E-ISSN: 2717-3453 | P-ISSN: 2783-1442

Journal of Fuzzy Extension and Applications

www.journal-fea.com

J. Fuzzy. Ext. Appl. Vol. 2, No. 4 (2021) 364–376.

Paper Type: Research Paper

Solution Procedure for Multi-Objective Fractional Programming Problem under Hesitant Fuzzy Decision Environment

Madineh Farnam[1], Majid Darehmiraki [2,*]

[1] Department of Electrical Engineering, Shohadaye Hoveizeh Campus of Technology, Shahid Chamran University of Ahvaz, Dasht-e Azadegan, Khuzestan, Iran; m.farnam@scu.ac.ir.

[2] Department of Mathematics, Behbahan Khatam Alanbia University of Technology, Behbahan, Khuzestan, Iran; darehmiraki@bkatu.ac.ir.

Citation:

Farnam, M., & Darehmiraki, M. (2021). Solution procedure for multi-objective fractional programming problem under hesitant fuzzy decision environment. *Journal of fuzzy extension and application*, 2(4), 364-376.

Received: 29/05/2021 Reviewed: 18/06/2021 Revised: 21/09/2021 Accept: 22/09/2021

Abstract

For the three last decades, the multi-objective fractional programming problem has attracted the attention of many researchers due to various applications in production planning, financial field, and inventory management, and so on. The main aim of this study is to introduce a new application of hesitant fuzzy sets in real-life modeling. We intend to model multi-objective linear fractional programming problems under a hesitant fuzzy environment and present a procedure to solve them. the increasing applications of multi-objective linear fractional programming problems and the lack of research papers in this field under a hesitant fuzzy environment are the main motivations of this study. In a hesitant fuzzy set, the membership degree of an element belongs to the set can be represented by several possible values in [0,1]. These values can be chosen by different experts that cannot reach a single opinion in determining a membership degree. so, in our model several evaluations for each of the goals established by decision makers based on their attitudes. The generalization of the fuzzy decision-making principle and some new concepts provide an effective solution procedure for the problem. Finally, a practical example is extended to illustrate the applicability of the proposed method.

Keywords: Hesitant fuzzy sets, Linear fractional programming problem, Multi-objective linear fractional programming problem, Hesitant fuzzy efficient solution.

1 | Introduction

Optimizations of the problem that the objective function is defined with the ratio of two linear functions are named fractional programming [26]–[28]. Optimization of linear fractional programming problem initially was proposed by the Charnes and Cooper [2] and after that, it was developed by Craven [4]. When we want to optimize the number of fractional goals simultaneously, such problem is called multi-objective linear fractional programming problem.

It has been considered by many researchers as one of the most widely used issues in recent decades. In particular, by introducing the fuzzy theory in decision making by Bellman and Zadeh [1], Using fuzzy perspective, appropriate methods created to solve this type of problem. For instance, the method of linguistic variables [15], [17], interactive method [21], goal planning [14], [18]

Corresponding Author: darehmiraki@bkatu.ac.ir

http://dx.doi.org/10.22105/jfea.2021.288198.1152

and the Taylor series method [12] are expressed by several researchers. In this way, often they solve the problem by considering the expectation levels for the objective functions by the decision maker and obtaining membership functions such as each of the objectives. For example, in goal planning, first a specific number is set as an ideal for each of the goals, and then the multi-objective linear fractional planning problem becomes a single-objective problem by determining the objective function related to the goal planning. Then we find a solution from the feasible area in a way that satisfies the set ideals as much as possible. Das et al. [6], [7] proposed the concepts of an effective ranking method between two triangular fuzzy numbers. Then, formulate an equivalent tri-objective linear fractional programming problem for the main problem to calculate the upper, middle and lower bounds of the main problem. So, they can construct the optimal values with obtaining upper, middle and lower bounds. A novel scheme which is constructed from a combination of Charnes−Cooper method and multi-objective linear programming problem proposed by [8], [9]. Veeramani and Sumathi [25] extended a new Algorithm to solve the Fuzzy Linear Fractional Programming Problem (FLFPP). Recently, Dong and Wan [10] proposed a new approach to solve fuzzy multi-objective optimization. In their model, the cost of the objective function, the resources and the technological coefficients was triangular fuzzy numbers. The main problem transformed into an equivalent deterministic Multi Objective Linear Fractional Programming Problem (MOLFPP), then solved them each objective function. Other methods for this type of problem can also be found in [3], [5], [11], [13], [15].

In this kind of problem, we often face with situations where the objective functions are irreconcilable and inconsistent. It is reliable that the optimal solution of one objective function generally is not the optimal for the other objective functions of the main problem. Stanojević and Stanojević [22], inspired by what Lotfi et al. [16], presented a process based on goal planning to obtain efficient solutions to such problems.

Of course, the problem of multi-objective linear fractional programming can be considered in such a way that the numerical coefficients in the objective functions are expressed in fuzzy form. Among the solution methods available in this type of uncertainty, we can mention the method presented by Pramy [19] which used graded mean integration representation, and according to it the fuzzy multi-objective linear fractional programming problem converted to a crisp model of multi-objective linear fractional programming. Then the efficient solution of the problem by transforming several goals to the single goal was obtained.

However, fuzzy theory has shown successful expansion and performance in modeling and solving mathematical programming problems, especially linear and multi-objective fractional programming. But in some issues, we need to use the opinions of different experts to evaluate the goals, parameters and other variables in the problem. For this purpose, a new expansion of fuzzy sets called hesitant fuzzy sets has been proposed in [23], [24]. In this generalization, when the decision maker is hesitant to express the degree of membership of an element, it is possible to assign more than one degree of membership to an element. Also, for cases where more than one decision maker with different experiences and views tends to record their opinion about the membership of an element, using this type of set can be an effective way to cover this type of uncertainty in the issue. With the introduction of cumulative operators on hesitant fuzzy sets [29], the necessary condition has been prepared for the extensive development of these sets to decision and optimization issues.

During these years, many applications of hesitant fuzzy sets in various problems are discovered. Especially multi-criteria decision making, risk evaluation and clustering algorithms. Several of them can be found in [30], [31] and their references.

A few researches have been done on the applications of this type of set in linear and nonlinear programming problems of objective and multi-objective types. Among the limited researches presented in this field, we can point out to [20].

In this study, we want to model and solve the problem of multi-objective linear fractional programming in a hesitant fuzzy decision space. For this purpose, the rest of this research is organized as follows: in Section 2, some basic concepts and essential definitions of hesitant fuzzy sets are given. In Section 3, the mathematical formulation of multi-objective linear fractional programming in hesitant fuzzy environment is provided and the process of solving it is given in Section 4. In Section 5, the accuracy of the solution process is examined by giving a practical example of fractional multi-objective production planning in a hesitant fuzzy environment. Finally, some results are mentioned.

2 | Preliminaries and Notations

In this section, some basic notions of hesitant fuzzy set and their operations are reviewed.

Definition 1. [29]. A hesitant fuzzy set over universal set X , is defined as follows:

$$\tilde{\tilde{A}} = \{(x, \tilde{h}_{\tilde{\tilde{A}}}(x)) | x \in X\},$$

where $\tilde{h}_{\tilde{\tilde{A}}}(x)$ is a set of amounts in $[0,1]$ and indicates the possible membership degrees of an element x belong to X. In fact, $\tilde{h}_{\tilde{\tilde{A}}}(x)$ is characterized with HFEs. Two important operations on HFEs are given as follows:

I. $\tilde{h}_1(x) \cup \tilde{h}_2(x) = \bigcup_{\gamma_1 \in \tilde{h}_1(x), \gamma_2 \in \tilde{h}_2(x)} max\{\gamma_1, \gamma_2\},$

II. $\tilde{h}_1(x) \cap \tilde{h}_2(x) = \bigcap_{\gamma_1 \in \tilde{h}_1(x), \gamma_2 \in \tilde{h}_2(x)} min\{\gamma_1, \gamma_2\}.$

In the next definition, hesitant fuzzy decision space is proposed as an extension of Bellman and Zadeh decision making theory which can be useful to solve problems with the hesitant fuzzy environment [3]. The basis of most developed models in the fuzzy space is the well-known fuzzy decision making (see[1]). For this model, it is assumed that all goals and constraints can consider as fuzzy numbers. As a result, the structure of the decision space should be introduced as follows:

$$\tilde{D} = \tilde{G} \cap \tilde{C} = (\tilde{G}_1 \cap \tilde{G}_2 \cap ... \cap \tilde{G}_p) \cap (\tilde{C}_1 \cap \tilde{C}_2 \cap ... \cap \tilde{C}_m).$$

Where p and m are the numbers of goals and constraints, respectively.

Definition 2. [20]. Assume we have a hesitant fuzzy goal $\tilde{\tilde{G}}$ and a hesitant constraint $\tilde{\tilde{C}}$ in X, then the hesitant fuzzy decision $\tilde{\tilde{D}}$ can be made by the combinations of $\tilde{\tilde{G}}$ and $\tilde{\tilde{C}}$, which means a hesitant fuzzy decision involves the intersection of $\tilde{\tilde{G}}$ and $\tilde{\tilde{C}}$. In this case, we can write

$$\tilde{\tilde{D}} = \tilde{\tilde{G}} \cap \tilde{\tilde{C}},$$

and for hesitant fuzzy function we have:

$$\tilde{h}_{\tilde{\tilde{D}}} = \tau(\tilde{h}_{\tilde{\tilde{G}}}, \tilde{h}_{\tilde{\tilde{C}}}),$$

where τ as a T-norm is employed to obtain the membership degree amounts for the intersection of hesitant fuzzy components. Also, we have

$$\tilde{h}_{\tilde{\tilde{G}}} = \left\{ \tilde{h}_{\tilde{\tilde{G}}}^{1}, \tilde{h}_{\tilde{\tilde{G}}}^{2}, ..., \tilde{h}_{\tilde{\tilde{G}}}^{q_G} \right\}, \quad \tilde{h}_{\tilde{\tilde{C}}} = \left\{ \tilde{h}_{\tilde{\tilde{C}}}^{1}, \tilde{h}_{\tilde{\tilde{C}}}^{2}, ..., \tilde{h}_{\tilde{\tilde{C}}}^{q_C} \right\},$$

where the number of the decision makers which established different aspiration levels for the objective functions and constraints are shown q_G and q_C respectively. This idea can extend for multi-objective programming. For this aim, assume we have p goals $\tilde{\tilde{G}}_1, \tilde{\tilde{G}}_2, ..., \tilde{\tilde{G}}_p$ and m constraints $\tilde{\tilde{C}}_1, \tilde{\tilde{C}}_2, ..., \tilde{\tilde{C}}_m$.

So, the proper decision is

$$\tilde{\tilde{D}} = \tilde{\tilde{G}} \cap \tilde{\tilde{C}} = (\tilde{\tilde{G}}_1 \cap \tilde{\tilde{G}}_2 \cap ... \cap \tilde{\tilde{G}}_p) \cap (\tilde{\tilde{C}}_1 \cap \tilde{\tilde{C}}_2 \cap ... \cap \tilde{\tilde{C}}_m).$$

In the space of hesitant fuzzy decision, each of the goals and constraints can consider as a hesitant fuzzy number, but in the current study, we considered only the goals as fuzzy hesitant numbers.

Definition 3. [31]. Assume $\tau : H^{(\mathfrak{M})} \times H^{(\mathfrak{M})} \to H^{(\mathfrak{M})}$, where τ is a hesitant triangular norm and $H^{(\mathfrak{M})}$ is a hesitant fuzzy set with $\mathfrak{M}$ elements, if $\tilde{h}_1, \tilde{h}_2, \tilde{h}_3 \in H^{(\mathfrak{M})}$ then the following axioms satisfy:

I. Commutative property; $\tau(\tilde{h}_1,\tilde{h}_2)=\tau(\tilde{h}_2,\tilde{h}_1)$,

II. Associative property; $\tau(\tilde{h}_1,\tau(\tilde{h}_2,\tilde{h}_3))=\tau(\tau(\tilde{h}_1,\tilde{h}_2),\tilde{h}_3)$,

III. Monotone property; $if\ \tilde{h}_2 \leq_{H(\mathfrak{M})} \tilde{h}_3\ then\ \tau(\tilde{h}_1,\tilde{h}_2) \leq_{H(\mathfrak{M})} \tau(\tilde{h}_1,\tilde{h}_3)$,

IV. Neutral property; $\tau(\tilde{h}_1,1_{H(\mathfrak{M})})=\tilde{h}_1$,

where $1_{H(\mathfrak{M})}=\{1,1,\ldots,1\}$ is a full *HFE* with $\mathfrak{M}$ elements.

In this study, we consider the minimum operator for the hesitant triangular norm on *HFE* k_i^{th} with following relation:

$$\tau(\tilde{h}_1,\tilde{h}_2)=\bigcup_{\gamma_1\in\tilde{h}_1(x),\gamma_2\in\tilde{h}_2(x)}\min\{\gamma_1,\gamma_2\}.$$

3 | Formulation of MOLFP Problem under Hesitant Fuzzy Decision Making Environment

In the current section, firstly, the classical linear fractional programming problem is defined. Then the important interpretation of the optimal solution is given in *Theorem 1*, and by developing that to multi-objective linear fractional programming problems, a test in *Theorem 2* is given to evaluate the efficiency of the feasible solution of this type of problems. Using *Algorithm 1* and based on *Theorem 2*, by starting from an arbitrary point, the efficient solution to the multi-objective linear fractional programming problem is obtained. *Theorem 3* gives us the guarantee that the sequence of points obtained from Algorithm 1 converges to the efficient solution of the linear fractional programming problem. After that, the general formulation of the MOLFP problem under the hesitant fuzzy environment is introduced and the solution process is developed efficiently, too.

3.1 | LFP Problem

The LFP problem is introduced generally as follows:

$$\begin{aligned}&\text{LFP:}\quad \max\frac{f(x)}{g(x)}=\frac{c^Tx+\alpha}{d^Tx+\beta}\\&\text{s.t. } x\in X(A,b)=\{x\in R^n: Ax\leq b, x\geq 0\},\end{aligned}\tag{1}$$

where $A\in R^{m\times n}$, $\alpha,\beta\in R$, $c\in R^n$, and $b\in R^m$. As we can see the objective function in *Eq. (1)* is expressed as a ratio of two linear functions. The feasible solution space, $X(A,b)$, is convex and bounded. So, if the denominator is non-zero in goal, the sign of the denominator does not change. (i.e. for all $x\in X(A,b)$, we can say: $d^Tx+\beta>0$ or $d^Tx+\beta<0$). Hence, the sign of the denominator always is positive or negative. Without reducing the generality of LFP problem, in this study we consider the sign of the dominator is positive. (If the denominator is negative, we can multiply the numerator and the denominator by a negative, thus we can $d^Tx+\beta>0$, for all $x\in X(A,b)$). There is an important interpretation for the optimal solution of the linear fractional programming problem, which can be expressed in the following theorem.

Theorem 1. [15]. The point $x^*\in X(A,b)$, is an optimal solution of *Eq. (1)* if and only if the optimal value of the following problem is zero,

$$
\begin{aligned}
& f_{t_0}^{*} = \max(\xi^{-} + \xi^{+}) \\
\text{s.t} \quad & c^{T}x + \alpha - \xi^{+} = f(x^{*})\theta, \\
& d^{T}x + \beta + \xi^{-} = g(x^{*})\theta, \\
& f(x^{*}) = c^{T}x + \alpha, \\
& g(x^{*}) = d^{T}x + \beta, \\
& \theta \geq 0, \xi^{-} \geq 0, \xi^{+} \geq 0.
\end{aligned} \tag{2}
$$

In the next subsection, by expanding this theorem, we can find the efficient solution to multi-objective programming problems.

3.2 | MOLFP Problem

In practical issues, the simultaneous optimization of several objective functions is often considered by decision makers. In this section, we examine the problem of linear fractional programming with more than one goal. The general form of the multi-objective linear fractional programming problem can be

$$
\text{MOLFP}: \quad \max_{x \in X(A,b)} \left(\left(\frac{f_1(x)}{g_1(x)}\right), \left(\frac{f_2(x)}{g_2(x)}\right), \ldots, \left(\frac{f_p(x)}{g_p(x)}\right) \right), \tag{3}
$$

expressed as follows:

where,

I. $X(A,b) = \left\{ x \in R^{n} : Ax(\leq = \geq)b, x \geq 0 \right\}$, is a convex and bounded set, A is a technological coefficient matrix,

II. $p \geq 2$, (p shows the number of objectives),

III. $f_i(x) = c_i^{T}x + \alpha_i$ *and* $g_i(x) = d_i^{T}x + \beta_i$ *for* $\forall i = 1,2,\ldots,p,$

IV. $c_i, d_i \in {}^{n}$ *and* $\alpha_i, \beta_i \in$ *for* $\forall i = 1,2,\ldots,p,$

V. $d_i^{T}x + \beta_i \geq 0,$ *for* $\forall x \in X, \forall i = 1,2,\ldots,p.$

As we know, when we want to optimize several objective functions simultaneously, the optimal solution for each of the objective functions is not necessarily the optimal solution for the other objectives. Therefore, in this kind of problem, we try to find the type of answers that are called efficient solutions. Hence, the main purpose of the maximization *Problem (2)* is to obtain efficient solutions.

Definition 4. [16]. $x^{*} \in X(A,b)$, is an efficient solution of *Eq. (2)* if and only if there is not $x \in X(A,b)$, such that

$$\frac{f_i(x)}{g_i(x)} \geq \frac{f_i(x^{*})}{g_i(x^{*})}, \qquad \forall i = 1,2,\ldots,p,$$

and, $\dfrac{f_j(x)}{g_j(x)} > \dfrac{f_j(x^{*})}{g_j(x^{*})}$ for at least one $j \in \{1,2,\ldots,p\}$.

Theorem 2. [22]. Assume $x^{*} \in X(A,b)$, is an arbitrary solution of *Eq. (2)*, then x^{*} is an efficient solution if and only if the optimal value of the following problem is zero,

Farnam and Darehmiraki |J. Fuzzy. Ext. Appl. 2(4) (2021) 364-376

$$
\begin{aligned}
& f_{tse}^{*}=\max \sum_{i=1}^{p}(\xi_i^{k-}+\xi_i^{k+}) \\
& \text{s.t} \quad c_i^{T}x+\alpha_i-\xi_i^{+}=f_i(x^{*})\theta_i, \forall i=1,2,\dots,p, \\
& \qquad d_i^{T}x+\beta_i+\xi_i^{-}=g(x^{*})\theta_i, \forall i=1,2,\dots,p, \\
& \qquad \theta_i\geq 0,\xi_i^{-}\geq 0,\xi_i^{+}\geq 0, \quad \forall i=1,2,\dots,p, \\
& \qquad\qquad x\in X.
\end{aligned} \tag{4}
$$

According to the following Algorithm, and employing the problem presented in *Theorem 3*, the efficient solution of the multi-objective linear fractional programming problem can be achieved correctly.

Algorithm 1. Obtaining efficient solution [22].

Step 1: Set $x^{*}=x^{0}$, $k=1$.

Step 2: Solve Problem (4), assume $(x^{k},t^{k},\theta^{k},\xi_i^{k-},\xi_i^{k+})$ be an optimal solution,

and set $t^{k}=\sum_{i=1}^{p}(\xi_i^{k-}+\xi_i^{k+})$.

Step 3: If $t^{k}=0$, then stop. x^{k}, is an efficient solution for MOLFP problem, else set $x^{*}=x^{k}$, $k=k+1$ and go to Step 2.

Theorem 3. [22]. The sequence $\{x_k\}$ which is generated by *Algorithm (1)* converges to the efficient solution of the multi-objective linear fractional programming problem.

3.3 | HFMOLFP Problem

As we know, facing various uncertainties in modeling and receiving information from the problem in the face of everyday problems is inevitable. Hesitant fuzzy sets are a useful tool to show decision makers' uncertainty and hesitation in assigning membership value to each element of the set. Theory and applications of hesitant fuzzy sets significantly developed during the last decade. In HFMOLFP, evaluations of several experts based on their experiences establish to make a better decision. According to this attitude we can define hesitant fuzzy multi-objective linear fractional programming problem by:

$$
\text{HFMOLFP}: \quad \underset{x\in X(A,b)}{\widetilde{\widetilde{\max}}}\left(\left(\frac{f_1(x)}{g_1(x)}\right),\left(\frac{f_2(x)}{g_2(x)}\right),\dots,\left(\frac{f_p(x)}{g_p(x)}\right)\right) \tag{5}
$$

or

$$
\text{HFMOLFP}: \quad \underset{x\in X(A,b)}{\widetilde{\widetilde{\max}}}\left(\left(\frac{f(x)}{g(x)}\right)_1,\left(\frac{f(x)}{g(x)}\right)_2,\dots,\left(\frac{f(x)}{g(x)}\right)_p\right),
$$

where the symbol "$\widetilde{\widetilde{max}}$" means that we have to maximize the problem under hesitant fuzzy decision environment. We have different membership functions for each goal that are presented by the evaluations of several experts. Hence, for each objective function of *Eq. (5)* the HFS $\widetilde{\widetilde{\left(\frac{f(x)}{g(x)}\right)}}_i$ as a hesitant fuzzy goal can be expressed by:

$$(\widetilde{\widetilde{\frac{f(x)}{g(x)}}})_i=\left\{\left(x,\tilde{h}_{(\widetilde{\widetilde{\frac{f(x)}{g(x)}}})_i}(x)\right)\middle|\, x\in X(A,b)\right\}. \tag{6}$$

If we assume q_i is the number of experts which are given their comments about the i^{th} goal, then the hesitant fuzzy membership function of the i^{th} goal, can be defined by:

$$\tilde{h}_{(\widetilde{\widetilde{\frac{f(x)}{g(x)}}})_i}(x)=\left\{\mu_{\left(\widetilde{\frac{f(x)}{g(x)}}\right)_i^1}(x),\mu_{\left(\widetilde{\frac{f(x)}{g(x)}}\right)_i^2}(x),\dots,\mu_{\left(\widetilde{\frac{f(x)}{g(x)}}\right)_i^{q_i}}(x)\right\}. \tag{7}$$

If $\mu_{\left(\widetilde{\frac{f(x)}{g(x)}}\right)_i^{k_i}}(x)$ is the corresponded membership function of each objective for all $i=1,2,\dots,p$ and $k_i=1,2,\dots,q_i$, then it can be presented by:

$$\mu_{\left(\widetilde{\frac{f(x)}{g(x)}}\right)_i^{k_i}}(x)=\begin{cases}1 & (\frac{f(x)}{g(x)})_i\ge z_i^{k_i},\\ \dfrac{(\frac{f(x)}{g(x)})_i-d_{z_i^{k_i}}}{z_i^{k_i}-d_{z_i^{k_i}}} & d_{z_i^{k_i}}\le(\frac{f(x)}{g(x)})_i\le z_i^{k_i},\\ 0 & (\frac{f(x)}{g(x)})_i\le d_{z_i^{k_i}}.\end{cases} \tag{8}$$

It should consider, $z_i^{k_i}$ and $d_{z_i^{k_i}}$, are the desired value and maximum permissible deviation for i^{th} goal respectively which is determined by the k_i^{th} decision maker, $k_i=1,2,\dots,q_i$.

$$\tilde{h}_p(x)=\left\{\tilde{h}_{(\widetilde{\widetilde{\frac{f(x)}{g(x)}}})_i}(x)\middle|\, i=1,2,\dots,p\right\}, \tag{9}$$

$$=\left\{\mu_{\left(\widetilde{\frac{f(x)}{g(x)}}\right)_i^{k_i}}(x)\middle|\, i=1,2,\dots,p,\ k_i=1,2,\dots,q_i\right\},$$

where $\tilde{h}_p(x)$ is the hesitant fuzzy decision space of the objectives.

Before presenting a method to solve the model, it is necessary to provide a proper definition of the concept of response in this type of problem. For this purpose, the definition of the hesitant fuzzy efficiency solution for the multi-objective linear fractional programming problem is given below.

Definition 5. $\langle x^*,\tilde{h}_p(x^*)\rangle$ with $x^*\in X(A,b)$, is called hesitant fuzzy efficient solution of *Eq. (5)* if there is not $x\in X(A,b)$, such that

$\mu_{\left(\widetilde{\frac{f(x)}{g(x)}}\right)_i^{k_i}}(x)\ge\mu_{\left(\widetilde{\frac{f(x)}{g(x)}}\right)_i^{k_i}}(x^*),\ \text{for all}\ i=1,2,\dots,p,\ k_i=1,2,\dots,q_i,$

and,

$\mu_{\left(\widetilde{\frac{f(x)}{g(x)}}\right)_j^{k_j}}(x)>\mu_{\left(\widetilde{\frac{f(x)}{g(x)}}\right)_j^{k_j}}(x^*)$ for at least one $j\in\{1,2,\dots,p\}$ and $k_j=1,2,\dots,q_j$.

Theorem 4. $x^* \in X(A,b)$ is an efficient solution of *Eq. (3)* if and only if x^* is a hesitant fuzzy efficient solution of the *Problem (5)*.

Proof. Assume that $x^* \in X(A,b)$ is not hesitant fuzzy efficient solution of *Eq. (5)*. So, we can state $\exists x \in X(A,b)$ $s.t$ $\mu_{\left(\frac{f(x)}{g(x)}\right)_i^{k_i}}(x) \geq \mu_{\left(\frac{f(x)}{g(x)}\right)_i^{k_i}}(x^*)$, $for\ all\ i = 1,2,...,p,\ k_i = 1,2,...,q_i$,

$\mu_{\left(\frac{f(x)}{g(x)}\right)_j^{k_j}}(x) > \mu_{\left(\frac{f(x)}{g(x)}\right)_j^{k_j}}(x^*)$ for at least one $j \in \{1,2,...,p\}$ and $k_j = 1,2,...,q_j$.

From $\mu_{\left(\frac{f(x)}{g(x)}\right)_j^{k_j}}(x) > \mu_{\left(\frac{f(x)}{g(x)}\right)_j^{k_j}}(x^*)$, and according to *Eq. (8)* we have

$$\frac{(\frac{f(x)}{g(x)})_j - d_{z_j^{k_j}}}{z_j^{k_j} - d_{z_j^{k_j}}} > \frac{(\frac{f(x^*)}{g(x^*)})_j - d_{z_j^{k_j}}}{z_j^{k_j} - d_{z_j^{k_j}}}$$

$$\Rightarrow (\frac{f(x)}{g(x)})_j - d_{z_j^{k_j}} > (\frac{f(x^*)}{g(x^*)})_j - d_{z_j^{k_j}}$$

$$\Rightarrow (\frac{f(x)}{g(x)})_j > (\frac{f(x^*)}{g(x^*)})_j,$$

which is in contradiction to efficiency x^* for the *Problem (3)*.

Conversely, suppose that $x^* \in X(A,b)$ is not efficient solution of *Eq. (3)*. So, we can state

$\exists x \in X(A,b)$ $s.t$ $\dfrac{f_i(x)}{g_i(x)} \geq \dfrac{f_i(x^*)}{g_i(x^*)}$, $\forall\ i = 1,2,...,p,$

$\dfrac{f_j(x)}{g_j(x)} > \dfrac{f_j(x^*)}{g_j(x^*)}$ for at least one $j \in \{1,2,...,p\}$.

We assume $z_j^{k_j}$ and $d_{z_j^{k_j}}$, are the desired value and maximum permissible deviation for j[th] objective function respectively which is determined by the k_j^{th} decision maker, $k_j = 1,2,...,q_j$. So, from $\dfrac{f_j(x)}{g_j(x)} > \dfrac{f_j(x^*)}{g_j(x^*)}$ we can write

$$(\frac{f(x)}{g(x)})_j - d_{z_j^{k_j}} > (\frac{f(x^*)}{g(x^*)})_j - d_{z_j^{k_j}}$$

$$\Rightarrow \frac{(\frac{f(x)}{g(x)})_j - d_{z_j^{k_j}}}{z_j^{k_j} - d_{z_j^{k_j}}} > \frac{(\frac{f(x^*)}{g(x^*)})_j - d_{z_j^{k_j}}}{z_j^{k_j} - d_{z_j^{k_j}}}$$

$$\Rightarrow \mu_{\left(\frac{f(x)}{g(x)}\right)_j^{k_j}}(x) > \mu_{\left(\frac{f(x)}{g(x)}\right)_j^{k_j}}(x^*),$$

which is in contradiction to efficiency x^* for the *Problem (5)*.

Now we can describe our solution method by using the above mentioned concepts.

4 | Solution Procedure

At first, we should determine the hesitant fuzzy decision space of HFMOLFP *Model (5)*. For this purpose, according to *Definition 2*, the hesitant decision space caused by all goals can be presented by:

$$\tilde{\tilde{D}} = \left(\widetilde{\widetilde{\frac{f(x)}{g(x)}}}\right)_1 \cap \left(\widetilde{\widetilde{\frac{f(x)}{g(x)}}}\right)_2 \cap \ldots \cap \left(\widetilde{\widetilde{\frac{f(x)}{g(x)}}}\right)_p . \tag{10}$$

Furthermore, using the membership function in *Eq. (8)* and according to *Definition 3*, for each $x \in X(A,b)$, we have:

$$\begin{aligned} \tilde{h}_{\tilde{\tilde{D}}} &= \bigcup \mu_{\left(\widetilde{\frac{f(x)}{g(x)}}\right)_i^{\alpha_1}}(x), \mu_{\left(\widetilde{\frac{f(x)}{g(x)}}\right)_i^{\alpha_2}}(x), \ldots, \mu_{\left(\widetilde{\frac{f(x)}{g(x)}}\right)_i^{\alpha_p}}(x), \\ &= \min_x \{\mu_{\left(\widetilde{\frac{f(x)}{g(x)}}\right)_i^{\alpha_1}}(x), \mu_{\left(\widetilde{\frac{f(x)}{g(x)}}\right)_i^{\alpha_2}}(x), \ldots, \mu_{\left(\widetilde{\frac{f(x)}{g(x)}}\right)_i^{\alpha_p}}(x)\}, \\ &= \{\min_x \{\mu_{\left(\widetilde{\frac{f(x)}{g(x)}}\right)_i^{\alpha_{ir}}}(x)\}_{i=1}^{p}\}_{r=1}^{\beta}, \end{aligned} \tag{11}$$

where $\beta = q_1, q_2, \ldots, q_p$ and $\alpha_{ir} \in \{1,2,\ldots,q_i\}$. The corresponding MOLFP problem of the r^{th} member of *Eq. (11)* can be introduced by the following formulation:

$$\begin{aligned} (P)^r: \quad & \min \{\mu_{\left(\widetilde{\frac{f(x)}{g(x)}}\right)_i^{\alpha_{ir}}}(x)\}_{i=1}^{p} \\ \text{s.t.} \quad & x \in X(A,b). \end{aligned} \tag{12}$$

Now, by introducing a new variable and according to Bellman and Zadeh principle we can state equivalent nonlinear programming as follows:

$$\begin{aligned} (NLP)^r: \quad & \max \lambda \\ \text{s.t.} \quad & \{\mu_{\left(\widetilde{\frac{f(x)}{g(x)}}\right)_i^{\alpha_{ir}}}(x)\} \geq \lambda, \ i = 1,2,\ldots,p, \\ & \lambda \in [0,1] \\ & x \in X(A,b). \end{aligned} \tag{13}$$

To get the efficient solution of the *Problem (13)*, start from the obtained point then employ *Algorithm 1*. After solving *Model (13)*, we obtain x^{r*} that denotes the r^{th} maximizing solution with the corresponded membership degree λ^{r*} of the HFMOLFP *Problem (5)*. Thus, by solving β MOLFP problem as *Model (13)*, we obtain, $\{x^{1*}, x^{2*}, \ldots, x^{\beta*}\}$ with the corresponded membership degree $\{\lambda^{1*}, \lambda^{2*}, \ldots, \lambda^{\beta*}\}$.

As described, after solving the problem, the decision maker can choose the desired value among a set of optimal solutions.

5 | Practical Example

In this part, we show the efficiency of the proposed model and method by solving a develoed practical example of production planning [12].

Example. A company manufactures two products of A_1 and A_2 . Assume that the increasing costs and capital demands, required are proportional to the individual activities. Also, regardless of the main product program 4000 dollars is considered as a fixed capital demand. in order to ensure conventional level determination, the inventory is considered more than 10% of the all production. Safety inventory is carried out, because the demand is uncertain and a product shortage may result if actual demand exceeds the forecasted demand. Furthermore, management of the company is determined by the inventory constraints for each product, which are 10% and 5% of the total production quantity for each product, respectively. *Table 1* shows the data for production, which are fixed. The company considers maximization for both the profitability of the owned /employed capital and inventory turnover ratio. When production quantities of A_1 and A_2 are x_1 and x_2 respectively, inventory quantities of them are y_1 and y_2. Then, multi-objective fractional programming of the above problem is as follows:

$$\max \frac{12x_1 + 13x_2}{12x_1 + 13x_2 + 4000}$$

$$\max \frac{12x_1 + 13x_2}{1.5y_1 + 1.6y_2 + 2}$$

s.t.

$$\begin{aligned}
2x_1 + x_2 &\le 250,\\
5x_1 + 4x_2 &\le 500,\\
45x_1 + 30x_2 &\le 1200,\\
0.1(x_1 + x_2) &\le y_1 + y_2,\\
0.05x_2 &\le y_2,\\
x_1 &\ge y_1,\\
x_2 &\ge y_2,\\
x_1, x_2, y_1, y_2 &\ge 0.
\end{aligned}$$

Table 1. Information for the production of the example.

Capacity available	**Demand per unit of each product**	
	A_1	**A_2**
Raw material (unites of quality): 250	2	1
Machines (hours): 500	5	4
Owned capital (dollars): 5200	40	55
Profit per unit (dollars)	12	13
Inventory cost per unit (dollars)	1.5	1.6

In addition, $z_i^{k_i}$ and $d_{z_i^{k_i}}$, $(i = 1,2 \text{ and } k_i = 1,2,3)$ are the desired value and maximum permissible deviation for i^{th} goal respectively which is determined by the k_i^{th} decision maker (*Table 2*).

Table 2. Information of the decision makers.

DMs	$\left(z_1^{k_i}, d_{z_1^{k_i}}\right)$	$\left(z_2^{k_i}, d_{z_2^{k_i}}\right)$
DM_1	$(z_1^1, d_{z_1^1}) = (0.1, -0.1)$	$(z_2^1, d_{z_2^1}) = (70, 20)$
DM_2	$(z_1^2, d_{z_1^2}) = (0.12, -0.06)$	$(z_2^2, d_{z_2^2}) = (74, 18)$
DM_3	$(z_1^3, d_{z_1^3}) = (0.13, -0.09)$	$(z_2^3, d_{z_2^3}) = (72, 25)$

Now, we can write the membership functions corresponding to each of the goals based on their decision maker's views by applying *Eq. (8)*. For example, if we use the first decision maker information, the two membership functions for the first and second goals are:

$$,\quad \mu_{\left(\frac{f(x)}{g(x)}\right)_1^1}(x)=\begin{cases} 1 & (\frac{f(x)}{g(x)})_1 \geq 01, \\ \dfrac{\dfrac{12x_1+13x_2}{12x_1+13x_2+4000}-(-0.1)}{0.1-(-0.1)} & -0.1 \leq (\frac{f(x)}{g(x)})_1 \leq 0.1, \\ 0 & (\frac{f(x)}{g(x)})_1 \leq -0.1. \end{cases}$$

$$\mu_{\left(\frac{f(x)}{g(x)}\right)_2^1}(x)=\begin{cases} 1 & (\frac{f(x)}{g(x)})_2 \geq 70, \\ \dfrac{\dfrac{12x_1+13x_2}{1.5y_1+1.6y_2+2}-20}{70-20} & 20 \leq (\frac{f(x)}{g(x)})_2 \leq 70, \\ 0 & (\frac{f(x)}{g(x)})_2 \leq 20. \end{cases}$$

Other membership functions can be written in a similar way. Hence, the hesitant fuzzy decision space of the problem comprises three membership functions for the first goal and three membership functions for the second goal. So, 9 fuzzy subspaces (9 fuzzy subproblems) can be considered for it. The results for 9 solved problems are summarized in *Table 3*.

Table 3. Information for the solution of the example.

$(NLP)^r$	λ^{r*}	x^{r*}
$(NLP)^1$	0.8567048	(2.051282 , 36.92308 , 2.051282 , 1.846154)
$(NLP)^2$	0.8006294	(2.051282 , 36.92308 , 2.051282 , 1.846154)
$(NLP)^3$	0.8050052	(2.051282 , 36.92308 , 2.051282 , 1.846154)
$(NLP)^4$	0.7992829	(0.00000 , 40.0000 , 0.00000 , 4.108629)
$(NLP)^5$	0.7945699	(1.324434 , 38.01335 , 1.324434 , 2.609345)
$(NLP)^6$	0.7954362	(1.08807 , 38.37579 , 1.082807 , 2.863053)
$(NLP)^7$	0.7903226	(0.00000 , 40.00000 , 0.00000 , 4.000000)
$(NLP)^8$	0.7886210	(0.5874268 , 39.11886 , 0.5874268 , 3.383202)
$(NLP)^9$	0.7891099	(0.4191001 , 39.37135 , 0.4191001 , 3.559945)

As seen in *Table 3*, a set of solutions is got by solving the subproblems. Therefore, the decision maker with an optimistic, pessimistic or balanced view of the values in *Table 3* can choose the desired solution.

6 | Conclusion

In this study, multi-objective linear fractional programming problem in hesitant fuzzy decision making is modeled in which more than one decision maker record their evaluations for the objectives. After modeling, we define the concept of hesitant fuzzy efficient solution for hesitant fuzzy multi-objective linear fractional programming problem. Then, by using hesitant fuzzy decision space, the main problem is converted to the equivalent problem. After that, by tracking simple stages, hesitant fuzzy efficient solution can be found in a proper way. A decision maker can use the obtained solutions from solving the hesitant fuzzy problem based on the desired achievement degree. A practical oriented production planning is modeled and solved to show the validity and efficiency of the proposed solution process.

Conflicts of Interest

The authors declare that they have no known competing financial interests, conflict interest, or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] Bellman, R. E., & Zadeh, L. A. (1970). Decision-making in a fuzzy environment. *Management science, 17*(4), B-141.

[2] Charnes, A., & Cooper, W. W. (1962). Programming with linear fractional functionals. *Naval Research logistics quarterly, 9*(3-4), 181-186.

[3] Choo, E. U., & Atkins, D. R. (1980). An interactive algorithm for multicriteria programming. *Computers and operations research, 7*(1-2), 81-87.

[4] Craven, B. D. (1988). *Fractional programming, (Vol. 4)*. Berlin, Heldermann.

[5] Darehmiraki, M. (2019). A novel parametric ranking method for intuitionistic fuzzy numbers. *Iranian journal of fuzzy systems, 16*(1), 129-143.

[6] Das, S. K., Edalatpanah, S. A., & Mandal, T. (2018). A proposed model for solving fuzzy linear fractional programming problem: Numerical Point of View. *Journal of computational science, 25*, 367-375.

[7] Das, S. K., Mandal, T., & Edalatpanah, S. A. (2017). A new approach for solving fully fuzzy linear fractional programming problems using the multi-objective linear programming. *RAIRO-operations research, 51*(1), 285-297.

[8] Das, S. K., & Mandal, T. (2017). A new model for solving fuzzy linear fractional programming problem with ranking function. *Journal of applied research on industrial engineering, 4*(2), 89-96.

[9] Das, S. K., Edalatpanah, S. A., & Mandal, T. (2019). A new method for solving linear fractional programming problem with absolute value functions. *International journal of operational research, 36*(4), 455-466.

[10] Dong, J., & Wan, S. (2019). A new method for solving fuzzy multi-objective linear programming problems. *Iranian journal of fuzzy systems, 16*(3), 145-159.

[11] Gupta, P., & Bhatia, D. (2001). Sensitivity analysis in fuzzy multiobjective linear fractional programming problem. *Fuzzy sets syst, 122*(2), 229-236.

[12] Guzel, N., & Sivri, M. (2005). Taylor series solution of multi-objective linear fractional programming problem. *Trakya university journal science, 6*(2), 80-87.

[13] Hannan, E. L. (1981). Linear programming with multiple fuzzy goals. *Fuzzy sets and systems, 6*(3), 235-248.

[14] Kornbluth, J. S., & Steuer, R. E. (1981). Goal programming with linear fractional criteria. *European journal of operational research, 8*(1), 58-65.

[15] Kornbluth, J. S., & Steuer, R. E. (1981). Multiple objective linear fractional programming. *Management science, 27*(9), 1024-1039.

[16] Lotfi, F. H., Noora, A. A., Jahanshahloo, G. R., Khodabakhshi, M., & Payan, A. (2010). A linear programming approach to test efficiency in multi-objective linear fractional programming problems. *Applied mathematical modelling, 34*(12), 4179-4183.

[17] Luhandjula, M. K. (1984). Fuzzy approaches for multiple objective linear fractional optimization. *Fuzzy sets and systems, 13*(1), 11-23.

[18] Pal, B. B., Moitra, B. N., & Maulik, U. (2003). A goal programming procedure for fuzzy multi-objective linear fractional programming problem. *Fuzzy sets and systems, 139*(2), 395-405.

[19] Pramy, F. A. (2018). An approach for solving fuzzy multi-objective linear fractional programming problems. *International journal of mathematical, engineering and management sciences, 3*(3), 280-293.

[20] Ranjbar, M., & Effati, S. (2019). Symmetric and right-hand-side hesitant fuzzy linear programming. *IEEE transactions on fuzzy systems, 28*(2), 215-227.

[21] Sakawa, M., & Yano, H. (1988). An interactive fuzzy satisficing method for multi-objective linear fractional programming problems. *Fuzzy sets and systems, 28*(2), 129-144.

[22] Stanojević, B., & Stanojević, M. (2013). On the efficiency test in multi-objective linear fractional programming problems by Lotfi et al. 2010. *Applied mathematical modelling, 37*(10-11), 7086-7093.

[23] Torra, V., & Narukawa, Y. (2009, August). On hesitant fuzzy sets and decision. *2009 IEEE international conference on fuzzy systems* (pp. 1378-1382). IEEE.

[24] Torra, V. (2010). Hesitant fuzzy sets. *International journal of intelligent systems, 25*(6), 529-539.

[25] Veeramani, C., & Sumathi, M. (2016). A new method for solving fuzzy linear fractional programming problems. *Journal of intelligent & fuzzy systems, 31*(3), 1831-1843.

[26] Wan, S. P., & Dong, J. Y. (2014). Possibility linear programming with trapezoidal fuzzy numbers. *Applied mathematical modelling, 38*(5-6), 1660-1672.

[27] Wan, S. P., & Li, D. F. (2013). Atanassov's intuitionistic fuzzy programming method for heterogeneous multiattribute group decision making with Atanassov's intuitionistic fuzzy truth degrees. *IEEE transactions on fuzzy systems, 22*(2), 300-312.

[28] Wan, S. P., & Li, D. F. (2015). Fuzzy mathematical programming approach to heterogeneous multiattribute decision-making with interval-valued intuitionistic fuzzy truth degrees. *Information sciences, 325*, 484-503.

[29] Xia, M., Xu, Z., & Chen, N. (2013). Some hesitant fuzzy aggregation operators with their application in group decision making. *Group decision and negotiation, 22*(2), 259-279.

[30] Xu, G. L., Wan, S. P., & Dong, J. Y. (2016). A hesitant fuzzy programming method for hybrid MADM with incomplete attribute weight information. *Informatica, 27*(4), 863-892.

[31] Zhang, R., Xu, Z., & Gou, X. (2021). An integrated method for multi-criteria decision-making based on the best-worst method and Dempster-Shafer evidence theory under double hierarchy hesitant fuzzy linguistic environment. *Applied intelligence, 51*(2), 713-735.