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An Interval-Valued Atanassov's Intuitionistic Fuzzy Multi-attribute Group Decision Making Method Based on the Best Representation of the WA and OWA Operators

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Abstract

In this paper we extend the notion of interval representation for interval-valued Atanassov's intuitionistic representations, in short Lx-representations, and use this notion to obtain the best possible one, of the Weighted Average (WA) and Ordered Weighted Average (OWA) operators. A main characteristic of this extension is that when applied to diagonal elements, i.e. fuzzy degrees, they provide the same results as the WA and OWA operators, respectively. Moreover, they preserve the main algebraic properties of the WA and OWA operators. A new total order for interval-valued Atanassov's intuitionistic fuzzy degrees is also introduced in this paper which is used jointly with the best Lx-representation of the WA and OWA, in a method for multi-attribute group decision making where the assesses of the experts, in order to take in consideration uncertainty and hesitation, are interval-valued Atanassov's intuitionistic fuzzy degrees. A characteristic of this method is that it works with interval-valued Atanassov's intuitionistic fuzzy values in every moments, and therefore considers the uncertainty on the membership and non-membership in all steps of the decision making. We apply this method in two illustrative examples and compare our result with other methods.

Keywords: Interval-Valued Atanassov's intuitionistic fuzzy sets, WA and OWA operators, Lx-representations, Total orders, Multi-Attribute group decision making.

1 | Introduction

From the seminal paper [71] on fuzzy set theory, several extensions for this theory have been proposed [18]. Among them, we stress "Interval-valued Fuzzy Sets Theory" [10], [19], [72] and "Atanassov's Intuitionistic Fuzzy Sets Theory" [2], [5], [25], [26]. Although they are mathematically equivalents, they capture dif- ferent kinds of uncertainty in the membership degrees, i.e. they have different semantics [61]. The first one takes in account the intrinsic difficulty to determine the exact membership degree of an object to some linguistic term; in this case, an expert provides an interval which expresses his uncertainty on such degree. The second one adds an extra degree to the usual fuzzy sets in order to model the hesitation and uncertainty about the membership degree. In fuzzy set theory, the non-membership degree is by default the complement of the membership degree, i.e.

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1- $\mu_A(x)$, meaning that there is no doubt or hesitation in the membership degree. In [3], both extensions are mixed by considering that we can also have an uncertainty or imprecision in the membership and non-membership degrees if we model them with intervals. This results in other extension of fuzzy set theory, known as Interval-Valued Atanassov's Intuitionistic Fuzzy Sets (IVAIFS). Several applications of IVAIFS, and extensions of usual fuzzy notions to the IVAIFS framework have been made, see for example [4], [7], [21], [32], [51], [64].

Besides, Group Decision Making (GDM) and Multi-attribute Decision Making (MADM) are the most well know branches of decision making. GDM consists in choosing of one or more alternatives among several ones by a group of decision makers (experts), probably with a weight of confidence [24]. MADM choosing one or more alternatives among several ones based in the assesses of an expert his opinion of how much the alternative fulfills a criteria or satisfies an attribute. Usually, a weighting vector for the attributes is associated, in order to represent the importance of an attribute in the overall decision problem. Nevertheless, complex decision making problems usually need to consider a group of experts as well as a set of criteria or attributes, i.e. a Multi-attribute Group Decision Making (MAGDM) [28], [43], [55], [59].

Fuzzy logic, by their nature, has played an important role in the field of decision making, since decision makers can be subject to uncertainty expressed in terms of fuzzy degrees [46], [47], [55], [57]. An important mathematical tool for fuzzy decision-making are Weighted Average (WA) and the Ordered Weighted Average (OWA) operators introduced in [69], which have triggered their "extension" for Interval-Valued Atanassov's Intuitionistic Fuzzy Values (IVAIFV) – see for example [65], [67]. However, in the cited cases, the proposed interval-valued Atanassov's intuitionistic OWA, although of preserve some algebraic properties of the OWA (monotonicity, idempotency, symmetry and boundedness [16]), have not the same behaviour as the OWA when applied to diagonals elements.

In [11], [54], in order to formalize the principle of correctness of interval computation [37], it was introduce the notion of interval representation of real functions. In addition, the best of the interval representations of a real function models the notion of optimality in interval computing. This notion has been used in the context of interval-valued fuzzy functions, to obtain interval-valued t-norms (t-conorms, overlap functions, fuzzy negations and fuzzy implications) from t- norms (t-conorms, overlap functions, fuzzy negations and fuzzy implications) in [1], [8], [14], [34]. In this paper we extend the notion of interval-valued representation and the best interval-valued representation of fuzzy functions for the interval-valued Atanassov's intuitionistic representations of fuzzy and interval- valued fuzzy functions. In particular, we provide a novel extension of the WA and OWA operator for IVAIFS, based on the best interval-valued Atanassov's intu- itionistic fuzzy representation, which preserve the main properties of the OWA operators and when restrict to the diagonals elements it is an OWA in $0, 1$. This new IVAIFAF OWA together with some total orders for IVAIFV are used to develop a method to rank alternatives from the individual interval-valued Atanassov's intuitionistic decision matrices of a group of experts reflecting how much each alternative satisfy each attribute. Two illustrative examples are considered in order to show the use of the method and to show that the final ranking of alternatives obtained by the method is adequate.

This paper is organized as follows: Section 2 introduces Atanassov intuitionisc and interval-valued fuzzy sets, the score and accuracy index and the notion of representation in particular in the interval-valued and Atanassov intuitionisc best representation of the WA and OWA operators. In Section 3 it is consider the notion of interval-valued intuitionistic fuzzy set and some orders for interval-valued. Atanassov's intuitionistic fuzzy values are presented. In particular, based in a novel notion of membership and subsets, interval-valued intuitionistic fuzzy degrees are seen as an interval of interval-valued fuzzy degrees and based in this

point of view a new total order for IVAIFV is provided. In Section 4 it is introduced the notion of IVAIFV representation and it is provide a canonical way of obtain the best representation of an interval-valued fuzzy function and of a fuzzy func- tion, which is used to obtain the best IVAIFV representation of the WA and OWA operators. In Section 5 the total orders on IVAIFV and the best IVAIFV representation of the WA and OWA are used to develop a method to solve MAGDMP and this method is used in two illustrative examples. Finally in Section 6 some final remarks on the paper are provided.

2| Preliminaries

Atanassov in [2] extended the notion of fuzzy sets, by adding an extra degree to model the hesitation or uncertainty in the membership degree. This second degree is called non-membership degree. In fuzzy set theory, by default, this non- membership degree is given by the complement of the membership degree, i.e. one minus the membership degree, and therefore is fixed whereas in Atanassov intuitionistic fuzzy sets the non-membership degree may take any value between zero and one minus the membership degree.

Definition 1. [2]. Let X be a non-empty set and two functions $\mu_A, \nu_A : X \to [0,1]$. Then

$$A = \{(x, \mu_A(x), \nu_A(x)) / x \in X\},$$

is an Atanassov Intuitionistic Fuzzy Set (AIFS) over X if $\mu_A(x) + \nu_A(x) \le 1$ for each $x \in X$.

The functions μ_A and ν_A provide the membership and non-membership degrees of elements in X to the AIFS A. Let $L^* = \{(x, y) \in [0,1]^2 / x + y \le 1\}$. Elements of L^* are called L^* -values. We define the projections $l, r : L^* \to [0,1]$ by $l(x,y)=x$ and $r(x,y)=y$, but by notational simplicity, we will denote $\underset{\sim}{x}$ and $\tilde{x}$ instead of $l(x)$ and $r(x)$, respectively.

The usual partial order on L^* is the following:

$$x \le_{L^*} y \text{ if } \underset{\sim}{x} \le \underset{\sim}{y} \text{ and } \tilde{y} \le \tilde{x}.$$

Deschrijver and Kerre [33] proved that $\langle L^*, \le_{L^*} \rangle$ is a complete lattice and therefore that AIFS are a particular kind of L-fuzzy set, in the sense of Goguen [35].

Let A be an AIFS over X. The intuitionistic fuzzy index[1] of an element $x \in X$ to A is given by $\pi^*_A(x) = 1 - \mu_A(x) - \nu_A(x)$. In particular, the intuitionistic fuzzy index of $x \in L^*$ is defined in a similar way, i.e. $\pi^*_A(x) = 1 - l(x) - r(x)$. This index measures the hesitation degree in each $x \in L^*$.

In [27], Chen and Tan, introduce the notion of score of a L^* -value as the function $S^* : L^* \to [-1,1]$ defined by

$$S^*(x) = \underset{\sim}{x} - \tilde{x} \tag{1}$$

In [38], Hong and Choi, introduce the notion of accuracy function for an L^* -value as the function $h^* : L^* \to [0,1]$ defined by

$$h^*(x) = \underset{\sim}{x} + \tilde{x} \tag{2}$$

[1] In the seminal paper on AIFS, i.e. in [2], this index was called degree of indeterminacy of an element $x \in X$ to A.

Xu and Yager in [68], based on the score and accuracy index on L^* and with the goal of rank L^* -values, introduce the total order on L^* defined by

$$x \leq_{YY} y \text{ if } s^*(x) < s^*(y) \text{ or } (s^*(x) = s^*(y) \text{ and } h^*(x) \leq h^*(y)). \quad (3)$$

In [36], [40], [52], [72] and in an independent way, fuzzy set theory was extended by considering subintervals of the unit interval [0,1] instead of a single value in [0,1]. The main goal was to represent the uncertainty in the process of assigning the membership degrees.

Definition 2. Let X be a non-empty set and $L = \{[a,b] / 0 \leq a \leq b \leq 1\}$ be the set of closed subintervals of [0,1]. An Interval-Valued Fuzzy Set (IVFS) A over X is an expression

$$A = \{(x, \mu_A(x)) / x \in X\}.$$

Where $\mu_A : X \to L$.

Define the projections[1] $\nabla, \Delta : L \to [0,1]$ by $\nabla([a,b]) = a$ and $\Delta([a,b]) = b$.

For notational simplicity, for an arbitrary $X \in L$, we will denote $\nabla(x)$ and $\Delta(x)$ by $\underline{X}$ and $\overline{X}$, respectively. An interval $X \in L$ is degenerate if $\underline{X} = \overline{X}$, i.e. $X=[x,x]$ for some $x \in [0,1]$. Given $X \in L$, we denote its standard complement $[1-\overline{X}, 1-\underline{X}]$ by X. A more general notion of complement (or negation) for L^* -values can be found in [8].

We can consider the following partial order on L,

$$X \leq_L Y \text{ iff } \underline{X} \leq \underline{Y} \text{ and } \overline{X} \leq \overline{Y}.$$

As it is well-known, $\langle L, \leq_L \rangle$ is a complete lattice and so it can be seen as a Goguen L-fuzzy set.

As pointed by Moore in [45], an interval has a dual nature: as a set of real numbers and as a new kind of number (an ordered pair of real numbers with the restriction that the first component is smaller than or equal to the second one). The order $\leq_L$ is an order which stresses the nature of ordered pair for elements in L whereas the inclusion of sets stresses the nature of set for elements in L. Nevertheless, the inclusion order on L can also be expressed using the ordered pair nature as follows:

$$X \not\subset Y \text{ iff } \underline{Y} \leq \underline{X} \leq \overline{X} \leq \overline{Y}.$$

The score and accuracy function for interval fuzzy values, i.e. of an arbitrary interval $X \in L$ are defined as follows:

$$s(X) = v(X) \text{ -1 and } h(X) = 1\text{-}w(x).$$

Where $v(X) = \underline{X} + \overline{X}$ and $w(X) = \overline{X} - \underline{X}$.

[1] These projections are particular cases of Atanassov's K_α-operator for intervals [19], [48].

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As it is well known, the lattices $\langle L^*, \leq_{L^*} \rangle$ and $\langle L, \leq_L \rangle$ are isomorphic. The map $\rho: L \to L^*$ defined by $\rho(X): (\underline{X}, 1\text{-}\overline{X})$ is a such an isomorphism. Although both lattices are algebraically equivalent, from a semantical point of view they are different [61].

Remark 1. Note that, the score and the accuracy indexes on L and L^* are related as follows: $s = s^* \ o \ \rho$ and $h = h^* \ o \ \rho$. Notice that the partial order $\leq_{XY}$ on L obtained from the partial order $\leq_{XY}$ in *Eq.(3)* by using this isomorphism, i.e. $X \leq_{XY} Y$ iff $\rho(X) \leq_{XY} \rho(Y)$, can be equivalently obtained as following:

$$X \leq_{XY} Y \text{ iff } s(X) \prec_{XY} s(Y) \text{ or } (s(X) = s(Y) \text{ and } h(X) \leq h(Y)). \tag{4}$$

Bustince et al. [22] introduced the notion of admissible orders in the context of interval-valued fuzzy functions in order to always be possible to compare intervals which is important in some kind of applications [23]. An order $\leq$ on L is admissible if it refines $\leq_L \leq_L$, i.e. $X \leq Y$ whenever $X \leq_L Y$. In particular $\leq_{XY}$ is an admissible order. Other examples of admissible orders can be found in [53]. In addition, when we translate the notion of intuitionistic fuzzy index for interval values, we get the interval-valued fuzzy index $\Pi(X) = \pi^*(\rho(X)) = \overline{X} - \underline{X} = w(X)$ for each $X \in L$. Thus, the length of an interval is a measure of their indeterminacy or imprecision.

2.1| The Best L and L* Representation of the OWA Operator

In [13], it was adapted the notion of interval representation of [11], [54] in the context of interval-valued fuzzy sets theory for the particular case of the intervalvalued t-norms. Interval representation captures, in a formal way, the property of correctness of interval functions in the sense of [37]. From then, interval representations of several other connectives and fuzzy constructions (see for example [8], [12], [49]) have been studied. Here we are interested in considering the case of n-ary increasing fuzzy functions. Let's start recalling some notions.

Definition 3. Let $f: [0,1]^n \to [0,1]$ be an n-ary function. A function $F: L^n \to L$ is an interval representation or *L*-representation of *f* if for each $X_1,.....,X_n \in L$ and $x_i \in X_i$ with $i=1,....,n$ we have that $f(x_1,.....,x_n) \in F(X_1,.....,X_n)$.

Let $F, G: L^n \to L$. We write $F \subseteq \to_L G$, if for any $X_1,.....,X_n \in L$, $G(X_1,.....,X_n) \subseteq F(X_1,.....,X_n)$. Notice that if $X, Y \in L$ and $X \subseteq Y$ then $h(X) \geq h(Y)$. Thus, $F \subseteq \to_L G$ means that G is always more accurate than *F*, i.e. $h(F(X_1,.....,X_n)) \leq h(G(X_1,.....,X_n))$ for any $X_1,.....,X_n \in L$. Notice also that if *G* is an *L*-representation of a function *f* and $F \subseteq \to_L G$ then *F* is also an *L*-representation of *f*, but less accurate than *G*. Therefore, *G* is a better *L*-representation of *f* than *F*.

Proposition 1. [34]. Let $f: [0,1]^n \to [0,1]$ be an n-ary increasing fuzzy function. Then the function $\hat{f}: L^n \to L$ defined by

$$\hat{f}(X_1,....,X_n) = [f(\underline{X}_1,....,\underline{X}_n), f(\overline{X}_1,....,\overline{X}_n)], \tag{5}$$

is an *L*-representation of *f*. Moreover, for any other *L*-representation *F* of *f*, $F \subseteq_L \hat{f}$.

$\hat{f}$ is therefore the more accurate L-representation of f, i.e. the best L-representation w.r.t. the $\subseteq_L$ order. So $\hat{f}$ has the property of optimality in the sense of [37].

Remark 2. [10]. An important characteristic of the best L-representation is that when we identify points and degenerate intervals, via the merging $m(x)=[x,x]$, f and $\hat{f}$ have the same behavior, i.e. $m(f(x_1,...,x_n))=\hat{f}(m(x_1),...,m(x_n))$. Another property of the best L-representation of some increasing function is that it is isotone with respect to both, the inclusion order and the $\leq L$ order, i.e. if $X_i, Y_i \in L$ and $i=1,...,n$ *then* $\hat{f}(X_1,\dots,X_n)\subseteq\hat{f}(Y_1,\dots,Y_n)$ *and, analogously, if* $X_i\leq_{\mathbb{L}} Y_i$ *for each* $i=1,\dots,n$ *then* $\hat{f}(X_1,\dots,X_n)\leq_{\mathbb{L}}\hat{f}(Y_1,\dots,Y_n)$.

Let $\Lambda=(\lambda_1,\dots,\lambda_n)\in[0,1]^n$ be an n-ary weighting vector, i.e. $\sum_{i=1}^n\lambda_i=1$. The weighted average (WA) operator is defined by

$$\mathrm{wa}_\Lambda(x_1,\dots,x_n)=\sum_{i=1}^{n}\lambda_i x_i.$$

The Ordered Weighted Averaging (OWA) operator introduced by Yager [69] is defined by

$$\mathrm{owa}_\Lambda(x_1,\dots,x_n)=\sum_{i=1}^{n}\lambda_i x_{\sigma(i)}.$$

Where $\sigma:\{1,\dots,n\}\to\{1,\dots,n\}$ is the permutation such that $x_{\sigma(i)}\geq x_{\sigma(i+1)}$ for any $i=1,\dots,n-1$, i.e. it orders in decreasing way a n-tuple of values in [0,1] and so $x_\sigma(i)$ is the ith greatest element of $\{x_1,\dots,x_n\}$. Notice that,

$$\mathrm{owa}_\Lambda(x_1,\dots,x_n)=\mathrm{wa}_\Lambda(x_{\sigma(1)},\dots,x_{\sigma(n)}). \tag{6}$$

Several interval-valued and Atanassov intuitionistic extensions of the OWA operator have been proposed (see for example [15], [44], [70]), but most of them are not $\mathbb{L}$ (L^*)-representations of the OWA operator and do not reduce to the fuzzy OWA operator when applied to degenerate intervals.

The best $\mathbb{L}$-representation of owa_Λ is the interval-valued function $\widehat{owa_\Lambda}:\mathbb{L}^n\to\mathbb{L}$ defined by

$$\widehat{\mathrm{owa}_\Lambda}(X_1,\dots,X_n)=\left[\mathrm{owa}_\Lambda\left(\underline{X_1},\dots,\underline{X_n}\right),\mathrm{owa}_\Lambda\left(\overline{X_1},\dots,\overline{X_n}\right)\right]=\sum_{i=1}^{n}\lambda_i X_{\tau(i)}.$$

Where $X_{\tau i}=[\underline{X}_{\tau_1(i)},\overline{X}_{\tau_2(i)}]$; $\tau_1,\tau_2:\{1,\dots,n\}\to\{1,\dots,n\}$ are permutations such that $\underline{X}_{\tau_1(i)}\geq\underline{X}_{\tau_1(i+1)}$ and $\overline{X}_{\tau_2(i)}\geq\overline{X}_{\tau_2(i+1)}$ for any $i=1,\dots,n-1$; the scalar product is the usual in interval mathematics (see [45]), i.e. for any $\lambda\in[0,1]$ and $X,Y\in\mathbb{L}$, $\lambda X=[\lambda\underline{X},\lambda\overline{X}]$ and the sum is w.r.t. the limited addition defined by $X[+]Y=[\min(\underline{X}+\underline{Y},1),\min(\overline{X}+\overline{Y},1)]$. Notice that, in this case, because $\sum_{i=1}^n\lambda_i=1$,

$$\left[\sum_{i=1}^{n}\right]\lambda_i X_i=\left[\min\left(\sum_{i=1}^{n}\lambda_i\underline{X_i},1\right),\min\left(\sum_{i=1}^{n}\lambda_i\overline{X_i},1\right)\right]=\left[\sum_{i=1}^{n}\lambda_i\underline{X_i},\sum_{i=1}^{n}\lambda_i\overline{X_i}\right]=\sum_{i=1}^{n}\lambda_i X_i. \tag{7}$$

Where $[\sum_{i=1}^n]$ is the sommatory with respect to $[+]$ and $\sum_{i=1}^n$ is the sommatory with respect the usual addition between intervals (see [45]).

Note that for each term in the sum above, lower and upper bounds from different intervals may be considered for a given weight λ_i. For example, for $\lambda_1=0.2$, $\lambda_2=0.3$, $\lambda_3=0.5$, $X_1=[0.6,0.8]$, $X_2=[0.7,0.9]$ and $X_3=[0.5,1]$ we have that $\left[\sum_{i=1}^3\right]\lambda_i X_i=[min(0.2\cdot0.6+0.3\cdot0.7+0.5\cdot0.5,1),min(0.2\cdot0.8+0.3\cdot0.9+0.5\cdot1,1)]=[0.58,0.93]=[0.2\cdot0.6+0.3\cdot0.7+0.5\cdot0.5,0.2\cdot0.8+0.3\cdot0.9+0.5\cdot1]=\sum_{i=1}^3\lambda_i X_i$.

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Analogously, a function $F:(L^*)^n \to L^*$ is an L^*-representation of a function $f:[0,1]^n \to [0,1]$ if for each $\mathbf{x}_i \in L^*$ and $x_i \in [\mathbf{x}_i, 1-\widetilde{\mathbf{x}_i}]$, with $i = 1, \dots, n$,

$$\mathrm{F}(\mathrm{x}_1, \dots, \mathrm{x}_n) \leq \mathrm{f}(\mathrm{x}_1, \dots, \mathrm{x}_n) \leq 1 - \mathrm{F}(\widetilde{\mathrm{x}_1, \dots, \mathrm{x}_n}). \tag{8}$$

Let $F, G: (L^*)^n \to L^*$. We denote by $F \sqsubseteq_{L^*} G$, if for any $\mathrm{x}_1, \dots, \mathrm{x}_n \in L^*$, $G(\mathrm{x}_1, \dots, \mathrm{x}_n) \subseteq_{L^*} F(\mathrm{x}_1, \dots, \mathrm{x}_n)$, where $x \subseteq_{L^*} y$ *if* $x \leq y$ and $\tilde{x} \leq \tilde{y}$. Notice that although of this order be the usual on R^2, considering the mathematical equivalence of L^* and $\mathbb{L}$, we have that $x \subseteq_{L^*} y$ iff $\rho^{-1}(X) \subseteq \rho^{-1}(Y)$. Thus, $F \sqsubseteq_{L^*} G$ means than the result of G is always more accurate than the result of F, i.e. $h^*(F(x_1, \dots, x_n)) \leq h^*(G(x_1, \dots, x_n))$ for any $x_1, \dots, x_n \in L^*$.

Proposition 2. Let $f:[0,1]^n \to [0,1]$ be an increasing function. Then the function $f:(L^*)^n \to L^*$ defined by

$$\mathrm{f}(\mathrm{x}_1, \dots, \mathrm{x}_n) = \left(\mathrm{f}(\mathrm{x}_1, \dots, \mathrm{x}_n), 1 - \mathrm{f}(1-\widetilde{\mathrm{x}_1}, \dots, 1-\widetilde{\mathrm{x}_n})\right).$$

is the greatest L^*-representation of f w.r.t. $\sqsubseteq_{L^*}$ order and so is the best one.

Proof. If $x_i \in [\mathbf{x}_i, 1-\widetilde{\mathbf{x}_i}]$ for each $i = 1, \dots, n$, then because f is increasing we have that $f(x_1, \dots, x_n) \leq f(x_1, \dots, x_n) \leq f(1-\widetilde{x_1}, \dots, 1-\widetilde{x_n})$ and therefore, $f(x_1, \dots, x_n) \leq f(x_1, \dots, x_n) \leq 1 - f(\widetilde{x_1, \dots, x_n})$. So, $f(x_1, \dots, x_n)$ is an L^*-representation of f.

Now, suppose that F is another L^*-representation of f, then by *Eq. (8)* and because f is increasing, we have that $F(x_1, \dots, x_n) \leq f(x_1, \dots, x_n) \leq f(1-\widetilde{x_1}, \dots, 1-\widetilde{x_n}) \leq 1 - F(\widetilde{x_1, \dots, x_n})$. Therefore, $f(x_1, \dots, x_n) \subseteq_{L^*} F(x_1, \dots, x_n)$, *i.e.* $F \sqsubseteq^* f$.

Moreover, if f is an aggregation function then f is also an L^*-valued aggregation function [42] (*Lemma 1*). Clearly, $f = \rho \circ \hat{f} \circ \rho^{-1}$, or equivalently, $\hat{f} = \rho^{-1} \circ f \circ \rho$. Therefore, owa_Λ it is the best L^*-representation of owa_Λ .

Proposition 3. Let $\mathrm{f}, \mathrm{g}: [0,1]^\mathrm{n} \to [0,1]$. If $\mathrm{f} \leq \mathrm{g}$ then $\hat{\mathrm{f}} \leq \hat{\mathrm{g}}$ and $\mathrm{f} \leq \mathrm{g}$.

Proof. Straightforward.

Remark 3. owa as well as $\widehat{\mathrm{owa}}$ are interval-valued and Atanassov intuitionistic aggregation functions in the sense of [42]. Moreover, both are symmetric and idempotent, and as a consequence of the above proposition, they are bounded by $\widehat{\mathrm{owa}_{(0,\dots,0,1)}}$ $(\mathrm{owa}_{(0,\dots,0,1)})$, i.e. $\widehat{\min}$ (min) and $\widehat{\mathrm{owa}_{(1,0,\dots,0)}}$ $(\mathrm{owa}_{(1,0,\dots,0)})$, i.e. $\widehat{\max}$ (max).

3| Interval-Valued Atanassov's Intuitionistic Fuzzy Sets

Definition 4. [3]. An IVAIFS A over a nonempty set X is an expression given by

$$\mathrm{A} = \{(\mathrm{x}, \mu_\mathrm{A}(\mathrm{x}), \nu_\mathrm{A}(\mathrm{x}))/\mathrm{x} \in \mathrm{X}\},$$

where $\mu_A, \nu_A: X \to \mathbb{L}$ with the condition $\overline{\mu_A(x)} + \overline{\nu_A(x)} \leq 1$.

Deschrijver and Kerre [33] provide an alternative approach for Atanassov intuitionistic fuzzy sets in term of L-fuzzy sets in the sense of Goguen [35]. Analogously, we can also see IVAIFS as a particular case of L-fuzzy set by considering the complete lattice $\langle \mathbb{L}^*, \leq_{\mathbb{L}^*} \rangle$ where

$$\mathbb{L}^* = \{(X, Y) \in \mathbb{L} \times \mathbb{L}/\overline{X} + \overline{Y} \leq 1\}.$$

And

$$(X_1, X_2) \leq_{\mathbb{L}^*} (Y_1, Y_2) \text{ iff } X_1 \leq_{\mathbb{L}} Y_1 \text{ and } Y_2 \leq_{\mathbb{L}} X_2.$$

Notice that $0_{\mathbb{L}^*} = ([0,0], [1,1])$ and $1_{\mathbb{L}^*} = ([1,1], [0,0])$. Analogously to the case of L^*, we define the projections $l, r: \mathbb{L}^* \to \mathbb{L}$ by

$$l(X_1, X_2) = X_1 \text{ and } r(X_1, X_2) = X_2,$$

and for each $X \in \mathbb{L}^*$, we denote $l(X)$ and $r(X)$ by X and $\widetilde{X}$, respectively.

Elements of $\mathbb{L}^*$ will be called $\mathbb{L}^*$-values. An $\mathbb{L}^*$-value X is a semi-diagonal element if X and $\widetilde{X}$ are degenerate intervals. $X \in \mathbb{L}^*$ is a diagonal element if $X + \widetilde{X} = [1,1]$ i.e. if $X = ([x, x], [1 - x, 1 - x])$ for some $x \in [0,1]$. We denote by $\mathscr{D}_S$ and $\mathscr{D}$ the sets of semi-diagonal and diagonal elements of $\mathbb{L}^*$, respectively. Clearly, $\mathscr{D} \subseteq \mathscr{D}_S$ and there is a bijection between $[0,1]$ and $\mathscr{D}$ ($\phi(x) = ([x, x], [1 - x, 1 - x])$), between L^* and $\mathscr{D}_S$ ($\psi(x) = ([x, x], [\tilde{x}, \tilde{x}])$) and between $\mathbb{L}$ and $\mathscr{D}_S$ ($\varphi(X) = ([\underline{X}, \underline{X}], [\overline{X}, \overline{X}]^c)$, i.e. $\varphi = \psi \circ \rho$) [29].

3.1 | Some indexes for $\mathbb{L}^*$-Values

In [50] the Atanassov intuitionistic fuzzy index was extended for IVAIFS, in order to provide an interval measure of the hesitation degree in IVAIFS. Let A be an IVAIFS over a set X. The interval-valued Atanassov intuitionistic fuzzy index of an element $x \in X$ for the IVAIFS A is determined by the expression $\Pi^*(x) = [1,1] - \mu_A(x) - \nu_A(x)$. In an analogous way the interval-valued Atanassov intuitionistic fuzzy index of an $(X, Y) \in \mathbb{L}^*$ is defined by

$$\Pi^*(X, Y) = [1,1] - X - Y. \tag{9}$$

The Chen and Tan score measure was extended for $\mathbb{L}^*$ in [66] [1] and [41].

In this paper we consider Xu's definition: Let $S: \mathbb{L}^* \to [-1,1]$ be defined by

$$S(X) = \frac{v(X) - v(\widetilde{X})}{2}.$$

For each $X \in \mathbb{L}^*$, $S(X)$ is called the score of X.

Remark 4. S when applied to semi-diagonal elements is the same, up to an isomorphism ψ, as s^*, i.e. $S(\psi(x)) = s^*(x)$ for any $x \in L^*$. Analogously, S when applied to semi-diagonal elements is the same, up to an isomorphism φ, as s, i.e. $S(\varphi(X)) = s(X)$ for any $X \in \mathbb{L}$. Moreover, the range of S ($[-1,1]$) is the same as that of s^* and S can be obtained from s and s^*, as shown by the *Eq. (10)*.

$$S(X) = \frac{s^*\left(s(X), s(\widetilde{X})\right)}{2}. \tag{10}$$

Since we can have two different $\mathbb{L}^*$-values with the same score, for example $S([0.2,0.3], [0.4,0.5]) = S([0.1,0.2], [0.3,0.4]) = -0.2$, the score determines just a pre-order on $\mathbb{L}^*$:

$$X \leq_S Y \text{ iff } S(X) \leq S(Y).$$

[1] Because this reference is in Chinese, we are based on the definition in [64].

Since $\leq_S$ is a pre-order, it defines the following natural equivalence relation: $X \equiv_S Y$ *iff* $X \leq_S Y$ and $Y \leq_S X$

Another important index for $\mathbb{L}^*$-values is the extension of the accuracy function. Nevertheless, in the literature several non-equivalent such "extensions" have been proposes. In [29], [30], it was made an analysis of five of such proposals concluding that the more reasonable would be the new accuracy function proposed in that paper and the one proposed in [66]. Here we will consider Xu's accuracy function:

$$\mathrm{H}(\mathrm{X}) = \frac{\mathrm{v}(\mathrm{X}) + \mathrm{v}(\tilde{\mathrm{X}})}{2}.$$

because, analogously to the case of S, the Xu's accuracy function when applied to semi-diagonal elements is the same, up to an isomorphisms ψ and φ, as h^* and h, respectively, i.e. $H(\psi(x)) = h^*(x)$ for any $x \in L^*$ and $H(\varphi(X)) = h(X)$ for any $X \in \mathbb{L}$. In addition, the range of H, h and h^* are the same.

3.2| Order for $\mathbb{L}^*$-Values

In [56], it was introduced the notion of n-dimensional fuzzy interval and it was observed that 4-dimensional fuzzy sets are isomorphic to IVAIFS. The degrees in an n-dimensional fuzzy interval take values in $L_n([0,1]) = \{(x_1, \dots, x_n) \in [0,1]^n / x_i \leq x_{i+1}$ for each $i = 1, \dots, n-1\}$. In [9] the elements of $L_n([0,1])$ are called n-dimensional intervals and the bijection $\varrho: \mathbb{L}^* \to L_4([0,1])$ defined by $\varrho(X) = (\nabla(X), \triangle(X), 1 - \triangle(\tilde{X}), 1 - \nabla(\tilde{X}))$ was provided. One of the possible interpretations considered in [9] for the 4-dimensional intervals (x_1, x_2, x_3, x_4) is that the intervals $[x_1, x_2]$ and $[x_3, x_4]$ represent an interval uncertainty in the bounds of an interval-valued degree, i.e. of an element $[x, y] \in \mathbb{L}$, and so $x \in [x_1, x_2]$ and $y \in [x_3, x_4]$. Having it in mind, we introduce the notion of membership of $\mathbb{L}$-values in $\mathbb{L}^*$-values.

Definition 5. Let $\mathrm{X} \in \mathbb{L}$ and $\mathcal{X} \in \mathbb{L}^*$. We say that $X \Subset \mathcal{X}$ if $\underline{X} \in \mathcal{X}$ and $\overline{X} \in \tilde{\mathcal{X}}^c$.

Observe that this notion is strongly related to the notion of nesting given in [6], [7] and therefore also can be used as a representation of IVAIFS by pairs of AIFS.

Notice that, for each $X, Y, Z \in \mathbb{L}$,

I. if $X \subseteq Y \subseteq Z$ and $X, Z \Subset \mathcal{Y}$ for some $\mathcal{Y} \in \mathbb{L}^*$, then $Y \Subset \mathcal{Y}$;
II. if $X \leq_{\mathbb{L}} Y \leq_{\mathbb{L}} Z$ and $X, Z \Subset \mathcal{Y}$ for some $\mathcal{Y} \in \mathbb{L}^*$, then $Y \Subset \mathcal{Y}$;
III. $Y \Subset \varphi(X)$ iff $Y = X$.

For any $\mathcal{X} \in \mathbb{L}^*$ we will denote

$$\vec{\mathcal{X}} = [\nabla(\mathcal{X}), \nabla(\tilde{\mathcal{X}}^c)] \quad \text{and} \quad \overleftarrow{\mathcal{X}} = [\triangle(\mathcal{X}), \triangle(\tilde{\mathcal{X}}^c)], \tag{11}$$

i.e. $\vec{\mathcal{X}} = [\nabla(\mathcal{X}), 1 - \triangle(\tilde{\mathcal{X}})]$ and $\overleftarrow{\mathcal{X}} = [\triangle(\mathcal{X}), 1 - \nabla(\tilde{\mathcal{X}})]$. Notice that, the set $S_{\mathcal{X}} = \{X \in \mathbb{L} / X \Subset \mathcal{X}\}$ is bounded, i.e. for any $X \in S_{\mathcal{X}}$, $\vec{\mathcal{X}} \leq_{\mathbb{L}} X \leq_{\mathbb{L}} \overleftarrow{\mathcal{X}}$ and $\vec{\mathcal{X}}, \overleftarrow{\mathcal{X}} \in S_{\mathcal{X}}$. Thus, $S_{\mathcal{X}}$ is a closed interval $([\vec{\mathcal{X}}, \overleftarrow{\mathcal{X}}])$ of $\mathbb{L}$-values and hence, analogously to $\mathbb{L}$-values, $\mathbb{L}^*$-values also have a dual nature: as an ordered pair of $\mathbb{L}$-values with some condition and as a set (an interval) of $\mathbb{L}$-values.

3.2.1 | Subset order for $\mathbb{L}^*$-values

Since the usual membership relation is used to introduce the subset relation in set theory, the relation $\Subset$ will allow us to introduce a notion of subset between $\mathbb{L}^*$-values. Let $\mathcal{X}, \mathcal{Y} \in \mathbb{L}^*$, we say that $\mathcal{X} \Subset \mathcal{Y}$ if for each $X \Subset \mathcal{X}$ we have that $X \Subset \mathcal{Y}$. Analogously to the case of $\mathbb{L}$-values, we can also define this inclusion relation via the bounds of the interval associated to $\mathbb{L}^*$-values.

Proposition 4. Let $X, Y \in \mathbb{L}^*$. Then the following expression are equivalents

I. $X \subseteq Y$;
II. $S_X \subseteq S_Y$;
III. $\vec{Y} \leq_{\mathbb{L}} \vec{X} \leq_{\mathbb{L}} \overleftarrow{X} \leq_{\mathbb{L}} \overleftarrow{Y}$;
IV. $X \subseteq Y$ *and* $\tilde{X} \subseteq \tilde{Y}$.

Proof.

I. 1⇒2: If *X_(⊆)* Y then for each *X_(∈)* X also *X_(∈)* Y, and so *S_X⊆S_Y*.
II. 2 ⇒ 3: Straightforward once that $S_X = [\vec{X}, \overleftarrow{X}]$.
III. 3 ⇒ 4: If $\vec{Y} \leq_{\mathbb{L}} \vec{X} \leq_{\mathbb{L}} \overleftarrow{X} \leq_{\mathbb{L}} \overleftarrow{Y}$ then by definition $[\nabla(Y), \nabla(\tilde{Y}^c)] \leq_{\mathbb{L}} [\nabla(X), \nabla(\tilde{X}^c)] \leq_{\mathbb{L}} [\triangle(X), \triangle(\tilde{X}^c)] \leq_{\mathbb{L}} [\triangle(Y), \triangle(\tilde{Y}^c)]$ So, $\nabla(Y) \leq \nabla(X) \leq \triangle(X) \leq \triangle(Y)$ and $\nabla(\tilde{Y}^c) \leq \nabla(\tilde{X}^c) \leq \triangle(\tilde{X}^c) \leq \triangle(\tilde{Y}^c)$, *i.e.* $1 - \triangle(\tilde{Y}) \leq 1 - \triangle(\tilde{X}) \leq 1 - \nabla(\tilde{X}) \leq 1 - \nabla(\tilde{Y})$. Therefore $X \subseteq Y$ *and* $\tilde{X} \subseteq \tilde{Y}$.
IV. 4 ⇒ 1: If $X \underline{\in} X$ then $\underline{X} \in X$ and $\overline{X} \in \tilde{X}^c$. So, because $X \subseteq Y$ and $\tilde{X} \subseteq \tilde{Y}$, then $\underline{X} \in Y$ and $\overline{X} \in \tilde{Y}^c$. Therefore, $X \underline{\in} Y$ and hence $X \underline{\subseteq} Y$.

Remark 5. Some properties of _($\subseteq$):

i. It is a partial order on $\mathbb{L}^*$-values;

ii. For each $X, Y \in \mathbb{L}$, $\varphi(X) \underline{\subseteq} \varphi(Y)$ iff $X = Y$;

iii. For *each* $x, y \in [0,1]$, $\phi(x) \underline{\subseteq} \phi(y)$ iff $x = y$;

iv. Defining the complement of $\mathbb{L}^*$-values by $X^c = (\tilde{X}, X)$, then $X \underline{\subseteq} Y$ iff $X^c \underline{\subseteq} Y^c$.

3.2.2| Extension of $\leq_{XY}$ total order for $\mathbb{L}^*$-values

In order to rank any possible set of $\mathbb{L}^*$-values it is necessary to provide a total order on $\mathbb{L}^*$, as made in [68] for L^*-values which was based on the score and accuracy index. Following the same idea, we define the next binary relation on $\mathbb{L}^*$-values:

$$X \leq_{S,H} Y \quad \text{iff} \quad \begin{cases} X <_S Y & \text{or} \\ X \equiv_S Y & \text{and} \quad H(X) \leq H(Y) \end{cases} \tag{12}$$

for any $X, Y \in \mathbb{L}^*$, where $X <_S Y$ iff $X \leq_S Y$ and $X \not\equiv_S Y$.

Nevertheless, as noted in [64], this relation is not an order. However, in [64] it was provided the next total order[1] for $\mathbb{L}^*$:

$$X \preccurlyeq Y \quad \text{iff} \quad \begin{cases} X <_S Y & \text{or} \\ X \equiv_S Y & \text{and} \quad H(X) < H(Y) \quad \text{or} \\ X \equiv_S Y & \text{and} \quad H(X) = H(Y) \quad \text{and} \quad T(X) < T(Y) \quad \text{or} \\ X \equiv_S Y & \text{and} \quad H(X) = H(Y) \quad \text{and} \quad T(X) = T(Y) \quad \text{and} \quad G(X) \leq G(Y) \end{cases} \tag{13}$$

for any $X, Y \in \mathbb{L}^*$, where $T(X) = w(X) - w(\tilde{X})$ and $G(X) = w(X) + w(\tilde{X})$.

[1] In [64] was not claimed this, but from Proposition 4.1. of [31], it is possible to conclude that this order is total.

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In [29], it was defined a new total order for $\mathbb{L}^*$-values, denoted here by $\lesssim$, which is based on the total order for L^*-values of Xu and Yager given by *Eq. (3)*.

Theorem 1. [29] The binary relation $\lesssim$ on $\mathbb{L}^*$, defined for any $X, Y \in \mathbb{L}^*$ by

$$\mathrm{X} \lesssim \mathrm{Y} \ \text{ iff } \ \mathrm{X} <_{\mathrm{XY}} \mathrm{Y} \ \text{ or } \ \left(\mathrm{X} = \mathrm{Y} \ \text{ and } \ \tilde{\mathrm{X}} \leq_{\mathrm{XY}} \tilde{\mathrm{Y}}\right), \tag{14}$$

is a total order.

Observe that the order $\lesssim$ is a particular instance of the admissible orders on $\mathbb{L}^*$ introduced in [30], [31] (see also [32]), i.e. is total and refines $\leq_{\mathbb{L}^*}$.

Here, we propose a new total order, with the same principle as (14), but by considering other intervals:

Theorem 2. The binary relation $\lessapprox$ on $\mathbb{L}^*$, defined for any $X, Y \in \mathbb{L}^*$, by

$$\mathrm{X} \lessapprox \mathrm{Y} \ \text{ iff } \ \vec{\mathrm{X}} <_{\mathrm{XY}} \vec{\mathrm{Y}} \ \text{ or } \ \left(\vec{\mathrm{X}} = \vec{\mathrm{Y}} \ \text{ and } \ \overleftarrow{\mathrm{X}} \leq_{\mathrm{XY}} \overleftarrow{\mathrm{Y}}\right). \tag{15}$$

is a total order.

Proof. Trivially, $\lessapprox$ is reflexive and antisymmetric. The transitivity of $\lessapprox$ follows from the transitivity of $\leq_{XY}$ and equality. Analogously, the totallity of $\lessapprox$ follows from the totality of $\leq_{XY}$.

4 | $\mathbb{L}^*$-Representation of OWA

4.1 | $\mathbb{L}^*$-Representations of $\mathbb{L}$-Functions

The notion of membership on $\mathbb{L}^*$-values also allows us to adapt the notion of interval representation for $\mathbb{L}^*$ in the following way.

Definition 6. Let $F\colon \mathbb{L}^n \to \mathbb{L}$ and $\mathscr{F}\colon (\mathbb{L}^*)^n \to \mathbb{L}^*$. $\mathcal{F}$ is an $\mathbb{L}^*$-representation of F if for each $X_i \in \mathbb{L}^*$, and $X_i \,\underline{\in}\, X_i$, with $i = 1, \dots, n$, $\mathrm{F}(X_1, \dots, X_n) \,\underline{\in}\, \mathscr{F}(X_1, \dots, X_n)$.

Let $\mathscr{G}, \mathscr{F}\colon (\mathbb{L}^*)^n \to \mathbb{L}^*$. We say that $\mathscr{F}$ is narrower than $\mathscr{G}$, denoted by $\mathscr{G} \sqsubseteq_{\mathbb{L}^*} \mathscr{F}$, if for any $X_i \in \mathbb{L}^*$ with $i = 1, \dots, n$, $\mathscr{F}(X_1, \dots, X_n) \,\underline{\subseteq}\, \mathscr{G}(X_1, \dots, X_n)$. Analogously to the case of $\mathbb{L}$-representation, we say that an $\mathbb{L}^*$-representation $\mathscr{F}$ of a function $F\colon \mathbb{L}^n \to \mathbb{L}$ is better than another $\mathbb{L}^*$-representation $\mathscr{G}$ of F if $\mathscr{G} \sqsubseteq_{\mathbb{L}^*} \mathscr{F}$.

Theorem 3. Let $F\colon \mathbb{L}^n \to \mathbb{L}$ be an isotone function. Then $\ddot{F}\colon (\mathbb{L}^*)^n \to \mathbb{L}^*$ defined by

$$\ddot{\mathrm{F}}(\mathrm{X}_1, \dots, \mathrm{X}_n) = \left(\left[\underline{\mathrm{F}\left(\vec{\mathrm{X}}_1, \dots, \vec{\mathrm{X}}_n\right)}, \underline{\mathrm{F}\left(\overleftarrow{\mathrm{X}}_1, \dots, \overleftarrow{\mathrm{X}}_n\right)}\right], \left[1 - \overline{\mathrm{F}\left(\overleftarrow{\mathrm{X}}_1, \dots, \overleftarrow{\mathrm{X}}_n\right)}, 1 - \overline{\mathrm{F}\left(\vec{\mathrm{X}}_1, \dots, \vec{\mathrm{X}}_n\right)}\right]\right), \tag{16}$$

is an $\mathbb{L}^*$-representation of F. Moreover, if $\mathscr{F}$ is another $\mathbb{L}^*$-representation of F then $\mathscr{F} \sqsubseteq_{\mathbb{L}^*} \ddot{F}$.

Proof. Let $X_i \in \mathbb{L}^*$ with $i = 1, \dots, n$. Since, F is isotone w.r.t. $\leq_{\mathbb{L}}$, then for each $X_i \in X_i$ with $i = 1, \dots, n$, $F(\vec{X}_1, \dots, \vec{X}_n) \leq_{\mathbb{L}} F(X_1, \dots, X_n) \leq_{\mathbb{L}} F(\overleftarrow{X}_1, \dots, \overleftarrow{X}_n)$ and so $\underline{F(\vec{X}_1, \dots, \vec{X}_n)} \leq \underline{F(X_1, \dots, X_n)} \leq \underline{F(\overleftarrow{X}_1, \dots, \overleftarrow{X}_n)}$ and $\overline{F(\vec{X}_1, \dots, \vec{X}_n)} \leq \overline{F(X_1, \dots, X_n)} \leq \overline{F(\overleftarrow{X}_1, \dots, \overleftarrow{X}_n)}$. Therefore,

$\underline{F(X_1,\dots,X_n)} \in [\underline{F(\overrightarrow{X_1},\dots,\overrightarrow{X_n})}, \underline{F(\overleftarrow{X_1},\dots,\overleftarrow{X_n})}] = \ddot{F}(X_1,\dots,X_n)$ and $\overline{F(X_1,\dots,X_n)} \in$ $[\overline{F(\overrightarrow{X_1},\dots,\overrightarrow{X_n})}, \overline{F(\overleftarrow{X_1},\dots,\overleftarrow{X_n})}] = \ddot{F}(\widetilde{X_1,\dots,X_n})^c$

Hence, $F(X_1,\dots,X_n) \underline{\in} \ddot{F}(X_1,\dots,X_n)$.

If $\mathscr{F}: (\mathbb{L}^*)^n \to \mathbb{L}^*$ is another $\mathbb{L}^*$-representation of F, then for each $X_i \in \mathbb{L}^*$, and $X_i \underline{\in} X_i$, with $i = 1,\dots,n$, $F(X_1,\dots,X_n) \underline{\in} \mathscr{F}(X_1,\dots,X_n)$. In particular, $F(\overrightarrow{X_1},\dots,\overrightarrow{X_n}), F(\overleftarrow{X_1},\dots,\overleftarrow{X_n}) \underline{\in} \mathscr{F}(X_1,\dots,X_n)$. So, by definition of $\underline{\in}$, $\underline{F(\overrightarrow{X_1},\dots,\overrightarrow{X_n})}, \underline{F(\overleftarrow{X_1},\dots,\overleftarrow{X_n})} \in \mathscr{F}(X_1,\dots,X_n)$ and $\overline{F(\overleftarrow{X_1},\dots,\overleftarrow{X_n})}, \overline{F(\overrightarrow{X_1},\dots,\overrightarrow{X_n})} \in \mathscr{F}(\widetilde{X_1,\dots,X_n})^c$, i.e. $1 - \overline{F(\overleftarrow{X_1},\dots,\overleftarrow{X_n})}, 1 - \overline{F(\overrightarrow{X_1},\dots,\overrightarrow{X_n})} \in \mathscr{F}(\widetilde{X_1,\dots,X_n})$. Therefore, $\ddot{F}(X_1,\dots,X_n) \subseteq \mathscr{F}(X_1,\dots,X_n)$ and $\ddot{F}(\widetilde{X_1,\dots,X_n}) \subseteq \mathscr{F}(\widetilde{X_1,\dots,X_n})$ and so, by Proposition 4, $\ddot{F}(X_1,\dots,X_n) \underline{\subseteq} \mathscr{F}(X_1,\dots,X_n)$. Hence, $\mathscr{F} \sqsubseteq_{\mathbb{L}^*} \ddot{F}$.

Corollary 1. Let $\mathrm{F}: \mathbb{L}^n \to \mathbb{L}$ be an isotone function. Then

$$\overrightarrow{\ddot{\mathrm{F}}(\mathrm{X}_1,\dots,\mathrm{X}_n)} = \mathrm{F}\left(\overrightarrow{\mathrm{X}_1},\dots,\overrightarrow{\mathrm{X}_n}\right) \text{ and } \overleftarrow{\ddot{\mathrm{F}}(\mathrm{X}_1,\dots,\mathrm{X}_n)} = \mathrm{F}\left(\overleftarrow{\mathrm{X}_1},\dots,\overleftarrow{\mathrm{X}_n}\right). \tag{17}$$

Proof. Straightforward from *Theorem 3* and *Eq. (11)*.

Corollary 2. Let $\mathrm{f}: [0,1]^n \to [0,1]$ be an isotone function. Then

$$\hat{\mathrm{f}}(\mathrm{X}_1,\dots,\mathrm{X}_n) = (\hat{\mathrm{f}}(\mathrm{X}_1,\dots,\mathrm{X}_n), \hat{\mathrm{f}}\left(\widetilde{\mathrm{X}_1}^c,\dots,\widetilde{\mathrm{X}_n}^c)^c\right). \tag{18}$$

Proof. Straightforward from *Theorem 3* and *eq. (11)*.

Corollary 3. Let $f, g: [0,1]^n \to [0,1]$ be isotone functions such that $f \leq g$. Then, $\hat{f} \leq \hat{g}$, i.e. $\hat{f}(X_1,\dots,X_n) \leq_{\mathbb{L}^*} \hat{g}(X_1,\dots,X_n)$ for each $X_i \in \mathbb{L}^*$ with $i = 1,\dots,n$.

Proof. Straightforward from *Corollary 2* and definition of $\leq_{\mathbb{L}^*}$.

Proposition 5. Let $\mathrm{F}: \mathbb{L}^n \to \mathbb{L}$ be an isotone function. Then $\ddot{\mathrm{F}}(\mathscr{D}_S) \subseteq \mathscr{D}_S$ and $\ddot{\mathrm{F}}(\mathscr{D}) \subseteq \mathscr{D}$

Proof. For any $i = 1,\dots,n$, let $X_i \in \mathscr{D}_S$. Then $X_i = ([x_i, x_i], [y_i, y_i])$ for some $x_i, y_i \in [0,1]$ such that $x_i + y_i \leq 1$. Since $\overrightarrow{X_i} = [x_i, 1 - y_i] = \overleftarrow{X_i}$ then, by *Eq. (16)*, $\ddot{F}(X_1,\dots,X_n)$ and $\ddot{F}(\widetilde{X_1,\dots,X_n})$ are degenerate intervals and so $\ddot{F}(X_1,\dots,X_n) \in \mathscr{D}_S$.

For any $i = 1,\dots,n$, let $X_i \in \mathscr{D}$. Then $X_i = ([x_i, x_i], [1 - x_i, 1 - x_i])$ for some $x_i \in [0,1]$. Since $\overrightarrow{X_i} = [x_i, x_i] = \overleftarrow{X_i}$ then, by equation (16), $\ddot{F}(X_1,\dots,X_n)$ and $\ddot{F}(\widetilde{X_1,\dots,X_n})$ are degenerate intervals and $\ddot{F}(X_1,\dots,X_n) = \ddot{F}(\widetilde{X_1,\dots,X_n})^c$. So $\ddot{F}(X_1,\dots,X_n) \in \mathscr{D}$.

Lemma 1. Let $\mathrm{X}, \mathrm{Y} \in \mathbb{L}^*$. Then $\overrightarrow{\mathrm{X}} \subseteq \overrightarrow{\mathrm{Y}}$ and $\overleftarrow{\mathrm{X}} \subseteq \overleftarrow{\mathrm{Y}}$ iff $\mathrm{X} \leq \mathrm{Y}$ and $\tilde{\mathrm{X}} \leq \tilde{\mathrm{Y}}$. Dually, $\mathrm{X} \subseteq \mathrm{Y}$ and $\tilde{\mathrm{X}} \subseteq \tilde{\mathrm{Y}}$ iff $\overrightarrow{\mathrm{X}} \leq \overrightarrow{\mathrm{Y}}$ and $\overleftarrow{\mathrm{X}} \leq \overleftarrow{\mathrm{Y}}$.

Proof. $\overrightarrow{\mathrm{X}} \subseteq \overrightarrow{\mathrm{Y}}$ and $\overleftarrow{\mathrm{X}} \subseteq \overleftarrow{\mathrm{Y}}$ iff $\nabla(\mathrm{Y}) \leq \nabla(\mathrm{X})$, $\triangle(\mathrm{Y}) \leq \triangle(\mathrm{X})$, $\nabla(\tilde{\mathrm{X}}^c) \leq \nabla(\tilde{\mathrm{Y}}^c)$ and $\triangle(\tilde{\mathrm{X}}^c) \leq \triangle(\tilde{\mathrm{Y}}^c)$ iff $\mathrm{X} \leq \mathrm{Y}$ and $\tilde{\mathrm{X}}^c \leq \tilde{\mathrm{Y}}^c$ iff $\mathrm{X} \leq \mathrm{Y}$ and $\tilde{\mathrm{X}} \leq \tilde{\mathrm{Y}}$. The other case is analogous.

Proposition 6. Let $\mathrm{F}: \mathbb{L}^n \to \mathbb{L}$ be an isotone function. Then

$$\ddot{\mathrm{F}}(\mathrm{X}_1,\dots,\mathrm{X}_n) = \mathrm{F}(\mathrm{X}_1,\dots,\mathrm{X}_n) \text{ and } \ddot{\mathrm{F}}(\widetilde{\mathrm{X}_1,\dots,\mathrm{X}_n}) = \mathrm{F}\left(\widetilde{\mathrm{X}_1},\dots,\widetilde{\mathrm{X}_n}\right). \tag{19}$$

Proof. Straightforward from *Lemma 1* and *Corollary 1*.

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Proposition 7. Let $\mathrm{F}, \mathrm{G}: \mathbb{L}^n \to \mathbb{L}$ be isotone functions. If $\mathrm{F} \sqsubseteq_{\mathbb{L}} \mathrm{G}$ then $\ddot{\mathrm{F}} \sqsubseteq_{\mathbb{L}^*} \ddot{\mathrm{G}}$.

Proof. Let $\mathrm{X}_i \in \mathbb{L}^*$ for any $i = 1, \dots, n$. Since, $F \sqsubseteq_{\mathbb{L}} G$, then $G(\overrightarrow{\mathrm{X}_1}, \dots, \overrightarrow{\mathrm{X}_n}) \subseteq F(\overrightarrow{\mathrm{X}_1}, \dots, \overrightarrow{\mathrm{X}_n})$ and $G(\overleftarrow{\mathrm{X}_1}, \dots, \overleftarrow{\mathrm{X}_n}) \subseteq F(\overleftarrow{\mathrm{X}_1}, \dots, \overleftarrow{\mathrm{X}_n})$ and so $\underline{F(\overrightarrow{\mathrm{X}_1}, \dots, \overrightarrow{\mathrm{X}_n})} \leq \underline{G(\overrightarrow{\mathrm{X}_1}, \dots, \overrightarrow{\mathrm{X}_n})} \leq \overline{G(\overrightarrow{\mathrm{X}_1}, \dots, \overrightarrow{\mathrm{X}_n})} \leq \overline{F(\overrightarrow{\mathrm{X}_1}, \dots, \overrightarrow{\mathrm{X}_n})}$ and $\underline{F(\overleftarrow{\mathrm{X}_1}, \dots, \overleftarrow{\mathrm{X}_n})} \leq \underline{G(\overleftarrow{\mathrm{X}_1}, \dots, \overleftarrow{\mathrm{X}_n})} \leq \overline{G(\overleftarrow{\mathrm{X}_1}, \dots, \overleftarrow{\mathrm{X}_n})} \leq \overline{F(\overleftarrow{\mathrm{X}_1}, \dots, \overleftarrow{\mathrm{X}_n})}$. Therefore, by *Theorem 3*, $\ddot{F}(\mathrm{X}_1, \dots, \mathrm{X}_n) \underline{\subseteq} \ddot{\mathrm{G}}(\mathrm{X}_1, \dots, \mathrm{X}_n)$. Hence, $\ddot{F} \sqsubseteq_{\mathbb{L}^*} \ddot{G}$.

Note that, considering the interval point of view for $\mathbb{L}^*$-values, we have that

$$\ddot{\mathrm{F}}(\mathrm{X}_1, \dots, \mathrm{X}_n) \simeq [\mathrm{F}(\overrightarrow{\mathrm{X}_1}, \dots, \overrightarrow{\mathrm{X}_n}), \mathrm{F}(\overleftarrow{\mathrm{X}_1}, \dots, \overleftarrow{\mathrm{X}_n})].$$

4.2 | $\mathbb{L}^*$-Representations of $[0, 1]$-Functions

Let $x \in [0,1]$ and $\mathrm{X} \in \mathbb{L}^*$. Then $x \in^{**} \mathrm{X}$ if $\phi(x) \underline{\subseteq} \mathrm{X}$, i.e. if $1 - \nabla(\tilde{\mathrm{X}}) \leq x \leq \triangle(\mathrm{X})$. There is a close relation between $\underline{\in}$ and $\in^{**}$ as can we see in the next proposition.

Proposition 8. Let $X \in \mathbb{L}^*$ and $X \in \mathbb{L}$. $X \underline{\in} X$ if and only if $\underline{X} \in^{**} X$ and $\overline{X} \in^{**} X$

Proof. Since, trivially, $\overrightarrow{\phi(x)} = [x, x] = \overleftarrow{\phi(x)}$ for any $x \in \mathbb{L}$, then

$$\begin{array}{llll}
X \underline{\in} \mathrm{X} & iff & \overrightarrow{\mathrm{X}} \leq_{\mathbb{L}} X \leq_{\mathbb{L}} \overleftarrow{\mathrm{X}} & \\
 & iff & \overrightarrow{\mathrm{X}} \leq_{\mathbb{L}} [\underline{X}, \underline{X}] \leq_{\mathbb{L}} [\overline{X}, \overline{X}] \leq_{\mathbb{L}} \overleftarrow{\mathrm{X}} & \\
 & iff & \phi(\underline{X}) \underline{\subseteq} \mathrm{X} \;\; and \;\; \phi(\overline{X}) \underline{\subseteq} \mathrm{X} & by\ Prop.\ 4 \\
 & iff & \underline{X} \in^{**} \mathrm{X} \;\; and \;\; \overline{X} \in^{**} \mathrm{X} & by\ def.\ of\ \in^{**}
\end{array}$$

With this notion of membership, we can naturally extend the notion of $\mathbb{L}$-representation of fuzzy function for the $\mathbb{L}^*$-representation of fuzzy function and introduce a new notion of inclusion for $\mathbb{L}^*$-values.

Definition 7. Let $\mathrm{f}: [0,1]^n \to [0,1]$ and $\mathscr{F}: (\mathbb{L}^*)^n \to \mathbb{L}^*$. $\mathcal{F}$ is an $\mathbb{L}^*$-representation of f if for each $\mathrm{X}_i \in \mathbb{L}^*$ and $\mathrm{x}_i \in^{**} \mathrm{X}_i$, with $\mathrm{i} = 1, \dots, \mathrm{n}$, we have that $\mathrm{f}(\mathrm{x}_1, \dots, \mathrm{x}_n) \in^{**} \mathscr{F}(\mathrm{X}_1, \dots, \mathrm{X}_n)$

Let $\mathrm{X}, \mathrm{Y} \in \mathbb{L}^*$. Then $\mathrm{X} \subseteq^{**} \mathrm{Y}$ if for each $x \in^{**} \mathrm{X}$, also $x \in^{**} \mathrm{Y}$. However, $\subseteq^{**}$ is not a partial order (it is not antisymmetric – e.g. consider $\mathrm{X} = ([0.2,0.3], [0.4,0.5])$ and $\mathrm{Y} = ([0.1,0.3], [0.2,0.5])$). Therefore, we just consider $\underline{\subseteq}$ as the extension of inclusion order for $\mathbb{L}^*$.

Analogously to the case of $\mathbb{L}$-representation, we said that an $\mathbb{L}^*$-representation $\mathscr{F}$ of a function $f: [0,1]^n \to [0,1]$ is better than another $\mathbb{L}^*$-representation $\mathscr{G}$ of f if $\mathscr{G} \sqsubseteq_{\mathbb{L}^*} \mathscr{F}$.

Proposition 9. Let $\mathrm{f}: [0,1]^n \to [0,1]$ and $\mathrm{F}: \mathbb{L}^n \to \mathbb{L}$ be isotone functions. If F is an $\mathbb{L}$-representation of f then $\ddot{\mathrm{F}}$ is an $\mathbb{L}^*$-representation of f.

Proof. If $x_i \in^{**} \mathrm{X}_i$ for any $i = 1, \dots, n$, then $\phi(x_i) = ([x_i, x_i], [1 - x, 1 - x]) \underline{\subseteq} \mathrm{X}_i$ and so, by *Proposition 4*, $[x_i, x_i] \subseteq \mathrm{X}_i$ and $[x_i, x_i]^c \subseteq \widetilde{\mathrm{X}_\iota}$, or equivalently, $[x_i, x_i] \subseteq \widetilde{\mathrm{X}_\iota}^c$. Therefore, $x_i \in \widetilde{\mathrm{X}_\iota}$ and $x_i \in \widetilde{\mathrm{X}_\iota}^c$. Thus, since F is an $\mathbb{L}$-representation of f, $f(x_1, \dots, x_n) \in F(\mathrm{X}_1, \dots, \mathrm{X}_n)$ and so $[f(x_1, \dots, x_n), f(x_1, \dots, x_n)] \subseteq F(\mathrm{X}_1, \dots, \mathrm{X}_n)$ and $[f(x_1, \dots, x_n), f(x_1, \dots, x_n)]^c \subseteq F(\widetilde{\mathrm{X}_1}^c, \dots, \widetilde{\mathrm{X}_n}^c)^c$. Hence, by *Corollary 2*, $[f(x_1, \dots, x_n), f(x_1, \dots, x_n)] \subseteq \ddot{F}(\mathrm{X}_1, \dots, \mathrm{X}_n)$ and $[f(x_1, \dots, x_n), f(x_1, \dots, x_n)]^c \subseteq \ddot{F}(\widetilde{\mathrm{X}_1, \dots, \mathrm{X}_n})$. Therefore, $\phi(f(x_1, \dots, x_n)) \underline{\subseteq} \ddot{F}(\mathrm{X}_1, \dots, \mathrm{X}_n)$, i.e. $f(x_1, \dots, x_n) \in^{**} \ddot{F}(\mathrm{X}_1, \dots, \mathrm{X}_n)$. So, $\ddot{F}$ is an $\mathbb{L}^*$-representation of f.

Theorem 4. Let $f: [0,1]^n \to [0,1]$ be an isotone function. $\hat{f}$ is the best, w.r.t. $\sqsubseteq_{\mathbb{L}^*}$, $\mathbb{L}^*$-representation of f.

Proof. From *Propositions 1* and *9* and *Remark 2* it follows that $\hat{f}$ is an $\mathbb{L}^*$-representation of f. Thus, it only remains to prove that is the best one.

Let $\mathscr{F}: (\mathbb{L}^*)^n \to \mathbb{L}^*$ be another $\mathbb{L}^*$-representation of f and $X_i \in \mathbb{L}^*$ for $i = 1, \dots, n$. If $X_i \subseteq X_i$, for any $i = 1, \dots, n$, then by *Proposition 8* $\underline{X_i} \in^{**} X_i$ and $\overline{X_i} \in^{**} X_i$. So, because $\mathscr{F}$ is $\mathbb{L}^*$-representation of f, $f(\underline{X_1}, \dots, \underline{X_n}) \in^{**} \mathscr{F}(X_1, \dots, X_n)$ and $f(\overline{X_1}, \dots, \overline{X_n}) \in^{**} \mathscr{F}(X_1, \dots, X_n)$. Thus, by equation (5), $\underline{\hat{f}(X_1, \dots, X_n)} \in^{**} \mathscr{F}(X_1, \dots, X_n)$ and $\underline{\hat{f}(X_1, \dots, X_n)} \in^{**} \mathscr{F}(X_1, \dots, X_n)$. Therefore, by Proposition 8, $\hat{f}(X_1, \dots, X_n) \subseteq \mathscr{F}(X_1, \dots, X_n)$, i.e. $\mathscr{F}$ is an $\mathbb{L}^*$-representation of $\hat{f}$. Hence, by *Theorem 3*, $F \sqsubseteq_{\mathbb{L}^*} \hat{f}$, and so $\hat{f}$ is a better $\mathbb{L}^*$-representation of f than $\mathscr{F}$.

4.3| The Best $\mathbb{L}^*$-Representation of the OWA Operator

Aggregation functions play an important role in fuzzy sets theory, so it is natural to extend this definition for IVAIFS.

Definition 8. An n-ary function $\mathscr{A}: (\mathbb{L}^*)^n \to \mathbb{L}^*$ is an n-ary interval-valued Atanassov's intuitionistic aggregation function if

I. If $X_i \leq_{\mathbb{L}^*} Y_i$, for each $i = 1, \dots, n$, then $\mathscr{A}(X_1, \dots, X_n) \leq_{\mathbb{L}^*} \mathscr{A}(Y_1, \dots, Y_n)$;
II. $\mathscr{A}(0_{\mathbb{L}^*}, \dots, 0_{\mathbb{L}^*}) = 0_{\mathbb{L}^*}$ and $\mathscr{A}(1_{\mathbb{L}^*}, \dots, 1_{\mathbb{L}^*}) = 1_{\mathbb{L}^*}$.

Theorem 5. Let $A: [0,1]^n \to [0,1]$ be an n-ary aggregation function. Then $\widehat{A}$ is an n-ary interval-valued Atanassov's intuitionistic aggregation function. Moreover, if A is idempotent and/or symmetric, then $\widehat{A}$ is also idempotent and/or symmetric.

Proof. Straightforward from *Corollary 2* and *Remark 2*.

In order to motivate the next section, we will need some arithmetic operations on $\mathbb{L}^*$.

Scalar product. The multiplication $\odot$ of an scalar $\lambda \in [0,1]$ by $X \in \mathbb{L}^*$ is defined by

$$\lambda \odot X = \left(\lambda X, \lambda \tilde{X}\right). \tag{20}$$

Division by a positive integer. Let $n \in \mathbb{Z}^+$ be a positive integer, then $\frac{X}{n} = \frac{1}{n} \odot X$

Limited addition. Let $X, Y \in L^*$. Then

$$X \oplus Y = (X[+]Y, \tilde{X}[+]\tilde{Y}). \tag{21}$$

It is clear that these operations are well defined, i.e. they always provide an element of $\mathbb{L}^*$.

Definition 9. Let Λ be an n-ary weighting vector, i.e. $\Lambda = (\lambda_1, \dots, \lambda_n) \in [0,1]^n$ such that $\sum_{i=1}^{n} \lambda_i = 1$. The n-dimensional interval-valued intuitionistic weighted average $\mathbb{L}^* - WA_\Lambda$ is given by

$$\mathbb{L}^* - WA_{\Lambda}(X_1, \dots, X_n) = \sum_{i=1}^{n} \lambda_i \odot X_i, \tag{22}$$

where the sum is w.r.t. the limited addition.

Lemma 2. Let $X, Y \in \mathbb{L}^*$ and $\lambda_1, \lambda_2 \in [0,1]$ such that $\lambda_1 + \lambda_2 \leq 1$. Then $\lambda_1 \odot X \oplus \lambda_2 \odot Y = (\lambda_1 X + \lambda_2 Y, \lambda_1 \tilde{X} + \lambda_2 \tilde{Y})$.

Proof. Straightforward from *Eqs. (7)*, *(20)* and *(21)*.

Lemma 3. Let Λ be a weighting vector. Then, $\widehat{\mathrm{wa}}_\Lambda(\mathrm{X}_1^c, \dots, \mathrm{X}_n^c)^c = \widehat{\mathrm{wa}}_\Lambda(\mathrm{X}_1, \dots, \mathrm{X}_n)$.

Proof. Straightforward from *Proposition 1* and the fact that $1 - wa_\Lambda(1 - x_1, \dots, 1 - x_n) = wa_\Lambda(x_1, \dots, x_n)$.

Theorem 6. Let Λ be a weighting vector. Then $\mathbb{L}^* - WA_\Lambda = \widehat{wa}_\Lambda$, i.e. is the best $\mathbb{L}^*$-representation of the weighted average operator.

Proof. First note that by the monotonicity of the weighted average operator, $\widehat{wa}_\Lambda(X_1, \dots, X_n) = [wa_\Lambda(\underline{X_1}, \dots, \underline{X_n}), wa_\Lambda(\overline{X_1}, \dots, \overline{X_n})] = \sum_{i=1}^n \lambda_i X_i$. So,

$$\begin{aligned}
\widehat{wa}_\Lambda(\mathrm{X}_1, \dots, \mathrm{X}_n) &= (\widehat{wa}_\Lambda(\mathrm{X}_1, \dots, \mathrm{X}_n), \widehat{wa}_\Lambda(\widetilde{\mathrm{X}_1^c}, \dots, \widetilde{\mathrm{X}_n^c})^c) \quad by\ Cor.\ 2 \\
&= (\widehat{wa}_\Lambda(\mathrm{X}_1, \dots, \mathrm{X}_n), \widehat{wa}_\Lambda(\widetilde{\mathrm{X}_1}, \dots, \widetilde{\mathrm{X}_n})) \quad by\ Lemma\ 23 \\
&= (\textstyle\sum_{i=1}^n \lambda_i \mathrm{X}_i, \sum_{i=1}^n \lambda_i \widetilde{\mathrm{X}_\imath}) \quad by\ Prop.\ 231 \\
&= \textstyle\sum_{i=1}^n \lambda_i \odot \mathrm{X}_i \quad by\ Lemma\ 2312\,.
\end{aligned}$$

Definition 10. Let Λ be an n-ary weighting vector, i.e. $\Lambda = (\lambda_1, \dots, \lambda_n) \in [0,1]^n$ such that $\sum_{i=1}^n \lambda_i = 1$. The n-dimensional interval-valued intuitionistic weighted addition $\mathbb{L}^* - OWA_\Lambda$ is given by

$$\mathbb{L}^* - \mathrm{OWA}_\Lambda(\mathrm{X}_1, \dots, \mathrm{X}_n) = \sum_{i=1}^n \lambda_i \odot \mathrm{X}_{\gamma(i)}, \tag{23}$$

where the sum is w.r.t. the limited addition and

$$\mathrm{X}_{\gamma(i)} = ([\nabla(\mathrm{X}_{\gamma_1(i)}), \triangle(\mathrm{X}_{\gamma_2(i)})], [\nabla(\widetilde{\mathrm{X}_{\gamma_3(\imath)}}), \triangle(\widetilde{\mathrm{X}_{\gamma_4(\imath)}})]). \tag{24}$$

with $\gamma_j\colon \{0,1 \dots, n\} \to \{0,1 \dots, n\}$ for $j = 1, \dots, 4$, being permutations such that $\nabla(\mathrm{X}_{\gamma_1(i)}) \geq \nabla(\mathrm{X}_{\gamma_1(i+1)})$, $\triangle(\mathrm{X}_{\gamma_2(i)}) \geq \triangle(\mathrm{X}_{\gamma_2(i+1)})$, $\nabla(\widetilde{\mathrm{X}_{\gamma_3(\imath)}}) \leq \nabla(\widetilde{\mathrm{X}_{\gamma_3(\imath+1)}})$ and $\triangle(\widetilde{\mathrm{X}_{\gamma_4(\imath)}}) \leq \triangle(\widetilde{\mathrm{X}_{\gamma_4(\imath+1)}})$ for any $i = 1, \dots, n-1$.

Lemma 4. Let Λ be a weighting vector. Then, $\widehat{owa}_\Lambda(X_1^c, \dots, X_n^c)^c = \widehat{owa}_{\Lambda^r}(X_1, \dots, X_n)$ where $\Lambda^r = (\lambda_n, \dots, \lambda_1)$.

Proof. Straightforward from *Proppsition 1* and the fact that $1 - owa_\Lambda(1 - x_1, \dots, 1 - x_n) = owa_{\Lambda^r}(x_1, \dots, x_n)$.

Theorem 7. Let Λ be a weighting vector. Then $\mathbb{L}^* - OWA_\Lambda = \widehat{owa}_\Lambda$, i.e. is the best $\mathbb{L}^*$-representation of the ordered weighted average operator.

Proof.

$$
\begin{array}{lll}
& \mathbb{L}^* - \mathrm{OWA}_{\Lambda}(\mathrm{X}_1, \dots, \mathrm{X}_n) & \\
= & \sum_{i=1}^{n} \lambda_i \odot \mathrm{X}_{\gamma(i)} & \text{by eq. (23)} \\
= & \mathbb{L}^* - \mathrm{WA}_{\Lambda}(\mathrm{X}_{\gamma(1)}, \dots, \mathrm{X}_{\gamma(n)}) & \text{by eq. (2322)} \\
= & \widehat{\mathrm{wa}}_{\Lambda}(\mathrm{X}_{\gamma(1)}, \dots, \mathrm{X}_{\gamma(n)}) & \text{by Thm. 23226} \\
= & (\widehat{\mathrm{wa}}_{\Lambda}(\mathrm{X}_{\gamma(1)}, \dots, \mathrm{X}_{\gamma(n)}), \widehat{\mathrm{wa}}_{\Lambda}(\widetilde{\mathrm{X}_{\gamma(1)}}^{c}, \dots, \widetilde{\mathrm{X}_{\gamma(n)}}^{c})^{c}) & \text{by eq. (2322618)} \\
= & (\widehat{\mathrm{wa}}_{\Lambda}(\mathrm{X}_{\gamma(1)}, \dots, \mathrm{X}_{\gamma(n)}), \widehat{\mathrm{wa}}_{\Lambda}(\widetilde{\mathrm{X}_{\gamma(1)}}, \dots, \widetilde{\mathrm{X}_{\gamma(n)}})) & \text{by Lemma 23226183} \\
= & (\widehat{\mathrm{wa}}_{\Lambda}([\nabla(\mathrm{X}_{\gamma_1(1)}), \triangle(\mathrm{X}_{\gamma_2(1)})], \dots, [\nabla(\mathrm{X}_{\gamma_1(n)}), \triangle(\mathrm{X}_{\gamma_2(n)})], & \\
& \widehat{\mathrm{wa}}_{\Lambda}([\nabla(\widetilde{\mathrm{X}_{\gamma_3(1)}}), \triangle(\widetilde{\mathrm{X}_{\gamma_4(1)}})], \dots, [\nabla(\widetilde{\mathrm{X}_{\gamma_3(n)}}), \triangle(\widetilde{\mathrm{X}_{\gamma_4(n)}})])) & \text{by eq. (2322618324)} \\
= & ([\mathrm{wa}_{\Lambda}(\nabla(\mathrm{X}_{\gamma_1(1)}), \dots, \nabla(\mathrm{X}_{\gamma_1(n)})), \mathrm{wa}_{\Lambda}(\triangle(\mathrm{X}_{\gamma_2(1)}), \dots, \triangle(\mathrm{X}_{\gamma_2(n)}))], & \\
& [\mathrm{wa}_{\Lambda}(\nabla(\widetilde{\mathrm{X}_{\gamma_3(1)}}), \dots, \nabla(\widetilde{\mathrm{X}_{\gamma_3(n)}})), \mathrm{wa}_{\Lambda}(\triangle(\widetilde{\mathrm{X}_{\gamma_4(1)}}), \dots, \triangle(\widetilde{\mathrm{X}_{\gamma_4(n)}}))]) & \text{by eq. (23226183245)} \\
= & ([\mathrm{wa}_{\Lambda}(\nabla(\mathrm{X}_{\gamma_1(1)}), \dots, \nabla(\mathrm{X}_{\gamma_1(n)})), \mathrm{wa}_{\Lambda}(\triangle(\mathrm{X}_{\gamma_2(1)}), \dots, \triangle(\mathrm{X}_{\gamma_2(n)}))], & \\
& [\mathrm{wa}_{\Lambda^r}(\nabla(\widetilde{\mathrm{X}_{\gamma_3(n)}}), \dots, \nabla(\widetilde{\mathrm{X}_{\gamma_3(1)}})), \mathrm{wa}_{\Lambda^r}(\triangle(\widetilde{\mathrm{X}_{\gamma_4(n)}}), \dots, \triangle(\widetilde{\mathrm{X}_{\gamma_4(1)}}))]) & \\
= & ([\mathrm{owa}_{\Lambda}(\nabla(\mathrm{X}_{\gamma(1)}), \dots, \nabla(\mathrm{X}_{\gamma(n)})), \mathrm{owa}_{\Lambda}(\triangle(\mathrm{X}_{\gamma(1)}), \dots, \triangle(\mathrm{X}_{\gamma(n)}))], & \\
& [\mathrm{owa}_{\Lambda^r}(\nabla(\widetilde{\mathrm{X}_{\gamma(1)}}), \dots, \nabla(\widetilde{\mathrm{X}_{\gamma(n)}})), \mathrm{owa}_{\Lambda^r}(\triangle(\widetilde{\mathrm{X}_{\gamma(1)}}), \dots, \triangle(\widetilde{\mathrm{X}_{\gamma(n)}}))]) & \text{by eq. (232261832456)} \\
= & (\widehat{\mathrm{owa}}_{\Lambda}(\mathrm{X}_{\gamma(1)}, \dots, \mathrm{X}_{\gamma(n)}), \widehat{\mathrm{owa}}_{\Lambda^r}(\widetilde{\mathrm{X}_{\gamma(1)}}, \dots, \widetilde{\mathrm{X}_{\gamma(n)}})) & \text{by eq. (2322618324565)} \\
= & (\widehat{\mathrm{owa}}_{\Lambda}(\mathrm{X}_1, \dots, \mathrm{X}_n), \widehat{\mathrm{owa}}_{\Lambda}(\widetilde{\mathrm{X}_1}^{c}, \dots, \widetilde{\mathrm{X}_n}^{c})^{c}) & \text{by Lemma 23226183245654} \\
= & \widehat{\mathrm{owa}}_{\Lambda}(\mathrm{X}_1, \dots, \mathrm{X}_n). & \text{by eq. (2322618324565418)}
\end{array}
$$

Corollary 4. $\mathbb{L}^* - OWA_{\Lambda}$ is an idempotent and symmetric n-ary interval-valued Atanassov's intuitionistic aggregation function. In addition, $\mathbb{L}^* - OWA_{\Lambda}$ is bounded, i.e. $\widehat{min} \leq_{\mathbb{L}^*} \mathbb{L}^* - OWA_{\Lambda} \leq_{\mathbb{L}^*} \widehat{max}$

Proof. Straightforward from *Theorems 7* and *5* and *Corollary 3*.

5| A Method for Multi-attribute Group Decision Making Based Interval-Valued Atanassov's Intuitionistic Decision Matrices

Let $E = \{e_1, \dots, e_m\}$ be a set of experts, $X = \{x_1, \dots, x_n\}$ be a finite set of alternatives, and $A = \{a_1, \dots, a_p\}$ be a set of attributes or criteria. The decision makers determines a weighting vector $W = (w_1, \dots, w_p)^T$ for the attributes. A method for MAGDM based on IVAIDM is an algorithm which determines a ranking of the alternatives in X based in the opinion of each expert in E of how much the alternatives attend each attribute. In particular we consider the case where the evaluation of the experts contains imprecision and hesitation which is represented by interval-valued Atanassov's intuitionistic degrees.

We propose the next method (algorithm) to obtain such ranking:

X, W, and for every $l = 1, \dots, m$ an $\mathbb{L}^*$-valued decision matrix R^l of dimension $n \times p$ where each position (i, j) in R^l, denoted by R^l_{ij}, contains the interval-valued Atanassov's intuitionistic value which reflects how much the alternative x_i attends the attribute (or criterium[1]) a_j.

A ranking $r: X \to \{1, \dots, n\}$, denoting that an alternative $x \in X$ is better than an alternative $y \in X$ whenever $r(x) \leq r(y)$ and when $r(x) = r(y)$ meaning that the method is not able of determine if x is better or worst alternative than y[2].

[1] For the case of the cost criteria is considered the usal complement of these interval-valued Atanassov's intuitionistic values.

[2] The most decision making methods admits cases for which the method is unable of discriminate between two different alternatives which is better.

Step 1. Aggregate the IVAIDM of all experts in a single IVAIDM $\mathcal{RC}$, for each $i = 1, \dots, n$ and $j = 1, \dots, p$, as follows:

$$\mathscr{RC}_{ij} = \widehat{\mathrm{owa}}_{\Lambda}\left(\mathrm{R}_{ij}^{1}, \dots, \mathrm{R}_{ij}^{m}\right). \tag{25}$$

where $\Lambda = (\lambda_1, \dots, \lambda_m)$ is the following weighting vector:

1. Case m is even: $\lambda_i = \frac{1}{2^{\frac{m}{2}+2-i}} + \frac{1}{m2^{\frac{m}{2}}}$ for each $i = 1, \dots, \frac{m}{2}$, and $\lambda_i = \lambda_{m+1-i}$ for each $i = \frac{m}{2} + 1, \dots, m$.
2. Case m is odd: $\lambda_i = \frac{1}{2^{\frac{m+1}{2}+2-i}} + \frac{1}{m2^{\frac{m+1}{2}}} + \frac{1}{4m}$ for each $i = 1, \dots, \frac{m+1}{2}$, and $\lambda_i = \lambda_{m+1-i}$ for each $i = \frac{m+1}{2} + 1, \dots, m$.

Table 1. Assesses of expert $\mathbf{p_1}$.

$\mathbf{R^1}$	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_3}$	$\mathbf{a_4}$
A_1	([0.4,0.8],[0.0,0.1])	([0.3,0.6],[0.0,0.2])	([0.2,0.7],[0.2,0.3])	([0.3,0.4],[0.4,0.5])
A_2	([0.5,0.7],[0.1,0.2])	([0.3,0.5],[0.2,0.4])	([0.4,0.7],[0.0,0.2])	([0.1,0.2],[0.7,0.8])
A_3	([0.5,0.7],[0.2,0.3])	([0.6,0.8],[0.1,0.2])	([0.4,0.7],[0.1,0.2])	([0.6,0.8],[0.0,0.2])

$\mathscr{RC}$ is the IVAIDM of consensus of all expert opinions[1].

Step 2. For each alternative x_i, with $i = 1, \dots, n$, using $\widehat{wa}_W$, determine the collective overall index $\mathbb{L}^*$-value O_i as follows:

$$\mathrm{O_i} = \widehat{\mathrm{wa}}_{\mathrm{W}}(\mathscr{RC}_{\mathrm{i1}}, \dots, \mathscr{RC}_{\mathrm{in}}). \tag{26}$$

Step 3. Rank the alternatives by considering a total order on their collective overall index $\mathbb{L}^*$-values and choosing the greatest one. Thus, the output function $r: X \to \{1, \dots, n\}$ is defined by $r(x_i) = j$ iff O_i is the jth greatest collective overall index. Notice that if two or more alternatives, e.g. x and y, have the same collective overall index, then $r(x) = r(y)$.

Example 1. Consider the air-condition system selection problem used as example in [62]. This problem considers three air-condition systems (alternatives) $\{A_1, A_2, A_3\}$; four attributes: a_1 (economical), a_2 (function), a_3 (being operative) and a_4 (longevity); and three experts $\{p_1, p_2, p_3\}$. By using statistical methods, for each expert p_l, alternative A_i and atribute a_j an interval-valued membership degree and an interval-valued non-membership degree, i.e. an IVAIFV, is provided. These IVAIFV are summarized in the *Tables 1*, *2* and *3* (the same used in [62]). We consider the weighting vector $W = (0.2134, 0.1707, 0.2805, 0.3354)$ for the attributes[2] .

Since we have three experts ($m = 3$), then the weighting vector Λ is calculated as following:

$$\lambda_1 = \frac{1}{2^3} + \frac{1}{3 \cdot 2^2} + \frac{1}{4 \cdot 3} = \frac{1}{8} + \frac{1}{6} = 0.291\overline{6},$$

$$\lambda_2 = \frac{1}{2^2} + \frac{1}{3 \cdot 2^2} + \frac{1}{4 \cdot 3} = \frac{1}{4} + \frac{1}{6} = 0.41\overline{6},$$

$$\lambda_3 = \frac{1}{2^3} + \frac{1}{3 \cdot 2^2} + \frac{1}{4 \cdot 3} = \frac{1}{8} + \frac{1}{6} = 0.291\overline{6}.$$

[1] It is not hard of prove that when $n > 1$, Λ is a weighting vector.

[2] In [62] was considered the weights $V = (0.35, 0.28, 0.46, 0.55)$ which not satisfy the condition that the sums of the weights must be equal to 1. W is the weighting vector obtained normalizing V in order to satisfy this condition.

Table 2. Assesses of expert p_2.

R^2	a_1	a_2	a_3	a_4
A_1	([0.5,0.9],[0.0,0.1])	([0.4,0.5],[0.3,0.5])	([0.5,0.8],[0.0,0.1])	([0.4,0.7],[0.1,0.2])
A_2	([0.7,0.8],[0.1,0.2])	([0.5,0.6],[0.2,0.3])	([0.5,0.8],[0.0,0.2])	([0.5,0.6],[0.3,0.4])
A_3	([0.5,0.6],[0.1,0.4])	([0.6,0.7],[0.1,0.2])	([0.4,0.8],[0.1,0.2])	([0.2,0.6],[0.2,0.3])

Table 3. Assesses of expert p_3.

R^3	a_1	a_2	a_3	a_4
A_1	([0.3,0.9],[0.0,0.1])	([0.2,0.5],[0.1,0.4])	([0.4,0.7],[0.1,0.2])	([0.3,0.6],[0.3,0.4])
A_2	([0.3,0.8],[0.1,0.2])	([0.5,0.6],[0.1,0.3])	([0.2,0.8],[0.0,0.2])	([0.3,0.5],[0.2,0.3])
A_3	([0.2,0.6],[0.1,0.2])	([0.2,0.6],[0.2,0.3])	([0.3,0.6],[0.1,0.3])	([0.4,0.7],[0.1,0.2])

The *Table 4* present the collective reflexive IvIFPR obtained from *Tables 1*, *2* and *3* by consider the *Eq. (25)*.

The collective overall preference obtained by using the calculation in *Eq. (26)*, is the following:

$$O_1 = ([0.3509555488,0.6721],[0.140916,0.2651]),$$

$$O_2 = ([0.3867014634,0.6262],[0.180441,0.3184]),$$

$$O_3 = ([0.4086795732,0.6848],[0.111192,0.2443]).$$

Thus, considering this collective overall preference and the total orders shows in section III.B, we have the ranking of the alternatives in the *Table 5*. Therefore, all the ranking obtained with this method, for the different the orders considered, agree with four of the five ranking obtained in [39], [62], [63], for this same illustrative example.

Example 2. Consider the investment choice problem used as example in [59], [60]. This problem considers an investment company which would like to invest a sum of money in the best option among the following five possible alternatives to invest the money: A_1 is a car company; A_2 is a food company; A_3 is a computer company; A_4 is an arms company; and A_5 is a TV company. The choice of the best investmente must be made taking into account the following four benefit criteria: c_1 is the profit ability; c_2 is the growth analysis; c_3 is the social-political impact; and c_4 is the enterprise culture. The five possible alternatives will be evaluated considering the interval-valued intuitionistic fuzzy information given by three decision makers e_1, e_2 and e_3, who evaluate how much the alternative satisfies each one of the criterias. These informations are summarized in the *Tables 6*, *7* and *8* (the same considered in [58], [59], [60]).

Since, in [59], [60] it was not considered a weight for the criteria, here we consider that all criteria have the same weight, i.e. we consider $W = (0.25,0.25,0.25,0.25)$. The ranking obtained by using our method considering the four total orders and the obtained by [59], [60] is summarized in the *Table 9*.

Table 4. Collective reflexive IvIFPR.

RC	a_1	a_2	a_3	a_4
A_1	([0.4,0.871],[0.0,0.1])	([0.3,0.53],[0.13,0.371])	([0.371,0.73],[0.1,0.2])	([0.33,0.571],[0.271,0.371])
A_2	([0.5,0.771],[0.1,0.2])	[0.4416,0.571],[0.171,0.33])	[0.371,0.771],[0.0,0.2])	[0.3,0.4416],[0.3875,0.4874])
A_3	[0.4125,0.63],[0.13,0.3])	([0.4833,0.7],[0.13,0.23])	([0.371,0.7],[0.1,0.23])	([0.4,0.7],[0.1,0.23])

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Table 5. Ranking obtained for the alternatives considering several total orders and the obtained in [62], [63].

$\lessapprox$	$\precsim$	$\preccurlyeq$	[62]	[63] (a), (b) and (d)	[63] (c)
A_3	A_3	A_3	A_3	A_3	A_2
A_1	A_1	A_1	A_1	A_1	A_1
A_2	A_2	A_2	A_2	A_2	A_3

Table 6. Assesses of expert e_1.

R^1	a_1	a_2	a_3	a_4
A_1	([0.4,0.5],[0.3,0.4])	([0.4,0.6],[0.2,0.4])	([0.1,0.3],[0.5,0.6])	([0.3,0.4],[0.3,0.5])
A_2	([0.6,0.7],[0.2,0.3])	([0.6,0.7],[0.2,0.3])	([0.4,0.7],[0.1,0.2])	([0.5,0.6],[0.1,0.3])
A_3	([0.6,0.7],[0.1,0.2])	([0.5,0.6],[0.3,0.4])	([0.5,0.6],[0.1,0.3])	([0.4,0.5],[0.2,0.4])
A_4	([0.3,0.4],[0.2,0.3])	([0.6,0.7],[0.1,0.3])	([0.3,0.4],[0.1,0.2])	([0.3,0.7],[0.1,0.2])
A_5	([0.7,0.8],[0.1,0.2])	([0.3,0.5],[0.1,0.3])	([0.5,0.6],[0.2,0.3])	([0.3,0.4],[0.5,0.6])

Table 7. Assesses of expert e_2.

R^2	a_1	a_2	a_3	a_4
A_1	([0.3,0.4],[0.4,0.5])	([0.5,0.6],[0.1,0.3])	([0.4,0.5],[0.3,0.4])	([0.4,0.6],[0.2,0.4])
A_2	([0.3,0.6],[0.3,0.4])	([0.4,0.7],[0.1,0.2])	([0.5,0.6],[0.2,0.3])	([0.6,0.7],[0.2,0.3])
A_3	([0.6,0.8],[0.1,0.2])	([0.5,0.6],[0.1,0.2])	([0.5,0.7],[0.2,0.3])	([0.1,0.3],[0.5,0.6])
A_4	([0.4,0.5],[0.3,0.5])	([0.5,0.8],[0.1,0.2])	([0.2,0.5],[0.3,0.4])	([0.4,0.7],[0.1,0.2])
A_5	([0.6,0.7],[0.2,0.3])	([0.6,0.7],[0.1,0.2])	([0.5,0.7],[0.2,0.3])	([0.6,0.7],[0.1,0.3])

Table 8. Assesses of expert e_3.

R^3	a_1	a_2	a_3	a_4
A_1	([0.2,0.5],[0.3,0.4])	([0.4,0.5],[0.1,0.2])	([0.3,0.6],[0.2,0.3])	([0.3,0.7],[0.1,0.3])
A_2	([0.2,0.7],[0.2,0.3])	([0.3,0.6],[0.2,0.4])	([0.4,0.7],[0.1,0.2])	([0.5,0.8],[0.1,0.2])
A_3	([0.5,0.6],[0.3,0.4])	([0.7,0.8],[0.1,0.2])	([0.5,0.6],[0.2,0.3])	([0.4,0.5],[0.3,0.4])
A_4	([0.3,0.6],[0.2,0.4])	([0.4,0.6],[0.2,0.3])	([0.1,0.4],[0.3,0.6])	([0.3,0.7],[0.1,0.2])
A_5	([0.6,0.7],[0.1,0.3])	([0.5,0.6],[0.3,0.4])	([0.5,0.6],[0.2,0.3])	([0.5,0.6],[0.2,0.4])

Table 9. Ranking obtained for the alternatives considering several total orders and the obtained in [62].

Proposed Method			The methods proposed in [59], [60]				
$\lessapprox$	$\precsim$	$\preccurlyeq$	[60]	[59] $\gamma < 0.378$	[59] $\gamma = 0.378$	[59] $0.378 < \gamma < 1$	[59] $\gamma = 1$
A_5	A_5	A_2	A_5	A_3	$A_3 \sim A_5$	A_5	A_5
A_2	A_2	A_5	A_2	A_5		A_3	$A_3 \sim A_2$
A_3	A_3	A_3	A_3	A_2	A_2	A_2	
A_4	A_4	A_4	A_4	A_4	A_4	A_4	A_4
A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1

Thus, making an analysis of these rankings of the alternatives we have that there is an absolute consensus that the worst alternative is *A1* and the second worst alternative is *A4*. On the other hand, if we consider, for the other alternatives, the amount of times that an alternative was better ranked than the others (which is summarized in the *Table 10*) we can conclude that the more rasonable ranking of the alternatives would be$A_5 > A_2 > A_3 > A_4 > A_1$

which agrees with the ranking obtained in [60] and also for the proposed method with the orders $\lessapprox$ and $\precsim$.

This way of aggregate or fuses many rankings of a set of alternatives corresponds to the ranking fusion function M2 of [20].

6| Final remarks

This paper proposes a new extension of the OWA and WA operators in the context of interval-valued intuitionistic fuzzy values, which has as main characteristic by the best $\mathbb{L}^*$-representation of the usual OWA and WA operators. Therefore, when applied to the diagonal elements these new operators have the same behaviour as the OWA and WA. This paper also extended the notion of interval representations introduced in [54] for $\mathbb{L}^*$-representations, and has introduced a new notion of inclusion for $\mathbb{L}^*$-values which is based in a notion of membership. Besides, we introduced a new total order for $\mathbb{L}^*$-values and provide new extensions of the OWA operator for $\mathbb{L}$ and L^*-values.

Table 10. Comparing based on the Table 9.

R^3	A_2	A_3	A_5
A_2	–	5	1
A_3	3	–	1
A_5	7	5	–

We have shown the validity of our theoretical develpments by means of an illustrative decision-making example. In [32] was introduced an interval-valued Atanassov's intuitionistic extension of OWA's where the weights are assigned by decreasingly ordering the inputs with respect to an admissible order. The problem with this OWA is that in general it is not increasing with respect to the admissible order. So, as future work we intend to investigate OWAs on $\mathbb{L}^*$ which are increasing with respect to a fixed admissible order. In addition, based on [17], we will use such OWAs in a method to select the most important vertice of an Interval-Valued Intuitionistic Fuzzy Graph [7].

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