# On Algebraic Aspects of $\eta$-Fuzzy Subgroups

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**Abstract**

In this paper, we define the term of $\eta$-fuzzy subgroup and show that every fuzzy subgroup is a $\eta$-fuzzy subgroup. We define some of the algebraic properties of the concept of $\eta$-fuzzy cosets. Furthermore, we initiate the study of $\eta$-fuzzy normal subgroup and quotient group with respect to the $\eta$-fuzzy normal subgroup and demonstrate some of their various group theoretical properties.

**Keywords:** Fuzzy subgroup, $\eta$-fuzzy subgroup, $\eta$-fuzzy coset, $\eta$-fuzzy normal subgroup.



## 1. Introduction

Fuzzy sets were first studied by Zadeh in [12], and since there has been an incredible attention in this particular branch of mathematics because of its many applications in fields like engineering and computer science as well as the analysis of social and economic behaviour. In 1971, Rosenfeld introduced the concept of fuzzy groups on fuzzy sets in [8] and developed number of basic results for fuzzy groups. In fact, the fuzzy subgroups admit many algebraic properties of the groups. For more details, we refer to [9] and [10]. Anthony in [2] redefined the concept of fuzzy subgroup. Later, Das modified Zadeh and Rosenfeld's work in [4] by defining the level subgroups of a given group. The concept of fuzzy homomorphism between two groups was defined by Chakrabatty and Khare, they also examined how it affected fuzzy subgroups in

[3]. Additionally, Ajmal [1] presented the concept of the typical kernel of a group homomorphism in fuzzy subgroups. The most recent research on the use of fuzzy sets in various algebraic structures may be found in [11], [13], [14], [15] and [16]. In [5], Gupta developed the notion of T operators on fuzzy sets. The theory of fuzzy operators plays a key role in various disciplines, specifically in the field of engineering and artificial intelligence. This significant application of fuzzy operators motivates us to familiarize the concept of a fuzzy set based on these operators.

In this paper, a fuzzy set is defined in relation to a $N_T$-operator. With the help of fuzzy subset, we propose a new version of fuzzy subgroup called it $\eta$-fuzzy subgroup and analyse its supplementary theory, derive analogues for several fundamental group theoretic results. Using the classical homomorphism, we demonstrate that the homomorphic image (pre-image) of $\eta$-fuzzy subgroup is $\eta$-fuzzy subgroup. Furthermore, we introduce the concept of $\eta$–fuzzy cosets and fuzzy normal subgroup. We also define isomorphism between the quotient group with respect to the normal subgroup $G_\rho \eta$. Since $\eta$ −fuzzy subgroup is more abstract structure then the fuzzy subgroup and the results in this version are more general than the existing results in the literature. Throughout in this paper, we will refer to $FS(G)$ and $FNS(G)$ as the fuzzy subgroup and fuzzy normal subgroup of a group $G$, respectively.

## 2. Preliminaries

We review some of these core concepts which are relevant to the rest of our discussion.

**Definition 2.1 [6]:** Let $E$ be a nonempty set. A mapping $\rho: E \longrightarrow [0, 1]$ is called a fuzzy subset of $E$.

**Definition 2.2 [6]:** Let $\rho$ and $\sigma$ be fuzzy sets of a set $E$. Their intersection $\rho \cap \sigma$ and union $\rho \cup \sigma$ are fuzzy sets of $E$ defined by

i. $(\rho \cap \sigma)(a_1) = \min\{\rho(a_1), \sigma(a_1)\}, \quad \forall \; a_1 \in E.$

ii. $(\rho \cup \sigma)(a_1) = \max\{\rho(a_1), \sigma(a_1)\}, \quad \forall \; a_1 \in E.$

**Definition 2.3 [6]:** Let $\rho$ be a fuzzy set of a set $E$. For $\gamma \in [0,1]$, the set $\rho_\gamma = \{a_1 : a_1 \in E, \rho(a_1) \geq \gamma\}$ is called level subset of $\rho$.

**Definition 2.4 [6]:** Let $\rho$ be a fuzzy subset of a group $G$. Then $\rho$ is called a $FS(G)$ if

i. $\rho(a_1 a_2) \geq \min\{\rho(a_1), \rho(a_2)\}, \quad \forall \, a_1, a_2 \in G,$

ii. $\rho(a_1{}^{-1}) \geq \rho(a_1), \quad \forall \, a_1 \in G.$

**Lemma 2.5 [6]:** Let $\rho : G \to [0, 1]$ be a $FS(G)$, for all $a_1 \in G$, we have

i. $\rho(e) \geq \rho(a_1),\ \forall\, a_1 \in G,$

ii. $\rho(a_1^{-1}) = \rho(a_1).$

**Theorem 2.6 [4]:** Let $\rho$ be a fuzzy subset of group $G$ then $\rho$ is $FS(G)$ if and only if the level subset $\rho_\gamma$, for $\gamma \in [0,1]$, $\rho(e) \geq \gamma$, is subgroup of $G$, where $e$ is an identity of $G$.

**Definition 2.7 [7]:** A $FS(G)$ $\rho$ is called a $FNS(G)$ if

$$\rho(a_1a_2) = \rho(a_2a_1), \quad \forall\, a_1, a_2 \in G.$$

**Definition 2.8 [7]:** Let $\rho$ be a $FS(G)$. For any $a_1 \in G$, define a map $\rho_{a_1} : G \rightarrow [0,1]$ as follows

$$\rho_{a_1}(g) = \rho(g a_1^{-1}), \quad \forall\, g \in G.$$

$\rho_{a_1}$ is called fuzzy coset of $G$ determined by $a_1$ and $\rho$.

**Definition 2.9 [5]:** A map T: $[0,1] \times [0,1] \longrightarrow [0,1]$ define by $(a_1, a_2) \longmapsto \min\{a_1, a_2\}$ is T−norm iff $\forall\, a_1,\ a_2, a_3, a_4 \in [0,1]$

i. $T(a_1, a_2) = T(a_2, a_1),$

ii. $T(a_1, T(a_2, a_3)) = T(T(a_1, a_2), a_3),$

iii. $T(a_1, 1) = T(1, a_1) = 1,$

iv. If $a_1 \leq a_3$ and $a_2 \leq a_4$ then $T(a_1, a_2) \leq T(a_3, a_4)$.

## 3. $\eta$-Fuzzy Subsets and their properties

**Definition 3.1:** Let $N_T$ be an operator defined as $N_T : [0,1] \times [0,1] \longrightarrow [0,1]$ by

$$N_T(a_1, a_2) = \min\{1 - a_1, 1 - a_2\}, \quad \forall\, a_1,\ a_2 \in [0,1]$$

Infect, $N_T$ admits the properties below, $\forall\, a_1,\ a_2, a_3, a_4 \in [0,1]$

i. $N_T(a_1, a_2) = N_T(a_2, a_1),$

ii. $N_T(a_1, 1) = N_T(1, a_1) = 0,$

iii. If $a_1 \leq a_3$ and $a_2 \leq a_4$, then $N_T(a_1, a_2) \geq N_T(a_3, a_4)$.

The operator $N_L$ is non-associative.

**Definition 3.2:** Let $\rho: X \rightarrow [0,1]$ be fuzzy subset of $X$ and $\eta \in [0,1]$, the fuzzy subset $\rho^\eta$ of $X$ (w. r. t fuzzy set $\rho: X \rightarrow [0,1]$) denotes the $\eta$-fuzzy subset of $X$ and is defined as follows

$$\rho^\eta(a_1) = \min\{1 - \rho(a_1), 1 - \eta\},\ \forall\, a_1 \in X.$$

**Example 3.3:** Let $A = \{\text{set of young people}\}$ define $\rho$ fuzzy set on $A$ as follows

$$\rho_A(a_1) = \begin{cases} 1 & if \quad a_1 < 25 \\ \dfrac{40 - a_1}{15} & if \quad 25 \leq a_1 \leq 40 \\ 0 & if \quad a_1 > 40. \end{cases}$$

Take $\eta = 0.6$, now for $a_1 = 20$, we have ${\rho^\eta}_A(a_1) = 0$. For $a_2 = 30$ we have ${\rho^\eta}_A(a_2) = 0.333$ and for $a_3 = 45$ we have ${\rho^\eta}_A(a_3) = 0.4$.

**Remark 3.4:** It is important to note that one can obtain the negation of classical fuzzy subset $\rho(a_1)$ by choosing the value of $\eta = 0$ in above definition whereas the case become crisp for the choice of $\eta = 1$. These algebraic facts lead to note that the case illustrates the $\eta$-fuzzy version with respect to any fuzzy subset for the value of $\eta$, when $\eta \in (0, 1)$.

**Definition 3.5:** Let $\xi: G \to G'$ where $G,\ G'$ are groups and $\rho$ and $\sigma$ be $\eta$-fuzzy subsets of $G$ and $G'$ respectively. Then $\xi(\rho^\eta)$ and $\xi^{-1}(\sigma^\eta)$ are the image of $\eta$-fuzzy subset $\rho^\eta$ and the inverse image of $\eta$-fuzzy subset $\sigma^\eta$ respectively, defined as

i. $\xi(\rho^\eta)(a_2) = \begin{cases} \sup \rho^\eta(a_1) : a_1 \in \xi^{-1}(a_2), & if\ \xi^{-1}(a_2) \neq \emptyset \\ 0\,, & if\ \xi^{-1}(a_2) = \emptyset \end{cases}$

ii. $\xi^{-1}(\sigma^\eta)(a_1) = \sigma^\eta(\xi(a_1)),\ \forall\ a_1 \in G.$

**Example 3.6:** Let $\xi: V_4 \to \mathbb{R}$ where $V_4 = \{1,\ a_1,\ a_2,\ a_1a_2\}$ defined as follows

$\xi(1) = 1,\ \xi(a_1) = 2,\ \xi(a_2) = -2$ and $\xi(a_1a_2) = 4$. Define fuzzy set $\rho$ on $V_4$ given by $\rho(1) = 1,\ \rho(a_1) = 0.8,\ \rho(a_2) = 0.4$ and $\rho(a_1a_2) = 0.5$ define fuzzy set $\sigma$ on $\mathbb{R}$ as follows

$$\sigma(a_1) = \frac{1}{|a_1|}.$$

Take $\eta = 0.3$, So $\xi\left(\rho^\eta(a_2)\right) = \{0,\ 0.2,\ 0.7,\ 0.5\}$ and $\xi^{-1}\left(\sigma^\eta(a_1)\right) = \{0,\ 0.5,\ 0.7\}$.

**Theorem 3.7:**

1) Let $\rho$ and $\sigma$ be any two fuzzy subsets of a set $E$ then $(\rho\ \cap \sigma)^\eta = \rho^\eta \cap \sigma^\eta$.
2) Let $\rho$ and $\sigma$ be two fuzzy subsets of a set $P$ and $Q$ respectively and $\xi : P \longrightarrow Q$ be a mapping, then

i. $\xi(\rho^\eta) = \left(\xi(\rho)\right)^\eta$.

ii. $\xi^{-1}(\rho^\eta) = \left(\xi^{-1}(\rho)\right)^\eta$.

**Proof:** (1) By definition (3.2), we have

$(\rho\ \cap \sigma)^\eta(a_1) = \min\{1 - (\rho\ \cap \sigma)(a_1), 1 - \eta\}$, where $a_1 \in E$ and $\eta \in [0, 1]$.

$= \min\left\{1 -\ \min\{\rho(a_1),\ \sigma(a_1)\}, 1 - \eta\right\}$

$= \min\left\{ \min\{1 - \rho(a_1),\ 1-\eta\},\ \min\{1 - \sigma(a_1), 1-\eta\}\right\}$

$= \min\{ \rho^{\eta}(a_1), \sigma^{\eta}(a_1)\} = (\rho^{\eta} \cap \sigma^{\eta})(a_1),\ \forall\, a_1 \in E.$

Consequently, $(\rho^{\eta} \cap \sigma^{\eta})(a_1) = \rho^{\eta} \cap \sigma^{\eta}$.

(2). (i). $\xi(\rho^{\eta})(a_2) = \sup\{ \rho^{\eta}(a_1) : \xi(a_1) = a_2 \}$

$= \sup\{\min\{1 - \rho(a_1), 1-\eta\}\}$

$= \min\{\sup\{1 - \rho(a_1), 1-\eta\}\}$

$= \min\{1 - \xi(\rho)(a_2), 1-\eta\}$

$= \left(\xi(\rho)\right)^{\eta},\ \forall\, a_2 \in Q.$

Hence, $\xi(\rho^{\eta}) = \left(\xi(\rho)\right)^{\eta}$.

(ii) From definition (3.2), we have

$$\xi^{-1}(\rho^{\eta})(a_1) = (\rho^{\eta})\xi(a_1) = \min\left\{1 - \rho\left(\xi(a_1)\right), 1-\eta\right\} = \min\{1 - \xi^{-1}\rho(a_1), 1-\eta\}$$

$$= \left(\xi^{-1}(\rho)\right)^{\eta}(a_1),\ \forall\, a_1 \in P.$$

Hence, $\xi^{-1}(\rho^{\eta}) = \left(\xi^{-1}(\rho)\right)^{\eta}$.

## 4. $\eta$-Fuzzy Subgroups

This section deals with the concept of $\eta$-$FS(G)$ and $\eta$-$FNS(G)$. We prove that every $FS(G)$ $\left(FNS(G)\right)$is also $\eta$-$FS(G)$ ($FNS(G)$) but converse need not to be true. The concept of $\eta$-fuzzy coset is defined and discussed deeply. Moreover, applying the idea of $\eta - FNS(G)$, we introduced the quotient group with respect to $FNS(G)$. This leads us to develop a natural homomorphism with respect to $\eta - FNS(G)$ from a group $G$ to its quotient group. Additionally, we discover the homomorphic image and pre-image of $\eta - FS(G)$ ($\eta - FNS(G)$. We conclude this section by establishing an isomorphism between the quotient group $\frac{G}{\rho^{\eta}}$ and $\frac{G}{G_{\rho^{\eta}}}$.

**Definition 4.1:** Let $G$ be a group and $\rho: G \longrightarrow [0,1]$ be a fuzzy subset $G$. Let $\eta \in [0,1]$, then $\rho$ is called $\eta$-$FS(G)$ if $\rho^{\eta}$ is $FS(G)$. In other words, $\rho$ is $\eta$-$FS(G)$ if $\rho^{\eta}$ admits the following properties, $\forall\, a_1,\ a_2 \in G$

i. $\rho^{\eta}(a_1 a_2) \geq \min\{ \rho^{\eta}(a_1), \rho^{\eta}(a_2)\}$,
ii. $\rho^{\eta}({a_1}^{-1}) = \rho^{\eta}(a_1)$.

**Example 4.2:** Let $\rho$ be a fuzzy subset of the group $G = V_4 = \{1, a_1, a_2, a_1 a_2\}$ defined as

$$\rho(1) = 0.7 \text{ and } \rho(a_1) = \rho(a_2) = \rho(a_1 a_2) = 0.9.$$

Define $\eta$-fuzzy subset $\rho^{\eta}$ of $G$ for $\eta = 0.8$ as follows

$$\rho^{\eta}(1) = 0.2 \text{ and } \rho^{\eta}(a_1) = \rho^{\eta}(a_2) = \rho^{\eta}(a_1 a_2) = 0.1.$$

Clearly, $\rho^{\eta}$ is $\eta$-$FS(G)$.

**Remark 4.3:** Note that, $\rho$ is $\eta$-$FS(G)$ for any choice of $\eta$ in each of the following case

i. $\rho(a_1a_2) \geq \eta >$min $\{\rho(a_1), \rho(a_2)\}$,

ii. $\eta \geq \rho(a_1a_2) >$min $\{\rho(a_1), \rho(a_2)\}$,

iii. $\rho(a_1a_2) >$min $\{\rho(a_1), \rho(a_2)\} > \eta$.

iv. For $\eta = 0$, we get the complement of classical fuzzy subgroup.

**Proposition 4.4:** Let $\rho$ be $\eta$-$FS(G)$. Then the following statements hold

i. $\rho^\eta(a_1) \leq \rho^\eta(e)$, $\forall\, a_1 \in G$ and $e$ is identity element of $G$.

ii. $\rho^\eta(a_1{a_2}^{-1}) = \rho^\eta(e)$ gives $\rho^\eta(a_1) = \rho^\eta(a_2)$, $\forall\, a_1,\ a_2 \in G$.

**Proof:** (i) Since $\rho^\eta(a_1{a_1}^{-1}) = \rho^\eta(e)$ and also $\rho^\eta(a_1{a_1}^{-1}) = \min\{\rho^\eta(a_1), \rho^\eta({a_1}^{-1})\}$

$$= \min\{\rho^\eta(a_1), \rho^\eta(a_1)\} = \rho^\eta(a_1).$$

This implies that $\rho^\eta(a_1) \leq \rho^\eta(e)$, $\forall\, a_1 \in G$.

(ii) Since we have $\rho^\eta(a_1) = \rho^\eta(a_1{a_2}^{-1}a_2) \geq \min\{\rho^\eta(a_1{a_2}^{-1}), \rho^\eta(a_2)\}$.

Then by our assumption we have $\rho^\eta(a_1) \geq \min\{\rho^\eta(e), \rho^\eta(a_2)\}$ which implies that

$$\rho^\eta(a_1) \geq \rho^\eta(a_2).$$

Similarly, $\rho^\eta(a_2) = \rho^\eta(a_2{a_1}^{-1}a_1) \geq \min\{\rho^\eta(a_2{a_1}^{-1}), \rho^\eta(a_1)\}$ then by our assumption we have

$\rho^\eta(a_2) \geq \min\{\rho^\eta(e), \rho^\eta(a_1)\} = \rho^\eta(a_1)$, which implies that $\rho^\eta(a_2) \geq \rho^\eta(a_1)$.

Hence, $\rho^\eta(a_1) = \rho^\eta(a_2)$.

The next result leads to note that every $FS(G)$ is $\eta$-$FS(G)$.

**Proposition 4.5:** Every $FS(G)$ is also $\eta - FS(G)$.

**Proof:** Let $\rho$ be a $FS(G)$. Consider,

$\rho^\eta(a_1a_2) = \min\{1 - \rho(a_1a_2), 1 - \eta\}$, where $\eta \in [0, 1]$ and $a_1, a_2 \in G$.

$\geq \min\{1 - \min\{\rho(a_1), \rho(a_2)\}, 1 - \eta\}$

$= \min\{\min\{1 - \rho(a_1),\ 1 - \eta\},\ \min\{1 - \rho(a_2), 1 - \eta\}\} = \min\{\rho^\eta(a_1), \rho^\eta(a_2)\}$.

Thus, we have $\rho^\eta(a_1a_2) \geq \min\{\rho^\eta(a_1), \rho^\eta(a_2)\}$.

Moreover, $\rho^\eta({a_1}^{-1}) = \min\{1 - \rho({a_1}^{-1}), 1 - \eta\} = \min\{1 - \rho(a_1), 1 - \eta\} = \rho^\eta(a_1)$.

This implies that $\rho$ is $\eta$-$FS(G)$.

**Remark 4.6:** The converse of the aforementioned proposition must not be true.

**Example 4.7:** Let $G = S_3 = \{(1), (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$,

$$\rho((1)) = 0.4, \rho((1\ 2)) = \rho((1\ 3)) = \rho((2\ 3)) = 0.5,$$

and

$$\rho((1\ 2\ 3)) = \rho((1\ 3\ 2)) = 0.6.$$

Consider the $\eta$-fuzzy set for $\eta = 0.55$ as follows

$$\rho^{\eta}\big((1)\big) = 0.45, \rho^{\eta}\big((1\ 2)\big) = \rho^{\eta}\big((1\ 3)\big) = \rho^{\eta}\big((2\ 3)\big) = 0.45,$$

and

$$\rho^{\eta}\big((1\ 2\ 3)\big) = \rho^{\eta}\big((1\ 3\ 2)\big) = 0.4.$$

Clearly, the fuzzy subset $\rho$ is $\eta - FS(G)$ .

Moreover, $\rho$ is not $FS(G)$ because all possible level subset

$\rho_{0.4} = \{\ (1), (1\ 2), (1\ 3), (2\ 3)\ \}$, $\rho_{0.5} = \{\ (1\ 2), (1\ 3), (2\ 3)\ \}$ and $\rho_{0.6} = \{\ \ (1\ 2\ 3), (1\ 3\ 2)\ \}$.

$\rho_{0.4}, \rho_{0.5}$ and $\rho_{0.6}$ are not subgroups of $S_3 = G$.

**Proposition 4.8:** Let $G$ be a group and $\rho$ be its fuzzy subset such that $\rho(a_1) = \rho({a_1}^{-1}),\ \forall\ a_1 \in G$. Let $\eta \geq m$, where $\eta \in [0, 1]$ and $m = \sup\{\rho(a_1), a_1 \in G\}$, then $\rho$ is also $\eta$-$FS(G)$.

**Proof:** Since, we have $\eta \geq m$. So, $\eta \geq \sup\{\rho(a_1), a_1 \in G\}$, which implies $\eta \geq\ \rho(a_1)$, $\forall\ a_1 \in G$.

So, we have $\rho^{\eta}(a_1) = \min\{1 - \rho(a_1), 1 - \eta\ \} = 1 - \eta$, $\forall\ a_1 \in G$.

This implies that $\rho^{\eta}(a_1 a_2) \geq \min\ \{\rho^{\eta}(a_1), \rho^{\eta}(a_2)\}$, $\forall\ a_1, a_2 \in G$.

Also, $\rho^{\eta}({a_1}^{-1}) =\ \rho^{\eta}(a_1)$.

Hence, $\rho$ is $\eta$-$FS(G)$.

**Proposition 4.9:** Let $\rho$ and $\sigma$ be any two $\eta$-$FS(G)$. Then $\rho\ \cap \sigma$ is also $\eta$-$FS(G)$.

**Proof:** Let $\rho$ and $\sigma$ be two $\eta$-$FS(G)$ of a group $G$ and let $a_1, a_2 \in G$.

Since, $(\rho\ \cap \sigma)^{\eta}(a_1) = (\rho^{\eta} \cap \sigma^{\eta})(a_1)$ hold, then we have

$$(\rho\ \cap \sigma)^{\eta}(a_1 a_2) = (\rho^{\eta} \cap \sigma^{\eta})(a_1 a_2).$$

So, $(\rho^{\eta} \cap \sigma^{\eta})(a_1 a_2) = \min\ \{\ \rho^{\eta}(a_1 a_2)\ , \sigma^{\eta}(a_1 a_2)\}$

$$\geq \min\{\min\{\ \rho^{\eta}(a_1), \rho^{\eta}(a_2)\}, \min\{\ \ \sigma^{\eta}(a_1),\qquad \sigma^{\eta}(a_2)\ \}\}$$

$$= \min\{\min\{\ \rho^{\eta}(a_1), \sigma^{\eta}(a_1)\}, \min\{\ \ \rho^{\eta}(a_2),\ \ \sigma^{\eta}(a_2)\ \}\}$$

$$= \min\{(\rho^{\eta} \cap \sigma^{\eta})(a), (\rho^{\eta} \cap \sigma^{\eta})(a_2)\}.$$

This implies that$(\rho^{\eta} \cap \sigma^{\eta})(ab) \geq \min\{(\rho^{\eta} \cap \sigma^{\eta})(a), (\rho^{\eta} \cap \sigma^{\eta})(a_2)\}$.

Moreover, $(\rho\ \cap \sigma)^{\eta}({a_1}^{-1}) = (\rho^{\eta} \cap \sigma^{\eta})({a_1}^{-1}) = \min\{\rho^{\eta}(a_1),$

$$\rho^{\eta}({a_1}^{-1})\} = \min\{\rho^{\eta}(a_1), \rho^{\eta}(a_1)\}.$$

We have $(\rho\ \cap \sigma)^{\eta}({a_1}^{-1}) = (\rho\ \cap \sigma)^{\eta}(a_1)$.

Consequently, $\rho\ \cap \sigma$ is $\eta$-$FS(G)$.

**Corollary 4.10:** The intersection of any finite number of $\eta$-$FS(G)$ is also $\eta$-$FS(G)$.

**Proposition 4.11:** Let $\rho$ and $\sigma$ be any two $\eta$-$FS(G)$. Then $\rho\ \cup \sigma$ need not to be $\eta$-$FS(G)$.

**Example 4.12:** Let $G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\ \}$. Take two subgroups of $G$ that are

$H_1 = \{\ \pm 1, \pm i\}$ and $H_2 = \{\ \pm 1, \pm j\}$.

Let $\rho$ and $\sigma$ be two fuzzy subsets of $G$ as

$$\rho(a_1) = \begin{cases} 0.2 & if\ a_1 \in H_1 \\ 0.9 & otherwise \end{cases} \quad \text{and} \quad \sigma(a_1) = \begin{cases} 0.3 & if\ a_1 \in H_2 \\ 1 & otherwise \end{cases}$$

Since, $\rho^{\eta}(a)$= min $\{1 - \rho(a), 1 - \eta\}$.

Define $\eta$-fuzzy subsets $\rho^{\eta}$ and $\sigma^{\eta}$ for $\eta = 0$, as follows

$$\rho^0(a_1) = \begin{cases} 0.8 & if\ a_1 \in H_1 \\ 0.1 & otherwise \end{cases} \quad \text{and} \quad \sigma^0(a_1) = \begin{cases} 0.7 & if\ a_1 \in H_2 \\ 0 & otherwise \end{cases}$$

It is easy to check that $\rho^0$ and $\sigma^0$ are 0-$FS(G)$.

Now, we define $\rho^0 \cup \sigma^0$ as

$$(\rho^0 \cup \sigma^0)(a_1) = \max\{\rho^0(a_1), \sigma^0(a_1)\}$$

So, we have

$$(\rho^0 \cup \sigma^0)(a_1) = \begin{cases} 0.8 & if\ a_1 \in H_1 \\ 0.7 & if\ a_1 \in H_2 \backslash H_1 \\ 0.1 & otherwise \end{cases}$$

Let $a_1 = i$ and $a_2 = j$,

Observe that $(\rho^0 \cup \sigma^0)(i) = 0.8$ and $(\rho^0 \cup \sigma^0)(j) = 0.7$,

So, min $\{(\rho^0 \cup \sigma^0)(i), (\rho^0 \cup \sigma^0)(j)\} = 0.7$, but $(\rho^0 \cup \sigma^0)(ij) = (\rho^0 \cup \sigma^0)(k) = 0.1$.

This implies that

$$(\rho^0 \cup \sigma^0)(ij) \leq \min\{(\rho^0 \cup \sigma^0)(i), (\rho^0 \cup \sigma^0)(j)\}.$$

Hence, $\rho^0 \cup \sigma^0$ is not 0-$FS(G)$.

**Example 4.13:** Let $G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. Take a subgroup of $G$ that is $H_1 = \{\pm 1, \pm i\}$ .

Let $\rho$ and $\sigma$ be two fuzzy subsets of $G$ as

$$\rho(a_1) = \begin{cases} 0.3 & if\ a_1 \in H_1 \\ 0.9 & otherwise \end{cases} \quad \text{and} \quad \sigma(a_1) = \begin{cases} 0.2 & if\ a_1 \in H_1 \\ 1 & otherwise. \end{cases}$$

Then we have $\rho^0$ and $\sigma^0$ as follows

$$\rho^0(a_1) = \begin{cases} 0.7 & if\ a_1 \in H_1 \\ 0.1 & otherwise \end{cases} \quad \text{and} \quad \sigma^0(a_1) = \begin{cases} 0.8 & if\ a_1 \in H_1 \\ 0 & otherwise. \end{cases}$$

Then we have

$$(\rho^0 \cup \sigma^0)(a_1) = \begin{cases} 0.8 & if\ a_1 \in H_1 \\ 0.1 & otherwise. \end{cases}$$

It can be easily seen that $\rho^0 \cup \sigma^0$ is $0 - FS(G)$.

**Definition 4.14:** Let $\rho$ be $\eta$-$FS(G)$, for any $a_1 \in G$ define $\eta$-fuzzy left coset $a_1\rho^{\eta}$ of $\rho$ in $G$ as follows

$$a_1\rho^{\eta}(x) = \min\{1 - \rho(a_1^{-1}x), 1 - \eta\}, \forall\ a_1, x \in G.$$

Similarly, we define $\eta$-fuzzy right coset $\rho^{\eta}a_1$ of $\rho$ in $G$ as follows

$$\rho^{\eta}a_1(x) = \min\{1 - \rho(xa_1^{-1}), 1 - \eta\}, \forall\ a_1, x \in G.$$

**Example 4.15:** Let $\rho$ be a fuzzy subset of the group $G = Z_4 = \{0, 1, 2, 3\}$ defined as

$\rho(0) = 0.2, \rho(2) = 0.4$ and $\rho(1) = \rho(3) = 0.4.$

Define $\eta$-fuzzy subset $\rho^{\eta}$ of $G$ for $\eta = 0.5$ as follows

$\rho^{\eta}(0) = 0.5, \rho^{\eta}(2) = 0.5$ and $\rho^{\eta}(1) = \rho^{\eta}(3) = 0.4.$

Clearly, $\rho^{\eta}$ is $\eta$-$FS(G)$. Consider $\eta$-fuzzy left coset of $\rho$ by the element $2 \in Z_4$ as follows

$$2 + \rho^{\eta}(a_1) = \min\{1 - \rho(2 + a_1), 1 - \eta\} = \begin{cases} 0.4 & if\ a_1 \in \{0, 2\} \\ 0.5 & otherwise. \end{cases}$$

Similarly, define $\eta$-fuzzy right coset of $G$.

**Proposition 4.16:** Let $\rho$ be $\eta$- $FS(G)$. Then $\rho$ be $\eta - FNS(G)$ if and only if $a_1\rho^{\eta}(x) = \rho^{\eta}a_1(x)$, $\forall\ a_1 \in G$.

Note: $a_1\rho^{\eta}(x) = \rho^{\eta}({a_1}^{-1}x)$ and $\rho^{\eta}a_1(x) = \rho^{\eta}(x{a_1}^{-1})$, $\forall\ x \in G$.

The following result leads to note that every $FNS(G)$ is $\eta$- $FNS(G)$.

**Example 4.17:** In view of Example (4.15) $\eta$- $FS(G)$ is a $\eta$-$FNS(G)$, because its all $\eta$-fuzzy left cosets and $\eta$-fuzzy right cosets are equal. For instance, consider

$$2 + \rho^{\eta}(a_1) = \rho^{\eta}(a_1) + 2 = \begin{cases} 0.4 & if\ a_1 \in \{0, 2\} \\ 0.5 & otherwise \end{cases}$$

**Proposition 4.18:** Every $FNS(G)$ is also $\eta$-$FNS(G)$.

**Proof:** Suppose that $\rho$ is $FNS(G)$ which implies that $a_1\rho = \rho a_1$.

Then for any $x \in G$ we have $\rho({a_1}^{-1}x) = \rho(x{a_1}^{-1})$. So, we have

$\min\{1 - \rho({a_1}^{-1}x), 1 - \eta\} = \min\{1 - \rho(x{a_1}^{-1}), 1 - \eta\}$.

This implies that $a_1\rho^{\eta}(x) = \rho^{\eta}a_1(x)$, $\forall\ x \in G$. Consequently, $\rho$ is $\eta$-$FNS(G)$.

Note that the converse of above result need not to be true.

**Example 4.19:** Let $G = D_3 = < a, b :\ a^3 = b^2 = e,\ ba = a^2b >$.

Define a fuzzy subset $\rho$ of $G$ as follows

$$\rho(a) = \begin{cases} 0.3 & if\ a \in < b > \\ 0.1 & otherwise. \end{cases}$$

Take $\eta = 0.6$ then we have $\rho^{\eta}(a) = 1 - \eta$, $\forall\ a \in G$.

$$a_1\rho^{\eta}(g) = \min\{1 - \rho({a_1}^{-1}g), 1 - \eta\} = 1 - \eta$$
$$= \min\{1 - \rho(g{a_1}^{-1}), 1 - \eta\} = \rho^{\eta}a_1(g).$$

Then $a_1\rho^{\eta}(g) = \rho^{\eta}a_1(g)$ which implies that $\rho$ is $\eta$-$FNS(G)$. But it can be seen that $\rho$ is not $FNS(G)$. This is because

$\rho\big((a^2)(ab)\big) = 0.3$ and $\rho\big((ab)(a^2)\big) = 0.1$. i.e. $\rho(a^{-1}g) = \rho(ga^{-1})$ not hold.

**Proposition 4.20:** Let $\rho$ be $\eta$-FNS(G). Then $\rho^{\eta}(b^{-1}a_1b) = \rho^{\eta}(a_1)$ or $\rho^{\eta}(a_1a_2) = \rho^{\eta}(a_2a_1)$ hold, $\forall\ a_1, a_2 \in G$.

**Proof:** Since, we have $\rho$ be $\eta - FNS(G)$ then we have $a_1\rho^{\eta} = \rho^{\eta}a_1$, $\forall\ a_1 \in G$.

This implies that $a_1\rho^{\eta}(a_2{}^{-1}) = \rho^{\eta}a_1(a_2{}^{-1}), \forall\, a_2{}^{-1} \in G$.

$=\ \min\ \{1-\rho(a_1{}^{-1}a_2{}^{-1}), 1-\eta\} =\ \min\ \{1-\rho(a_2{}^{-1}a_1{}^{-1}), 1-\eta\} = \rho^{\eta}(a_1{}^{-1}a_2{}^{-1}) = \rho^{\eta}(a_2{}^{-1}a_1{}^{-1})$.

Consequently, we have $\rho^{\eta}((a_2a_1)^{-1}) = \rho^{\eta}((a_1a_2)^{-1})$.

Hence, $\rho^{\eta}(a_1a_2) = \rho^{\eta}(a_2a_1)$.

**Theorem 4.21:** Let $\rho$ be $\eta$-$FS(G)$. Then following statements are equivalent

i. $\rho^{\eta}(a_2a_1) = \rho^{\eta}(a_1a_2), \forall\, a_1, a_2 \in G$,

ii. $\rho^{\eta}(a_1a_2a_1{}^{-1}) = \rho^{\eta}(a_2), \forall\, a_1, a_2 \in G$

iii. $\rho^{\eta}(a_1a_2a_1{}^{-1}) \geq \rho^{\eta}(a_2), \forall\, a_1, a_2 \in G$

iv. $\rho^{\eta}(a_1a_2a_1{}^{-1}) \leq \rho^{\eta}(a_2), \forall\, a_1, a_2 \in G$.

**Proposition 4.22:** Let $\rho$ be $\eta$-$FS(G)$. Let $\eta \geq m$, where $\eta \in [0,1]$ and $m = \sup\{\rho(a_1), a_1 \in G\}$, then $\rho$ is also $\eta$-$FNS(G)$.

**Proof:** Since, we have $\eta \geq m$, so $\eta \geq \sup\{\rho(a_1), a_1 \in G\}$, which implies that $\eta \geq \rho(a_1)$, $\forall\, a_1 \in G$.

So, we have $\rho^{\eta}(a_1) = \min\{1-\rho(a_1), 1-\eta\} = 1-\eta$

$\rho^{\eta}a_1(g) \geq \min\{1-\rho(ga_1{}^{-1}), 1-\eta\} = 1-\eta$.

Similarly, $a_1\rho^{\eta}(g) \geq \min\{1-\rho(a_1{}^{-1}g), 1-\eta\} = 1-\eta$.

This implies that $a_1\rho^{\eta} = \rho^{\eta}a_1, \forall\, a_1 \in G$.

Hence, $\rho$ is also $\eta$-$FNS(G)$.

The following result illustrate that the set $G_{\rho^{\eta}}$ is infect a normal subgroup of $G$.

**Proposition 4.23:** Let $\rho$ be $\eta$-$FNS(G)$. Then the set define as $G_{\rho^{\eta}} = \{a_1 \in G : \rho^{\eta}(a_1) = \rho^{\eta}(e)\} \trianglelefteq G$.

**Proof:** Since, $G_{\rho^{\eta}}$ is nonempty because $e \in G$. Let $a_1, a_2 \in G_{\rho^{\eta}}$

$\rho^{\eta}(a_1a_2{}^{-1}) \geq \min\{\rho^{\eta}(a_1), \rho^{\eta}(a_2{}^{-1})\}\ \forall\, a_1, a_2 \in G$

$= \min\{\rho^{\eta}(a_1), \rho^{\eta}(a_2)\}\ \forall\, a_1, a_2 \in G$

$= \min\{\rho^{\eta}(e), \rho^{\eta}(e)\}\ \forall\, a_1, a_2 \in G$

$= \rho^{\eta}(e)$.

This implies that $\rho^{\eta}(a_1a_2{}^{-1}) \geq \rho^{\eta}(e)$.

Since, $\rho^{\eta}$ is $FS(G)$ which implies that $\rho^{\eta}(a_1a_2{}^{-1}) \leq \rho^{\eta}(e)$.

Hence, $\rho^{\eta}(a_1a_2{}^{-1}) = \rho^{\eta}(e)$ implies that $G_{\rho^{\eta}}$ is subgroup of $G$.

Now we prove it is normal subgroup of $G$. Let $a_1 \in G_{\rho^{\eta}}$ and $a_2 \in G$, then we have

$\rho^{\eta}(a_2{}^{-1}a_1a_2) = \rho^{\eta}(a_1) = \rho^{\eta}(e)$.

This implies that $a_2{}^{-1}a_1a_2 \in G_{\rho^\eta}$ .

Consequently, we have $G_{\rho^\eta} \trianglelefteq G$.

**Proposition 4.24:** Let $\rho$ be $\eta$-$FNS(G)$ of a group $G$. Then the following statements hold

i. $a_1\rho^\eta = a_2\rho^\eta \iff a_1{}^{-1}a_2 \in G_{\rho^\eta}$.

ii. $\rho^\eta a_1 = \rho^\eta a_2 \iff a_1a_2{}^{-1} \in G_{\rho^\eta}$.

**Proof:** (i) Suppose that $a_1\rho^\eta = a_2\rho^\eta$, then we have

$$\rho^\eta(a_1{}^{-1}a_2) = \min\{1-\rho(a_1{}^{-1}a_2), 1-\eta\}$$
$$= a_1\rho^\eta(a_2) = a_2\rho^\eta(a_2)$$
$$= \min\{1-\rho(a_2{}^{-1}a_2), 1-\eta\}$$
$$= \min\{1-\rho(e), 1-\eta\}$$
$$= \rho^\eta(e).$$

Thus $\rho^\eta(a_1{}^{-1}a_2) = \rho^\eta(e)$ implies that $a_1{}^{-1}a_2 \in G_{\rho^\eta}$.

Conversely,

$$a_1\rho^\eta(a_3) = \min\{1-\rho(a_1{}^{-1}a_3), 1-\eta\}.$$

So, $\rho^\eta(a_1{}^{-1}a_3) = \rho^\eta(a_1{}^{-1}a_2 . a_2{}^{-1}a_3) \geq \min\{\rho^\eta(a_1{}^{-1}a_2), \rho^\eta(a_2{}^{-1}a_3)\}$

$=\min\{\rho^\eta(e), \rho^\eta(a_2{}^{-1}a_3)\}$

$= \rho^\eta(a_2{}^{-1}a_3) = a_2\rho^\eta(a_3)$.

By interchanging the $a_1$ and $a_2$, we have $a_1\rho^\eta(a_3) = a_2\rho^\eta(a_3)$ for all $a_3 \in G$.

Hence, $a_1\rho^\eta = a_2\rho^\eta$.

(ii) Similar as above proof.

**Proposition 4.25:** Let $\rho$ be $\eta$-$FNS(G)$ of a group $G$ and $a_1, a_2, x, y \in G$. If $a_1\rho^\eta = x\rho^\eta$ and $a_2\rho^\eta = y\rho^\eta$ then $a_1a_2\rho^\eta = xy\rho^\eta$.

**Proof:** Given that $a_1\rho^\eta = x\rho^\eta$ and $a_2\rho^\eta = y\rho^\eta$, which implies that $a_1{}^{-1}x,\ a_2{}^{-1}y \in G_{\rho^\eta}$.

Now $(a_1a_2)^{-1}xy = a_2{}^{-1}(a_1{}^{-1}x)y = a_2{}^{-1}(a_1{}^{-1}x)(a_2a_2{}^{-1})y$

$= [a_2{}^{-1}(a_1{}^{-1}x)a_2](a_2{}^{-1}y) \in G_{\rho^\eta} \quad \because G_{\rho^\eta} \trianglelefteq G$.

This implies that $(a_1a_2)^{-1}xy \in G_{\rho^\eta}$. Hence, $a_1a_2\rho^\eta = xy\rho^\eta$.

**Proposition 4.26:** Let $\frac{G}{\rho^\eta}$ be the collection of all $\eta$-fuzzy cosets of a $\eta - FS(G)$. This form a group under the binary operation $\circledast$ define on the set $\frac{G}{\rho^\eta}$ as follows $\rho^\eta a_1 \circledast \rho^\eta a_2 = \rho^\eta a_1a_2$, $\forall$ $a_1, a_2 \in G$.

**Proof:** As we know that $\frac{G}{\rho^\eta} = \{\rho^\eta a_1 : a_1 \in G\}$.

Let $\rho^\eta a_1 = \rho^\eta a_1$ and $\rho^\eta a_2 = \rho^\eta a_2'$, $\forall$ $a_1, a_1', a_2, a_2' \in G$.

Let $g \in G$ then $(\rho^{\eta} a_1 \circledast \rho^{\eta} a_2)(g) = \rho^{\eta} a_1 a_2(g) = \min\{1 - \rho(g((a_1 a_2)^{-1}), 1 - \eta\}$

$= \min\{1 - \rho((g a_2{}^{-1}) a_1{}^{-1}), 1 - \eta\}$

$= \rho^{\eta} a_1 (g a_2{}^{-1}) = \rho^{\eta} a_1{}' (g a_2{}^{-1})$

$= \min\{1 - \rho((g a_2{}^{-1}) a_1{}^{-1}), 1 - \eta\}$

$= \min\{1 - \rho((a'^{-1} g) b^{-1}), 1 - \eta\}$

$= \rho^{\eta} a_2 (a_1{}'^{-1} g) = \rho^{\eta} a_2{}' (a_1{}'^{-1} g)$

$= \min\left\{1 - \rho\left((a_1{}'^{-1} g) a_2{}'^{-1}\right), 1 - \eta\right\}$

$= \min\left\{1 - \rho\left(a_2{}'^{-1} (a_1{}'^{-1} g)\right), 1 - \eta\right\}$

$= \min\left\{1 - \rho\left((a_2{}'^{-1} a_1{}'^{-1})\right), 1 - \eta\right\}$

$= \min\{1 - \rho(((a_1{}' a_2{}')^{-1} g)), 1 - \eta\}$

$= \min\{1 - \rho(g(a_1{}' a_2{}')^{-1})), 1 - \eta\}$

$= \rho^{\eta} a_1{}' a_2{}'$.

Hence, $\circledast$ is well define operation on the set $\frac{G}{\rho^{\eta}}$.

The set $\frac{G}{\rho^{\eta}}$ under this binary operation admits the associative law. The element $\rho^{\eta} e$ of $\frac{G}{\rho^{\eta}}$ is the identity element and the inverse of an element $\rho^{\eta} a_1$ is $\rho^{\eta} a_1{}^{-1}$.

**Example 4.27:** In view of Example (4.15) consider $\rho$ as $\eta$- $FNS(G)$.

The set $\frac{G}{\rho^{\eta}} = \{\rho^{\eta}, 2 + \rho^{\eta}\}$ forms a group under the following binary operation defined on $\frac{G}{\rho^{\eta}}$ as $(a_1 + \rho^{\eta}) + (a_2 + \rho^{\eta}) = ((a_1 + a_2) + \rho^{\eta})$.

Note that $\rho^{\eta}(a_1)$ is identity element of this group and inverse of $a_1 + \rho^{\eta}$ is $(-a_1) + \rho^{\eta}$.

**Definition 4.28:** The group $\frac{G}{\rho^{\eta}}$ of $\eta$-fuzzy cosets of a $\eta - FNS(G)$ is called the quotient group of $G$ by $\rho^{\eta}$.

**Theorem 4.29:** Let $G$ be a group and $\frac{G}{\rho^{\eta}}$ be quotient group with respect to $\eta - FNS(G)$. There exist a natural epimorphism from $G$ to $\frac{G}{\rho^{\eta}}$ which is defined as $\xi(a_1) = \rho^{\eta} a_1$ with $Ker\, \xi = G_{\rho^{\eta}}$.

**Proof:** Let $a_1, a_2 \in G$ be any elements. Then $\xi(a_1 a_2) = \rho^{\eta} a_1 a_2 = \rho^{\eta} a_1\, \rho^{\eta} a_2 = \xi(a_1)\xi(a_2)$.

Therefore, $\xi$ is homomorphism. For each $\rho^{\eta} a_1 \in G_{\rho^{\eta}}$ we have $a_1 \in G$ such that $\xi(a_1) = \rho^{\eta} a_1$. This implies that $\xi$ is onto homomorphism.

Now $Ker\, \xi = \{a_1 \in G : \xi(a_1) = \rho^{\eta} e\}$

$= \{a_1 \in G : \rho^{\eta} a_1 = \rho^{\eta} e\}$

$$= \{ a_1 \in G : a_1 e^{-1} \in G_{\rho^\eta} \}$$
$$= \{ a_1 \in G : a_1 \in G_{\rho^\eta} \} = G_{\rho^\eta}.$$

## 5. Homomorphism of $\eta$-Fuzzy Subgroups

Anthony and Sherwood [17] observed that using a minimum in the Rosenfield definition of a fuzzy subgroup constrains the concept, rendering it useless in a variety of fuzzy situations. They introduced the concept of an T-norm and redefined the fuzzy subgroup by substituting a T-norm for a minimum. They investigated the impact of a simple homomorphism on fuzzy subgroups. Here, we present the results of homomorphism in frame work of our proposed definition.

**Theorem 5.1:** Let $\xi : G \longrightarrow G'$ be a bijective homomorphism of a group $G$ into a group $G'$. If $\rho$ is $\eta$- $FS(G)$ then the homomorphic image $\xi(\rho)$ is $\eta$-$FS(G')$.

**Proof:** Given that $\rho$ be $\eta$-$FS(G')$. Let $a'_1, a'_2 \in G'$ be any element then we have unique elements $a_1, a_2 \in G$, such that $\varphi(a_1) = a'_1$ and $\varphi(a_2) = a'_2$.

Further, $\left(\xi(\rho)\right)^\eta (a'_1 a'_2)$

$$= \min \{1 - \xi(\rho)(a'_1 a'_2),\ 1 - \eta \}$$
$$= \min \left\{1 - \xi(\rho)\left(\xi(a_1)\xi(a_2)\right),\ 1 - \eta \right\}$$
$$= \min \left\{1 - \xi(\rho)\left(\xi(a_1 a_2)\right),\ 1 - \eta \right\}$$
$$= \min \{1 - \xi(\rho)(a_1 a_2),\ 1 - \eta \}$$
$$= \rho^\eta (a_1 a_2)$$
$$\geq \min \{\rho^\eta(a_1), \rho^\eta(a_2) \}, \forall\ a_1, a_2 \in G$$
$$= \min \{\xi(\rho)^\eta \xi(a_1), \xi(\rho)^\eta \xi(a_2) \}$$
$$= \min \left\{\xi(\rho)^\eta (a'_1), \xi(\rho)^\eta (a'_2) \right\}.$$

Consequently,

$$(\xi(\rho))^\eta (a'_1 a'_2) \geq \min\left\{(\xi(\rho))^\eta (a'_1),\ (\xi(\rho))^\eta (a'_2) \right\}.$$

Also, $\left(\xi(\rho)\right)^\eta (a'^{-1}) = \xi(\rho^\eta)(a'^{-1}) = \xi(\rho^\eta)\left(\varphi(a^{-1})\right) = \rho^\eta(a^{-1}) = \rho^\eta(a)$

$$= \xi(\rho^\eta)\left(\xi(a)\right) = \left(\xi(\rho)\right)^\eta (a')$$

Thus, $\left(\xi(\rho)\right)^\eta (a'^{-1}) = \left(\xi(\rho)\right)^\eta (a')$.

Consequently, $\xi(\rho)$ is $\eta$-$FS(G')$.

**Theorem 5.2:** Let $\xi : G \longrightarrow G'$ be a bijective homomorphism of a group $G$ into $G'$. If $\rho$ is $\eta$-$FNS(G)$ then the homomorphic image $\xi(\rho)$ is $\eta$-$FNS(G')$.

**Proof:** Given that $\rho$ be $\eta$- $FNS(G)$. Let $a'_1, a'_2 \in G'$ be any element then we have unique elements $a_1, a_2 \in G$, such that $\xi(a_1) = a'_1$ and $\xi(a_2) = a'_2$.

$$\begin{aligned}
\left(\xi(\rho)\right)^{\eta}(a'_1 a'_2) &= \min\{1 - \xi(\rho)(a'_1 a'_2),\ 1-\eta\} \\
&= \min\left\{1 - \xi(\rho)\left(\xi(a_1)\xi(a_2)\right),\ 1-\eta\right\} \\
&= \min\left\{1 - \xi(\rho)\left(\xi(a_1 a_2)\right),\ 1-\eta\right\} \\
&= \min\left\{1 - \xi(\rho)\left(\xi(a_2 a_1)\right),\ 1-\eta\right\} \\
&= \min\left\{1 - \xi(\rho)\left(\xi(a_2)\xi(a_1)\right),\ 1-\eta\right\} \\
&= \min\left\{1 - \xi(\rho)\left(a'_2 a'_1\right),\ 1-\eta\right\} \\
&= \left(\xi(\rho)\right)^{\eta}(a'_1 a'_2).
\end{aligned}$$

Consequently, $\xi(\rho)$ is $\eta - FNS(G')$.

**Theorem 5.3:** Let $\xi : G \longrightarrow G'$ be a homomorphism of a group $G$ into $G'$. If $\sigma$ is $\eta - FS(G')$ then the pre-image $\xi^{-1}(\sigma)$ is $\eta$-$FS(G)$.

**Proof:** Given that $\sigma$ be $\eta$-$FS(G')$. Let $a_1, a_2 \in G$ be any element then we have

$$\begin{aligned}
(\xi^{-1}(\sigma))^{\eta}(a_1 a_2) &= \xi^{-1}(\sigma^{\eta})(a_1 a_2) = \sigma^{\eta}\left(\xi(a_1 a_2)\right) = \sigma^{\eta}\left(\xi(a_1)\xi(a_2)\right) \\
&\geq \min\left\{\sigma^{\eta}\left(\xi(a_1)\right),\ \sigma^{\eta}\left(\xi(a_2)\right)\right\} \\
&= \min\{\xi^{-1}(\sigma^{\eta})(a_1),\ \xi^{-1}(\sigma^{\eta})(a_1)\}.
\end{aligned}$$

Thus,

$(\xi^{-1}(\sigma))^{\eta}(a_1 a_2) \geq \min\{((\xi^{-1}\sigma)^{\eta})(a_1), ((\xi^{-1}\sigma)^{\eta})(a_2)\}$.

Also $(\xi^{-1}(\sigma))^{\eta}(a^{-1}) = \xi^{-1}(\sigma^{\eta})(a^{-1}) = \sigma^{\eta}\left(\xi(a^{-1})\right) = \sigma^{\eta}\left(\xi(a)\right) = \xi^{-1}(\sigma^{\eta})(a)$.

Thus, $(\xi^{-1}(\sigma))^{\eta}(a^{-1}) = (\xi^{-1}(\sigma))^{\eta}(a)$.

Hence, $\xi^{-1}(\sigma)$ is $\eta$-$FS(G)$.

**Theorem 5.4:** Let $\xi : G \longrightarrow G'$ be a homomorphism of a group $G$ into $G'$. If $\sigma$ is $\eta$-$FNS(G')$ then the pre-image $\xi^{-1}(\sigma)$ is $\eta$-$FNS(G)$.

**Proof:** Let $\sigma$ be $\eta - FNS(G')$. Let $a_1, a_2 \in G$ be any element then we have

$$\begin{aligned}
(\xi^{-1}(\sigma))^{\eta}(a_1 a_2) &= \xi^{-1}(\sigma^{\eta})(a_1 a_2) = \sigma^{\eta}\left(\xi(a_1 a_2)\right) = \sigma^{\eta}\left(\xi(a_1)\xi(a_2)\right) = \sigma^{\eta}\left(\xi(a_2 a_1)\right) \\
&= (\xi^{-1}(\sigma))^{\eta}(a_2 a_1).
\end{aligned}$$

Thus,

$(\xi^{-1}(\sigma))^{\eta}(a_1 a_2) = (\xi^{-1}(\sigma))^{\eta}(a_2 a_1)$.

Hence $\varphi^{-1}(\sigma)$ is $\eta$- $FNS(G)$.

**Theorem 5.5:** Let $\rho$ be a $\eta$-$FNS(G)$ and $a_1, a_2 \in G$ be any element. If $a_1\rho^{\eta} = a_2\rho^{\eta}$ then $\rho^{\eta}(a_1) = \rho^{\eta}(a_2)$.

**Proof:** Suppose that $a_1\rho^{\eta} = a_2\rho^{\eta}$ then by proposition (4.20) we have ${a_1}^{-1}a_2 \in G_{\rho^{\eta}}$ and ${a_2}^{-1}a_1 \in G_{\rho^{\eta}}$. Since, $\eta$-$FNS(G)$, this implies that

$\rho^{\eta}(a_1) = \rho^{\eta}({a_2}^{-1}a_1a_2) \geq \min\{\rho^{\eta}({a_2}^{-1}a_1),\ \rho^{\eta}(a_2)\} = \min\{\rho^{\eta}(e),\ \rho^{\eta}(a_2)\} = \rho^{\eta}(a_2)$.

Therefore, we have $\rho^{\eta}(a_1) \geq \rho^{\eta}(a_2)$. Similarly, we have $\rho^{\eta}(a_1) \leq \rho^{\eta}(a_2)$.

Hence, $\rho^{\eta}(a_1) = \rho^{\eta}(a_2)$.

**Theorem 5.6:** Let $\rho$ be a $\eta$-$FNS(G)$. Then $\frac{G}{\rho^{\eta}} \cong \frac{G}{G_{\rho^{\eta}}}$.

**Proof:** Define a map $\varphi : \frac{G}{\rho^{\eta}} \longrightarrow \frac{G}{G_{\rho^{\eta}}}$ by the rule

$\xi(a_1\rho^{\eta}) = a_1G_{\rho^{\eta}}$ , for all $a_1 \in G$. In view of proposition (4.20) $\xi$ is well define.

The application of proposition (4.20) leads to note that $\xi$ is injective. $\xi$ is obviously surjective.

Now consider, for $a_1\rho^{\eta}, a_2\rho^{\eta} \in \frac{G}{\rho^{\eta}}$ we have $\xi\big((a_1\rho^{\eta})(a_2\rho^{\eta})\big) = \varphi(a_1a_2\rho^{\eta}) = a_1a_2G_{\rho^{\eta}} = a_1G_{\rho^{\eta}}.a_2G_{\rho^{\eta}} = \xi(a_1\rho^{\eta})\xi(a_2\rho^{\eta})$.

So, $\xi$ is homomorphism. Since $\xi$ is a bijective mapping, which implies this is an isomorphism. Hence, $\frac{G}{\rho^{\eta}} \cong \frac{G}{G_{\rho^{\eta}}}$.

## Conclusion

In this paper, we introduced the idea of $\eta - FS(G)$ and $\eta$-fuzzy cosets for a given group. We used the concept of $\eta$-$FNS(G)$ and discussed various related results and properties. We also studied the effect on the image and inverse image of η-$FS(G)$ $(FNS(G))$ under group homomorphism.We shall extend this concept to intuitionistic fuzzy sets in the upcoming studie s and look into its numerous algebraic features. Moreover, we used the concept of $\eta$ −fuzzy subset in classical field theory.



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