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Pythagorean Fuzzy Weak Bi-Ideals of Γ- Near Ring

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Abstract

We present the notion of Pythagorean Fuzzy Weak Bi-Ideals (PFWBI) and interval valued Pythagorean fuzzy weak bi-ideals of Γ-near-rings and studies some of its properties. We present the notion of interval valued Pythagorean fuzzy weak bi-ideal and establish some of its properties. We study interval valued Pythagorean fuzzy weak bi-ideals of Γ-near-ring using homomorphism.

Keywords: Pythagorean fuzzy, Fuzzy ideals, Homomorphism, Near ring.

1 | Introduction

Zadeh [26] defined Fuzzy Set (FS) to deal with uncertainty. Atanassov [2] presented the notion of Intuitionistic FS (IFS) and studied some of its properties. Later, Yager [24], [25] defined and studied the properties of Pythagorean Fuzzy Set (PFS) and also used PFS to solve Multi-Criteria Decision-Making (MCDM) problems. Booth [3] presented the properties of Γ-near-rings. Chinnadurai and Kadalarasi [7] studied the near-ring properties of Fuzzy Weak Bi-Ideals (FWBI). Chinnadurai et al. [4],[5] studied the Γ-near-rings characterization of fuzzy weak bi-ideal and interval-valued fuzzy weak bi-ideal. Later, Chinnadurai et al. [6] discussed the Γ-near-rings properties of interval-valued fuzzy ideals.

Akram [1] established the properties of fuzzy lie algebras. Kim and Kim [13] studied the near-rings concept of fuzzy ideals. Kaviyarasu et al. [10]-[12] studied the different type of ideals in INK- algebras. Jun et al. [9] presented the notion of fuzzy ideals and studied their properties in Γ-near-rings. Manikantan [14] defined and studied some of the near-rings properties of fuzzy bi-ideals. Meenakumari and Chelvam [15] presented the Γ-near-ringsproperties of fuzzy bi-ideals. Narayanan and Manikatan [16] introduced the near-rings notions of fuzzy subnear-ring, fuzzy ideal, and fuzzy quasi-ideal.

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Pilz [17] introduced the concept of anti fuzzy soft gamma rings and studied their properties. Rao and Swaminathan [20] presented the notion of anti-homomorphism between two fuzzy rings and established its properties. Rao and Venkateswarlu [21] studied the properties of anti fuzzy ideal and pre-image of fuzzy ideal. Satyanarayana [18] dealt with the theory of near-rings. Salah Abou-Zaid [19] studied fuzzy ideals of a near-ring. Chelvam and Meenakumari [22] obtained the characterization for gamma near-fields. Thillaigovindan et al. [23] introduced the notion of generalized T-fuzzy bi-ideals of a gamma-semigroup. Cho et al. [8] presented the notion of bi-ideals in near-rings and used it in near-fields.

We introduce the notion of Pythagorean fuzzy weak bi-ideal of Γ-near-rings and interval valued Pythagorean fuzzy weak bi-ideal of Γ-near-rings. We discuss and present some properties of homomorphism of Pythagorean fuzzy weak bi-ideal and homomorphism of an interval valued Pythagorean fuzzy weak bi-ideal in gamma near-ring.

2| Preliminaries

Definition 1. [16]. A fuzzy set π of a Γ-near-ring M is called a fuzzy left (resp. right) ideal of M if

I. $\pi(k-l) \geq min\{\pi(k), \pi(l)\}$, *for all* $k, l \in M$,
II. $\pi(y+x-y) \geq \pi(x)$, *for all* $x, y \in M$,
III. $\pi(u\alpha(x+v)-u\alpha v) \geq \pi(x)$, (*resp.* $\pi(x\alpha u) \geq \pi(x)$) *for all* $x, u, v \in M$ *and* $\alpha \in \Gamma$.

Definition 2. [15]. A fuzzy set π of M is called a fuzzy bi-ideal of M if

I. $\pi(x-y) \geq min\{\pi(x), \pi(y)\}$ for all $x, y \in M$,
II. $\pi(x\alpha y\beta z) \geq min\{\pi(x), \pi(z)\}$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

Definition 3. [2]. An intuitionistic fuzzy set A is a nonempty set X is an object having the form $A = \{x, (\pi_A(x), \vartheta_A(x))\colon x \in X\}$ where the functions $\pi_A : X \to [0,1]$ and $\vartheta_A : X \to [0,1]$ define the degree of membership and non-membership of the element $x \in X$ to the set A, which is a subset of X respectively $0 \leq \pi_A(x) + \vartheta_A(x) \leq 1$ we use the simple $A = (\pi_A, \vartheta_A)$.

Definition 4. [25]. A Pythagorean fuzzy subset P is a nonempty set X is an object having the form $P = \{(x, \pi_P(x), \vartheta_P(x))/x \in X\}$, where the functions $\pi_P\colon X \to [0,1]$ and $\vartheta_P\colon X \to [0,1]$ denote the degree of membership and non membership of each element $x \in X$ to the set P, respectively, and $0 \leq (\pi_P(x))^2 + (\vartheta_P(x))^2 \leq 1$ for all $x \in X$. For the sake of simplicity, for the Pythagorean fuzzy subset $P = \{(x, \pi_P(x), \vartheta_P(x))\ /x \in X\}$.

3| Pythagorean Fuzzy Weak Bi-Ideals of Γnear Ring

In this section, we initiate the notion of Pythagorean fuzzy weak bi-ideal of M and discuss some of its properties.

Definition 4. A subgroup W of $(M, +)$ is said to be a weak bi-ideal of M if $W\Gamma W\Gamma W \subseteq W$.

Definition 5. A Pythagorean fuzzy set$P = (\pi_P, \vartheta_P)$ of M is called a Pythagorean fuzzy weak bi-ideal of M, if

I. $\pi_P(x-y) \geq min\{\pi_P(x), \pi_P(y)\}$.
II. $\vartheta_P(x-y) \leq max\{\vartheta_P(x), \vartheta_P(y)\}$.
III. $\pi_P(x\gamma y\gamma z) \geq min\{\pi_P(x), \pi_P(y), \pi_P(z)\}$.
IV. $\vartheta_P(x\gamma y\gamma z) \leq max\{\vartheta_P(x), \vartheta_P(y), \vartheta_P(z)\}$ *for all* $x, y, z \in M$ *and* $\alpha, \beta \in \Gamma$.

Example 1. Let $M = \{w, x, y, z\}$ be a nonempty set with binary operation ` $+$ ' and $\Gamma = \{\gamma\}$ be a nonempty set of binary operations as the following tables:

+	w	x	y	z
w	w	x	y	z
x	x	w	z	y
y	y	z	w	x
z	z	y	x	w

and

γ	w	x	y	z
w	w	x	w	x
x	w	x	w	x
y	w	x	y	z
z	w	x	y	z

Let $\pi_P : M \to [0,1]$ be a Pythagorean fuzzy subset defined by $\pi_P(w) = 0.7, \pi_P(x) = 0.6, \pi_P(y) = \pi_P(z) = 0.5$. and $\vartheta_P(w) = 0.3, \vartheta_P(x) = 0.5, \vartheta_P(y) = 0.8 = \vartheta_P(z)$. Then $P = (\pi_P, \vartheta_P)$ is a Pythagorean fuzzy weak bi-ideal of M.

Theorem 1. Let $P = (\pi_P, \vartheta_P)$ be a Pythagorean fuzzy subgroup of M. Then $P = (\pi_P, \vartheta_P)$ is a Pythagorean fuzzy weak bi-ideal of M if and only if $\pi_P \star \pi_P \star \pi_P \subseteq \pi_P$ and $\vartheta_P \star \vartheta_P \star \vartheta_P \supseteq \vartheta_P$.

Proof. Assume that $P = (\pi_P, \vartheta_P)$ be a Pythagorean fuzzy weak bi-ideal of M. Let $x, y, z, y_1, y_2 \in M$ and $\alpha, \beta \in \Gamma$ such that $x = y\alpha z$ and $y = y_1\beta y_2$. Then

$$(\pi_P \star \pi_P \star \pi_P)(x) = \sup_{x=y\alpha z} \{\min\{(\pi_P \star \pi_P)(y), \pi_P(z)\}\}$$

$$= \sup_{x=y\alpha z} \{\min\{ \sup_{y=y_1\beta y_2} \min\{\pi_P(y_1), \pi_P(y_2)\}, \pi_P(z)\}\}$$

$$= \sup_{x=y\alpha z} \sup_{y=y_1\beta y_2} \{\min\{\min\{\pi_P(y_1), \pi_P(y_2)\}, \pi_P(z)\}\}$$

$$= \sup_{x=y_1\beta y_2\alpha z} \left\{\min\left\{\pi_P(y_1), \pi_P(y_2), \pi_P(z)\right\}\right\},$$

since π_P is a fuzzy weak bi-ideal of M,

$$\pi_P(y_1\beta y_2\alpha z) \geq \min\{\pi_P(y_1), \pi_P(y_2), \pi_P(z)\}$$

$$\leq \sup_{x=y_1\beta y_2\alpha z} \pi_P(y_1\beta y_2\alpha z)$$

$$= \pi_P(x).$$

And

$$(\vartheta_P \star \vartheta_P \star \vartheta_P)(x) = \inf_{x=y\alpha z} \{\min\{(\vartheta_P \star \vartheta_P)(y), \vartheta_P(z)\}\},$$

$$= \inf_{x=y\alpha z} \{\max\{ \inf_{y=y_1\beta y_2} \min\{\vartheta_P(y_1), \vartheta_P(y_2)\}, \vartheta_P(z)\}\}$$

$$= \inf_{x=y\alpha z} \sup_{y=y_1\beta y_2} \{\max\{\max\{\vartheta_P(y_1), \vartheta_P(y_2)\}, \vartheta_P(z)\}\}$$

$$= \inf_{x=y_1\beta y_2\alpha z} \left\{\max\left\{\vartheta_P(y_1), \vartheta_P(y_2), \vartheta_P(z)\right\}\right\},$$

since ϑ_P is a fuzzy weak bi-ideal of M,

$$\vartheta_P(y_1\beta y_2\alpha z) \leq \max\{\vartheta_P(y_1), \vartheta_P(y_2), \vartheta_P(z)\}$$

$$\geq \inf_{x=y_1\beta y_2\alpha z} \vartheta_P(y_1\beta y_2\alpha z)$$

$$= \vartheta_P(x).$$

If x can not be expressed as $x = y\alpha z$, then $(\pi_P \star \pi_P \star \pi_P)(x) = 0 \leq \pi_P(x)$ and

$(\vartheta_P \star \vartheta_P \star \vartheta_P)(x) = 0 \geq \vartheta_P(x)$. In both cases $\pi_P \star \pi_P \star \pi_P \subseteq \pi_P$, and $\vartheta_P \star \vartheta_P \star \vartheta_P \supseteq \vartheta_P$.

Conversely, assume that $\pi_P \star \pi_P \star \pi_P \subseteq \pi_P$. For $x\prime, x, y, z \in M$ and $\alpha, \beta, \alpha_1, \beta_1 \in \Gamma$.

Let $x\prime$ be such that $x\prime = x\alpha y\beta z$.

Then $\pi_P(x\alpha y\beta z) = \pi_P(x\prime) \geq (\pi_P \star \pi_P \star \pi_P)(x\prime)$

$$= \sup_{x\prime=p\alpha_1 q} \{\min\{(\pi_P \star \pi_P)(p), \pi_P(q)\}\}$$

$$= \sup_{x\prime=p\alpha_1 q} \{\min\{\sup_{p=p_1\beta_1 p_2} \min\{\pi_P(p_1), \pi_P(p_2)\}, \pi_P(q)\}\}$$

$$= \sup_{x'=p_1\beta_1 p_2\alpha_1 q} \left\{\min\left\{\pi_P(p_1), \pi_P(p_2), \pi_P(q)\right\}\right\}$$

$$\geq \min\{\pi_P(x), \pi_P(y), \pi_P(z)\}.$$

$$\vartheta_P(x\alpha y\beta z) = \vartheta_P(x\prime) \leq (\vartheta_P * \vartheta_P * \vartheta_P)(x\prime)$$

$$= \inf_{x\prime=p\alpha_1 q} \{\max\{(\vartheta_P * \vartheta_P)(p), \vartheta_P(q)\}\}$$

$$= \inf_{x\prime=p\alpha_1 q} \{\max\{\inf_{p=p_1\beta_1 p_2} \min\{\vartheta_P(p_1), \vartheta_P(p_2)\}, \vartheta_P(q)\}\}$$

$$= \inf_{x\prime=p_1\beta_1 p_2\alpha_1 q} \{\max\{\vartheta_P(p_1), \vartheta_P(p_2), \vartheta_P(q)\}\}$$

$$\leq \max\{\vartheta_P(x), \vartheta_P(y), \vartheta_P(z)\}.$$

Hence $\pi_P(x\alpha y\beta z) \geq min\{\pi_P(x), \pi_P(y), \pi_P(z)\}$, and $\vartheta_P(x\alpha y\beta z) \leq max\{\vartheta_P(x), \vartheta_P(y), \vartheta_P(z)\}$.

Lemma 1. Let $\pi_P = (\pi_{P1}, \pi_{P2})$ and $\vartheta_P = (\vartheta_{P1}, \vartheta_{P2})$ be Pythagorean fuzzy weak bi-ideals of M. Then the products $\pi_P * \vartheta_P$ and $\vartheta_P * \pi_P$ are also Pythagorean fuzzy weak bi-ideals of M.

Proof. Let π_P and ϑ_P be a Pythagorean fuzzy weak bi-ideals of M and let $\alpha, \alpha_1, \alpha_2 \in \Gamma$. Then

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$$(\pi_{P1} \star \pi_{P2})(x-y) = \sup_{x-y=a\alpha b} \min\{\pi_{P1}(a), \pi_{P2}(b)\}$$

$$\geq \sup_{x-y=a_1\alpha_1 b_1 - a_2\alpha_2 b_2 < (a_1-a_2)(b_1-b_2)} \min\{\pi_{P1}(a_1-a_2), \pi_{P2}(b_1-b_2)\}$$

$$\geq \sup\min\{\min\{\pi_{P1}(a_1), \pi_{P1}(a_2)\}, \min\{\pi_{P2}(b_1), \pi_{P2}(b_2)\}\}$$

$$= \sup\min\{\min\{\pi_{P1}(a_1), \pi_{P2}(b_1)\}, \min\{\pi_{P1}(a_2), \pi_{P2}(b_2)\}\}$$

$$\geq \min\{\sup_{x=a_1\alpha_1 b_1} \min\{\pi_{P1}(a_1), \pi_{P2}(b_1)\}, \sup_{y=a_2\alpha_2 b_2} \min\{\pi_{P1}(a_2), \pi_{P2}(b_2)\}\}$$

$$= \min\{(\pi_{P1} \star \pi_{P2})(x), (\pi_{P1} \star \pi_{P2})(y)\}.$$

$$(\vartheta_{P1} \star \vartheta_{P2})(x-y) = \inf_{x-y=a\alpha b} \max\{\vartheta_{P1}(a), \vartheta_{P2}(b)\}$$

$$\leq \inf_{x-y=a_1\alpha_1 b_1 - a_2\alpha_2 b_2 < (a_1-a_2)(b_1-b_2)} \max\{\vartheta_{P1}(a_1-a_2), \vartheta_{P2}(b_1-b_2)\}$$

$$\leq \inf\max\{\max\{\vartheta_{P1}(a_1), \vartheta_{P1}(a_2)\}, \max\{\vartheta_{P2}(b_1), \vartheta_{P2}(b_2)\}\}$$

$$= \inf\max\{\max\{\vartheta_{P1}(a_1), \vartheta_{P2}(b_1)\}, \max\{\vartheta_{P1}(a_2), \vartheta_{P2}(b_2)\}\}$$

$$\leq \max\{\inf_{x=a_1\alpha_1 b_1} \max\{\vartheta_{P1}(a_1), \vartheta_{P2}(b_1)\}, \inf_{y=a_2\alpha_2 b_2} \max\{\vartheta_{P1}(a_2), \vartheta_{P2}(b_2)\}\}$$

$$= \max\{(\vartheta_{P1} \star \vartheta_{P2})(x), (\vartheta_{P1} \star \vartheta_{P2})(y)\}.$$

It follows that $\pi_P * \vartheta_P$ is a Pythagorean fuzzy subgroup of M. Further,

$$(\pi_P \star \vartheta_P) \star (\pi_P \star \vartheta_P) \star (\pi_P \star \vartheta_P) = \pi_P \star \vartheta_P \star (\pi_P \star \vartheta_P \star \pi_P) \star \vartheta_P$$

$$\subseteq \pi_P \star \vartheta_P \star (\vartheta_P \star \vartheta_P \star \vartheta_P) \star \vartheta_P$$

$$\subseteq \pi_P \star (\vartheta_P \star \vartheta_P \star \vartheta_P),$$

since P is a Pythagorean fuzzy weak bi-ideal of $M \subseteq \pi_P \star \vartheta_P$.

Therefore $\pi_P * \vartheta_P$ is a Pythagorean fuzzy weak bi-ideal of M. Similarly $\vartheta_P * \pi_P$ is a Pythagorean fuzzy weak bi-ideal of M.

Lemma 2. Every Pythagorean fuzzy ideal of M is a Pythagorean fuzzy bi-ideal of M.

Proof. Let $P = (\pi_P, \vartheta_P)$ be a Pythagorean fuzzy ideal of M. Then

$$\pi_P \star M \star \pi_P \subseteq \pi_P \star M \star M \subseteq \pi_P \star M \subseteq \pi_P,$$

$$\vartheta_P \star M \star \vartheta_P \supseteq \vartheta_P \star M \star M \supseteq \vartheta_P \star M \supseteq \vartheta_P,$$

since $P = (\pi_P, \vartheta_P)$ be a Pythagorean fuzzy ideal of M.

This implies that $\pi_P \star M \star \pi_P \subseteq \pi_P$ and $\vartheta_P \star M \star \vartheta_P \supseteq \vartheta_P$.

Therefore $P = (\pi_P, \vartheta_P)$ be a Pythagorean fuzzy bi-ideal of M.

Theorem 2. Every Pythagorean fuzzy bi-ideal of M is a Pythagorean fuzzy weak bi-ideal of M.

Proof. Assume that $P = (\pi_P, \vartheta_P)$ be a Pythagorean fuzzy bi-ideal of M.

Then $\pi_P \star \mathrm{M} \star \pi_P \subseteq \pi_P$ and $\vartheta_P \star \mathrm{M} \star \vartheta_P \supseteq \vartheta_P$.

We have $\pi_P \star \pi_P \star \pi_P \subseteq \pi_P \star \mathrm{M} \star \pi_P$ and $\vartheta_P \star \vartheta_P \star \vartheta_P \supseteq \vartheta_P \star \mathrm{M} \star \vartheta_P$.

This implies that $\pi_P \star \pi_P \star \pi_P \subseteq \pi_P \star \mathrm{M} \star \pi_P \subseteq \pi_P$

and $\vartheta_P \star \vartheta_P \star \vartheta_P \supseteq \vartheta_P \star \mathrm{M} \star \vartheta_P \supseteq \vartheta_P$.

Therefore $P = (\pi_P, \vartheta_P)$ is a Pythagorean fuzzy weak bi-ideal of M.

Example 2. Let $M = \{w, x, y, z\}$ be a nonempty set with binary operation $+$ and $\Gamma = \{\alpha\}$ be a nonempty set of binary operations as the following tables:

+	w	x	y	z
w	w	x	y	z
x	x	w	z	y
y	y	z	w	x
z	z	y	x	w

and

α	w	x	y	z
w	w	w	w	w
x	w	x	w	x
y	w	w	y	y
z	w	x	y	z

Let $\pi_P : M \to [0,1]$ be a fuzzy set defined by $\pi_P(w) = 0.9$, $\pi_P(x) = 0.4 = \pi_P(y)$ and $\pi_P(z) = 0.6$, and $\vartheta_P(w) = 0.1, \vartheta_P(x) = 0.5 = \vartheta_P(y), \vartheta_P(z) = 0.3$. Then π_P is a fuzzy weak bi-ideal of M. But π_P is not a fuzzy ideal and bi-ideal of M and $\pi_P(z\gamma y\gamma z) = \pi_P(y) = 0.4 > 0.6 = min\{\pi_P(z), \pi_P(z)\}$ and $\vartheta_P(x\alpha(z+w) - x\alpha w) \leq \vartheta_P(z) = 0.5 \nleq 0.3$ and $\vartheta_P(z\gamma x\gamma z) = \vartheta_P(x) = 0.5 \nleq 0.3 = \min\{\vartheta_P(z), \vartheta_P(z)\}$.

Theorem 3. Let $\{(\pi_{P_i}, \vartheta_{P_i}) | i \in \Omega\}$ be family of Pythagorean fuzzy weak bi-ideals of a near-ring M, then $\bigcap_{i\in\Omega} \pi_{P_i}$ and $\cup_{i\in\Omega} \vartheta_{P_i}$ are also a Pythagorean fuzzy weak bi-idea of M, where Ω is any index set.

Proof. Let $\{\pi_i\}_{i\in\Omega}$ be a family of Pythagorean fuzzy weak bi-ideals of M.

Let $x, y, z \in M, \alpha, \beta \in \Gamma$ and $\pi = \bigcap_{i\in\Omega} \pi_i$.

Then, $\bigcap_{i\in\Omega} \pi_{P_i}(x) = \bigcap_{i\in\Omega} \pi_{P_i}(x) = (\inf_{i\in\Omega} \pi_{P_i})(x) = \inf_{i\in\Omega} \pi_{P_i}(x)$ and

$\cup_{i\in\Omega} \vartheta_{P_i}(x) = \cup_{i\in\Omega} \vartheta_{P_i}(x) = (\sup_{i\in\Omega} \vartheta_{P_i})(x) = \sup_{i\in\Omega} \vartheta_{P_i}(x)$.

$$\bigcap_{i\in\Omega} \pi_{P_i}(x-y) = \inf_{i\in\Omega} \pi_{P_i}(x-y)$$

$$\geq \inf_{i\in\Omega}\min\{\pi_{P_i}(x), \pi_{P_i}(y)\}$$

$$= \min\{\inf_{i\in\Omega}\pi_{P_i}(x), \inf_{i\in\Omega}\pi_{P_i}(y)\}$$

$$= \min\{\bigcap_{i\in\Omega} \pi_{P_i}(x), \bigcap_{i\in\Omega} \pi_{P_i}(y)\}$$

$$\cup_{i\in\Omega} \vartheta_{P_i}(x-y) = \sup_{i\in\Omega}\pi_{P_i}(x-y)$$

$$\leq \sup_{i\in\Omega}\max\{\vartheta_{P_i}(x), \vartheta_{P_i}(y)\}$$

$$= \max\{\sup_{i\in\Omega}\vartheta_{P_i}(x), \sup_{i\in\Omega}\vartheta_{P_i}(y)\}$$

$$= \max\{\cup_{i\in\Omega} \vartheta_{P_i}(x), \cup_{i\in\Omega} \vartheta_{P_i}(y)\}.$$

And,

$$\bigcap_{i\in\Omega} \pi_{P_i}(x\alpha y\beta z) = \inf_{i\in\Omega}\pi_{P_i}(x\alpha y\beta z)$$

$$\geq \inf_{i\in\Omega}\min\{\pi_{P_i}(x), \pi_{P_i}(y), \pi_{P_i}(z)\}$$

$$= \min\{\inf_{i\in\Omega}\pi_{P_i}(x), \inf_{i\in\Omega}\pi_{P_i}(y), \inf_{i\in\Omega}\pi_{P_i}(z)\}$$

$$= \min\{\bigcap_{i\in\Omega} \pi_{P_i}(x), \bigcap_{i\in\Omega} \pi_{P_i}(y), \bigcap_{i\in\Omega} \pi_{P_i}(z)\}$$

$$\cup_{i\in\Omega} \vartheta_{P_i}(x\alpha y\beta z) = \sup_{i\in\Omega}\vartheta_{P_i}(x\alpha y\beta z)$$

$$\leq \sup_{i\in\Omega}\max\{\vartheta_{P_i}(x), \vartheta_{P_i}(y), \vartheta_{P_i}(z)\}$$

$$= \max\{\sup_{i\in\Omega}\vartheta_{P_i}(x), \sup_{i\in\Omega}\vartheta_{P_i}(y), \sup_{i\in\Omega}\vartheta_{P_i}(z)\}$$

$$= \max\{\cup_{i\in\Omega} \vartheta_{P_i}(x), \cup_{i\in\Omega} \vartheta_{P_i}(y), \cup_{i\in\Omega} \vartheta_{P_i}(z)\}.$$

Hence the set $\bigcap_{i\in\Omega} \pi_{P_i}$and $\cup_{i\in\Omega} \vartheta_{P_i}$ are also a family of Pythagorean fuzzy weak bi-ideals of M.

Theorem 4. Let $P = (\pi_P, \vartheta_P)$ be a Pythagorean fuzzy subset of M. Then $U(\pi_P;t)$ and $L(\vartheta_P;s)$ is a Pythagorean fuzzy weak bi-ideal of M if and only if π_{P_t} is a weak bi-ideal of M, for all $t \in [0,1]$.

Proof. Assume that $P = (\pi_P, \vartheta_P)$ is a Pythagorean fuzzy weak bi-ideal of M.

Let $s, t \in [0,1]$ such that $x, y \in U(\pi_P; t)$.

Then $\pi_P(x) \geq t$ and $\pi_P(y) \geq t$,

Then $\pi_P(x-y) \geq min\{\pi_P(x), \pi_P(y)\} \geq min\{t,t\} = t$ and

$\vartheta_P(x-y) \leq max\{\vartheta_P(x), \vartheta_P(y)\} \leq max\{s,s\} = s.$

Thus $x-y \in U(\pi_P \ \ t)$.Let $x, y, z \in \pi_{Pt}$ and $\alpha, \beta \in \Gamma$.

This implies that $\pi_P(x\alpha y\beta z) \geq min\{\pi_P(x), \pi_P(y), \pi_P(z)\} \geq min\{t, t, t\} = t$, and

$\vartheta_P(x\alpha y\beta z) \leq max\{\vartheta_P(x), \vartheta_P(y), \vartheta_P(z)\} \leq max\{s, \ s, \ s\} = s$.

Therefore $x\alpha y\beta z \in U(\pi_P; s)$.

Hence $U\ (\pi_P; t)$ and $(\vartheta_P; s)$ is a weak bi-ideal of M.

Conversely, assume that $U\ (\pi_P; t)$ and $(\vartheta_P; s)$ is a weak bi-ideal of M, for all $s, t \in [0,1]$.

Let $x, y \in M$. Suppose

$\pi_P(x-y) < min\{\pi_P(x), \pi_P(y)\} and\ \vartheta_P(x-y) > max\{\vartheta_P(x), \vartheta_P(y)\}$.

Choose t such that $\pi(x-y) < t < min\{\pi_P(x), \pi_P(y)\}$ and $\vartheta_P(x-y) > s > max\{\vartheta_P(x), \vartheta_P(y)\}$.

This implies that $\pi_P(x) > t, \pi_P(y) > t$ and $\pi_P(x-y) < t$.

Then we have $x, y \in \pi_{Pt}$ but $x-y \notin \pi_{Pt}$ and $\vartheta_P(x) < s, \vartheta_P(y) < s$ and $\vartheta_P(x-y) > s$, we have $x, y \in \vartheta_{Ps}$ but $x-y \notin \vartheta_{Ps}$ a contradiction.

Thus $\pi_P(x-y) \geq min\{\pi_P(x), \pi_P(y)\}$ and $\vartheta_P(x-y) \leq max\{\vartheta_P(x), \vartheta_P(y)\}$.

If there exist $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ such that $\pi_P(x\alpha y\beta z) < min\{\pi_P(x), \pi_P(y), \pi_P(z)\}$ and $\vartheta_P(x\alpha y\beta z) > max\{\vartheta_P(x), \vartheta_P(y), \vartheta_P(z)\}$.

Choose t such that $\pi_P(x\alpha y\beta z) < t < min\{\pi_P(x), \pi_P(y), \pi_P(z)\}$.

Choose s such that $\vartheta_P(x\alpha y\beta z) > s > max\{\vartheta_P(x), \vartheta_P(y), \vartheta_P(z)\}$.

Then $\pi_P(x) > t, \pi_P(y) > t, \pi_P(z) > t$ and $\vartheta_P(x) < s, \vartheta_P(y) < s, \vartheta_P(z) < s$ and $\pi_P(x\alpha y\beta z) < t$.

So, $x, y, z \in \pi_P >$ but $x\alpha y\beta z \notin \pi_{Pt}$, and $x\alpha y\beta z \notin \vartheta_{Ps}$, which is a contradiction.

Hence $\pi_P(x\alpha y\beta z) \geq min\{\pi_P(x), \pi_P(y), \pi_P(z)\}, \ \vartheta_P(x\alpha y\beta z) \leq max\{\vartheta_P(x), \vartheta_P(y), \vartheta_P(z)\}$.

Therefore $P = (\pi_P, \vartheta_P)$ is a Pythagorean fuzzy weak bi-ideal of M.

Theorem 5. Let $P = (\pi_P, \vartheta_P)$ be a Pythagorean fuzzy weak bi-ideal of M then the set $M_{\pi,\vartheta} = \{x \in M | \pi_P(x) = \pi_P(0) = \vartheta_P(x)\}$ is aweak bi-ideal of M.

Proof. Let $x, y \in M_{(\pi_P, \vartheta_P)}$. Then $\pi_P(x) = \pi_P(0), \pi_P(y) = \pi_P(0), \vartheta_P(x) = 0, \vartheta_P(y) = 0$ and

$$\pi_P(x-y) \geq min\{\pi_P(x), \pi_P(y)\}$$

$$= \min\{\pi_P(0), \pi_P(0)\}$$

$$= \pi_P(0), \text{ and}$$

$$\vartheta_P(x-y) \leq \max\{\vartheta_P(x), \vartheta_P(y)\}$$

$$= \max\{\vartheta_P(0), \vartheta_P(0)\}$$

$= \vartheta_P(0).$

So $\pi_P(x-y) = \pi_P(0), \vartheta_P(x-y) = \vartheta_P(0)$.

Thus $x-y \in M_{\pi_P}$, $x-y \in M_{\vartheta_P}$. For every $x, y, z \in M_{\pi_P}$ and $\alpha, \beta \in \Gamma$. We have

$$\pi_P(x\alpha y\beta z) \geq \min\{\pi_P(x), \pi_P(y), \pi_P(z)\}$$

$$= \min\{\pi_P(0), \pi_P(0), \pi_P(0)\}$$

$$= \pi_P(0).$$

And

$$\vartheta_P(x\alpha y\beta z) \leq \max\{\vartheta_P(x), \vartheta_P(y), \vartheta_P(z)\}$$

$$= \max\{\vartheta_P(0), \vartheta_P(0), \vartheta_P(0)\}$$

$$= \vartheta_P(0).$$

Thus $x\alpha y\beta z \in M_{\pi_P}$, $x\alpha y\beta z \in M_{\vartheta_P}$. Hence $M_{(\pi_P, \vartheta_P)}$ is a weak bi-ideal of M.

4| Homomorphism of Pythagorean Fuzzy Weak Bi-Ideals of Γ-Near-Rings

In this section, we characterize Pythagorean fuzzy weak bi-ideals of Γ-near-rings using homomorphism.

Definition 6. Let f be a mapping from a set M to a set S. Let $f = (\pi_P, \vartheta_P)$ be a Pythagorean fuzzy subsets of M and S, resp. then f is image of π_P and ϑ_P under f is a fuzzy subset of S defined by

$$f(\pi_P)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \pi_P(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

$$f(\vartheta_P)(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \vartheta_P(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 1 & \text{otherwise} \end{cases}$$

And the pre-image of π_P and ϑ_P under f is a fuzzy subset of M defined by

$f^{-1}(\pi_P(x)) = \pi_P(f(x))$, $f^{-1}(\vartheta_P(x)) = \vartheta_P(f(x))$ *for all* $x \in M$ *and* $f^{-1}(y) = \{x \in M | f(x) = y\}$.

Theorem 6. Let $f: M \to S$ be a homomorphism between Γ-near-rings M and S. If $P = (\pi_P, \vartheta_P)$ is a Pythagorean fuzzy weak bi-ideal of S, then $f^{-1}(P) = [f^{-1}(\pi_P, \vartheta_P)]$ is a Pythagorean fuzzy weak bi-ideal of M.

Proof. Let f be a Pythagorean fuzzy weak bi-ideal of M. Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then

$$f^{-1}(\pi_P)(x-y) = \pi_P(f(x-y))$$

$$= \pi_P(f(x) - f(y))$$

$$\geq \min\{\pi_P(f(x)), \pi_P(f(y))\}$$

$$= \min\{f^{-1}(\pi_P(x)), f^{-1}(\pi_P(y))\}.$$

$$f^{-1}(\vartheta_P)(x - y) = \vartheta_P(f(x - y))$$

$$= \vartheta_P(f(x) - f(y))$$

$$\leq \max\{\vartheta_P(f(x)), \vartheta_P(f(y))\}$$

$$= \max\{f^{-1}(\vartheta_P(x)), f^{-1}(\vartheta_P(y))\}.$$

$$f^{-1}(\pi_P)(x\alpha y\beta z) = \pi_P(f(x\alpha y\beta z))$$

$$= \pi_P(f(x)\alpha f(y)\beta f(z))$$

$$\geq \min\{\pi_P(f(x)), \pi_P(f(y)), \pi_P(f(z))\}$$

$$= \min\{f^{-1}(\pi_P(x)), f^{-1}(\pi_P(y)), f^{-1}(\pi_P(z))\}.$$

$$f^{-1}(\vartheta_P)(x\alpha y\beta z) = \vartheta_P(f(x\alpha y\beta z))$$

$$= \vartheta_P(f(x)\alpha f(y)\beta f(z))$$

$$\leq \max\{\vartheta_P(f(x)), \vartheta_P(f(y)), \vartheta_P(f(z))\}$$

$$= \max\{f^{-1}(\vartheta_P(x)), f^{-1}(\vartheta_P(y)), f^{-1}(\vartheta_P(z))\}.$$

Therefore $f^{-1}(P) = [f^{-1}(\pi_P, \vartheta_P)]$ is a Pythagorean fuzzy weak bi-ideal of M.

We can also state the converse of the *Theorem 7* by strengthening the condition on f as follows.

Theorem 7. Let $f: M \to S$ be an onto homomorphism of Γ-near-rings M and S. Let $P = (\pi_P, \vartheta_P)$ be a Pythagorean fuzzy subset of S. If $f^{-1}(P) = [f^{-1}(\pi_P), f^{-1}(\vartheta_P)$ is a Pythagorean fuzzy weak bi-ideal of M, then $P = (\pi_P, \vartheta_P)$ is a Pythagorean fuuzy weak bi-ideal of S.

Proof. Let $x, y, z \in S$. Then $f(j) = x$, $f(k) = y$ and $f(l) = z$ for some $j, k, l \in M$ and $\alpha, \beta \in \Gamma$. It follows that

$$\pi_P(x - y) = \pi_P(f(j) - f(k))$$

$$= \pi_P(f(j - k))$$

$$= f^{-1}(\pi_P)(j - k)$$

$$\geq \min\{f^{-1}(\pi_P)(j), f^{-1}(\pi_P)(k)\}$$

$$= \min\{\pi_P(f(j)), \pi_P(f(k))\}$$

$$= \min\{\pi_P(x), \pi_P(y)\}.$$

$$\vartheta_P(x - y) = \vartheta_P(f(j) - f(k))$$

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$$= \vartheta_P(f(j-k))$$

$$= f^{-1}(\vartheta_P)(j-k)$$

$$\le \max\{f^{-1}(\vartheta_P)(j), f^{-1}(\vartheta_P)(k)\}$$

$$= \max\{\vartheta_P(f(j)), \vartheta_P(f(k))\}$$

$$= \max\{\vartheta_P(x), \vartheta_P(y)\}.$$

And

$$\pi_P(x\alpha y\beta z) = \pi_P(f(j)\alpha f(k)\beta f(l))$$

$$= \pi_P(f(jkl))$$

$$= f^{-1}(\pi_P)(jkl)$$

$$\ge \min\{f^{-1}(\pi_P)(j), f^{-1}(\pi_P)(k), f^{-1}(\pi_P)(l)\}$$

$$= \min\{\pi_P(f(j)), \pi_P(f(k)), \pi_P(f(l))\}$$

$$= \min\{\pi_P(x), \pi_P(y), \pi_P(z)\}.$$

$$\vartheta_P(x\alpha y\beta z) = \vartheta_P(f(j)\alpha f(k)\beta f(l))$$

$$= \vartheta_P(f(jkl))$$

$$= f^{-1}(\vartheta_P)(jkl)$$

$$\le \max\{f^{-1}(\vartheta_P)(j), f^{-1}(\vartheta_P)(k), f^{-1}(\vartheta_P)(l)\}$$

$$= \max\{\vartheta_P(f(j)), \vartheta_P(f(k)), \vartheta_P(f(l))\}$$

$$= \max\{\vartheta_P(x), \vartheta_P(y), \vartheta_P(z)\}.$$

Hence P is a Pythagorean fuzzy weak bi-ideal of S.

Theorem 8. Let $f: M \to S$ be an onto Γ-near-ring homomorphism. If $P = (\pi_P, \vartheta_P)$ is a Pythagorean fuzzy weak bi-ideal of M, then $f(P) = f(\pi_P, \vartheta_P)$ is a Pythagorean fuzzy weak bi-ideal of M.

Proof. Let P be a Pythagorean fuzzy weak bi-ideal of M. Since $f(\pi_P)(x\prime) = \sup\limits_{f(x)=x'} (\pi_P(x))$, for $x\prime \in S$ and $f(\vartheta_P)(x\prime) = \inf\limits_{f(x)=x'} (\vartheta_P(x))$, for $x\prime \in S$hence $f(P)$ is nonempty. Let $x\prime, y\prime \in S$ and $\alpha, \beta \in \Gamma$. Then we have $\{x|x \in f^{-1}(x\prime - y) \supseteq \{x - y|x \in f^{-1}(x\prime)$ and $y \in f^{-1}(y\prime)\}$ and $\{x|x \in f^{-1}(x\prime y\prime)\} \supseteq \{x\alpha y|x \in f^{-1}(x\prime)$ and $y \in f^{-1}(y\prime)\}$.

$$f(\pi_P)(x\prime - y\prime) = \sup_{f(z)=(x\prime - y\prime)} \{\pi_P(z)\}$$

$$\ge \sup_{f(x)=x\prime, f(y)=y\prime} \{\pi_P(x-y)\}$$

$$\geq \sup_{f(x)=x',f(y)=y'} \{\min\{\pi_P(x), \pi_P(y)\}\}$$

$$= \min\{\sup_{f(x)=x'} \{\pi_P(x)\}, \sup_{f(y)=y'} \{\pi_P(y)\}\}$$

$$= \min\{f(\pi_P)(x'), \quad f(\pi_P)(y')\}.$$

And

$$f(\vartheta_P)(x' - y') = \inf_{f(z)=x'-y'} \{\vartheta_P(z)\}$$

$$\leq \inf_{f(x)=x',f(y)=y'} \{\vartheta_P(x - y)\}$$

$$\leq \inf_{f(x)=x',f(y)=y'} \{\max\{\vartheta_P(x), \vartheta_P(y)\}\}$$

$$= \max\{\inf_{f(x)=x'} \{\vartheta_P(x)\}, \inf_{f(y)=y'} \{\vartheta_P(y)\}\}$$

$$= \max\{f(\vartheta_P)(x'), \quad f(\vartheta_P)(y')\}.$$

Next,

$$f(\pi_P)(x'\alpha y'\beta z') = \sup_{f(w)=x'\alpha y'\beta z'} \{\pi_P(w)\}$$

$$\geq \sup_{f(x)=x',f(y)=y',f(z)=z'} \{\pi_P(x\alpha y\beta z)\}$$

$$\geq \sup_{f(x)=x',f(y)=y',f(z)=z'} \{\min\{\pi_P(x), \pi_P(y), \pi_P(z)\}\}$$

$$= \min\{\sup_{f(x)=x'} \{\pi_P(x)\}, \sup_{f(y)=y'} \{\pi_P(y)\}, \sup_{f(z)=z'} \{\pi_P(z)\}\}$$

$$= \min\{f(\pi_P)(x'), f(\pi_P)(y'), f(\pi_P)(z')\}.$$

And

$$f(\vartheta_P)(x'\alpha y'\beta z') = \inf_{f(w)=x'\alpha y'\beta z'} \{\vartheta_P(w)\}$$

$$\leq \inf_{f(x)=x',f(y)=y',f(z)=z'} \{\vartheta_P(x\alpha y\beta z)\}$$

$$\leq \inf_{f(x)=x',f(y)=y',f(z)=z'} \{\max\{\vartheta_P(x), \vartheta_P(y), \vartheta_P(z)\}\}$$

$$= \max\{\inf_{f(x)=x'} \{\vartheta_P(x)\}, \inf_{f(y)=y'} \{\vartheta_P(y)\}, \inf_{f(z)=z'} \{\vartheta_P(z)\}\}$$

$$= \max\{f(\vartheta_P)(x'), \quad f(\vartheta_P)(y'), \quad f(\vartheta_P)(z')\}.$$

Therefore $f(P)$ is a Pythagorean fuzzy weak bi-ideal of *S*.

5| Interval Valued Pythagorean Fuzzy Weak Bi-Ideals of Γ-Near-Rings

In this section, we initiate the notion of interval valued Pythagorean fuzzy weak bi-ideal of M and discuss some of its properties.

Definition 7. An interval valued Pythagorean fuzzy set $\overline{P}=(\overline{\pi}_P,\overline{\vartheta}_P)$ of M is called an interval valued Pythagorean fuzzy weak bi-ideal of M, if

I. $\overline{\pi}_P(x-y) \geq min\{\overline{\pi}_P(x),\overline{\pi}_P(y)\}$.
II. $\overline{\vartheta}_P(x-y) \leq max\{\overline{\vartheta}_P(x),\overline{\vartheta}_P(y)\}$.
III. $\overline{\pi}_P(x\gamma y\gamma z) \geq min\{\overline{\pi}_P(x),\overline{\pi}_P(y),\overline{\pi}_P(z)\}$.
IV. $\overline{\vartheta}_P(x\gamma y\gamma z) \leq max\{\overline{\vartheta}_P(x),\overline{\vartheta}_P(y),\overline{\vartheta}_P(z)\}$ *for all* $x,y,z \in M$ *and* $\alpha,\beta \in \Gamma$.

Example 3. Let $M=\{w,\ x,\ y,\ z\}$ be a nonempty set with binary operation `$+$' and $\Gamma=\{\gamma\}$ be a nonempty set of binary operations as the following tables:

+	w	x	y	z
w	w	x	y	z
x	x	w	z	y
y	y	z	w	x
z	z	y	x	w

and

γ	w	x	y	z
w	w	x	w	x
x	w	x	w	x
y	w	x	y	z
z	w	x	y	z

Let $\overline{\pi}_P : M \to D[0,1]$ and $\overline{\vartheta}_P : M \to D[0,1]$ be aninterval valued fuzzy subsets defined by $\overline{\pi}_P(w)=[0.6,0.7]$, $\overline{\pi}_p(x)=[0.5,0.6]$, $\overline{\pi}_P(y)=\overline{\pi}_P(z)=[0.4,0.5]$. And $\overline{\vartheta}_P(w)=[0.2,0.3]$, $\overline{\vartheta}_P(x)=[0.4,0.5]$, $\overline{\vartheta}_P(y)=[0.7,0.8]=\overline{\vartheta}_P(z)$. Then $\overline{P}=(\overline{\pi}_P,\overline{\vartheta}_P)$ is an interval valued Pythagorean fuzzy weak bi-ideal of M.

Theorem 9. Let $\overline{P}=(\overline{\pi}_P,\overline{\vartheta}_P)$ be an interval valued Pythagorean fuzzy subgroup of M. Then $\overline{P}=(\overline{\pi}_P,\overline{\vartheta}_P)$ is an interval valued Pythagorean fuzzy weak bi-ideal of M if and only if $\overline{\pi}_P \star \overline{\pi}_P \star \overline{\pi}_P \subseteq \overline{\pi}_P$ and $\overline{\vartheta}_P \star \overline{\vartheta}_P \star \overline{\vartheta}_P \supseteq \overline{\vartheta}_P$.

Proof. Assume that $\overline{P}=(\overline{\pi}_P,\overline{\vartheta}_P)$ be an interval valued Pythagorean fuzzy weak bi-ideal of M. Let $x,y,z,y_1,y_2 \in M$ and $\alpha,\beta \in \Gamma$ such that $x=y\alpha z$ and $y=y_1\beta y_2$. Then

$$(\overline{\pi}_P \star \overline{\pi}_P \star \overline{\pi}_P)(x) = \sup_{x=y\alpha z}\{\min\{(\overline{\pi}_P \star \overline{\pi}_P)(y),\overline{\pi}_P(z)\}\}$$

$$= \sup_{x=y\alpha z}\{\min\{\sup_{y=y_1\beta y_2}\min\{\overline{\pi}_P(y_1),\overline{\pi}_P(y_2)\}\},\overline{\pi}_P(z)\}\}$$

$$= \sup_{x=y\alpha z}\sup_{y=y_1\beta y_2}\{\min\{\min\{\overline{\pi}_P(y_1),\overline{\pi}_P(y_2)\},\overline{\pi}_P(z)\}\}$$

$$= \sup_{x=y_1\beta y_2\alpha z}\{\min\{\overline{\pi}_P(y_1),\overline{\pi}_P(y_2),\overline{\pi}_P(z)\}\}$$

$$\overline{\pi}_P(y_1\beta y_2\alpha z) \geq \min\{\overline{\pi}_P(y_1),\overline{\pi}_P(y_2),\overline{\pi}_P(z)\}$$

$$\leq \sup_{x=y_1\beta y_2\alpha z}\overline{\pi}_P(y_1\beta y_2\alpha z)$$

$$= \overline{\pi}_P(x).$$

And

$$(\overline{\vartheta}_P \star \overline{\vartheta}_P \star \overline{\vartheta}_P)(x) = \inf_{x=y\alpha z}\{\min\{(\overline{\vartheta}_P \star \overline{\vartheta}_P)(y), \overline{\vartheta}_P(z)\}\}$$

$$= \inf_{x=y\alpha z}\{\max\{\inf_{y=y_1\beta y_2}\min\{\overline{\vartheta}_P(y_1), \overline{\vartheta}_P(y_2)\}, \overline{\vartheta}_P(z)\}\}$$

$$= \inf_{x=y\alpha z y=y_1\beta y_2}\inf\{\max\{\max\{\overline{\vartheta}_P(y_1), \overline{\vartheta}_P(y_2)\}, \overline{\vartheta}_P(z)\}\}$$

$$= \inf_{x=y_1\beta y_2\alpha z}\{\max\{\overline{\vartheta}_P(y_1), \overline{\vartheta}_P(y_2), \overline{\vartheta}_P(z)\}\}$$

$$\overline{\vartheta}_P(y_1\beta y_2\alpha z) \leq \max\{\overline{\vartheta}_P(y_1), \overline{\vartheta}_P(y_2), \overline{\vartheta}_P(z)\}$$

$$\geq \inf_{x=y_1\beta y_2\alpha z}\overline{\vartheta}_P(y_1\beta y_2\alpha z)$$

$$= \overline{\vartheta}_P(x).$$

Since $\overline{P}$ is an interval valued Pythagorean fuzzy weak bi-ideal of M,

If x can not be expressed as $x = y\alpha z$, then $(\overline{\pi}_P \star \overline{\pi}_P \star \overline{\pi}_P)(x) = 0 \leq \overline{\pi}_P(x)$ and

$(\overline{\vartheta}_P \star \overline{\vartheta}_P \star \overline{\vartheta}_P)(x) = 0 \geq \overline{\vartheta}_P(x)$. In both cases $\overline{\pi}_P \star \overline{\pi}_P \star \overline{\pi}_P \subseteq \overline{\pi}_P$ and $\overline{\vartheta}_P \star \overline{\vartheta}_P \star \overline{\vartheta}_P \supseteq \overline{\vartheta}_P$.

Conversely, assume that $\overline{\pi}_P \star \overline{\pi}_P \star \overline{\pi}_P \subseteq \overline{\pi}_P$. For $x\prime, x, y, z \in M$ and $\alpha, \beta, \alpha_1, \beta_1 \in \Gamma$.

Let $x\prime$ be such that $x\prime = x\alpha y\beta z$.

Then $\overline{\pi}_P(x\alpha y\beta z) = \overline{\pi}_P(x\prime) \geq (\overline{\pi}_P \star \overline{\pi}_P \star \overline{\pi}_P)(x\prime)$

$$= \sup_{x'=p\alpha_1 q}\{\min\left\{(\overline{\pi}_P \star \overline{\pi}_P)(p), \overline{\pi}_P(q)\right\}\}$$

$$= \sup_{x\prime=p\alpha_1 q}\{\min\{\sup_{p=p_1\beta_1 p_2}\min\{\overline{\pi}_P(p_1), \overline{\pi}_P(p_2)\}, \overline{\pi}_P(q)\}\}$$

$$= \sup_{x\prime=p_1\beta_1 p_2\alpha_1 q}\{\min\{\overline{\pi}_P(p_1), \overline{\pi}_P(p_2), \overline{\pi}_P(q)\}\}$$

$$\geq \min\{\overline{\pi}_P(x), \overline{\pi}_P(y), \overline{\pi}_P(z)\}.$$

$$\overline{\vartheta}_P(x\alpha y\beta z) = \overline{\vartheta}_P(x\prime) \leq (\overline{\vartheta}_P * \overline{\vartheta}_P * \overline{\vartheta}_P)(x\prime)$$

$$= \inf_{x\prime=p\alpha_1 q}\{\max\{(\overline{\vartheta}_P * \overline{\vartheta}_P)(p), \overline{\vartheta}_P(q)\}\}$$

$$= \inf_{x\prime=p\alpha_1 q}\{\max\{\inf_{p=p_1\beta_1 p_2}\min\{\overline{\vartheta}_P(p_1), \overline{\vartheta}_P(p_2)\}, \overline{\vartheta}_P(q)\}\}$$

$$= \inf_{x\prime=p_1\beta_1 p_2\alpha_1 q}\{\max\{\overline{\vartheta}_P(p_1), \overline{\vartheta}_P(p_2), \overline{\vartheta}_P(q)\}\}$$

$$\leq \max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y), \overline{\vartheta}_P(z)\}.$$

Hence $\overline{\pi}_P(x\alpha y\beta z) \geq min\{\overline{\pi}_P(x), \overline{\pi}_P(y), \overline{\pi}_P(z)\}$ and $\overline{\vartheta}_P(x\alpha y\beta z) \leq max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y), \overline{\vartheta}_P(z)\}$.

Lemma 3. Let $\overline{\pi}_P = (\overline{\pi}_{P1}, \overline{\pi}_{P2})$ and $\overline{\vartheta}_P = (\overline{\vartheta}_{P1}, \overline{\vartheta}_{P2})$ be an interval valued Pythagorean fuzzy weal bi-ideals of M. Then the products $\overline{\pi}_P * \overline{\vartheta}_P$ *and* $\overline{\vartheta}_P * \overline{\pi}_P$ are also interval valued Pythagorean fuzzy weak bi-ideals of M.

Proof. Let $\overline{\pi}_P$ and $\overline{\vartheta}_P$ be an interval valued Pythagorean fuzzy weak bi-ideals of M and let $\alpha, \alpha_1, \alpha_2 \in \Gamma$. Then

$$(\overline{\pi}_1 \star \overline{\pi}_2)(x-y) = \sup_{x-y=a\alpha b}\min\{\overline{\pi}(a), \overline{\pi}(b)\}$$

$$\geq \sup_{x-y=a_1\alpha_1 b_1 - a_2\alpha_2 b_2 < (a_1-a_2)(b_1-b_2)}\min\{\overline{\pi}(a_1-a_2), \overline{\pi}(b_1-b_2)\}$$

$$\geq \sup \min\{\min\{\overline{\pi}(a_1), \overline{\pi}(a_2)\}, \min\{\overline{\pi}(b_1), \overline{\pi}(b_2)\}\}$$

$$= \sup\min\{\min\{\overline{\pi}(a_1), \overline{\pi}(b_1)\}, \min\{\overline{\pi}(a_2), \overline{\pi}(b_2)\}\}$$

$$\geq \min\{\sup_{x=a_1\alpha_1 b_1}\min\{\overline{\pi}(a_1), \overline{\pi}(b_1)\}, \sup_{y=a_2\alpha_2 b_2}\min\{\overline{\pi}(a_2), \overline{\pi}(b_2)\}\}$$

$$= \min\{(\overline{\pi} \star \overline{\pi})(x), (\overline{\pi} \star \overline{\pi})(y)\}.$$

$$(\overline{\vartheta}_1 \star \overline{\vartheta}_2)(x-y) = \inf_{x-y=a\alpha b}\max\{\overline{\vartheta}(a), \overline{\vartheta}(b)\}$$

$$\leq \inf_{x-y=a_1\alpha_1 b_1 - a_2\alpha_2 b_2 < (a_1-a_2)(b_1-b_2)}\max\{\overline{\vartheta}(a_1-a_2), \overline{\vartheta}(b_1-b_2)\}$$

$$\leq \inf\max\{\max\{\overline{\vartheta}(a_1), \overline{\vartheta}(a_2)\}, \max\{\overline{\vartheta}(b_1), \overline{\vartheta}(b_2)\}\}$$

$$= \inf\max\{\max\{\overline{\vartheta}(a_1), \overline{\vartheta}(b_1)\}, \max\{\overline{\vartheta}(a_2), \overline{\vartheta}(b_2)\}\}$$

$$\leq \max\{\inf_{x=a_1\alpha_1 b_1}\max\{\overline{\vartheta}(a_1), \overline{\vartheta}(b_1)\}, \inf_{y=a_2\alpha_2 b_2}\max\{\overline{\vartheta}(a_2), \overline{\vartheta}(b_2)\}\}$$

$$= \max\{(\overline{\vartheta} \star \overline{\vartheta})(x), (\overline{\vartheta} \star \overline{\vartheta})(y)\}.$$

It follows that $\overline{\pi} \star \overline{\vartheta}$ is an interval valued Pythagorean fuzzy subgroup of M. Further,

$(\overline{\pi} \star \overline{\vartheta}) \star (\overline{\pi} \star \overline{\vartheta}) \star (\overline{\pi} \star \overline{\vartheta}) = \overline{\pi} \star \overline{\vartheta} \star (\overline{\pi} \star \overline{\vartheta} \star \overline{\pi}) \star \overline{\vartheta} \subseteq \overline{\pi} \star \overline{\vartheta} \star (\overline{\vartheta} \star \overline{\vartheta} \star \overline{\vartheta}) \star \overline{\vartheta} \subseteq \overline{\pi} \star (\overline{\vartheta} \star \overline{\vartheta} \star \overline{\vartheta})$, since $\overline{\vartheta}$ is an interval valued Pythagorean fuzzy weak bi-ideal of $M \subseteq \overline{\pi} \star \overline{\vartheta}$.

Therefore $\overline{\pi}_P \star \overline{\vartheta}_P$ is an interval valued Pythagorean fuzzy weak bi-ideal of M. Similarly $\overline{\vartheta}_P \star \overline{\pi}_P$ is an interval valued Pythagorean fuzzy weak bi-ideal of M.

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Lemma 4. Every interval valued Pythagorean fuzzy ideal of M is an interval valued Pythagorean fuzzy bi-ideal of M.

Proof. Let $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ be an interval valued Pythagorean fuzzy ideal of M. Then

$$\overline{\pi}_P \star \mathrm{M} \star \overline{\pi}_P \subseteq \overline{\pi}_P \star \mathrm{M} \star \mathrm{M} \subseteq \overline{\pi}_P \star \mathrm{M} \subseteq \overline{\pi}_P,$$

$$\overline{\vartheta}_P \star \mathrm{M} \star \overline{\vartheta}_P \supseteq \overline{\vartheta}_P \star \mathrm{M} \star \mathrm{M} \supseteq \overline{\vartheta}_P \star \mathrm{M} \supseteq \overline{\vartheta}_P,$$

since $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ be an interval valued Pythagorean fuzzy ideal of M.

This implies that $\overline{\pi}_P \star M \star \overline{\pi}_P \subseteq \overline{\pi}_P$ and $\overline{\vartheta}_P \star M \star \overline{\vartheta}_P \supseteq \overline{\vartheta}_P$.

Therefore $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ be an interval valued Pythagorean fuzzy bi-ideal of M.

Theorem 10. Every interval valued Pythagorean fuzzy bi-ideal of M is an interval valued Pythagorean fuzzy weak bi-ideal of M.

Proof. Assume that $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ be an interval valued Pythagorean fuzzy bi-ideal of M.

Then $\overline{\pi}_P \star M \star \overline{\pi}_P \subseteq \overline{\pi}_P$ and $\overline{\vartheta}_P \star M \star \overline{\vartheta}_P \supseteq \overline{\vartheta}_P$.

We have $\overline{\pi}_P \star \overline{\pi}_P \star \overline{\pi}_P \subseteq \overline{\pi}_P \star M \star \overline{\pi}_P$ and $\overline{\vartheta}_P \star \overline{\vartheta}_P \star \overline{\vartheta}_P \supseteq \overline{\vartheta}_P \star M \star \overline{\vartheta}_P$.

This implies that $\overline{\pi}_P \star \overline{\pi}_P \star \overline{\pi}_P \subseteq \overline{\pi}_P \star M \star \overline{\pi}_P \subseteq \overline{\pi}_P$ and $\overline{\vartheta}_P \star \overline{\vartheta}_P \star \overline{\vartheta}_P \supseteq \overline{\vartheta}_P \star M \star \overline{\vartheta}_P \supseteq \overline{\vartheta}_P$.

Therefore $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ is an interval valued Pythagorean fuzzy weak bi-ideal of M.

Example 4. Let $M = \{w, x, y, z\}$ be a nonempty set with binary operation $+$ and $\Gamma = \{\alpha\}$ be a nonempty set of binary operations as the following tables:

+	**w**	**x**	**y**	**z**
w	w	x	y	z
x	x	w	z	y
y	y	z	w	x
z	z	y	x	w

and

α	**w**	**x**	**y**	**z**
w	w	w	w	w
x	w	x	w	x
y	w	w	y	y
z	w	x	y	z

Let $\overline{P} : M \to D[0,1]$ be aninterval valued Pythagorean fuzzy set defined by $\overline{\pi}_P(w) = [0.8,0.9], \overline{\pi}_P(x) = [0.3,0.4] = \overline{\pi}_P(y)$ and $\overline{\pi}_P(z) = [0.5,0.6]$, and $\overline{\vartheta}_P(w) = [0,0.1], \overline{\vartheta}_P(x) = [0.4,0.5] = \overline{\vartheta}_P(y), \overline{\vartheta}_P(z) = [0.2,0.3]$. Then $\overline{\pi}_P$ is an interval valued fuzzy weak bi-ideal of M. But $\overline{\pi}_P$ is not a fuzzy ideal and bi-ideal of M and

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$\overline{\pi}_P(z\gamma y\gamma z) = \overline{\pi}_P(y) = [0.3,0.4] > [0.5,0.6] = min\{\overline{\pi}_P(z), \overline{\pi}_P(z)\}$ and $\overline{\vartheta}_P(x\alpha(z+w) - x\alpha w) \leq \overline{\vartheta}_P(z) = [0.4,0.5] \not\leq [0.2,0.3]$ and $\overline{\vartheta}_P(z\gamma x\gamma z) = \overline{\vartheta}_P(x) = [0.4,0.5] \not\leq [0.2,0.3] = \min\{\overline{\vartheta}_P(z), \overline{\vartheta}_P(z)\}$.

Theorem 11. Let $\{(\overline{\pi}_{P_i}, \overline{\vartheta}_{P_i}) | i \in \Omega\}$ be family of interval valued Pythagorean fuzzy weak bi-ideals of a near-ring M, then $\bigcap_{i\in\Omega} \overline{\pi}_{P_i}$ and $\cup_{i\in\Omega} \overline{\vartheta}_{P_i}$ are also an interval valued Pythagorean fuzzy weak bi-ideal of M, where Ω is any index set.

Proof. Let $\{\overline{\pi}_{P_i}, \overline{\vartheta}_{P_i}\}_{i\in\Omega}$ be a family of interval valued Pythagorean fuzzy weak bi-ideals of M.

Let $x, y, z \in M, \alpha, \beta \in \Gamma$ and $\overline{\pi}_{P_i} = \bigcap_{i\in\Omega} \overline{\pi}_i$, $\overline{\vartheta}_{P_i} = \cup_{i\in\Omega} \overline{\vartheta}_{P_i}$

Then, $\overline{\pi}_{P_i}(x) = \bigcap_{i\in\Omega} \overline{\pi}_{P_i}(x) = (\inf_{i\in\Omega} \overline{\pi}_{P_i})(x) = \inf_{i\in\Omega} \overline{\pi}_{P_i}(x)$

and $\overline{\vartheta}_{P_i}(x) = \cup_{i\in\Omega} \overline{\vartheta}_{P_i}(x) = (\sup_{i\in\Omega} \overline{\vartheta}_{P_i})(x) = \sup_{i\in\Omega} \overline{\vartheta}_{P_i}(x)$.

$$\overline{\pi}_{P_i}(x-y) = \inf_{i\in\Omega} \overline{\pi}_{P_i}(x-y)$$

$$\geq \inf_{i\in\Omega} \min\{\overline{\pi}_{P_i}(x), \overline{\pi}_{P_i}(y)\}$$

$$= \min\{\inf_{i\in\Omega} \overline{\pi}_{P_i}(x), \inf_{i\in\Omega} \overline{\pi}_{P_i}(y)\}$$

$$= \min\{\bigcap_{i\in\Omega} \overline{\pi}_{P_i}(x), \bigcap_{i\in\Omega} \overline{\pi}_{P_i}(y)\}$$

$$= \min\{\overline{\pi}_{P_i}(x), \overline{\pi}_{P_i}(y)\}.$$

$$\overline{\vartheta}_{P_i}(x-y) = \sup_{i\in\Omega} \overline{\vartheta}_{P_i}(x-y)$$

$$\leq \sup_{i\in\Omega} \max\{\overline{\vartheta}_{P_i}(x), \overline{\vartheta}_{P_i}(y)\}$$

$$= \max\{\sup_{i\in\Omega} \overline{\vartheta}_{P_i}(x), \sup_{i\in\Omega} \overline{\vartheta}_{P_i}(y)\}$$

$$= \max\{\cup_{i\in\Omega} \overline{\vartheta}_{P_i}(x), \cup_{i\in\Omega} \overline{\vartheta}_{P_i}(y)\}$$

$$= \max\{\overline{\vartheta}_{P_i}(x), \overline{\vartheta}_{P_i}(y)\}.$$

And,

$$\overline{\pi}_{P_i}(x\alpha y\beta z) = \inf_{i\in\Omega} \overline{\pi}_{P_i}(x\alpha y\beta z)$$

$$\geq \inf_{i\in\Omega} \min\{\overline{\pi}_{P_i}(x), \overline{\pi}_{P_i}(y), \overline{\pi}_{P_i}(z)\}$$

$$= \min\{\inf_{i\in\Omega} \overline{\pi}_{P_i}(x), \inf_{i\in\Omega} \overline{\pi}_{P_i}(y), \inf_{i\in\Omega} \overline{\pi}_{P_i}(z)\}$$

$$= \min\{\bigcap_{i\in\Omega} \overline{\pi}_{P_i}(x), \bigcap_{i\in\Omega} \overline{\pi}_{P_i}(y), \bigcap_{i\in\Omega} \overline{\pi}_{P_i}(z)\}$$

$= \min\{\overline{\pi}_{P_i}(x), \overline{\pi}_{P_i}(y), \overline{\pi}_{P_i}(z)\}.$

$\overline{\vartheta}_{P_i}(x\alpha y\beta z) = \sup_{i\in\Omega} \overline{\vartheta}_{P_i}(x\alpha y\beta z)$

$\leq \sup_{i\in\Omega}\max\{\overline{\vartheta}_{P_i}(x), \overline{\vartheta}_{P_i}(y), \overline{\vartheta}_{P_i}(z)\}$

$= \max\{\sup_{i\in\Omega}\overline{\vartheta}_{P_i}(x), \sup_{i\in\Omega}\overline{\vartheta}_{P_i}(y), \sup_{i\in\Omega}\overline{\vartheta}_{P_i}(z)\}$

$= \max\{\cup_{i\in\Omega} \overline{\vartheta}_{P_i}(x), \cup_{i\in\Omega} \overline{\vartheta}_{P_i}(y), \cup_{i\in\Omega} \overline{\vartheta}_{P_i}(z)\}$

$= \max\{\overline{\vartheta}_{P_i}(x), \overline{\vartheta}_{P_i}(y), \overline{\vartheta}_{P_i}(z)\}.$

Theorem 12. Let $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ be an interval valued Pythagorean fuzzy subset of M. Then $U(\overline{\pi}_P; t)$ and $L(\overline{\vartheta}_P; s)$ is an interval valued Pythagorean fuzzy weak bi-ideal of M if and only if $\overline{\pi}_{Pt}$ is a weak bi-ideal of M, for all $t \in [0,1]$.

Proof. Assume that $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ is an interval valued Pythagorean fuzzy weak bi-ideal of R.

Let $s, t \in [0,1]$ such that $x, y \in U(\overline{\pi}_P; t)$.

Then $\overline{\pi}_P(x) \geq t$ and $\overline{\pi}_P(y) \geq t$,then $\overline{\pi}_P(x-y) \geq \min\{\overline{\pi}_P(x), \overline{\pi}_P(y)\} \geq \min\{t, t\} = t$ and

$\overline{\vartheta}_P(x-y) \leq \max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y)\} \leq max\{s, s\} = s.$

Thus $x - y \in U(\overline{\pi}_P\ \ t)$.Let $x, y, z \in \overline{\pi}_{Pt}$ and $\alpha, \beta \in \Gamma$.

This implies that $\overline{\pi}_P(x\alpha y\beta z) \geq \min\{\overline{\pi}_P(x), \overline{\pi}_P(y), \overline{\pi}_P(z)\} \geq \min\{t, t, t\} = t$, and

$\overline{\vartheta}_P(x\alpha y\beta z) \leq \max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y), \overline{\vartheta}_P(z)\} \leq \max\{s,\ s,\ s\} = s.$

Therefore $x\alpha y\beta z \in U(\overline{\pi}_P; s)$.

Hence U $(\overline{\pi}_P;$ t) and $(\overline{\vartheta}_P; s)$ is an interval valued Pythagorean fuzzy weak bi-ideal of M.

Conversely, assume that U $(\overline{\pi}_P;$ t) and $(\overline{\vartheta}_P; s)$ is an interval valuedPythagorean fuzzy weak bi-ideal of M, for all $s, t \in [0,1]$.

Let $x, y \in M$. Suppose $\overline{\pi}_P(x-y) < min\{\overline{\pi}_P(x), \overline{\pi}_P(y)\}$and $\overline{\vartheta}_P(x-y) > max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y)\}$.

Choose t such that $\overline{\pi}_P(x-y) < t < min\{\overline{\pi}_P(x), \overline{\pi}_P(y)\}$ and $\overline{\vartheta}_P(x-y) > s > max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y)\}$.

This implies that $\overline{\pi}_P(x) > t, \overline{\pi}_P(y) > t$ and $\overline{\pi}_P(x-y) < t$.

Then we have $x, y \in \overline{\pi}_{Pt}$ but $x - y \notin \overline{\pi}_{Pt}$ and $\overline{\vartheta}_P(x) < s, \overline{\vartheta}_P(y) < s$ and $\overline{\vartheta}_P(x-y) > s$, we have $x, y \in \overline{\vartheta}_{Ps}$ but $x - y \notin \overline{\vartheta}_{Ps}$ a contradiction.

Thus $\overline{\pi}_P(x-y) \geq \min\{\overline{\pi}_P(x), \overline{\pi}_P(y)\}$ and $\overline{\vartheta}_P(x-y) \leq \max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y)\}$.

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If there exist $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ such that $\overline{\pi}_P(x\alpha y\beta z) < min\{\overline{\pi}_P(x), \overline{\pi}_P(y), \overline{\pi}_P(z)\}$ and $\overline{\vartheta}_P(x\alpha y\beta z) > max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y), \overline{\vartheta}_P(z)\}$.

Choose t such that $\overline{\pi}_P(x\alpha y\beta z) < t < min\{\overline{\pi}_P(x), \overline{\pi}_P(y), \overline{\pi}_P(z)\}$.

Choose s such that $\overline{\vartheta}_P(x\alpha y\beta z) > s > max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y), \overline{\vartheta}_P(z)\}$.

Then $\overline{\pi}_P(x) > t, \overline{\pi}_P(y) > t, \overline{\pi}_P(z) > t$ and $\overline{\vartheta}_P(x) < s, \overline{\vartheta}_P(y) < s, \overline{\vartheta}_P(z) < s$ and $\overline{\pi}_P(x\alpha y\beta z) < t$.

So, $x, y, z \in \overline{\pi}_P >$ but $x\alpha y\beta z \notin \overline{\pi}_{Pt}$, and $x\alpha y\beta z \notin \overline{\vartheta}_{Ps}$, which is a contradiction.

Hence $\overline{\pi}_P(x\alpha y\beta z) \geq \min\{\overline{\pi}_P(x), \overline{\pi}_P(y), \overline{\pi}_P(z)\}, \overline{\vartheta}_P(x\alpha y\beta z) \leq \max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y), \overline{\vartheta}_P(z)\}$.

Therefore $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ is an interval valued Pythagorean fuzzy weak bi-ideal of M.

Theorem 13. Let $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ be an interval valued Pythagorean fuzzy weak bi-ideal of M then the set $M_{\overline{\pi}_P, \overline{\vartheta}_P} = \{x \in M | \overline{\pi}_P(x) = \overline{\pi}_P(0) = \overline{\vartheta}_P(x)\}$ is a interval valued Pythagorean fuzzy weak bi-ideal of M.

Proof. Let $x, y \in M_{(\overline{\pi}_P, \overline{\vartheta}_P)}$.

Then $\overline{\pi}_P(x) = \overline{\pi}_P(0), \overline{\pi}_P(y) = \overline{\pi}_P(0), \overline{\vartheta}_P(x) = 0, \overline{\vartheta}_P(y) = 0$ and

$\overline{\pi}_P(x - y) \geq \min\{\overline{\pi}_P(x), \overline{\pi}_P(y)\} = \min\{\overline{\pi}_P(0), \overline{\pi}_P(0)\} = \overline{\pi}(0)$, and

$\overline{\vartheta}_P(x - y) \leq \max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y)\} = \max\{\overline{\vartheta}_P(0), \overline{\vartheta}_P(0)\} = \overline{\vartheta}_P(0)$.

So $\overline{\pi}_P(x - y) = \overline{\pi}_P(0), \overline{\vartheta}_P(x - y) = \overline{\vartheta}_P(0)$.

Thus $x - y \in M_{\overline{\pi}_P}$, $x - y \in M_{\overline{\vartheta}_P}$. For every $x, y, z \in M_{\overline{\pi}_P}$ and $\alpha, \beta \in \Gamma$. We have

$$\overline{\pi}_P(x\alpha y\beta z) \geq \min\{\overline{\pi}_P(x), \overline{\pi}_P(y), \overline{\pi}_P(z)\},$$

$$= \min\{\overline{\pi}_P(0), \overline{\pi}_P(0), \overline{\pi}_P(0)\} = \overline{\pi}_P(0),$$

and

$$\overline{\vartheta}_P(x\alpha y\beta z) \leq \max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y), \overline{\vartheta}_P(z)\}$$

$$= \max\{\overline{\vartheta}_P(0), \overline{\vartheta}_P(0), \overline{\vartheta}_P(0)\} = \overline{\vartheta}_P(0).$$

Thus $x\alpha y\beta z \in M_{\overline{\pi}_P}, x\alpha y\beta z \in M_{\overline{\vartheta}_P}$. Hence $M_{(\overline{\pi}_P, \overline{\vartheta}_P)}$ is an interval valuedPythagorean fuzzy weak bi-ideal of M.

6 | Homomorphism of Interval Valued Pythagorean Fuzzy Weak Bi-Ideals of Γ-Near-Rings

In this section, we characterize interval valued Pythagorean fuzzy weak bi-ideals of Γ-near-rings using homomorphism.

Definition 8. Let f be a mapping from a set M to a set S. Let $f = (\overline{\pi}_P, \overline{\vartheta}_P)$ be an interval valued Pythagorean fuzzy subsets of M and S, resp. then f is image of $\overline{\pi}_P$ and $\overline{\vartheta}_P$ under f is a fuzzy subset of S defined by

$$f(\overline{\pi}_P)(y) = \begin{cases} \sup\limits_{x\in f^{-1}(y)} \overline{\pi}_P(x) & \text{if } f^{-1}(y) \neq \varnothing \\ 0 & \text{otherwise.} \end{cases}$$

$$f(\overline{\vartheta}_P)(y) = \begin{cases} \inf\limits_{x\in f^{-1}(y)} \overline{\vartheta}_P(x) & \text{if } f^{-1}(y) \neq \varnothing \\ 1 & \text{otherwise} \end{cases}$$

And the pre-image of $\overline{\pi}_P$ and $\overline{\vartheta}_P$ under f is a fuzzy subset of M defined by

$$f^{-1}(\overline{\pi}_P(x)) = \overline{\pi}_P(f(x)), f^{-1}(\overline{\vartheta}_P(x)) = \overline{\vartheta}_P(f(x)) \text{ for all } x \in M \text{ and } f^{-1}(y) = \{x \in M | f(x) = y\}.$$

Theorem 14. Let $f: M \to S$ be a homomorphism between Γ-near-rings M and S. If $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ is an interval valued Pythagorean fuzzy weak bi-ideal of S, then $f^{-1}(\overline{P}) = [f^{-1}(\overline{\pi}_P, \overline{\vartheta}_P)]$ is an interval valued fuzzy weakbi-ideal of M.

Proof. Let f be an interval valued Pythagorean fuzzy weak bi-ideal of S. Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then

$$f^{-1}(\overline{\pi}_P)(x-y) = \overline{\pi}_P(f(x-y))$$

$$= \overline{\pi}_P(f(x) - f(y))$$

$$\geq \min\{\overline{\pi}_P(f(x)), \overline{\pi}_P(f(y))\}$$

$$= \min\{f^{-1}(\overline{\pi}_P(x)), f^{-1}(\overline{\pi}_P(y))\}.$$

$$f^{-1}(\overline{\vartheta}_P)(x-y) = \overline{\vartheta}_P(f(x-y))$$

$$= \overline{\vartheta}_P\big(f(x) - f(y)\big)$$

$$\leq \max\{\overline{\vartheta}_P(f(x)), \overline{\vartheta}_P(f(y))\}$$

$$= \max\{f^{-1}(\overline{\vartheta}_P(x)), f^{-1}(\overline{\vartheta}_P(y))\}.$$

$$f^{-1}(\overline{\pi}_P)(x\alpha y\beta z) = \overline{\pi}_P(f(x\alpha y\beta z))$$

$$= \overline{\pi}_P(f(x)\alpha f(y)\beta f(z))$$

$$\geq \min\{\overline{\pi}_P(f(x)), \overline{\pi}_P(f(y)), \overline{\pi}_P(f(z))\}$$

$$= \min\{f^{-1}(\overline{\pi}_P(x)), f^{-1}(\overline{\pi}_P(y)), f^{-1}(\overline{\pi}_P(z))\}.$$

$$f^{-1}(\overline{\vartheta}_P)(x\alpha y\beta z) = \overline{\vartheta}_P(f(x\alpha y\beta z))$$

$$= \overline{\vartheta}_P(f(x)\alpha f(y)\beta f(z))$$

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$\leq \max\{\overline{\vartheta}_P(f(x)), \overline{\vartheta}_P(f(y)), \overline{\vartheta}_P(f(z))\}$

$= \max\{f^{-1}(\overline{\vartheta}_P(x)), f^{-1}(\overline{\vartheta}_P(y)), f^{-1}(\overline{\vartheta}_P(z))\}.$

Therefore $f^{-1}(\overline{P}) = [f^{-1}(\overline{\pi}_P, \overline{\vartheta}_P)]$ is an interval valued Pythagorean fuzzy weak bi-ideal of M.

We can also state the converse of the *Theorem 7* by strengthening the condition on f as follows.

Theorem 15. Let $f: M \to S$ be an onto homomorphism of Γ-near-rings M and S. Let $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ be an interval valued Pythagorean fuzzy subset of S. If $f^{-1}(\overline{P}) = [f^{-1}(\overline{\pi}_P, \overline{\vartheta}_P)]$ is an interval valued Pythagorean fuzzy weak bi-ideal of M, then $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ is a Pythagorean fuzzy weak bi-ideal of S.

Proof. Let $x, y, z \in S$. Then $f(j) = x, f(k) = y$ and $f(l) = z$ for some $j, k, l \in M$ and $\alpha, \beta \in \Gamma$. It follows that

$\overline{\pi}_P(x - y) = \overline{\pi}_P(f(j) - f(k))$

$= \overline{\pi}_P(f(j - k))$

$= f^{-1}(\overline{\pi}_P)(j - k)$

$\geq \min\{f^{-1}(\overline{\pi}_P)(j), f^{-1}(\overline{\pi}_P)(k)\}$

$= \min\{\overline{\pi}_P(f(j)), \overline{\pi}_P(f(k))\}$

$= \min\{\overline{\pi}_P(x), \overline{\pi}_P(y)\}.$

$\overline{\vartheta}(x - y) = \overline{\vartheta}(f(j) - f(k))$

$= \overline{\vartheta}(f(j - k))$

$= f^{-1}(\overline{\vartheta})(j - k)$

$\leq \max\{f^{-1}(\overline{\vartheta})(j), f^{-1}(\overline{\vartheta})(k)\}$

$= \max\{\overline{\vartheta}(f(j)), \overline{\vartheta}(f(k))\}$

$= \max\{\overline{\vartheta}(x), \overline{\vartheta}(y)\}.$

And

$\overline{\pi}_P(x\alpha y\beta z) = \overline{\pi}_P(f(j)\alpha f(k)\beta f(l))$

$= \overline{\pi}_P(f(jkl))$

$= f^{-1}(\overline{\pi}_P)(jkl)$

$\geq \min\{f^{-1}(\overline{\pi}_P)(j), f^{-1}(\overline{\pi}_P)(k), f^{-1}(\overline{\pi}_P)(l)\}$

$= \min\{\overline{\pi}_P(f(j)), \overline{\pi}_P(f(k)), \overline{\pi}_P(f(l))\}$

$= \min\{\overline{\pi}_P(x), \overline{\pi}_P(y), \overline{\pi}_P(z)\}.$

$\overline{\vartheta}(x\alpha y\beta z) = \overline{\vartheta}(f(j)\alpha f(k)\beta f(l))$

$= \overline{\vartheta}(f(jkl))$

$= f^{-1}(\overline{\vartheta})(jkl)$

$\leq \max\{f^{-1}(\overline{\vartheta})(j), f^{-1}(\overline{\vartheta})(k), f^{-1}(\overline{\vartheta})(l)\}$

$= \max\{\overline{\vartheta}(f(j)), \overline{\vartheta}(f(k)), \overline{\vartheta}(f(l))\}$

$= \max\{\overline{\vartheta}(x), \overline{\vartheta}(y), \overline{\vartheta}(z)\}.$

Hence $\overline{P}$ is an interval valued Pythagorean fuzzy weak bi-ideal of S.

Theorem 16. Let $f: M \rightarrow S$ be an onto Γ-near-ring homomorphism. If $\overline{P} = (\overline{\pi}_P, \overline{\vartheta}_P)$ is an interval valued Pythagorean fuzzy weak bi-ideal of M, then $f(\overline{P}) = [f(\overline{\pi}_P, \overline{\vartheta}_P)]$ is an interval valued Pythagorean fuzzy weak bi-ideal of S.

Proof. Let $\overline{P}$ be an interval valued Pythagorean fuzzy weak bi-ideal of M. Since $f(\overline{\pi}_P)(x') = \sup\limits_{f(x)=x'} (\overline{\pi}_P(x))$ and $f(\overline{\vartheta}_P)(x') = \inf\limits_{f(x)=x'} (\overline{\vartheta}_P(x))$, for $x' \in S$ and hence $f(\overline{P})$ is nonempty. Let $x', y' \in S$ and $\alpha, \beta \in \Gamma$. Then we have $\{x | x \in f^{-1}(x' - y) \supseteq \{x - y | x \in f^{-1}(x')$ and $y \in f^{-1}(y')\}$ and $\{x | x \in f^{-1}(x'y')\} \supseteq \{x\alpha y | x \in f^{-1}(x')$ $and\, y \in f^{-1}(y')\}$.

$f(\overline{\pi}_P)(x' - y') = \sup\limits_{f(z)=(x'-y')} \{\overline{\pi}_P(z)\}$

$\geq \sup\limits_{f(x)=x', f(y)=y'} \{\overline{\pi}_P(x - y)\}$

$\geq \sup\limits_{f(x)=x', f(y)=y'} \{\min\{\overline{\pi}_P(x), \overline{\pi}_P(y)\}\}$

$= \min\{\sup\limits_{f(x)=x'} \{\overline{\pi}_P(x)\}, \sup\limits_{f(y)=y'} \{\overline{\pi}_P(y)\}\}$

$= \min\{f(\overline{\pi}_P)(x'), \ f(\overline{\pi}_P)(y')\}.$

And

$f(\overline{\vartheta}_P)(x' - y') = \inf\limits_{f(z)=x'-y'} \{\overline{\vartheta}_P(z)\}$

$\leq \inf\limits_{f(x)=x', f(y)=y'} \{\overline{\vartheta}_P(x - y)\}$

$\leq \inf\limits_{f(x)=x', f(y)=y'} \{\max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y)\}\}$

$= \max\{\inf\limits_{f(x)=x'} \{\overline{\vartheta}_P(x)\}, \inf\limits_{f(y)=y'} \{\overline{\vartheta}_P(y)\}\}$

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$$= \max\{f(\overline{\vartheta}_P)(x'),\ f(\overline{\vartheta}_P)(y')\}.$$

Next,

$$f(\overline{\pi})(x'\alpha y'\beta z') = \sup_{f(w)=x'\alpha y'\beta z'} \{\overline{\pi}(w)\}$$

$$\geq \sup_{f(x)=x',f(y)=y',f(z)=z'} \{\overline{\pi}(x\alpha y\beta z)\}$$

$$\geq \sup_{f(x)=x',f(y)=y',f(z)=z'} \{\min\{\overline{\pi}(x),\overline{\pi}(y),\overline{\pi}(z)\}\}$$

$$= \min\{\sup_{f(x)=x'} \{\overline{\pi}(x)\},\ \sup_{f(y)=y'} \{\overline{\pi}(y)\},\ \sup_{f(z)=z'} \{\overline{\pi}(z)\}\}$$

$$= \min\{f(\overline{\pi})(x'), f(\overline{\pi})(y'), f(\overline{\pi})(z')\}.$$

And

$$f(\overline{\vartheta}_P)(x'\alpha y'\beta z') = \inf_{f(w)=x'\alpha y'\beta z'} \{\overline{\vartheta}_P(w)\}$$

$$\leq \inf_{f(x)=x',f(y)=y',f(z)=z'} \{\overline{\vartheta}_P(x\alpha y\beta z)\}$$

$$\leq \inf_{f(x)=x',f(y)=y',f(z)=z'} \{\max\{\overline{\vartheta}_P(x), \overline{\vartheta}_P(y), \overline{\vartheta}_P(z)\}\}$$

$$= \max\{\inf_{f(x)=x'} \{\overline{\vartheta}_P(x)\},\ \inf_{f(y)=y'} \{\overline{\vartheta}_P(y)\},\ \inf_{f(z)=z'} \{\overline{\vartheta}_P(z)\}\}$$

$$= \max\{f(\overline{\vartheta}_P)(x'), f(\overline{\vartheta}_P)(y'), f(\overline{\vartheta}_P)(z')\}.$$

Therefore $f(B)$ is an interval valued Pythagorean fuzzy weak bi-ideal of S.

7| Conclusion

In this paper, we discuss Pythagorean fuzzy weak ideal, Pythagorean fuzzy weak bi-ideal, Homomorphism of Pythagorean fuzzy weak ideal and weak bi-ideal. An interval valued Pythagorean fuzzy ideal, interval valued Pythagorean fuzzy weak bi-ideal, Homomorphism of interval valued Pythagorean fuzzy weak ideal and bi-ideal in gamma near ring are studied and investigated some properties with suitable examples.

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