subset \tau_c$ this implies $\tau_c = \tau_X$.

4 | Interior and Closure of n-Cylindrical Fuzzy Neutrosophic Sets

We will now define the interior and closure of a set in n-CyFNTS.

Chang gave the definition of a neighborhood of a fuzzy open set instead of a neighborhood of a point.

Definition 21. Let A and N be two n-cylindrical fuzzy neutrosophic subsets of an n-CyFNTS. N is called neighborhood of A if there exists an n-CyFNOS, O such that $A \subset O \subset N$.

Preposition 2. A ⊂ X is n-cylindrical fuzzy neutrosophic open in $(X, \tau_{\mathscr{C}})$ if and only if it carries a neighborhood of its each subset.

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Definition 22. Let (X, τ_X) be an n-CyFNTS and let P = {⟨ x, $\alpha_P(x)$, $\beta_P(x)$, $\gamma_P(x)$ ⟩ | x ∈ X} is an n-CyFNS in X. Then the n-Cylindrical fuzzy neutrosophic interior (int_{CyN}) is defined as the n-CyFN union of all n-CyFN open subsets of X ie, $int_{CyN}(P) = \cup \{A: A \in \tau_X \text{ and } A \subseteq P\}$.

Clearly int_{CyN}(P) is the biggest n-CyFN open set that is contained by P.

Example 3. Let X = {x, y} and $\tau_X = \{1_{cyN}, 0_{cyN}, P, Q, R, S\}$, where

P = {<x; 0.4, 0.5, 0.8>, <y; 0.3, 0.5, 0.3>}, Q = {<x; 0.5, 0.5, 0.3>, <y; 0.5, 0.5, 0.9>},

R = {<x; 0.5, 0.5, 0.3>, <y; 0.5, 0.5, 0.3>}, S = {<x; 0.4, 0.5, 0.8>, <y; 0.3, 0.5, 0.9>}.

Clearly (X, τ_X) is an n-CyFNTS.

Let G = {<x; 0.5, 0.5, 0.2 >, <y; 0.5, 0.5, 0.2>}, Clearly int_{CyN}(G) = R.

Theorem 4. Let (X, τ_X) be an n-CyFNTS and G ⊆ X. G is an n-CyFN open set if and only if G = int_{CyN}(G).

Proof: if G is an n-CyFN open set, then the largest n-CyFN open set contained by G is int_{CyN}(G). Hence G = int_{CyN}(G).

Conversely, int_{CyN}(G) is an n-CyFN open set and, if G = int_{CyN}(G) then G is n-CyFN open set.

Theorem 5. Let (X, τ_X) be an n-CyFNTS and G, H ⊆ X. Then,

I. $int_{CyN}(int_{CyN}(G)) = int_{CyN}(G)$.
II. $G \subseteq H \Longrightarrow int_{CyN}(G) \subseteq int_{CyN}(H)$.
III. $int_{CyN}(G) \cap int_{CyN}(H) = int_{CyN}(G \cap H)$.
IV. $int_{CyN}(G) \cup int_{CyN}(H) \subseteq int_{CyN}(G \cup H)$.

Proof:

I. Let int_{CyN}(G) = A. Then $A \in \tau_X$ if and only if A = int_{CyN}(A), therefore, $int_{CyN}(G) = int_{CyN}(int_{CyN}(G))$.
II. Let G ⊆ H. From the definition of n-CyFN interior, $int_{CyN}(G) \subseteq G$ and $int_{CyN}(H) \subseteq H$. Also int_{CyN}(H) is the largest n-CyFN open set contained by H. Hence $G \subseteq H \Longrightarrow int_{CyN}(G) \subseteq int_{CyN}(H)$.
III. By the definition of n-CyFN interior, $int_{CyN}(G) \subseteq G$ and $int_{CyN}(H) \subseteq H$. Then $int_{CyN}(G) \cap int_{CyN}(H) \subseteq G \cap H$. But $int_{CyN}(G \cap H)$ is the largest open set contained by G ∩ H, then $int_{CyN}(G) \cap int_{CyN}(H) \subseteq int_{CyN}(G \cap H)$. (1)

On the other hand, G ∩ H ⊆ G and G ∩ H ⊆ H then $int_{CyN}(G \cap H) \subseteq int_{CyN}(G)$ and $int_{CyN}(G \cap H) \subseteq int_{CyN}(H)$, and $int_{CyN}(G \cap H) \subseteq int_{CyN}(G) \cap int_{CyN}(H)$. (2)

From (1) and (2), the result follows.

IV. We have $int_{CyN}(G) \subseteq G$ and $int_{CyN}(H) \subseteq H$. Then $int_{CyN}(G) \cup int_{CyN}(H) \subseteq (G \cup H)$. But $int_{CyN}(G \cup H)$ is the largest open set contained by G ∪ H. Hence, $int_{CyN}(G) \cup int_{CyN}(H) \subseteq int_{CyN}(G \cup H)$.

Definition 23. Let (X, τ_X) be an n-CyFNTS and let P = {⟨ x, $\alpha_P(x)$, $\beta_P(x)$, $\gamma_P(x)$ ⟩ | x ∈ X} is an n-CyFNS in X. Then the n-Cylindrical fuzzy neutrosophic closure (cl_{CyN}) of P.

ie, cl_{CyN}(P) = ∩ {B: B is an n-CyFNCS in X and $B \subseteq P$}.

It is to be noted that, cl_{CyN}(P) is the smallest closed set that contains P.

Example 4. From *Example 3*,

The set of closed sets of τ_X is denoted as $\tau_X{}^c = \{\{1_{cyN}, 0_{cyN}, P^c, Q^c, R^c, S^c\}$

P^c = {<x; 0.8, 0.5, 0.4>, <y; 0.3, 0.5, 0.3>}, Q^c = {<x; 0.3, 0.5, 0.5>, <y; 0.9, 0.5, 0.5>},

R^c = {<x; 0.3, 0.5, 0.5>, <y; 0.3, 0.5, 0.5>}, S^c = {<x; 0.8, 0.5, 0.4>, <y; 0.9, 0.5, 0.3>},

Let H = {<x; 0.2, 0.5, 0.5 >, <y; 0.2, 0.5, 0.5>}, Clearly cl_{CyN}(H) =R^c.

Theorem 6. Let (X, τ_X) be an n-CyFNTS and H ⊆X. H is an n-CyFN closed set if and only if H = cl_{CyN}(H).

Proof: the proof is obvious.

Theorem 7. Let (X, τ_X) be an n-CyFNTS and H ⊆X. Then $int_{CyN}(H) \subseteq H \subseteq cl_{CyN}(H)$.

Proof: from the definition of n-CyFN interior, it is clear that int_{CyN}(H) is the largest n-CyFN open set contained by H. Hence $int_{CyN}(H) \subseteq H$.

Also from the definition of n-CyFN closure, cl_{CyN}(H) is the smallest closed set that contains H. Therefore $H \subseteq cl_{CyN}(H)$.

Hence proved.

Theorem 8. Let (X, τ_X) be an n-CyFNTS and G, H ⊆ X. Then,

I. $cl_{CyN}(cl_{CyN}(H)) = cl_{CyN}(H)$.
II. G ⊆H $\Longrightarrow cl_{CyN}(G) \subseteq cl_{CyN}(H)$.
III. $cl_{CyN}(G \cap H) \subseteq cl_{CyN}(G) \cap cl_{CyN}(H)$.
IV. $cl_{CyN}(G) \cup cl_{CyN}(H) = cl_{CyN}(G \cup H)$.
V. $(cl_{CyN}\ H))^c = int_{CyN}(H^c)$.

Proof:

I. Let cl_{CyN}(H) = K. Then K is an n-CyFN closed set. Then K = cl_{CyN}(K) Hence $cl_{CyN}(cl_{CyN}(H)) = cl_{CyN}(H)$.
II. Let G ⊆H. From the definition of n-CyFN closure, G ⊆ cl_{CyN}(G) and H ⊆ cl_{CyN}(H). Also cl_{CyN}(H) is the smallest n-CyFN closed set that contains H. Hence G ⊆ H $\Longrightarrow cl_{CyN}(G) \subseteq cl_{CyN}(H)$.
III. cl_{CyN}(G) and cl_{CyN}(H) are n-CyFN closed sets. So $cl_{CyN}(G) \cap cl_{CyN}(H)$ is an n-CyFN closed set. By the definition of n-CyFN closure, $G \subseteq cl_{CyN}(G)$ and H ⊆ cl_{CyN}(H). Then G∩H⊆ $cl_{CyN}(G) \cap cl_{CyN}(H)$. But cl_{CyN}(G∩H) is the smallest closed set that contains G∩H, then $cl_{CyN}(G \cap H) \subseteq cl_{CyN}(G) \cap cl_{CyN}(H)$.
IV. We have $G \subseteq cl_{CyN}(G)$ and $H \subseteq cl_{CyN}(H)$, then G∪H ⊆ $cl_{CyN}(G) \cup cl_{CyN}(H)$.

But cl_{CyN}(G∪H) is the smallest closed set that contains G∪H. Therefore $cl_{CyN}(G \cup H) \subseteq cl_{CyN}(G) \cup cl_{CyN}(H)$.

Conversely, $G \subseteq cl_{CyN}(G) \subseteq cl_{CyN}(G \cup H)$ and $H \subseteq cl_{CyN}(H) \subseteq cl_{CyN}(G \cup H)$. Then $cl_{CyN}(G) \cup cl_{CyN}(H) \subseteq cl_{CyN}(G \cup H)$.Hence the result holds.

V. If we take the basic definition of n-CyFN closure and n-CyFN interior, we get, $(cl_{CyN}\ \mathrm{H}))^c = (\cap\ G_j)^c$, $\mathrm{H} \subseteq G_j$ and $G_j{}^c \in \tau_X$.

But $(\cap\ G_j)^c = \cup\ G_j{}^c = int_{CyN}(H^c)$.

Theorem 9. Let (X, τ_X) and (Y, τ_Y) be two n-CyFNTS and let f: X→Y is an n-CyFN function. Then

I. f is n-CyFN open function if $\mathrm{f}(int_{CyN}(G)) \subseteq int_{CyN}(f\ G))$ for each n-CyFN set G over X.
II. f is n-CyFN closed function if $cl_{CyN}(\mathrm{f(H)}) \subseteq \mathrm{f}(cl_{CyN}(H))$, for each n-CyFN set H over X.

Proof: straight forward.

Definition 24. Let (X, τ_X) and (Y, τ_Y) be two n-CyFNTS and let f: X→Y is an n-CyFN function. Then f is n-CyFN homeomorphism if:

I. f is a bijection.
II. f is n-CyFN continuous.
III. Inverse of f, f^{-1} is also n-CyFN continuous.

Example 5. Let X = {a, b, c, d}. Now consider the n-CyFN subsets I, J, K, L of X as

I = {<a; 0.4, 0.5, 0.6>, <b; 0.6, 0.5, 0.3>, <c; 0.7, 0.5, 0.5>, <d; 0.7, 0.5, 0.6>},

J = {<a; 0.7, 0.5, 0.6>, <b; 0.6, 0.5, 0.3>, <c; 0.7, 0.5, 0.5>, <d; 0.4, 0.5, 0.6>},

K = {<a; 0.7, 0.5, 0.6>, <b; 0.6, 0.5, 0.3>, <c; 0.7, 0.5, 0.5>, <d; 0.7, 0.5, 0.6>},

L = {<a; 0.4, 0.5, 0.6>, <b; 0.6, 0.5, 0.3>, <c; 0.7, 0.5, 0.5>, <d; 0.4, 0.5, 0.6>}.

Let $\tau_X = \{0_{cyN}, 1_{cyN}, \mathrm{I}, \mathrm{J}, \mathrm{K}, \mathrm{L}\}$. Then (X, τ_X) is an n-CyFNTS. Consider θ: X→X defined by θ(a) = d, θ(b) = b, θ(c) = c and θ(d) = a. Clearly θ is an n-CyFN homeomorphism.

Theorem 10. Let (X, τ_X) and (Y, τ_Y) be two n-CyFNTS and let f: X→Y is an n-CyFN function. Then the following conditions are equivalent.

I. f is f is n-CyFN homeomorphism.
II. f is n-CyFN continuous and n-CyFN open.
III. f is n-CyFN continuous and n-CyFN closed.

Proof: from the definition and properties of n-CyFN continuous function, n-CyFN interior and n-CyFN closure, it follows.

5 | Conclusion

n-Cylindrical fuzzy neutrosophic sets are the latest extension of neutrosophic sets in which I as an independent neutrosophic component. In our previous work we defined the topological space in n-CyFNS context. So far, we have introduced n-CyFN base, n-CyFN subbase and related theorems. Through this paper our aim is to extend the important concepts like continuity, interiors, closure and related theorems to n-cylindrical fuzzy neutrosophic environment. Here, we define the membership functions of an image and its pre image in n-CyFNSs. Using this concept, we then introduced the n-Cylindrical fuzzy neutrosophic continuity of a function defined between two n-CyFNTS. In addition, we present some fundamental results related to this concept, as well as a characterization of the n-Cylindrical fuzzy neutrosophic continuity. Also defined are the n-Cylindrical fuzzy neutrosophic interior and n-Cylindrical fuzzy neutrosophic closure of n-CyFN subsets of n-CyFNTS. We examine some properties based on these concepts. The concept of n-CyFN open function, n-CyFN closed function and n-CyFN homeomorphism are also presented. To that extent, the study sheds more light on the subject, allowing for further investigation in other important concepts of topology like compactness, connectedness, separation axioms etc.

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Conflicts of Interest

The authors declare no conflict of interest.

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An Overview of Plithogenic Set and Symbolic Plithogenic Algebraic Structures

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Abstract

This paper is devoted to plithogeny, plithogenic Set, and its extensions. These concepts are branches of uncertainty and indeterminacy instruments of practical and theoretical interest. Starting with some examples, we proceed towards general structures. Then we present definitions and applications of the principal concepts derived from plithogeny, and relate them to complex problems.

Keywords: Plithogeny, Neutrosophy, Dialectics, Plithogenic set, Plithogenic logic, Plithogenic probability, Plithogenic statistics, Multivariate analysis, Symbolic Plithogenic algebraic structures.

1 | Etymology of Plithogeny

Plitho-geny etymologically comes from: (Gr.) πλήθος (plithos) = crowd, large number of, multitude, plenty of, and -geny < (Gr.) -γενιά (-geniá) = generation, the production of something & γένεια (géneia) = generations, the production of something < -γένεση (-génesi) = genesis, origination, creation, development, according to Translate Google dictionaries and Webster's new world dictionary of American English, third college edition, Simon & Schuster, Inc.

Therefore, Plithogeny is the genesis or origination, creation, formation, development, and evolution of new entities from dynamics and organic fusions of contradictory and/or neutrals and/or non-contradictory multiple old entities.

Plithogeny is a dynamics of many pairs of opposites ($<A_1>$, $<antiA_1>$), …, ($<A_n>$, $<antiA_n>$), and their neutralities (indeterminacies) $<neutA_1>$, …, $<neutA_n>$.

While Neutrosophy, as a particular case of Plithogeny, is the dynamics of only one pair of opposites ($<A>$, $<antiA>$) and its neutral (indeterminacy) $<neutA>$.

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Dialectics and Yin Yang are the dynamics of one pair of opposites <A> and <antiA>, without taking into consideration their neutral (indeterminacy) <neutA>.

Plithogenic means what is pertaining to Plithogeny.

All plithogeny, plithogenic set, plithogenic logic, plithogenic probability, plithogenic statistics, and symbolic plithogenic algebraic structures were [29]-[32].

This article is a synthesis of evolution of plithogenic domain during a six-year span, with articles published by many authors around the world [1]-[28].

2 | Plithogenic Set

A plithogenic set P is a set whose elements are characterized by one or more attributes, and each attribute may have many values. Each attribute's value v has a corresponding degree of appurtenance d(x,v) of the element x to the set P, with respect to some given criteria.

In order to obtain a better accuracy for the plithogenic aggregation operators, a contradiction (dissimilarity) degree is defined between each attribute value and the dominant (most important) attribute value. The contradiction degrees may similarly be: either fuzzy, intuitionistic fuzzy, neutrosophic, or any fuzzy extension, yet for simplicity we'll be using the fuzzy ones.

However, there are cases when such dominant attribute value may not be taking into consideration or may not exist [therefore it is considered zero by default], or there may be many dominant attribute values. In such cases, either the contradiction degree function is suppressed, or another relationship function between attribute values should be established.

The plithogenic aggregation operators (intersection, union, complement, inclusion, equality) are based on contradiction degrees between attributes' values, and the first two operators (intersection and union) are linear combinations of the fuzzy operators' tnorm ($\wedge_F$) and tconorm ($\vee_F$) where 'F' stands for 'fuzzy'.

Let $(\overset{0}{a_o}, \overset{c_1}{a_1}, \ldots, \overset{c_i}{a_i}, \ldots, \overset{c_n}{a_n})$ and $(\overset{0}{b_o}, \overset{c_1}{b_1}, \ldots, \overset{c_i}{b_i}, \ldots, \overset{c_n}{b_n})$ be the attribute value degrees of a generic element x from a plithogenic set *P*, assigned by an Expert A, and respectively an Expert B, where the contradiction degrees of the attribute values are: $0 \le c_1 \le \ldots \le c_i \le \ldots \le c_n \le 1$, i.e. $c(a_o, a_i) = c_i$ for $1 \le i \le n$. let $\wedge_P, \vee_P$ represent respectively the plithogenic intersection and plithogenic union. Then,

The plithogenic intersection:

$$x(\overset{0}{a_o}, \overset{c_1}{a_1}, \ldots, \overset{c_i}{a_i}, \ldots, \overset{c_n}{a_n}) \wedge_P x(\overset{0}{b_o}, \overset{c_1}{b_1}, \ldots, \overset{c_i}{b_i}, \ldots, \overset{c_n}{b_n}) = x(a_0 \wedge_F b_o, (1-c_1)\cdot(a_1 \wedge_F b_1) +$$
$$c_1 \cdot (a_1 \vee_F b_1), \ldots, (1-c_i)\cdot(a_i \wedge_F b_i) + c_i \cdot (a_i \vee_F b_i), \ldots, (1-c_n)\cdot(a_n \wedge_F b_n) + c_n \cdot (a_n \vee_F b_n)).$$

The plithogenic union:

$$x(\overset{0}{a_o}, \overset{c_1}{a_1}, \ldots, \overset{c_i}{a_i}, \ldots, \overset{c_n}{a_n}) \vee_P x(\overset{0}{b_o}, \overset{c_1}{b_1}, \ldots, \overset{c_i}{b_i}, \ldots, \overset{c_n}{b_n}) = x(a_0 \vee_F b_o, (1-c_1)\cdot(a_1 \vee_F b_1) +$$
$$c_1 \cdot (a_1 \wedge_F b_1), \ldots, (1-c_i)\cdot(a_i \vee_F b_i) + c_i \cdot (a_i \wedge_F b_i), \ldots, (1-c_n)\cdot(a_n \vee_F b_n) + c_n \cdot (a_n \wedge_F b_n)).$$

Plithogenic set is a generalization of the crisp set, fuzzy set, intuitionistic fuzzy set, neutrosophic set, since these these types of sets are characterized by a single attribute value (appurtenance): which has one value (membership), for the crisp set and fuzzy set, two values (membership, and nonmembership), for intuitionistic fuzzy set, or three values (membership, nonmembership and indeterminacy), for neutrosophic set.

2.1 | Example of Plithogenic Fuzzy Set

Let P be a plithogenic set, representing the students from a college. Let x belonging to P be a generic student that is characterized by three attributes:

I. altitude (a), whose values are {tall, short} = $\{a_1, a_2\}$.
II. weight (w), whose values are {obese, fat, medium, thin} = $\{w_1, w_2, w_3, w_4\}$.
III. hair color (h), whose values are {blond, reddish, brown} = $\{h_1, h_2, h_3\}$.

The uni-dimensional attribute contradiction degrees are:

$$c(a_1, a_2) = 1,$$
$$c(w_1, w_2) = 1/3;\ c(w_1, w_3) = 2/3;\ c(w_1, w_4) = 1,$$
$$c(h_1, h_2) = 1/2;\ c(h_1, h_3) = 1.$$

Dominant attribute values are: a_1, w_1, and h_1 respectively for each corresponding uni-dimensional attribute a, w, and h respectively.

The multi-attribute dimension is 9 attribute values.

P = { John(tall, short; obese, fat, medium, thin; blond, reddish, brown),

John(tall, short; obese, fat, medium, thin; blond, reddish, brown) }.

For example:

P = { John(0.8, 0.2; 0.1, 0.2, 0.4, 0.6; 0.9, 0.6, 0.0),

Richard (0.5, 0.4; 0.3, 0.21, 0.4, 0.7; 0.8, 0.5, 0.01) }.

2.2 | Example of Plithogenic Neutrosophic Set

P = { John((0.8, 0.1, 0.3), (0.2, 0.0, 0.1); (0.1, 0.5, 0.5), (0.2, 0.6, 0.6), (0.4, 0.4, 0.4),

(0.6, 0.7, 0.2); (0.9, 0.8, 0.8), (0.6, 0.3, 0.3), (0.0, 0.2, 0.2)),

Richard ((0.5, 0.2, 0.0), (0.6, 0.8, 0.8); (0.7, 0.5, 0.4), (0.1, 0.6, 0.1), (0.8, 0.4, 0.7),

(0.3, 0.4 0.2); (0.7, 0.8, 0.9), (0.5, 0.0, 0.3), (0.2, 0.8, 0.1) }.

2.3 | A Plithogenic Application to Images

A pixel x may be characterized by colors $k_1, k_2, ..., k_n$. We write $x(k_1, k_2, ..., k_n)$, where $n \geq 1$.

We may consider the degree of each color either fuzzy, intuitionistic fuzzy, or neutrosophic, or any fuzzy extension. For example:

Fuzzy degree:

$$x(0.4, 0.6, 0.1, ..., 0.3).$$

Intuitionistic fuzzy degree:

$$x(\ (0.1, 0.2), (0.3, 0.5), (0.0, 0.6), ..., (0.8, 0.9)\ ).$$

Neutrosophic degree:

$$x\big((0.0,\ 0.3,\ 0.6),\ (0.2,\ 0.8,\ 0.9),\ (0.7,\ 0.4,\ 0.2),\ \ldots,\ (0.1,\ 0.1,\ 0.9) \big).$$

Then, we can use the plithogenic (fuzzy, intuitionistic fuzzy, neutrosophic, other fuzzy extension) intersection operator to combine them.

But, we first establish the degrees of contradictions $c(k_i, k_j)$ between the colors k_i and k_j in order to find the linear combinations of t-norm and t-conorm that one applies to each color (similar to the indeterminacy above). The contradiction degrees may also be: either fuzzy, intuitionistic fuzzy, neutrosophic, or other fuzzy extension. In majority of cases we choose, for simplicity, the fuzzy one.

3 | Plithogenic Logic

A plithogenic logic proposition P is a proposition that is characterized by degrees of many truth-values with respect to the corresponding attributes' values. For each attribute's value v there is a corresponding degree of truth-value d(P, v) of P with respect to the attribute value v. Plithogenic logic is a generalization of the classical logic, fuzzy logic, intuitionistic fuzzy logic, and neutrosophic logic, since these four types of logics are characterized by a single attribute value (truth-value): which has one value (truth), for the classical logic and fuzzy logic, two values (truth, and falsehood), for intuitionistic fuzzy logic, or three values (truth, falsehood, and indeterminacy), for neutrosophic logic. A plithogenic logic proposition P, in general, may be characterized by more than four degrees of truth-values resulted from under various attributes.

3.1 | Example of Plithogenic Fuzzy Logic

Let P = "John is a knowledgeable person" be a logical proposition. It is evaluated in a fuzzy way (one component for each attribute value) by multiple experts (two) and under multiple attribute values (six), that's why it is called plithogenic.

The three attributes under which this proposition has to be evaluated about-according to the experts-are: Science (whose attribute values are: mathematics, physics, anatomy), literature (whose attribute values are: poetry, novel), and arts (whose only attribute value is: sculpture).

According to Expert A (lexander), the fuzzy truth-values of plithogenic proposition P are:

I. $P_A(0.7, 0.6, 0.4; 0.9, 0.2; 0.5)$, which means that John's fuzzy degree of truth (knowledge) in mathematics is 0.7, degree of truth (knowledge) in physics is 0.6, degree of truth (knowledge) in anatomy is 0.4.
II. Degree of truth (knowledge) in poetry is 0.9, degree of truth (knowledge) in novels is 0.2.
III. Degree of truth (knowledge) in sculpture is 0.5.

But, according to Expert B (arbara), the truth-values of plithogenic proposition P are:

$$P_{B\,=}(0.9,\ 0.6,\ 0.2;\ 0.8,\ 0.7;\ 0.3).$$

Assume that the experts consider that the attributes' values contradiction (dissimilarity) degrees are:

0	0.3	0.8	0	0.9	0
mathematics	physics	anatomy	poetry	novels	sculpture

And taking the fuzzy $t_{norm}(a, b) = a \wedge_F b = a{\cdot}b$, and fuzzy $t_{conorm}(a, b) = a \vee_F b = a + b - a{\cdot}b$

One gets:

$P_A(0.7, 0.6, 0.4; 0.9, 0.2; 0.5) \wedge_P P_B(0.9, 0.6, 0.2; 0.8, 0.7; 0.3) =$

$P_{A} \wedge_{P\ B}(0.7 \wedge_{P} 0.9, 0.6 \wedge_{P} 0.6, 0.4 \wedge_{P} 0.2; 0.9 \wedge_{P} 0.8, 0.2 \wedge_{P} 0.7; 0.5 \wedge_{P} 0.3) =$

$P_{A} \wedge_{P\ B}(0.7 \wedge_{F} 0.9, 0.6 \wedge_{P} 0.6, 0.4 \wedge_{P} 0.2; 0.9 \wedge_{F} 0.8, 0.2 \wedge P\ 0.7; 0.5 \wedge_{F} 0.3) =$

$P_{A} \wedge_{P\ B}(0.7 \cdot 0.9, (1-0.3) \cdot [0.6 \wedge_{F} 0.6] + 0.3 \cdot [0.6 \vee_{F} 0.6], (1-0.8) \cdot [0.4 \wedge_{F} 0.2] + 0.8 \cdot [0.4 \vee_{F} 0.2];$

$0.9 \cdot 0.8, (1-0.9) \cdot [0.2 \wedge_{F} 0.7] + 0.9 \cdot [0.2 \vee_{F} 0.7]; 0.5 \cdot 0.3) =$

$P_{A} \wedge_{P\ B}(0.630, 0.7 \cdot [0.6 \cdot 0.6] + 0.3 \cdot [0.6 + 0.6 - 0.6 \cdot 0.6],$

$0.2 \cdot [0.4 \cdot 0.2] + 0.8 \cdot [0.4 + 0.2 - 0.4 \cdot 0.2];$

$0.720, 0.1 \cdot [0.2 \cdot 0.7] + 0.9 \cdot [0.2 + 0.7 - 0.2 \cdot 0.7]; 0.150) =$

$P_{A} \wedge_{P\ B}(0.630, 0.504, 0.432; 0.720, 0.698; 0.150),$

Where $/\backslash_{P}$, $/\backslash_{F}$, $\backslash/_{P}$, $\backslash/_{P}$ represent the fuzzy intersection, fuzzy union, plithogenic intersection, and plithogenic union respectively.

3.2 | Example of Plithogenic Neutrosophic Logic

Let P = "John is a knowledgeable person" be a logical proposition. It is evaluated in a neutrosophic way (three components for each attribute value) by multiple attribute values (six), that's why it is called plithogenic .

P((0.7, 0.1, 0.5), (0.6, 0.0, 0.9), (0.4, 0.4, 0.4); (0.9,0.2, 0.3), (0.2, 0.6, 0.7); (0.5, 0.3, 0.1)), which means that John's neutrosophic degrees of truth (knowledge), indeterminate-truth, and falsehood (nonknowledge) in mathematics is 0.7, 0.1, and 0.5 respectively, and similarly for his neutrosophic degrees in physics, and anatomy; in the same way for Richard's neutrosophic degrees.

4 | Plithogenic Probability

Since in plithogenic probability each event E from a probability space U is characterized by many chances of the event to occur [not only one chance of the event E to occur: as in classical probability, imprecise probability, and neutrosophic probability], a plithogenic probability distribution function, PP(x), of a random variable x, is described by many plithogenic probability distribution sub-functions, where each sub-function represents the chance (with respect to a given attribute value) that value x occurs, and these chances of occurrence can be represented by classical, imprecise, or neutrosophic probabilities (depending on the type of degree of a chance).

4.1 | Example of Plithogenic Probabilistic

What is the plithogenic probability that Jennifer will graduate at the end of this semester in her program of electrical engineering, given that she is enrolled in and has to pass two courses of mathematics (non-linear differential equations, and stochastic analysis), and two courses of Mechanics (Fluid Mechanics, and Solid Mechanics)?

We have a 4 attribute-values of plithogenic probability.

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According to her adviser, Jennifer's plithogenic single-valued fuzzy probability of graduating at the end of this semester is:

$$J\left(0.5,\ 0.6;\ 0.8,\ 0.4 \right).$$

Which means 50% chance of passing the non-linear differential equations class, 60% chance of passing the stochastic analysis class (as part of mathematics), and 80% of passing the Fluid Mechanics class, and 40% of passing the Solid Mechanics class (as part of Physics).

Therefore, the plithogenic probability in this example is composed from 4 classical probabilities.

Also, according to her adviser, Jennifer's plithogenic single-valued neutrosophic probability of graduating at the end of this semester is:

$$J\left(\left(0.5,\ 0.1,\ 0.3\right),\ \left(0.6,\ 0.2,\ 0{,}7\right);\ \left(0.8,\ 0.5,\ 0.4\right),\ \left(0.4,\ 0.0,\ 0.9\right) \right).$$

Which means 50% chance of passing, 10% indeterminate-chance, and 30% chance of non-passing the non-linear differential equations class; similarly for the others.

While the plithogenic probability of an event E to occur is calculated with respect to many chances of the event E to occur (it is calculated with respect to each event's attribute/parameter chance of occurrence).

Therefore, the plithogenic probability is a multi-probability (i.e. multi-dimensional probability), unlike the classical, imprecise and neutrosophic probabilities that are uni-dimensional probabilities.

The Neutrosophic Probability (and similarly for classical probability, and for the imprecise probability) of an event E to occur is calculated with respect to the chance of the event E to occur (i.e. it is calculated with respect to only ONE chance of occurrence).

5 | Plithogenic Statistics

As a generalization of classical statistics and neutrosophic statistics, the plithogenic statistics is the analysis of events described by the plithogenic probability.

Plithogenic statistics is a multivariate statistics, characterized by multiple random variables, whose degrees may be classical, fuzzy, intuitioninstic fuzzy, neutrosophic, or any other fuzzy extension.

In neutrosophic statistics we have some degree of indeterminacy, incompleteness, inconsistency into the data or into the statistical inference methods.

5.1 | Example of Plithogenic Statistics

Let's consider the previous example of plithogenic Probability that Jenifer will graduate at the end of this semester in her program of electrical engineering.

We now graph four probability distribution functions, instead of one as in the case when considering the neutrosophic distribution as a uni-dimensional neutrosophic function. Therefore, plithogenic Statistics is an extension of the multi-variate statistics.

6 | Symbolic Plithogenic Algebraic Structures

6.1 | Definitions of Symbolic Plithogenic Set and Symbolic Plithogenic Algebraic Structures

Let SPS be a non-empty set, included in a universe of discourse U, defined as follows:

SPS = {x | $x = a_0 + a_1p_1 + a_2P_2 + \ldots + a_nP_n$, $n \geq 1$, all a_i belong to R, or to C, or belong to some given algebraic structure space, for $0 \leq i \leq n$}.

where R = the set of real numbers, C = the set of complex numbers, and all P_i are symbols (letters, or variables), and are called Symbolic (Literal) plithogenic components (Variables)}, where 1, P_1, P_2, …, P_n act like a base for the elements of the above set SPS. a_0, a_1, a_2, …, a_n are called coefficients.

SPS is called a Symbolic plithogenic Set. And the algebraic structures defined on this set are called Symbolic plithogenic algebraic structures.

In general, Symbolic (or Literal) plithogenic theory is referring to the use of abstract symbols {i.e. the letters/parameters) P_1, P_2, …, P_n, representing the plithogenic components (variables) as above} in some theory.

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A Note on Somewhat Fuzzy Continuity

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Abstract

Thangaraj and Balasubramanian introduced the so-called somewhat fuzzy semicontinuous and somewhat fuzzy semiopen functions. Two years later, the same authors defined two other types of functions called somewhat fuzzy continuous and somewhat fuzzy open without indicating connections between them. At first glance, we may easily conclude (from their definitions) that every somewhat fuzzy continuous (resp. open) function is slightly fuzzy semicontinuous (resp. semiopen) but not conversely. In this note, we show that they are equivalent. We further prove that somewhat fuzzy continuous functions are weaker than fuzzy semicontinuous functions.

Keywords: Fuzzy continuous, Somewhat continuous, Somewhat fuzzy continuous, Somewhat fuzzy semicontinuous, Somewhat fuzzy open.

1 | Introduction

One of the fundamental concepts of all mathematics, particularly topology and analysis, is continuity of functions, which means that "small" changes in the input result in "small" changes in the output. After introducing the fuzzy set, Chang [4] defined the notion of fuzzy continuity of functions between fuzzy topological spaces in 1968. Following that, several generalizations of fuzzy continuity were introduced. In 1981, Azad [3] presented some classes of generalized fuzzy continuous functions. Fuzzy semi-continuity is one of them that is weaker than fuzzy continuity. In 2001-2003, Thangaraj and Balasubramanian [6] introduced somewhat fuzzy semicontinuous and somewhat fuzzy continuous functions [7]. They only discussed their relationship with fuzzy continuous functions. In the present note, we show that somewhat fuzzy semi-continuity and somewhat fuzzy continuity are the same, and somewhat fuzzy continuity is weaker than fuzzy semi-continuity. It is worth stating somewhat continuity appeared in different topological structures [1], [5], [7]. These classes of functions have a significant role in characterizing (soft) Baire spaces [2], [8] and weakly equivalence spaces [5].

Let X be a universe (domain). A mapping μ from X to the unit interval $\mathbb{I}$ is named a fuzzy set in X. The value $\mu(x)$ is called the degree of the membership of x in μ for each $x \in X$. The support of μ is

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the set $\{x \in X: \mu(x) > 0\}$. The complement of μ is denoted by μ^c (or $1 - \mu$ if there is no confusion), given by $\mu^c(x) = 1 - \mu(x)$ for all $x \in X$. Let $\{\mu_j: j \in J\}$ be a collection of fuzzy sets in X, where J is any index set. Then $\bigvee \mu_j(x) = sup\{\mu_j(x): j \in J\}$ and $\bigwedge \mu_j(x) = inf\{\mu_j(x): j \in J\}$ for each $x \in X$.

Definition 1 ([4]). A collectio $\mathscr{T}$ n of fuzzy sets in X is said to have a fuzzy topology on X if:

I. $0,1 \in \mathscr{T}$.
II. $\mu \wedge \lambda \in T$ whenever $\mu, \lambda \in \mathscr{T}$.
III. $for\ all\ \mu_j \in \mathscr{T}$ for any subcollection $\{\mu_j: j \in J\}$ of $\mathscr{T}$.

The pair $(X, \mathscr{T})$ is called a fuzzy topological space. Members of $\mathscr{T}$ are called fuzzy open subsets of X, and their complements are called fuzzy closed sets.

Let λ be a fuzzy set in X and let $(X, \mathscr{T})$ be a fuzzy topological space. The fuzzy interior of λ is defined by $Int(\lambda) := sup\{\mu: \mu \leq \lambda, \mu \in T\}$ and the fuzzy closure of λ is given by $Cl(\lambda) := inf\{\mu: \lambda \leq \mu, 1 - \mu \in T\}$. The set λ is called fuzzy semiopen [3] if $\lambda \leq Cl(Int(\lambda))$. The complement of every fuzzy semiopen set is fuzzy semiclosed. Evidently, every fuzzy open set is fuzzy semiopen but not the opposite (see, *Example 1*). The fuzzy semi-interior of λ is defined by $Int_s(\lambda) := sup\{\mu: \mu \leq \lambda, \mu$ is fuzzy semiopen$\}$ and the fuzzy semi-closure of λ is given by $Cl_s(\lambda) := inf\{\mu: \lambda \leq \mu, \mu$ is fuzzy semiclosed$\}$.

Lemma 1 ([10]). Let λ be a fuzzy set in X and let $(X, \mathscr{T})$ be a fuzzy topological space. The following hold:

I. $Int(\lambda) \leq Int_s(\lambda)$.
II. $Cl_s(\lambda) \leq Cl(\lambda)$.

To make our task easier, we introduce the following notions:

Definition 2. Let λ be a fuzzy set in X and let $(X, \mathscr{T})$ be a fuzzy topological space. Then

I. λ is called a somewhat fuzzy open set if $\lambda = 0$ or $Int(\lambda) \neq 0$.
II. λ is called a somewhat fuzzy semiopen set if $\lambda = 0$ or $Int_s(\lambda) \neq 0$.

Definition 3. A function f from a fuzzy topological space $(X, \mathscr{T})$ into another fuzzy topological space $(Y, \mathscr{U})$ is said to be:

I. Fuzzy continuous [4] if $f^{-1}(\beta) \in \mathscr{T}$ for each $\beta \in \mathscr{U}$.
II. Fuzzy semicontinuous [3] if $f^{-1}(\beta)$ is fuzzy semiopen for each $\beta \in \mathscr{U}$.
III. Somewhat fuzzy continuous [7] if $f^{-1}(\beta)$ is somewhat fuzzy open for each $\beta \in \mathscr{U}$.
IV. Somewhat fuzzy semicontinuous [6] if $f^{-1}(\beta)$ is somewhat fuzzy semiopen for each $\beta \in \mathscr{U}$.

Definition 4. A function f from a fuzzy topological space $(X, \mathscr{T})$ into another fuzzy topological space $(Y, \mathscr{U})$ is said to be:

I. Fuzzy open [9] if $f(\alpha) \in \mathscr{U}$ for each $\alpha \in \mathscr{T}$.
II. Fuzzy semiopen [3] if $f(\alpha)$ is fuzzy semiopen for each $\alpha \in \mathscr{T}$.
III. Somewhat fuzzy open [7] if $f(\alpha)$ is somewhat fuzzy open for each $\alpha \in \mathscr{T}$.
IV. Somewhat fuzzy semiopen [6] if $f(\alpha)$ is somewhat fuzzy semiopen for each $\alpha \in \mathscr{T}$.

2 | Main Results

We start by proving several lemmas that help us achieve our goal.

Lemma 2. Let λ be a non-zero fuzzy set in a fuzzy topological space $(X, \mathscr{T})$. Then λ is fuzzy semiopen iff $Cl(\lambda) = Cl(Int(\lambda))$.

Proof: Given a fuzzy semiopen set λ such that $\lambda \neq 0$. Then $\lambda \leq Cl(Int(\lambda))$ and so $Cl(\lambda) \leq Cl(Int(\lambda))$. On the other hand, we always have $Int(\lambda) \leq \lambda$. Therefore $Cl(Int(\lambda)) \leq Cl(\lambda)$. Hence $Cl(\lambda) = Cl(Int(\lambda))$.

Conversely, assume that $Cl(\lambda) = Cl(Int(\lambda))$, but $\lambda \leq Cl(\lambda)$ always, so $\lambda \leq Cl(Int(\lambda))$. Thus λ is fuzzy semiopen.

Lemma 3. Let λ be a non-zero fuzzy set in the fuzzy topological space $(X, \mathscr{T})$. If λ is fuzzy semiopen, then $Int(\lambda) \neq 0$.

Proof: Assume, if possible, that λ is a fuzzy semiopen set for which $Int(\lambda) = 0$, by *Lemma 2*, $Cl(\lambda) = 0$ implies that $\lambda = 0$, a contradiction.

Lemma 4. Let λ be a fuzzy set in a fuzzy topological space $(X, \mathscr{T})$. Then λ is somewhat fuzzy open if it is somewhat fuzzy semiopen.

Proof: Without loss of generality, we assume $\lambda \neq 0$; otherwise, the equivalence of these concepts is apparent. Since, by *Lemma 1*, $Int(\lambda) \leq Int_s(\lambda)$, so this concludes that the somewhat fuzzy openness of λ implies somewhat fuzzy semiopenness.

Conversely, suppose that λ is somewhat fuzzy semiopen. By definition, for some $x \in \lambda$, there is a fuzzy semiopen set μ in X such that $x \in \mu \leq \lambda$. By *Lemma 3*, $Int(\mu) \neq 0$. Therefore, for some $x \in \lambda$, one can find a fuzzy open set β such that $x \in \beta \leq \mu \leq \lambda$. Hence, $Int(\lambda) \neq 0$. This shows tha λ t is somewhat fuzzy open.

From the above lemmas, perhaps a relationship among the stated classes of fuzzy open sets can be concluded:

fuzzy open $\Longrightarrow$ fuzzy semiopen $\Longrightarrow$ somewhat fuzzy open $\Longleftrightarrow$ somewhat fuzzy semiopen.

None of the arrows in the diagram above can be reversed, as shown in the following example:

Example 1. Let $\mu, \lambda, \sigma, \alpha, \beta$ be fuzzy sets on $\mathbb{I}$ defined as:

$$\mu\ x) = \begin{cases} 0, & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 2x - 1, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

$$\lambda\ x) = \begin{cases} 1, & \text{if } 0 \leq x \leq \frac{1}{4}, \\ -4x + 2, & \text{if } \frac{1}{4} \leq x \leq \frac{1}{2}, \\ 0, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

$$\sigma\ x) = \begin{cases} 1, & \text{if } 0 \leq x \leq \frac{1}{4}, \\ -4x + 2, & \text{if } \frac{1}{4} \leq x \leq \frac{1}{2}, \\ 2x - 1, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

$$\alpha(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq 1/4, \\ 4x/3 - 1/3, & \text{if } 1/4 \leq x \leq 1. \end{cases}$$

And

$$\beta(x) = \begin{cases} 2x, & \text{if } 0 \le x \le 1/2, \\ 1, & \text{if } 1/2 \le x \le 1. \end{cases}$$

The family $\mathscr{T} = \{0, \mu, \lambda, \sigma, 1\}$ forms a fuzzy topology on $\mathbb{I}$. The set α is fuzzy semiopen but not fuzzy open (see, *Example 4.5* in [3]). One can quickly check that the set β is somewhat fuzzy open but not fuzzy semiopen.

Thangaraj and Balasubramanian [6], [7] showed that fuzzy continuity (openness) of a function implies somewhat fuzzy continuity (openness), but in reality, more than that is true.

$$\begin{array}{c} \text{fuzzy continuous} \\ \text{(fuzzy open)} \end{array} \Longrightarrow \begin{array}{c} \text{fuzzy semicontinuous} \\ \text{(fuzzy semiopen)} \end{array} \Longrightarrow \begin{array}{c} \text{somewhat fuzzy continuous} \\ \text{(somewhat fuzzy open)} \end{array}$$

Fig 1. The connections between the generalized fuzzy functions.

Theorem 1. For a function $f\colon (X, \mathscr{T}) \to (Y, \mathscr{U})$, the following statements hold:

I. f is somewhat fuzzy continuous if it is somewhat fuzzy semicontinuous.
II. f is somewhat fuzzy open if it is somewhat fuzzy semiopen.

Proof: The *Lemma 4* guarantees that $Int(\mu) \neq 0$ iff $Int_s(\mu) \neq 0$ for every fuzzy set μ, and the proof can be concluded.

We finish this note by pointing out that the same result holds for the functions mentioned above in general and soft topology.

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Conflicts of Interest

The author declare that there is no conflict of interests regarding the publication of this article.

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