



## Paper Type: Research Paper

On Algebraic Aspects of  $\eta$ -Fuzzy SubgroupsMuhammad Waseem Asghar<sup>1</sup>, Khushdil Ahmad<sup>1\*</sup> <sup>1</sup> Department of Mathematics, Government College University, Lahore 54000, Pakistan; waseem.asghar@gcu.edu.pk; khushdil.ahmad@gcu.edu.pk.

## Citation:

Asghar, M. W., & Ahmad, K. (2023). On algebraic aspects of  $\eta$ -fuzzy subgroups. *Journal of fuzzy extension and applications*, 4(2), 65-80.

Received: 22/11/2022

Reviewed: 24/12/2022

Revised: 15/01/2023

Accepted: 08/02/2023

## Abstract

In this paper, we define the term " $\eta$ -fuzzy subgroup" and show that every fuzzy subgroup is a  $\eta$ -fuzzy subgroup. We define some of the algebraic properties of the concept of  $\eta$ -fuzzy cosets. Furthermore, we initiate the study of the  $\eta$ -fuzzy normal subgroup and the quotient group with respect to the  $\eta$ -fuzzy normal subgroup and demonstrate some of their various group theoretical properties.

**Keywords:** Fuzzy subgroup,  $\eta$ -fuzzy subgroup,  $\eta$ -fuzzy coset,  $\eta$ -fuzzy normal subgroup.

## 1 | Introduction

Fuzzy sets were first studied by Zadeh in [12], and since there has been an incredible attention in this particular branch of mathematics because of its many applications in fields like engineering and computer science as well as the analysis of social and economic behaviour. Rosenfeld [8] introduced the concept of fuzzy groups on fuzzy sets in and developed number of basic results for fuzzy groups. In fact, the fuzzy subgroups admit many algebraic properties of the groups. For more details, we refer to [9], [10]. Anthony in [2] redefined the concept of fuzzy subgroup. Later, Das [4] modified Zadeh and Rosenfeld's work by defining the level subgroups of a given group. The concept of fuzzy homomorphism between two groups was defined by Chakraborty and Khare [3], they also examined how it affected fuzzy subgroups. Additionally, Ajmal [1] presented the concept of the typical kernel of a group homomorphism in fuzzy subgroups. The most recent research on the use of fuzzy sets in various algebraic structures may be found in [11], [13]-[16]. Gupta and Qi [5] developed the notion of T operators on fuzzy sets. The theory of fuzzy operators plays a key role in various disciplines, specifically in the field of engineering and artificial intelligence. This significant application of fuzzy operators motivates us to familiarize the concept of a fuzzy set based on these operators.

In this paper, a fuzzy set is defined in relation to a  $N_T$ -operator. With the help of fuzzy subset, we propose a new version of fuzzy subgroup called it  $\eta$ -fuzzy subgroup and analyse its supplementary

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theory, derive analogues for several fundamental group theoretic results. Using the classical homomorphism, we demonstrate that the homomorphic image (pre-image) of  $\eta$ -fuzzy subgroup is  $\eta$ -fuzzy subgroup. Furthermore, we introduce the concept of  $\eta$ -fuzzy cosets and fuzzy normal subgroup. We also define isomorphism between the quotient group with respect to the normal subgroup  $G_\rho$ . Since  $\eta$ -fuzzy subgroup is more abstract structure than the fuzzy subgroup and the results in this version are more general than the existing results in the literature. Throughout in this paper, we will refer to  $FS(G)$  and  $FNS(G)$  as the fuzzy subgroup and fuzzy normal subgroup of a group  $G$ , respectively.

## 2 | Preliminaries

We review some of these core concepts which are relevant to the rest of our discussion.

**Definition 1 ([6]).** Let  $E$  be a nonempty set. A mapping  $\rho: E \rightarrow [0,1]$  is called a fuzzy subset of  $E$ .

**Definition 2 ([6]).** Let  $\rho$  and  $\sigma$  be fuzzy sets of a set  $E$ . Their intersection  $\rho \cap \sigma$  and union  $\rho \cup \sigma$  are fuzzy sets of  $E$  defined by

- I.  $(\rho \cap \sigma)(a_1) = \min\{\rho(a_1), \sigma(a_1)\}$  for all  $a_1 \in E$ .
- II.  $(\rho \cup \sigma)(a_1) = \max\{\rho(a_1), \sigma(a_1)\}$  for all  $a_1 \in E$ .

**Definition 3 ([6]).** Let  $\rho$  be a fuzzy set of a set  $E$ . For  $\gamma \in [0,1]$ , the set  $\rho_\gamma = \{a_1 : a_1 \in E, \rho(a_1) \geq \gamma\}$  is called level subset of  $\rho$ .

**Definition 4 ([6]).** Let  $\rho$  be a fuzzy subset of a group  $G$ . Then  $\rho$  is called a  $FS(G)$  if

- I.  $\rho(a_1 a_2) \geq \min\{\rho(a_1), \rho(a_2)\}$  for all  $a_1, a_2 \in G$ .
- II.  $\rho(a_1^{-1}) \geq \rho(a_1)$  for all  $a_1 \in G$ .

**Lemma 1 ([6]).** Let  $\rho: G \rightarrow [0,1]$  be a  $FS(G)$ , for all  $a_1 \in G$ , we have

- I.  $\rho(e) \geq \rho(a_1)$  for all  $a_1 \in G$ .
- II.  $\rho(a_1^{-1}) = \rho(a_1)$ .

**Theorem 1 ([4]).** Let  $\rho$  be a fuzzy subset of group  $G$  then  $\rho$  is  $FS(G)$  if and only if the level subset  $\rho_\gamma$ , for  $\gamma \in [0,1]$ ,  $\rho(e) \geq \gamma$ , is subgroup of  $G$ , where  $e$  is an identity of  $G$ .

**Definition 5 ([7]).** A  $FS(G)$   $\rho$  is called a  $FNS(G)$  if  $\rho(a_1 a_2) = \rho(a_2 a_1)$  for all  $a_1, a_2 \in G$ .

**Definition 6 ([7]).** Let  $\rho$  be a  $FS(G)$ . For any  $a_1 \in G$ , define a map  $\rho_{a_1}: G \rightarrow [0,1]$  as follows:

- I.  $\rho_{a_1}(g) = \rho(g a_1^{-1})$  for all  $g \in G$ .
- II.  $\rho_{a_1}$  is called fuzzy coset of  $G$  determined by  $a_1$  and  $\rho$ .

**Definition 7 ([5]).** A map  $T: [0,1] \times [0,1] \rightarrow [0,1]$  define by  $(a_1, a_2) \mapsto \min\{a_1, a_2\}$  is  $T$ -norm iff for all  $a_1, a_2, a_3, a_4 \in [0,1]$

- I.  $T(a_1, a_2) = T(a_2, a_1)$ ,
- II.  $T(a_1, T(a_2, a_3)) = T(T(a_1, a_2), a_3)$ ,
- III.  $T(a_1, 1) = T(1, a_1) = 1$ ,
- IV. If  $a_1 \leq a_3$  and  $a_2 \leq a_4$  then  $T(a_1, a_2) \leq T(a_3, a_4)$ .

### 3 | $\eta$ -Fuzzy Subsets and Their Properties

**Definition 8.** Let  $N_T$  be an operator defined as  $N_T : [0,1] \times [0,1] \rightarrow [0,1]$  by

$$N_T(a_1, a_2) = \min\{1 - a_1, 1 - a_2\} \text{ for all } a_1, a_2 \in [0,1].$$

Infect,  $N_T$  admits the properties below, for all  $a_1, a_2, a_3, a_4 \in [0,1]$ :

- I.  $N_T(a_1, a_2) = N_T(a_2, a_1)$ ,
- II.  $N_T(a_1, 1) = N_T(1, a_1) = 0$ ,
- III. If  $a_1 \leq a_3$  and  $a_2 \leq a_4$ , then  $N_T(a_1, a_2) \geq N_T(a_3, a_4)$ .

The operator  $N_L$  is non-associative.

**Definition 9.** Let  $\rho : X \rightarrow [0,1]$  be fuzzy subset of  $X$  and  $\eta \in [0,1]$ , the fuzzy subset  $\rho^\eta$  of  $X$  (w. r. t fuzzy set  $\rho : X \rightarrow [0,1]$ ) denotes the  $\eta$ -fuzzy subset of  $X$  and is defined as follows:

$$\rho^\eta(a_1) = \min\{1 - \rho(a_1), 1 - \eta\} \text{ for all } a_1 \in X.$$

**Example 1.** Let  $A = \{\text{set of young people}\}$  define  $\rho$  fuzzy set on  $A$  as follows:

$$\rho_A(a_1) = \begin{cases} 1, & \text{if } a_1 < 25, \\ \frac{40 - a_1}{15}, & \text{if } 25 \leq a_1 \leq 40, \\ 0, & \text{if } a_1 > 40. \end{cases}$$

Take  $\eta = 0.6$ , now for  $a_1 = 20$ , we have  $\rho_A^\eta(a_1) = 0$ . For  $a_2 = 30$ , we have  $\rho_A^\eta(a_2) = 0.333$  and for  $a_3 = 45$ , we have  $\rho_A^\eta(a_3) = 0.4$ .

Remark 1: It is important to note that one can obtain the negation of classical fuzzy subset  $\rho(a_1)$  by choosing the value of  $\eta = 0$  in above definition whereas the case become crisp for the choice of  $\eta = 1$ . These algebraic facts lead to note that the case illustrates the  $\eta$ -fuzzy version with respect to any fuzzy subset for the value of  $\eta$ , when  $\eta \in (0,1)$ .

**Definition 10.** Let  $\xi : G \rightarrow G'$  where  $G, G'$  are groups and  $\rho$  and  $\sigma$  be  $\eta$ -fuzzy subsets of  $G$  and  $G'$  respectively. Then  $\xi(\rho^\eta)$  and  $\xi^{-1}(\sigma^\eta)$  are the image of  $\eta$ -fuzzy subset  $\rho^\eta$  and the inverse image of  $\eta$ -fuzzy subset  $\sigma^\eta$  respectively, defined as:

- I.  $\xi(\rho^\eta)(a_2) = \begin{cases} \sup \rho^\eta(a_1) : a_1 \in \xi^{-1}(a_2), & \text{if } \xi^{-1}(a_2) \neq \emptyset, \\ 0, & \text{if } \xi^{-1}(a_2) = \emptyset. \end{cases}$
- II.  $\xi^{-1}(\sigma^\eta)(a_1) = \sigma^\eta(\xi(a_1))$  for all  $a_1 \in G$ .

**Example 2.** Let  $\xi : V_4 \rightarrow \mathbb{R}$  where  $V_4 = \{1, a_1, a_2, a_1a_2\}$  defined as follows:

$\xi(1) = 1, \xi(a_1) = 2, \xi(a_2) = -2$  and  $\xi(a_1a_2) = 4$ . Define fuzzy set  $\rho$  on  $V_4$  given by  $\rho(1) = 1, \rho(a_1) = 0.8, \rho(a_2) = 0.4$  and  $\rho(a_1a_2) = 0.5$  define fuzzy set  $\sigma$  on  $\mathbb{R}$  as follows:

$$\sigma(a_1) = \frac{1}{|a_1|}.$$

Take  $\eta = 0.3$ , so  $\xi(\rho^\eta(a_2)) = \{0, 0.2, 0.7, 0.5\}$  and  $\xi^{-1}(\sigma^\eta(a_1)) = \{0, 0.5, 0.7\}$

**Theorem 2.**

- I. Let  $\rho$  and  $\sigma$  be any two fuzzy subsets of a set  $E$  then  $(\rho \cap \sigma)^\eta = \rho^\eta \cap \sigma^\eta$ .
- II. Let  $\rho$  and  $\sigma$  be two fuzzy subsets of a set  $P$  and  $Q$  respectively and  $\xi : P \rightarrow Q$  be a mapping, then

- $\xi(\rho^\eta) = (\xi(\rho))^\eta$ .
- $\xi^{-1}(\rho^\eta) = (\xi^{-1}(\rho))^\eta$ .

Proof: By *Definition 9*, we have

$$\begin{aligned} (\rho \cap \sigma)^\eta(a_1) &= \min \{1 - (\rho \cap \sigma)(a_1), 1 - \eta\}, \text{ where } a_1 \in E \text{ and } \eta \in [0, 1]. \\ &= \min \left\{ 1 - \min \{ \rho(a_1), \sigma(a_1) \}, 1 - \eta \right\} \\ &= \min \left\{ \min \{ 1 - \rho(a_1), 1 - \eta \}, \min \{ 1 - \sigma(a_1), 1 - \eta \} \right\} \\ &= \min \{ \rho^\eta(a_1), \sigma^\eta(a_1) \} = (\rho^\eta \cap \sigma^\eta)(a_1) \text{ for all } a_1 \in E. \end{aligned}$$

Consequently,  $(\rho^\eta \cap \sigma^\eta)(a_1) = \rho^\eta \cap \sigma^\eta$ .

$$\begin{aligned} \text{I. } \xi(\rho^\eta)(a_2) &= \sup \{ \rho^\eta(a_1) : \xi(a_1) = a_2 \} \\ &= \sup \{ \min \{ 1 - \rho(a_1), 1 - \eta \} \} \\ &= \min \{ \sup \{ 1 - \rho(a_1), 1 - \eta \} \} \\ &= \min \{ 1 - \xi(\rho)(a_2), 1 - \eta \} \\ &= (\xi(\rho))^\eta \text{ for all } a_2 \in Q. \end{aligned}$$

Hence,  $\xi(\rho^\eta) = (\xi(\rho))^\eta$ .

II. From *Definition 9*, we have

$$\begin{aligned} \xi^{-1}(\rho^\eta)(a_1) &= (\rho^\eta)\xi(a_1) = \min \{ 1 - \rho(\xi(a_1)), 1 - \eta \} = \min \{ 1 - \xi^{-1}\rho(a_1), 1 - \eta \} \\ &= (\xi^{-1}(\rho))^\eta(a_1) \text{ for all } a_1 \in P. \end{aligned}$$

Hence,  $\xi^{-1}(\rho^\eta) = (\xi^{-1}(\rho))^\eta$ .

## 4 | $\eta$ -Fuzzy Subgroups

This section deals with the concept of  $\eta$ -FS(G) and  $\eta$ -FNS(G). We prove that every FS(G) (FNS(G)) is also  $\eta$ -FS(G) (FNS(G)) but converse need not to be true. The concept of  $\eta$ -fuzzy coset is defined and discussed deeply. Moreover, applying the idea of  $\eta$ -FNS(G), we introduced the quotient group with respect to FNS(G). This leads us to develop a natural homomorphism with respect to  $\eta$ -FNS(G) from a group G to its quotient group. Additionally, we discover the homomorphic image and pre-image of  $\eta$ -FS(G) ( $\eta$ -FNS(G)). We conclude this section by establishing an isomorphism between the quotient group  $\frac{G}{\rho^\eta}$  and  $\frac{G}{G_{\rho^\eta}}$ .

**Definition 11.** Let G be a group and  $\rho: G \rightarrow [0, 1]$  be a fuzzy subset G. Let  $\eta \in [0, 1]$ , then  $\rho$  is called  $\eta$ -FS(G) if  $\rho^\eta$  is FS(G). In other words,  $\rho$  is  $\eta$ -FS(G) if  $\rho^\eta$  admits the following properties, for all  $a_1, a_2 \in G$ :

- I.  $\rho^\eta(a_1a_2) \geq \min \{ \rho^\eta(a_1), \rho^\eta(a_2) \}$ .
- II.  $\rho^\eta(a_1^{-1}) = \rho^\eta(a_1)$ .

**Example 3.** Let  $\rho$  be a fuzzy subset of the group  $G = V_4 = \{1, a_1, a_2, a_1a_2\}$  defined as  $\rho(1) = 0.7$  and  $\rho(a_1) = \rho(a_2) = \rho(a_1a_2) = 0.9$ .

Define  $\eta$ -fuzzy subset  $\rho^\eta$  of  $G$  for  $\eta = 0.8$  as follows:

$$\rho^\eta(1) = 0.2 \text{ and } \rho^\eta(a_1) = \rho^\eta(a_2) = \rho^\eta(a_1a_2) = 0.1.$$

Clearly,  $\rho^\eta$  is  $\eta$ -FS( $G$ ).

Remark 2: Note that,  $\rho$  is  $\eta$ -FS( $G$ ) for any choice of  $\eta$  in each of the following case

- I.  $\rho(a_1a_2) \geq \eta > \min \{ \rho(a_1), \rho(a_2) \}$ .
- II.  $\eta \geq \rho(a_1a_2) > \min \{ \rho(a_1), \rho(a_2) \}$ .
- III.  $\rho(a_1a_2) > \min \{ \rho(a_1), \rho(a_2) \} > \eta$ .
- IV. For  $\eta = 0$ , we get the complement of classical fuzzy subgroup.

**Proposition 1.** Let  $\rho$  be  $\eta$ -FS( $G$ ). Then the following statements hold:

- $\rho^\eta(a_1) \leq \rho^\eta(e)$  for all  $a_1 \in G$  and  $e$  is identity element of  $G$ .
- $\rho^\eta(a_1a_2^{-1}) = \rho^\eta(e)$  gives  $\rho^\eta(a_1) = \rho^\eta(a_2)$  for all  $a_1, a_2 \in G$ .

Proof:

$$\begin{aligned} \text{I. Since } \rho^\eta(a_1a_1^{-1}) &= \rho^\eta(e) \text{ and also } \rho^\eta(a_1a_1^{-1}) = \min \{ \rho^\eta(a_1), \rho^\eta(a_1^{-1}) \} \\ &= \min \{ \rho^\eta(a_1), \rho^\eta(a_1) \} = \rho^\eta(a_1). \end{aligned}$$

This implies that  $\rho^\eta(a_1) \leq \rho^\eta(e)$ , for all  $a_1 \in G$ .

$$\text{II. Since we have } \rho^\eta(a_1) = \rho^\eta(a_1a_2^{-1}a_2) \geq \min \{ \rho^\eta(a_1a_2^{-1}), \rho^\eta(a_2) \}.$$

Then by our assumption we have  $\rho^\eta(a_1) \geq \min \{ \rho^\eta(e), \rho^\eta(a_2) \}$  which implies that

$$\rho^\eta(a_1) \geq \rho^\eta(a_2).$$

Similarly,  $\rho^\eta(a_2) = \rho^\eta(a_2a_1^{-1}a_1) \geq \min \{ \rho^\eta(a_2a_1^{-1}), \rho^\eta(a_1) \}$  then by our assumption, we have  $\rho^\eta(a_2) \geq \min \{ \rho^\eta(e), \rho^\eta(a_1) \} = \rho^\eta(a_1)$ , which implies that  $\rho^\eta(a_2) \geq \rho^\eta(a_1)$ .

Hence,  $\rho^\eta(a_1) = \rho^\eta(a_2)$ .

The next result leads to note that every FS( $G$ ) is  $\eta$ -FS( $G$ ).

**Proposition 2.** Every FS( $G$ ) is also  $\eta$ -FS( $G$ ).

Proof: Let  $\rho$  be a FS( $G$ ). Consider,  $\rho^\eta(a_1a_2) = \min \{ 1 - \rho(a_1a_2), 1 - \eta \}$ , where  $\eta \in [0, 1]$  and  $a_1, a_2 \in G$ .

$$\geq \min \{ 1 - \min \{ \rho(a_1), \rho(a_2) \}, 1 - \eta \}$$

$$= \min\{\min\{1 - \rho(a_1), 1 - \eta\}, \min\{1 - \rho(a_2), 1 - \eta\}\} = \min\{\rho^\eta(a_1), \rho^\eta(a_2)\}.$$

Thus, we have  $\rho^\eta(a_1 a_2) \geq \min\{\rho^\eta(a_1), \rho^\eta(a_2)\}$ .

Moreover,  $\rho^\eta(a_1^{-1}) = \min\{1 - \rho(a_1^{-1}), 1 - \eta\} = \min\{1 - \rho(a_1), 1 - \eta\} = \rho^\eta(a_1)$ .

This implies that  $\rho$  is  $\eta$ -FS(G).

Remark 3: The converse of the aforementioned proposition must not be true.

**Example 4.** Let  $G = S_3 = \{(1), (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$ ,

$$\rho(1) = 0.4, \rho((1\ 2)) = \rho((1\ 3)) = \rho((2\ 3)) = 0.5,$$

And

$$\rho((1\ 2\ 3)) = \rho((1\ 3\ 2)) = 0.6.$$

Consider the  $\eta$ -fuzzy set for  $\eta = 0.55$  as follows:

$$\rho^\eta(1) = 0.45, \rho^\eta((1\ 2)) = \rho^\eta((1\ 3)) = \rho^\eta((2\ 3)) = 0.45,$$

And

$$\rho^\eta((1\ 2\ 3)) = \rho^\eta((1\ 3\ 2)) = 0.4.$$

Clearly, the fuzzy subset  $\rho$  is  $\eta$ -FS(G).

Moreover,  $\rho$  is not FS(G) because all possible level subset  $\rho_{0.4} = \{(1), (1\ 2), (1\ 3), (2\ 3)\}$ ,  $\rho_{0.5} = \{(1\ 2), (1\ 3), (2\ 3)\}$  and  $\rho_{0.6} = \{(1\ 2\ 3), (1\ 3\ 2)\}$ .

$\rho_{0.4}$ ,  $\rho_{0.5}$  and  $\rho_{0.6}$  are not subgroups of  $S_3 = G$ .

**Proposition 3.** Let  $G$  be a group and  $\rho$  be its fuzzy subset such that  $\rho(a_1) = \rho(a_1^{-1})$  for all  $a_1 \in G$ . Let  $\eta \geq m$ , where  $\eta \in [0, 1]$  and  $m = \sup\{\rho(a_1), a_1 \in G\}$ , then  $\rho$  is also  $\eta$ -FS(G).

Proof: Since, we have  $\eta \geq m$ . So,  $\eta \geq \sup\{\rho(a_1), a_1 \in G\}$ , which implies  $\eta \geq \rho(a_1)$  for all  $a_1 \in G$ .

So, we have  $\rho^\eta(a_1) = \min\{1 - \rho(a_1), 1 - \eta\} = 1 - \eta$  for all  $a_1 \in G$ .

This implies that  $\rho^\eta(a_1 a_2) \geq \min\{\rho^\eta(a_1), \rho^\eta(a_2)\}$  for all  $a_1, a_2 \in G$ .

Also,  $\rho^\eta(a_1^{-1}) = \rho^\eta(a_1)$ .

Hence,  $\rho$  is  $\eta$ -FS(G).

**Proposition 4.** Let  $\rho$  and  $\sigma$  be any two  $\eta$ -FS(G). Then  $\rho \cap \sigma$  is also  $\eta$ -FS(G).

Proof: Let  $\rho$  and  $\sigma$  be two  $\eta$ -FS(G) of a group  $G$  and let  $a_1, a_2 \in G$ .

Since,  $(\rho \cap \sigma)^\eta(a_1) = (\rho^\eta \cap \sigma^\eta)(a_1)$  hold, then we have

$$(\rho \cap \sigma)^\eta(a_1 a_2) = (\rho^\eta \cap \sigma^\eta)(a_1 a_2).$$

So,  $(\rho^\eta \cap \sigma^\eta)(a_1 a_2) = \min \{ \rho^\eta(a_1 a_2), \sigma^\eta(a_1 a_2) \}$

$$\begin{aligned} &\geq \min\{\min \{ \rho^\eta(a_1), \rho^\eta(a_2) \}, \min\{ \sigma^\eta(a_1), \sigma^\eta(a_2) \} \} \\ &= \min\{\min\{ \rho^\eta(a_1), \sigma^\eta(a_1) \}, \min\{ \rho^\eta(a_2), \sigma^\eta(a_2) \} \} \\ &= \min \{ (\rho^\eta \cap \sigma^\eta)(a_1), (\rho^\eta \cap \sigma^\eta)(a_2) \}. \end{aligned}$$

This implies that  $(\rho^\eta \cap \sigma^\eta)(ab) \geq \min \{ (\rho^\eta \cap \sigma^\eta)(a), (\rho^\eta \cap \sigma^\eta)(a_2) \}$ .

Moreover,  $(\rho \cap \sigma)^\eta(a_1^{-1}) = (\rho^\eta \cap \sigma^\eta)(a_1^{-1}) = \min \{ \rho^\eta(a_1),$

$$\rho^\eta(a_1^{-1}) \} = \min \{ \rho^\eta(a_1), \rho^\eta(a_1) \}.$$

We have  $(\rho \cap \sigma)^\eta(a_1^{-1}) = (\rho \cap \sigma)^\eta(a_1)$ .

Consequently,  $\rho \cap \sigma$  is  $\eta$ -FS(G).

**Corollary 1.** The intersection of any finite number of  $\eta$ -FS(G) is also  $\eta$ -FS(G).

**Proposition 5.** Let  $\rho$  and  $\sigma$  be any two  $\eta$ -FS(G). Then  $\rho \cup \sigma$  need not to be  $\eta$ -FS(G).

**Example 5.** Let  $G = Q_8 = \{ \pm 1, \pm i, \pm j, \pm k \}$ . Take two subgroups of G that are

$$H_1 = \{ \pm 1, \pm i \} \text{ and } H_2 = \{ \pm 1, \pm j \}.$$

Let  $\rho$  and  $\sigma$  be two fuzzy subsets of G as:

$$\rho(a_1) = \begin{cases} 0.2, & \text{if } a_1 \in H_1, \\ 0.9, & \text{otherwise.} \end{cases}$$

And

$$\sigma(a_1) = \begin{cases} 0.3, & \text{if } a_1 \in H_2, \\ 1, & \text{otherwise.} \end{cases}$$

Since,  $\rho^\eta(a) = \min \{ 1 - \rho(a), 1 - \eta \}$ .

Define  $\eta$ -fuzzy subsets  $\rho^\eta$  and  $\sigma^\eta$  for  $\eta = 0$ , as follows:

$$\rho^0(a_1) = \begin{cases} 0.8, & \text{if } a_1 \in H_1, \\ 0.1, & \text{otherwise.} \end{cases}$$

And

$$\sigma^0(a_1) = \begin{cases} 0.7, & \text{if } a_1 \in H_2, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that  $\rho^0$  and  $\sigma^0$  are 0-FS(G).

Now, we define  $\rho^0 \cup \sigma^0$  as:

$$(\rho^0 \cup \sigma^0)(a_1) = \max\{ \rho^0(a_1), \sigma^0(a_1) \},$$

So, we have

$$(\rho^0 \cup \sigma^0)(a_1) = \begin{cases} 0.8, & \text{if } a_1 \in H_1, \\ 0.7, & \text{if } a_1 \in H_2 \setminus H_1, \\ 0.1, & \text{otherwise.} \end{cases}$$

Let  $a_1 = i$  and  $a_2 = j$ ,

Observe that  $(\rho^0 \cup \sigma^0)(i) = 0.8$  and  $(\rho^0 \cup \sigma^0)(j) = 0.7$ .

So,  $\min \{(\rho^0 \cup \sigma^0)(i), (\rho^0 \cup \sigma^0)(j)\} = 0.7$ , but  $(\rho^0 \cup \sigma^0)(ij) = (\rho^0 \cup \sigma^0)(k) = 0.1$ .

This implies that

$$(\rho^0 \cup \sigma^0)(ij) \leq \min \{(\rho^0 \cup \sigma^0)(i), (\rho^0 \cup \sigma^0)(j)\}.$$

Hence,  $\rho^0 \cup \sigma^0$  is not 0-FS(G).

**Example 6.** Let  $G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ . Take a subgroup of  $G$  that is  $H_1 = \{\pm 1, \pm i\}$ .

Let  $\rho$  and  $\sigma$  be two fuzzy subsets of  $G$  as:

$$\rho(a_1) = \begin{cases} 0.3, & \text{if } a_1 \in H_1, \\ 0.9, & \text{otherwise.} \end{cases}$$

And

$$\sigma(a_1) = \begin{cases} 0.2, & \text{if } a_1 \in H_1, \\ 1, & \text{otherwise.} \end{cases}$$

Then we have  $\rho^0$  and  $\sigma^0$  as follows:

$$\rho^0(a_1) = \begin{cases} 0.7, & \text{if } a_1 \in H_1, \\ 0.1, & \text{otherwise.} \end{cases}$$

And

$$\sigma^0(a_1) = \begin{cases} 0.8, & \text{if } a_1 \in H_1, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$(\rho^0 \cup \sigma^0)(a_1) = \begin{cases} 0.8, & \text{if } a_1 \in H_1, \\ 0.1, & \text{otherwise.} \end{cases}$$

It can be easily seen that  $\rho^0 \cup \sigma^0$  is 0-FS(G).

**Definition 12.** Let  $\rho$  be  $\eta$ -FS(G), for any  $a_1 \in G$  define  $\eta$ -fuzzy left coset  $a_1\rho^\eta$  of  $\rho$  in  $G$  as follows:

$$a_1\rho^\eta(x) = \min \{1 - \rho(a_1^{-1}x), 1 - \eta\} \text{ for all } a_1, x \in G.$$

Similarly, we define  $\eta$ -fuzzy right coset  $\rho^\eta a_1$  of  $\rho$  in  $G$  as follows:

$$\rho^\eta a_1(x) = \min \{1 - \rho(xa_1^{-1}), 1 - \eta\} \text{ for all } a_1, x \in G.$$

**Example 7.** Let  $\rho$  be a fuzzy subset of the group  $G = Z_4 = \{0, 1, 2, 3\}$  defined as

$$\rho(0) = 0.2, \rho(2) = 0.4 \text{ and } \rho(1) = \rho(3) = 0.4.$$

Define  $\eta$ -fuzzy subset  $\rho^\eta$  of  $G$  for  $\eta = 0.5$  as follows:

$$\rho^\eta(0) = 0.5, \rho^\eta(2) = 0.5 \text{ and } \rho^\eta(1) = \rho^\eta(3) = 0.4.$$



Clearly,  $\rho^\eta$  is  $\eta$ -FS( $G$ ). Consider  $\eta$ -fuzzy left coset of  $\rho$  by the element  $2 \in Z_4$  as follows:

$$2 + \rho^\eta(a_1) = \min \{1 - \rho(2 + a_1), 1 - \eta\} = \begin{cases} 0.4, & \text{if } a_1 \in \{0, 2\}, \\ 0.5, & \text{otherwise.} \end{cases}$$

Similarly, define  $\eta$ -fuzzy right coset of  $G$ .

**Proposition 6.** Let  $\rho$  be  $\eta$ -FS( $G$ ). Then  $\rho$  be  $\eta$ -FNS( $G$ ) if and only if  $a_1\rho^\eta(x) = \rho^\eta a_1(x)$  for all  $a_1 \in G$ .

Note:  $a_1\rho^\eta(x) = \rho^\eta(a_1^{-1}x)$  and  $\rho^\eta a_1(x) = \rho^\eta(xa_1^{-1})$  for all  $x \in G$ .

The following result leads to note that every FNS( $G$ ) is  $\eta$ -FNS( $G$ ).

**Example 8.** In view of Example 7  $\eta$ -FS( $G$ ) is a  $\eta$ -FNS( $G$ ), because its all  $\eta$ -fuzzy left cosets and  $\eta$ -fuzzy right cosets are equal. For instance, consider

$$2 + \rho^\eta(a_1) = \rho^\eta(a_1) + 2 = \begin{cases} 0.4, & \text{if } a_1 \in \{0, 2\}, \\ 0.5, & \text{otherwise.} \end{cases}$$

**Proposition 7.** Every FNS( $G$ ) is also  $\eta$ -FNS( $G$ ).

Proof: Suppose that  $\rho$  is FNS( $G$ ) which implies that  $a_1\rho = \rho a_1$ .

Then for any  $x \in G$  we have  $\rho(a_1^{-1}x) = \rho(xa_1^{-1})$ . So, we have

$$\min \{1 - \rho(a_1^{-1}x), 1 - \eta\} = \min \{1 - \rho(xa_1^{-1}), 1 - \eta\}.$$

This implies that  $a_1\rho^\eta(x) = \rho^\eta a_1(x)$  for all  $x \in G$ . Consequently,  $\rho$  is  $\eta$ -FNS( $G$ ).

Note that the converse of above result need not to be true.

**Example 9.** Let  $G = D_3 = \langle a, b : a^3 = b^2 = e, ba = a^2b \rangle$ .

Define a fuzzy subset  $\rho$  of  $G$  as follows:

$$\rho(a) = \begin{cases} 0.3, & \text{if } a \in \langle b \rangle, \\ 0.1, & \text{otherwise.} \end{cases}$$

Take  $\eta = 0.6$  then we have  $\rho^\eta(a) = 1 - \eta$  for all  $a \in G$ .

$$a_1\rho^\eta(g) = \min \{1 - \rho(a_1^{-1}g), 1 - \eta\} = 1 - \eta$$

$$= \min \{1 - \rho(ga_1^{-1}), 1 - \eta\} = \rho^\eta a_1(g).$$

Then  $a_1\rho^\eta(g) = \rho^\eta a_1(g)$  which implies that  $\rho$  is  $\eta$ -FNS( $G$ ). But it can be seen that  $\rho$  is not FNS( $G$ ). This is because  $\rho((a^2)(ab)) = 0.3$  and  $\rho((ab)(a^2)) = 0.1$ . i.e.  $\rho(a^{-1}g) = \rho(ga^{-1})$  not hold.

**Proposition 8.** Let  $\rho$  be  $\eta$ -FNS( $G$ ). Then  $\rho^\eta(b^{-1}a_1b) = \rho^\eta(a_1)$  or  $\rho^\eta(a_1a_2) = \rho^\eta(a_2a_1)$  hold, for all  $a_1, a_2 \in G$ .

Proof: Since, we have  $\rho$  be  $\eta$ -FNS( $G$ ) then we have  $a_1\rho^\eta = \rho^\eta a_1$  for all  $a_1 \in G$ .

This implies that  $a_1\rho^\eta(a_2^{-1}) = \rho^\eta a_1(a_2^{-1})$  for all  $a_2^{-1} \in G$ .

$$= \min \{1 - \rho(a_1^{-1}a_2^{-1}), 1 - \eta\} = \min \{1 - \rho(a_2^{-1}a_1^{-1}), 1 - \eta\} = \rho^\eta(a_1^{-1}a_2^{-1}) = \rho^\eta(a_2^{-1}a_1^{-1}).$$

Consequently, we have  $\rho^\eta((a_2a_1)^{-1}) = \rho^\eta((a_1a_2)^{-1})$ .

Hence,  $\rho^\eta(a_1a_2) = \rho^\eta(a_2a_1)$ .

**Theorem 3.** Let  $\rho$  be  $\eta$ -FS(G). Then following statements are equivalent:

- I.  $\rho^\eta(a_2a_1) = \rho^\eta(a_1a_2)$  for all  $a_1, a_2 \in G$ .
- II.  $\rho^\eta(a_1a_2a_1^{-1}) = \rho^\eta(a_2)$  for all  $a_1, a_2 \in G$ .
- III.  $\rho^\eta(a_1a_2a_1^{-1}) \geq \rho^\eta(a_2)$  for all  $a_1, a_2 \in G$ .
- IV.  $\rho^\eta(a_1a_2a_1^{-1}) \leq \rho^\eta(a_2)$  for all  $a_1, a_2 \in G$ .

**Proposition 9.** Let  $\rho$  be  $\eta$ -FS(G). Let  $\eta \geq m$ , where  $\eta \in [0,1]$  and  $m = \sup \{ \rho(a_1), a_1 \in G \}$ , then  $\rho$  is also  $\eta$ -FNS(G).

Proof: Since, we have  $\eta \geq m$ , so  $\eta \geq \sup \{ \rho(a_1), a_1 \in G \}$ , which implies that  $\eta \geq \rho(a_1)$  for all  $a_1 \in G$ .

So, we have  $\rho^\eta(a_1) = \min \{ 1 - \rho(a_1), 1 - \eta \} = 1 - \eta$ ,

$\rho^\eta a_1(g) \geq \min \{ 1 - \rho(ga_1^{-1}), 1 - \eta \} = 1 - \eta$ .

Similarly,  $a_1\rho^\eta(g) \geq \min \{ 1 - \rho(a_1^{-1}g), 1 - \eta \} = 1 - \eta$ .

This implies that  $a_1\rho^\eta = \rho^\eta a_1$  for all  $a_1 \in G$ .

Hence,  $\rho$  is also  $\eta$ -FNS(G).

The following result illustrate that the set  $G_{\rho^\eta}$  is infect a normal subgroup of G.

**Proposition 10.** Let  $\rho$  be  $\eta$ -FNS(G). Then the set define as  $G_{\rho^\eta} = \{ a_1 \in G : \rho^\eta(a_1) = \rho^\eta(e) \} \trianglelefteq G$ .

Proof: Since,  $G_{\rho^\eta}$  is nonempty because  $e \in G$ . Let  $a_1, a_2 \in G_{\rho^\eta}$ ,

$\rho^\eta(a_1a_2^{-1}) \geq \min \{ \rho^\eta(a_1), \rho^\eta(a_2^{-1}) \}$  for all  $a_1, a_2 \in G$

$= \min \{ \rho^\eta(a_1), \rho^\eta(a_2) \}$  for all  $a_1, a_2 \in G$

$= \min \{ \rho^\eta(e), \rho^\eta(e) \}$  for all  $a_1, a_2 \in G$

$= \rho^\eta(e)$ .

This implies that  $\rho^\eta(a_1a_2^{-1}) \geq \rho^\eta(e)$ .

Since,  $\rho^\eta$  is FS(G) which implies that  $\rho^\eta(a_1a_2^{-1}) \leq \rho^\eta(e)$ .

Hence,  $\rho^\eta(a_1a_2^{-1}) = \rho^\eta(e)$  implies that  $G_{\rho^\eta}$  is subgroup of G.

Now we prove it is normal subgroup of G. Let  $a_1 \in G_{\rho^\eta}$  and  $a_2 \in G$ , then we have

$\rho^\eta(a_2^{-1}a_1a_2) = \rho^\eta(a_1) = \rho^\eta(e)$ .

This implies that  $a_2^{-1}a_1a_2 \in G_{\rho^\eta}$ .

Consequently, we have  $G_{\rho^\eta} \trianglelefteq G$ .

**Proposition 11.** Let  $\rho$  be  $\eta$ -FNS( $G$ ) of a group  $G$ . Then the following statements hold

I.  $a_1\rho^\eta = a_2\rho^\eta \iff a_1^{-1}a_2 \in G_{\rho^\eta}$ .

II.  $\rho^\eta a_1 = \rho^\eta a_2 \iff a_1a_2^{-1} \in G_{\rho^\eta}$ .

Proof:

I. Suppose that  $a_1\rho^\eta = a_2\rho^\eta$ , then we have

$$\begin{aligned} \rho^\eta(a_1^{-1}a_2) &= \min \{1 - \rho(a_1^{-1}a_2), 1 - \eta\} \\ &= a_1\rho^\eta(a_2) = a_2\rho^\eta(a_2) \\ &= \min \{1 - \rho(a_2^{-1}a_2), 1 - \eta\} \\ &= \min \{1 - \rho(e), 1 - \eta\} \\ &= \rho^\eta(e). \end{aligned}$$

Thus  $\rho^\eta(a_1^{-1}a_2) = \rho^\eta(e)$  implies that  $a_1^{-1}a_2 \in G_{\rho^\eta}$ .

Conversely,

$$a_1\rho^\eta(a_3) = \min \{1 - \rho(a_1^{-1}a_3), 1 - \eta\}.$$

$$\begin{aligned} \text{So, } \rho^\eta(a_1^{-1}a_3) &= \rho^\eta(a_1^{-1}a_2 \cdot a_2^{-1}a_3) \geq \min \{ \rho^\eta(a_1^{-1}a_2), \rho^\eta(a_2^{-1}a_3) \} \\ &= \min \{ \rho^\eta(e), \rho^\eta(a_2^{-1}a_3) \} \\ &= \rho^\eta(a_2^{-1}a_3) = a_2\rho^\eta(a_3). \end{aligned}$$

By interchanging the  $a_1$  and  $a_2$ , we have  $a_1\rho^\eta(a_3) = a_2\rho^\eta(a_3)$  for all  $a_3 \in G$ .

Hence,  $a_1\rho^\eta = a_2\rho^\eta$ .

II. Similar as above proof.

**Proposition 12.** Let  $\rho$  be  $\eta$ -FNS( $G$ ) of a group  $G$  and  $a_1, a_2, x, y \in G$ . If  $a_1\rho^\eta = x\rho^\eta$  and  $a_2\rho^\eta = y\rho^\eta$  then  $a_1a_2\rho^\eta = xy\rho^\eta$ .

Proof: Given that  $a_1\rho^\eta = x\rho^\eta$  and  $a_2\rho^\eta = y\rho^\eta$ , which implies that  $a_1^{-1}x, a_2^{-1}y \in G_{\rho^\eta}$ .

$$\begin{aligned} \text{Now } (a_1a_2)^{-1}xy &= a_2^{-1}(a_1^{-1}x)y = a_2^{-1}(a_1^{-1}x)(a_2a_2^{-1})y \\ &= [a_2^{-1}(a_1^{-1}x)a_2](a_2^{-1}y) \in G_{\rho^\eta} \quad \because G_{\rho^\eta} \trianglelefteq G. \end{aligned}$$

This implies that  $(a_1a_2)^{-1}xy \in G_{\rho^\eta}$ . Hence,  $a_1a_2\rho^\eta = xy\rho^\eta$ .

**Proposition 13.** Let  $\frac{G}{\rho^\eta}$  be the collection of all  $\eta$ -fuzzy cosets of a  $\eta$ -FS(G). This form a group under the binary operation  $\otimes$  define on the set  $\frac{G}{\rho^\eta}$  as follows:

$$\rho^\eta a_1 \otimes \rho^\eta a_2 = \rho^\eta a_1 a_2 \text{ for all } a_1, a_2 \in G.$$

Proof: As we know that  $\frac{G}{\rho^\eta} = \{ \rho^\eta a_1 : a_1 \in G \}$ .

Let  $\rho^\eta a_1 = \rho^\eta a_1$  and  $\rho^\eta a_2 = \rho^\eta a_2'$  for all  $a_1, a_1', a_2, a_2' \in G$ .

$$\text{Let } g \in G \text{ then } (\rho^\eta a_1 \otimes \rho^\eta a_2)(g) = \rho^\eta a_1 a_2(g) = \min \{ 1 - \rho(g((a_1 a_2)^{-1}), 1 - \eta \}$$

$$= \min \{ 1 - \rho((g a_2^{-1}) a_1^{-1}), 1 - \eta \}$$

$$= \rho^\eta a_1 (g a_2^{-1}) = \rho^\eta a_1' (g a_2^{-1})$$

$$= \min \{ 1 - \rho((g a_2^{-1}) a_1^{-1}), 1 - \eta \}$$

$$= \min \{ 1 - \rho((a_1^{-1} g) b^{-1}), 1 - \eta \}$$

$$= \rho^\eta a_2 (a_1'^{-1} g) = \rho^\eta a_2' (a_1'^{-1} g)$$

$$= \min \{ 1 - \rho((a_1'^{-1} g) a_2'^{-1}), 1 - \eta \}$$

$$= \min \{ 1 - \rho(a_2'^{-1} (a_1'^{-1} g)), 1 - \eta \}$$

$$= \min \{ 1 - \rho((a_2'^{-1} a_1'^{-1})), 1 - \eta \}$$

$$= \min \{ 1 - \rho((a_1' a_2')^{-1} g), 1 - \eta \}$$

$$= \min \{ 1 - \rho(g (a_1' a_2')^{-1}), 1 - \eta \}$$

$$= \rho^\eta a_1' a_2'.$$

Hence,  $\otimes$  is well define operation on the set  $\frac{G}{\rho^\eta}$ .

The set  $\frac{G}{\rho^\eta}$  under this binary operation admits the associative law. The element  $\rho^\eta e$  of  $\frac{G}{\rho^\eta}$  is the identity element and the inverse of an element  $\rho^\eta a_1$  is  $\rho^\eta a_1^{-1}$ .

**Example 10.** In view of *Example 7* consider  $\rho$  as  $\eta$ -FNS(G).

The set  $\frac{G}{\rho^\eta} = \{ \rho^\eta, 2 + \rho^\eta \}$  forms a group under the following binary operation defined on  $\frac{G}{\rho^\eta}$  as  $(a_1 + \rho^\eta) + (a_2 + \rho^\eta) = ((a_1 + a_2) + \rho^\eta)$ .

Note that  $\rho^\eta(a_1)$  is identity element of this group and inverse of  $a_1 + \rho^\eta$  is  $(-a_1) + \rho^\eta$ .

**Definition 13.** The group  $\frac{G}{\rho^\eta}$  of  $\eta$ -fuzzy cosets of a  $\eta$ -FNS(G) is called the quotient group of G by  $\rho^\eta$ .

**Theorem 14.** Let  $G$  be a group and  $\frac{G}{\rho^\eta}$  be quotient group with respect to  $\eta$ -FNS( $G$ ). There exist a natural epimorphism from  $G$  to  $\frac{G}{\rho^\eta}$  which is defined as  $\xi(a_1) = \rho^\eta a_1$  with  $\text{Ker } \xi = G_{\rho^\eta}$ .

Proof: Let  $a_1, a_2 \in G$  be any elements. Then  $\xi(a_1 a_2) = \rho^\eta a_1 a_2 = \rho^\eta a_1 \rho^\eta a_2 = \xi(a_1) \xi(a_2)$ .

Therefore,  $\xi$  is homomorphism. For each  $\rho^\eta a_1 \in G_{\rho^\eta}$  we have  $a_1 \in G$  such that  $\xi(a_1) = \rho^\eta a_1$ .

This implies that  $\xi$  is onto homomorphism.

Now  $\text{Ker } \xi = \{ a_1 \in G : \xi(a_1) = \rho^\eta e \}$

$$= \{ a_1 \in G : \rho^\eta a_1 = \rho^\eta e \}$$

$$= \{ a_1 \in G : a_1 e^{-1} \in G_{\rho^\eta} \}$$

$$= \{ a_1 \in G : a_1 \in G_{\rho^\eta} \} = G_{\rho^\eta}.$$

## 5 | Homomorphism of $\eta$ -Fuzzy Subgroups

Anthony and Sherwood [2] observed that using a minimum in the Rosenfeld [8] definition of a fuzzy subgroup constrains the concept, rendering it useless in a variety of fuzzy situations. They introduced the concept of an T-norm and redefined the fuzzy subgroup by substituting a T-norm for a minimum. They investigated the impact of a simple homomorphism on fuzzy subgroups. Here, we present the results of homomorphism in frame work of our proposed definition.

**Theorem 4.** Let  $\xi : G \rightarrow G'$  be a bijective homomorphism of a group  $G$  into a group  $G'$ . If  $\rho$  is  $\eta$ -FS( $G$ ) then the homomorphic image  $\xi(\rho)$  is  $\eta$ -FS( $G'$ ).

Proof: Given that  $\rho$  be  $\eta$ -FS( $G'$ ). Let  $a'_1, a'_2 \in G'$  be any element then we have unique elements  $a_1, a_2 \in G$ , such that  $\varphi(a_1) = a'_1$  and  $\varphi(a_2) = a'_2$ .

Further,  $(\xi(\rho))^\eta(a'_1 a'_2)$

$$= \min \{ 1 - \xi(\rho)(a'_1 a'_2), 1 - \eta \}$$

$$= \min \{ 1 - \xi(\rho)(\xi(a_1) \xi(a_2)), 1 - \eta \}$$

$$= \min \{ 1 - \xi(\rho)(\xi(a_1 a_2)), 1 - \eta \}$$

$$= \min \{ 1 - \xi(\rho)(a_1 a_2), 1 - \eta \}$$

$$= \rho^\eta(a_1 a_2)$$

$$\geq \min \{ \rho^\eta(a_1), \rho^\eta(a_2) \} \text{ for all } a_1, a_2 \in G$$

$$= \min \{ \xi(\rho)^\eta \xi(a_1), \xi(\rho)^\eta \xi(a_2) \}$$

$$= \min \{ \xi(\rho)^\eta(a'_1), \xi(\rho)^\eta(a'_2) \}.$$

Consequently,

$$(\xi(\rho))^{\eta}(a'_1 a'_2) \geq \min\{(\xi(\rho))^{\eta}(a'_1), (\xi(\rho))^{\eta}(a'_2)\}.$$

$$\text{Also, } (\xi(\rho))^{\eta}(a'^{-1}) = \xi(\rho^{\eta})(a'^{-1}) = \xi(\rho^{\eta})(\varphi(a^{-1})) = \rho^{\eta}(a^{-1}) = \rho^{\eta}(a)$$

$$= \xi(\rho^{\eta})(\xi(a)) = (\xi(\rho))^{\eta}(a')$$

$$\text{Thus, } (\xi(\rho))^{\eta}(a'^{-1}) = (\xi(\rho))^{\eta}(a').$$

Consequently,  $\xi(\rho)$  is  $\eta$ -FS( $G'$ ).

**Theorem 5.** Let  $\xi : G \rightarrow G'$  be a bijective homomorphism of a group  $G$  into  $G'$ . If  $\rho$  is  $\eta$ -FNS( $G$ ) then the homomorphic image  $\xi(\rho)$  is  $\eta$ -FNS( $G'$ ).

Proof: Given that  $\rho$  be  $\eta$ -FNS( $G$ ). Let  $a'_1, a'_2 \in G'$  be any element then we have unique elements  $a_1, a_2 \in G$ , such that  $\xi(a_1) = a'_1$  and  $\xi(a_2) = a'_2$ .

$$(\xi(\rho))^{\eta}(a'_1 a'_2) = \min\{1 - \xi(\rho)(a'_1 a'_2), 1 - \eta\}$$

$$= \min\{1 - \xi(\rho)(\xi(a_1)\xi(a_2)), 1 - \eta\}$$

$$= \min\{1 - \xi(\rho)(\xi(a_1 a_2)), 1 - \eta\}$$

$$= \min\{1 - \xi(\rho)(\xi(a_2 a_1)), 1 - \eta\}$$

$$= \min\{1 - \xi(\rho)(\xi(a_2)\xi(a_1)), 1 - \eta\}$$

$$= \min\{1 - \xi(\rho)(a'_2 a'_1), 1 - \eta\}$$

$$= (\xi(\rho))^{\eta}(a'_1 a'_2).$$

Consequently,  $\xi(\rho)$  is  $\eta$ -FNS( $G'$ ).

**Theorem 6.** Let  $\xi : G \rightarrow G'$  be a homomorphism of a group  $G$  into  $G'$ . If  $\sigma$  is  $\eta$ -FS( $G'$ ) then the pre-image  $\xi^{-1}(\sigma)$  is  $\eta$ -FS( $G$ ).

Proof: Given that  $\sigma$  be  $\eta$ -FS( $G'$ ). Let  $a_1, a_2 \in G$  be any element then we have

$$(\xi^{-1}(\sigma))^{\eta}(a_1 a_2) = \xi^{-1}(\sigma^{\eta})(a_1 a_2) = \sigma^{\eta}(\xi(a_1 a_2)) = \sigma^{\eta}(\xi(a_1)\xi(a_2))$$

$$\geq \min\{\sigma^{\eta}(\xi(a_1)), \sigma^{\eta}(\xi(a_2))\}$$

$$= \min\{\xi^{-1}(\sigma^{\eta})(a_1), \xi^{-1}(\sigma^{\eta})(a_2)\}.$$

Thus,

$$(\xi^{-1}(\sigma))^{\eta}(a_1 a_2) \geq \min\{(\xi^{-1}(\sigma^{\eta}))(a_1), (\xi^{-1}(\sigma^{\eta}))(a_2)\}.$$

$$\text{Also } (\xi^{-1}(\sigma))^{\eta}(a^{-1}) = \xi^{-1}(\sigma^{\eta})(a^{-1}) = \sigma^{\eta}(\xi(a^{-1})) = \sigma^{\eta}(\xi(a)) = \xi^{-1}(\sigma^{\eta})(a).$$

Thus,  $(\xi^{-1}(\sigma))^\eta(a^{-1}) = (\xi^{-1}(\sigma))^\eta(a)$ .

Hence,  $\xi^{-1}(\sigma)$  is  $\eta$ -FNS(G).

**Theorem 7.** Let  $\xi : G \rightarrow G'$  be a homomorphism of a group  $G$  into  $G'$ . If  $\sigma$  is  $\eta$ -FNS( $G'$ ) then the pre-image  $\xi^{-1}(\sigma)$  is  $\eta$ -FNS( $G$ ).

Proof: Let  $\sigma$  be  $\eta$ -FNS( $G'$ ). Let  $a_1, a_2 \in G$  be any element then we have

$$\begin{aligned} (\xi^{-1}(\sigma))^\eta(a_1 a_2) &= \xi^{-1}(\sigma)^\eta(a_1 a_2) = \sigma^\eta(\xi(a_1 a_2)) = \sigma^\eta(\xi(a_1)\xi(a_2)) = \sigma^\eta(\xi(a_2 a_1)) \\ &= (\xi^{-1}(\sigma))^\eta(a_2 a_1). \end{aligned}$$

Thus,

$$(\xi^{-1}(\sigma))^\eta(a_1 a_2) = (\xi^{-1}(\sigma))^\eta(a_2 a_1).$$

Hence  $\varphi^{-1}(\sigma)$  is  $\eta$ -FNS( $G$ ).

**Theorem 8.** Let  $\rho$  be a  $\eta$ -FNS( $G$ ) and  $a_1, a_2 \in G$  be any element. If  $a_1 \rho^\eta = a_2 \rho^\eta$  then  $\rho^\eta(a_1) = \rho^\eta(a_2)$ .

Proof: Suppose that  $a_1 \rho^\eta = a_2 \rho^\eta$  then by Proposition 8 we have  $a_1^{-1} a_2 \in G_{\rho^\eta}$  and  $a_2^{-1} a_1 \in G_{\rho^\eta}$ . Since,  $\eta$ -FNS( $G$ ), this implies that

$$\rho^\eta(a_1) = \rho^\eta(a_2^{-1} a_1 a_2) \geq \min \{ \rho^\eta(a_2^{-1} a_1), \rho^\eta(a_2) \} = \min \{ \rho^\eta(e), \rho^\eta(a_2) \} = \rho^\eta(a_2).$$

Therefore, we have  $\rho^\eta(a_1) \geq \rho^\eta(a_2)$ . Similarly, we have  $\rho^\eta(a_1) \leq \rho^\eta(a_2)$ .

Hence,  $\rho^\eta(a_1) = \rho^\eta(a_2)$ .

**Theorem 9.** Let  $\rho$  be a  $\eta$ -FNS( $G$ ). Then  $\frac{G}{\rho^\eta} \cong \frac{G}{G_{\rho^\eta}}$ .

Proof: Define a map  $\varphi : \frac{G}{\rho^\eta} \rightarrow \frac{G}{G_{\rho^\eta}}$  by the rule

$\xi(a_1 \rho^\eta) = a_1 G_{\rho^\eta}$ , for all  $a_1 \in G$ . In view of Proposition 8  $\xi$  is well define.

The application of Proposition 8 leads to note that  $\xi$  is injective.  $\xi$  is obviously surjective.

Now consider, for  $a_1 \rho^\eta, a_2 \rho^\eta \in \frac{G}{\rho^\eta}$  we have  $\xi((a_1 \rho^\eta)(a_2 \rho^\eta)) = \varphi(a_1 a_2 \rho^\eta) = a_1 a_2 G_{\rho^\eta} = a_1 G_{\rho^\eta} a_2 G_{\rho^\eta} = \xi(a_1 \rho^\eta) \xi(a_2 \rho^\eta)$ .

So,  $\xi$  is homomorphism. Since  $\xi$  is a bijective mapping, which implies this is an isomorphism. Hence,  $\frac{G}{\rho^\eta} \cong \frac{G}{G_{\rho^\eta}}$ .

## 6 | Conclusion

In this paper, we introduced the idea of  $\eta$ -FNS( $G$ ) and  $\eta$ -fuzzy cosets for a given group. We used the concept of  $\eta$ -FNS( $G$ ) and discussed various related results and properties. We also studied the effect on the image and inverse image of  $\eta$ -FNS( $G$ ) ( $FNS(G)$ ) under group homomorphism. We shall extend this

concept to intuitionistic fuzzy sets in the upcoming studies and look into its numerous algebraic features. Moreover, we used the concept of  $\eta$ -fuzzy subset in classical field theory.

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