E-ISSN: 2717-3453 | P-ISSN: 2783-1442

Journal of Fuzzy Extension and Applications

J. Fuzzy. Ext. Appl

www.journal-fea.com

J. Fuzzy. Ext. Appl. Vol. 4, No. 1 (2023) 40–51.

Paper Type: Original Article

On Some Related Concepts of n-Cylindrical Fuzzy Neutrosophic Topological Spaces

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Citation:

Kumari R, S., Kalayathankal, S. J., George, M., & Smarandache, F. (2023). On some related concepts of n-cylindrical fuzzy neutrosophic topological spaces. *Journal of fuzzy extension and applications, 4*(1), 40-51.

Received: 20/10/2022 Reviewed: 22/11/2022 Revised: 09/12/2022 Accepted: 13/1/2023

Abstract

This work is the continuation of our recent work entitled n-Cylindrical Fuzzy Neutrosophic Topological Spaces (n-CyFNTS). In this paper, we present n-Cylindrical Fuzzy Neutrosophic (n-CyFN) continuous functions and related results. A characterization on n-CyFN continuous functions is proposed as well. Along with familiarising the n-CyFN interior and n-CyFN closure of sets in n-Cylindrical Fuzzy Neutrosophic Topology (n-CyFNT) and its basic theorems, we define n-CyFN open function, n-CyFN closed function and n-CyFN homeomorphism.

Keywords: n-CyFN continuous functions, CyFN interior, n-CyFN closure, n-CyFN open function, n-CyFN closed function, n-CyFN homeomorphism.

1 | Introduction

Following Zadeh's invention of fuzzy sets [14], Chang [6] introduced the idea of fuzzy topological space, and numerous researchers adapted general topological notions in the context of fuzzy topology, among them. Atanassov's Intuitionistic Fuzzy Sets (IFSs) [5] are one of the fuzzy set's generalisations. Later, Coker [7] introduced the useful concept of intuitionistic fuzzy topological space by utilising the concept of the IFS. Jeon et al. [11] defined and investigated the concepts of intuitionistic fuzzy-continuity and pre-continuity. After Smarandache [8], [9] introduced the concepts of neutrosophy and neutrosophic set, Salama and Alblowi [1] introduced the concepts of neutrosophic crisp set and neutrosophic crisp topological spaces. Neutrosophy has laid the groundwork for a new class of mathematical theories that generalise both their crisp and fuzzy counterparts. The neutrosophic set is a generalisation of the IFS. Salama and Alblowi [1] introduced the concept of neutrosophic topological spaces in 2012. They defined neutrosophic topological space as a generalisation of intuitionistic fuzzy topological space and a neutrosophic set in addition to each element's degree of membership, degree of indeterminacy, and degree of non-membership. Smarandache [7], [8] introduced the dependence degree of (also, the independence degree of) the fuzzy and neutrosophic components for the first time. Arokiarani et al. [4] pioneered the concept of

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https://doi.org/10.22105/jfea.2023.377520.1240

fuzzy neutrosophic set, defined as the sum of all three membership functions not exceeding 3. In 2017, Veereswari [16] proposed a Fuzzy Neutrosophic topological space and basic operations on it. Sarannya et al. [13] introduced n-Cylindrical Fuzzy Neutrosophic Sets (n-CyFNS), which have T and F as dependent components and I as independent components. Except for fuzzy neutrosophic sets, the n-CyFNS is the largest extension of fuzzy sets. In this case, the degree to which positive, neutral, and negative membership functions satisfy the condition, $0 \leq \beta A(x) \leq 1$ and $0 \leq \alpha A\ n(x) + \gamma An(x) \leq 1$, $n > 1$, is an integer. They also defined the distance between two n-CyFNS, as well as their properties and basic operations.

In this work, the membership functions of an image and its pre image in n-CyFNSs are defined at first. We then introduced the n-CyFN continuity of a function defined between two n-Cylindrical Fuzzy Neutrosophic Topological Spaces (n-CyFNTS) using this concept. In addition, we characterise the n-CyFN continuity and present some fundamental results associated with this idea. Additionally defined are n-CyFN interior and closure of n-CyFN subsets of n-CyFNTS. Based on these concepts, we investigate some properties. As well as this, the ideas of n-CyFN open function, n-CyFN closed function and n-CyFN homeomorphism are presented.

2 | Preliminaries

Throughout this paper, U denotes the universe of discourse.

Definition 1 ([17]). A fuzzy set A in U is defined by membership function μA: A⟶ [0, 1] whose membership value $\mu A(x)$ shows the degree to which $x \in U$ includes in the fuzzy set A, for all $x \in U$.

Definition 2 ([6]). A fuzzy topological space is a pair (X, T), where X is any set and T is a family of fuzzy sets in X satisfying following axioms:

I. $\Phi, X \in T$.
II. If $A, B \in T$, then $A \cap B \in T$.
III. If $A_i \in T$ for each $i \in I$, then $U_i A_i \in T$.

Definition 3 ([5]). An IFS A on U is an object of the form $A = \{(x, \alpha_A(x), \gamma_A(x) \mid x \in U)\}$ where $\alpha_A(x) \in [0,1]$ is called the degree of membership of x in A, $\gamma_A(x) \in [0,1]$ is called the degree of non-membership of x in A, and where α_A and γ_A satisfy (for all $x \in U$) $(\alpha_A(x) + \gamma_A(x) \leq 1)$ IFS(U) denote the set of all the IFSs on a universe U.

Definition 4 ([8]). A neutrosophic set A on U is $A = < x, T_A(x), I_A(x), F_A(x) >$; $x \in U$, where T_A, I_A, F_A: A→ $]^-0,1^+[$ and $^-0 < T_A(x) + I_A(x) + F_A(x) < 3^+$.

Definition 5 ([16]). A fuzzy neutrosophic set A on U is $A = < x, T_A(x), I_A(x), F_A(x) >$; $x \in U$, where T_A, I_A, F_A: A→ [0,1] and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 6 ([9]). A neutrosophic set A on U is an object of the form $A = \{(x, u_A(x), \zeta_A(x), v_A(x)): x \in U\}$, where $u_A(x), \zeta_A(x), v_A(x) \in [0,1]$, $0 \leq u_A(x) + \zeta_A(x) + v_A(x) \leq 3$, for all $x \in U$. $u_A(x)$ is the degree of truth membership, $\zeta_A(x)$ is the degree of indeterminacy and $v_A(x)$ is the degree of non-membership. Here (x) and (x) are dependent components and (x) is an independent component.

Definition 7 ([1]). A Neutrosophic Topology (NT) on a non-empty set X is a family τ of neutrosophic subsets in X satisfying the following axioms:

I. (NT1) $0_N, 1_N \in \tau$.
II. (NT2) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$.
III. (NT3) $\cup G_i \in \tau$, for all $\{G_i: i \in J\} \subseteq \tau$.

In this case the pair (X) is called a Neutrosophic Topological Space (NTS) and any neutrosophic set in τ is known as Neutrosophic Open Set (NOS) in X. The elements of τ are called open neutrosophic sets, A neutrosophic set F is closed if and only if it C(F) is neutrosophic open.

Definition 8 ([16]). A Fuzzy Neutrosophic Topology (FNT) a non-empty set X is a family τ of fuzzy neutrosophic subsets in satisfying the following axioms:

I. (FNT1) $0_N, 1_N \in \tau$.
II. (FNT2) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$.
III. (FNT3) $\cup G_i \in \tau$, for all $\{G_i : i \in J\} \subseteq \tau$.

In this case the pair (X) is called a Fuzzy Neutrosophic Topological Space (FNTS) and any fuzzy neutrosophic set in τ is known as Fuzzy Neutrosophic Open Set (FNOS) in. The elements of τ are called open fuzzy neutrosophic sets.

Definition 9 ([13]). A n-CyFNS A on Ʋ is an object of the form $A = \{\langle x, \alpha_A(x), \beta_A(x), \gamma_A(x)\rangle \mid x \in Ʋ\}$ where $\alpha_A(x) \in [0, 1]$, called the degree of positive membership of x in A, $\beta_A(x) \in [0,1]$, called the degree of neutral membership of x in A and $\gamma_A(x) \in [0, 1]$, called the degree of negative membership of x in A, which satisfies the condition, (for all $x \in Ʋ$) ($0 \leq \beta_A(x) \leq 1$ and $0 \leq \alpha_A{}^n(x) + \gamma_A{}^n(x) \leq 1$, $n > 1$, is an integer. Here T and F are dependent neutrosophic components and I is independent.

For the convenience, $\langle \alpha_A(x), \beta_A(x), \gamma_A(x)\rangle$ is called as n-Cylindrical Fuzzy Neutrosophic Number (n-CyFNN) and is denoted as $A = \langle \alpha_A, \beta_A, \gamma_A\rangle$.

Definition 10 ([13]). Let $\{A_i : i \in I\}$ be an arbitrary family of n-CyFNS in Ʋ. Then

- $\cap A_i = \{\langle x, inf(\alpha_{Ai}(x)), inf(\beta_{Ai}(x)), sup((\gamma_{Ai}(x))\rangle \mid x \in U\}$.
- $\bigcup A_i = \{\langle x, sup(\alpha_{Ai}(x)), sup(\beta_{Ai}(x)), inf((\gamma_{Ai}(x))\rangle \mid x \in U\}$.

Definition 11 ([13]). Let $\mathcal{T}_N$(Ʋ) denote the family of all n-CyFNSs on Ʋ.

1. Inclusion: for every two A, B $\in \mathcal{T}_N$(Ʋ), $A \subseteq B$ iff (for all $x \in Ʋ$, $\alpha_A(x) \leq \alpha_B(x)$ and $\beta_A(x) \leq \beta_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$) and $A = B$ iff ($A \subseteq B$ and $B \subseteq A$).
2. Union: for every two A, B $\in \tau_N$(Ʋ), the union of two n-CyFNSs A and B is:

$$A \cup B(x) = \{\langle x, \max(\alpha_A(x), \alpha_B(x)), \max(\beta_A(x), \beta_B(x)), \min(\gamma_A(x), \gamma_B(x))\rangle \mid x \in Ʋ\}.$$

3. Intersection: for every two A, B $\in \mathcal{T}_N$(Ʋ), the intersection of two n-CyFNSs A and B is:

$$A \cap B(x) = \{\langle x, \min(\alpha_A(x), \alpha_B(x)), \min(\beta_A(x), \beta_B(x)), \max(\gamma_A(x), \gamma_B(x))\rangle \mid x \in Ʋ\}.$$

4. Complementation: for every $A \in \mathcal{T}_N$(Ʋ), the complement of an n-CyFNS A is $A^c = \{\langle x, \gamma_A(x), \beta_A(x), \alpha_A(x)\rangle \mid x \in Ʋ\}$.

Definition 12 ([13]).

I. If $A \subseteq B$ & $B \subseteq C$ then $A \subseteq C$.
II. $A \cup B = B \cup A$ & $A \cap B = B \cap A$.
III. $(A \cup B) \cup C = A \cup (B \cup C)$ & $(A \cap B) \cap C = A \cap (B \cap C)$.
IV. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ & $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.
V. $A \cap A = A$ & $A \cup A = A$.
VI. De Morgan's Law for A & B ie, $(A \cup B) = A^c \cap B^c$ & $(A \cap B) = A^c \cup B$.

Definition 13.

- $0_{CyN1}=\{\langle x, 0, 1, 1\rangle \mid x \in U\}$ and $1_{CyN1}=\{\langle x, 1, 1, 0\rangle \mid x \in U\}$ or,
- $0_{CyN2}=\{\langle x, 0, 0, 1\rangle \mid x \in U\}$ and $1_{CyN2}=\{\langle x, 1, 0, 0\rangle \mid x \in U\}$.

Commonly it can be denoted as 0_{CyN} and 1_{CyN}.

Definition 14. An n-Cylindrical Fuzzy Neutrosophic Topology (n-CyFNT) on a non-empty set X is a family, τ_X, of n-cylindrical fuzzy neutrosophic sets in X which satisfies the following conditions:

I. $0_{cyN}, 1_{cyN} \in \tau_X$.
II. $A_1 \cap A_2 \in \tau_X$.
III. $\cup A_i \in \tau_X$, for any arbitrary family $A_i \in \tau_X$, $i \in I$.

The pair (X, τ_X) is called an n-CyFNTS and any n-CyFNS belongs to τ_X is called an n-Cylindrical Fuzzy Neutrosophic Open Set (n-CyFNOS) and the complement of n-CyFNOS is called n-Cylindrical Fuzzy Neutrosophic Closed Set (n-CyFNCS) in X. Like classical topological spaces and fuzzy topological spaces, the family $\{0_{cyN}, 1_{cyN}\}$ is called indiscrete n-CyFNTS and the topology containing all the n-CyFN subsets is called Discrete n-CyFNTs.

Definition 15. Let (X, τ_X) be a CyFNTS on X. Then, $\mathscr{B} \subseteq \tau_X$, a sub family of τ_X is called an n-CyFN base for (X, τ_X), if each member of τ_X may be expressed as the union of members in $\mathscr{B}$.

$\mathscr{S} \subseteq \tau_X$, a sub family of τ_X is called a n-CyFN sub-base for (X, τ_X), if the family of all finite intersections of $\mathcal{S}$ forms a base for (X, τ_X).

3 | n-Cylindrical Fuzzy Neutrosophic Continuous Functions

In this section we will introduce n-CyFN continuous functions and related results.

For that, we define the positive, neutral and negative membership functions of image and pre image of a function in n-CyFNSs.

Definition 16. Let X and Y be two non-empty sets and let f: X→Y be a function. If $A=\{\langle x, \alpha_A(x), \beta_A(x), \gamma_A(x)\rangle \mid x\in X\}$ is an n-CyFNS in X, then the membership functions of image of A under f, defined as:

$f_{\alpha A}(y)=\sup\{\alpha_A(x)): x\in f^{-1} y)\}$, $if\ f^{-1}\ y)$ is non empty

$=0$ if $f^{-1}\ y) = 0$.

$f_{\beta A}(y)=\sup\{\beta_A(x)): x\in f^{-1}\ y)\ \}$, $if\ f^{-1}\ y)$ is non empty

$=0$ if $f^{-1}\ y) = 0$.

$f_{\gamma A}(y)=\inf\ \{\gamma_A(x)): x\in f^{-1}\ y)\}$, $if\ f^{-1}(y)$ is non empty

$=1$ if $f^{-1}\ y) = 0$.

Definition 17. Let X and Y be two non-empty sets and let f: X→Y be a function. If $B=\{\langle y, \alpha_B(y), \beta_B(y)), \gamma_B(y)\rangle \mid y\in Y\}$, is an n-CyFNS in Y, then the membership functions of pre image of B under f are defined as:

I. $f^{-1}\alpha_B(x) = \alpha_B(f(x))$,
II. $f^{-1}\beta_B(x) = \beta_B(f(x))$,
III. $f^{-1}\gamma_B(x) = \gamma_B(f(x))$, respectively.

Now we define the image and pre image of n-CyFNSs.

Definition 18. Let X and Y be two non-empty sets and let f: X→Y be a function. If B = {⟨ y, $\alpha_B(y)$, $\beta_B(y)$), $\gamma_B(y)$ ⟩ | y ∈ Y}, is an n-CyFNS in Y, then the pre image of Y under f is denoted by f^{-1} B), is the n-CyFNS in X defined by:

$$f^{(-1)}(B)=\{\langle x, f^{(-1)}\alpha_B(x), f^{(-1)}\beta_B(x), f^{(-1)}\gamma_B(x)\rangle | x \in X\}.$$

Similarly, if A = {⟨ x, $\alpha_A(x)$, $\beta_A(x)$, $\gamma_A(x)$ ⟩ | x ∈ X} is an n-CyFNS in X, then the image of A under f, denoted by f(A), is the n-CyFNS in Y defined by f(A) = {⟨ y, $f\alpha_A(y)$, $f\beta_A(y)$, $f\gamma_A(y)$ ⟩ | y ∈ Y}.

It is evident that f(A) and f^{-1} B) are n-CyFNSs and f is called n-CyFN function.

Preposition 1. Let X and Y be two non-empty sets and let f: X→Y be an n-CyFN function.

Then

1. $f^{-1}\left(B^C\right) = f^{-1}[B]^C$ for any n-CyFN subset B of Y.
2. $f[A]^C \subset f[A^C]$ for any n-CyFN subset A of X.
3. If B_1 & B_2 are two n-CyFN subsets of Y & $B_1 \subset B_2$, then $f^{-1}(B_1) \subset f^{-1}(B_2)$.
4. If A_1 & A_2 , are two n-CyFN subsets of X & $A_1 \subset A_2$, then $f\ (A_1) \subset f(A_2)$.
5. For any n-CyFN subset B of Y, $f[f^{-1}\ B)] \subset B$.
6. For any n-CyFN subset A of X, $A \subset f^{-1}[f(A)]$.
7. $f^{-1}\left(\cup B_j\right) = \cup f^{-1}(B_j)$ & $f^{-1}\left(\cap B_j\right) = \cap f^{-1}(B_j)$.
8. $f\left(\cup B_j\right) = \cup f(B_j)$ & $f\left(\cap B_j\right) \subseteq \cap f(B_j)$, if f is one-one, then $f\left(\cap B_j\right) = \cap f(B_j)$.
9. $f^{-1}(0_{CyN}) = 0_{CyN}$ & $f^{-1}(1_{CyN}) = 1_{CyN}$.
10. f $(0_{CyN}) = 0_{CyN}$ & $f(1_{CyN}) = 1_{CyN}$, if f is onto.

Proof:

For (1): we have f^{-1} B) = {⟨ x, $f^{-1}\alpha_B(x)$, f^{-1} $\beta_B(x)$, f^{-1} $\gamma_B(x)$ ⟩ | x ∈ X}. Then $f^{-1}\left(B^C\right)$ = {⟨ x, f^{-1} $\gamma_B(x)$, f^{-1} $\beta_B(x)$, $f^{-1}\alpha_B(x)$ ⟩ | x ∈ X} = {⟨ x, $\gamma_B(f(x))$, $\beta_B(f(x))$, $\alpha_B(f(x))$ ⟩ | x ∈ X}.

$=\{f^{-1}[B]\}^C$.

For (2): we have f(A) = {⟨ y, $f\alpha_A(y)$, $f_{\beta A}(y)$, $f\gamma_A(y)$ ⟩ | y ∈ Y}. Then

$f[A]^C$ = {⟨ y, $f\gamma_A(y)$, $f\beta_A(y)$, $f\alpha_A(y)$ ⟩ | y ∈ Y} = {⟨ y, inf $\gamma_A(x)$, sup $\beta_A(x)$, sup $\alpha_A(x)$ ⟩ | y ∈ Y}

Now we have, $f_{\alpha_{A^C}}(y) = \sup \alpha_{A^C}(z)$, $z \in f^{-1}(y)$

$\geq \alpha_{A^C}(y)$

$= \gamma_A(y)$

$\geq \inf \gamma_A(z) = f\gamma_A(y)$

$= f_{\alpha_A}{}^{C}(y).$

Similarly, it follows for the other membership functions. Thus $f[A]^C \subset f[A^C]$.

For (3): we have $B_1 \subset B_2$, for any $x \in X$, $f^{-1}\alpha_{B1}(x) = \alpha_{B1}(f(x)) \le \alpha_{B2}(f(x)) = f^{-1}\alpha_{B2}(x)$,

Similarly, we can prove that $f^{-1}\ \beta_{B1}(x) \le f^{-1}\ \beta_{B2}(x)$ and $f^{-1}\ \gamma_{B1}(x) \ge f^{-1}\ \gamma_{B2}(x)$.

Hence $f^{-1}(B_1) \subset f^{-1}(B_2)$.

For (4): this is similar to 3.

For (5): for any $y \in Y$, such that $f(y) \neq \varnothing$, $f(f^{-1}\alpha_B(y)) = \sup\ (f^{-1}\alpha_B(z)), z \in f^{-1}(y)$

$$= \sup \alpha_B\big(f\ z)\big), z \in f^{-1}\ y)$$

$$= \alpha_B\ y).$$

If $f(y) = \varnothing$, then $f(f^{-1}\alpha_B(y)) = 0 \le \alpha_B\ y)$, also $f(f^{-1}\beta_B(y)) \le \beta_B\ y)$. Similarly, we have $f(f^{-1}\gamma_B(y)) \ge \gamma_B\ y)$.

For (6): for any $x \in X$ $f^{-1}(f\alpha_A\ x)) = f\alpha_A\ f(x))$

$$= \sup \alpha_A(z), z \in f^{-1}(f(x))$$

$$\ge \alpha_A(x).$$

Similarly, we have $f^{-1}(f\beta_A\ x)) \ge \beta_A(x)$ and $f^{-1}(f\gamma_A\ x)) \le \gamma_A(x)$.

For (7): it follows immediately.

For (8): it is evident since 7 holds.

For (9): we have $f^{-1}(0_{CyN}) = f^{-1}(\langle y, 0, 0, 1 \rangle) = \langle x, f^{-1}(0), f^{-1}(0), f^{-1}(1) \rangle = \langle x, 0, 0, 1 \rangle = 0_{CyN}$ $f^{-1}(1_{CyN}) = f^{-1}(\langle y, 1,0, 0 \rangle) = \langle x, f^{-1}(1), f^{-1}(0), f^{-1}(0) \rangle = \langle x, 1, 0, 0 \rangle = 1_{CyN}$.

For (10): this is similar to 7.

Now we'll look at how to define n-CyFN continuity of a function.

Definition 19. Let (X, τ_X) and (Y, τ_Y) be two n-cylindrical fuzzy neutrosophic topological spaces and let f: X→Y be an n-CyFN function. Then f is said to be n-CyFN continuous if for any n-Cylindrical fuzzy neutrosophic subset A of X and for any neighborhood V of f[A] there exists a neighborhood U of A such that $f[U] \subset V$.

The following theorem gives the characterization of n-cylindrical fuzzy neutrosophic continuity.

Theorem 1. Let (X, τ_X) and (Y, τ_Y) be two n-cylindrical fuzzy neutrosophic topological spaces and let f: X→Y be a function. Then the following are equivalent:

I. f is n-cylindrical fuzzy neutrosophic continuous.
II. For any n-cylindrical fuzzy neutrosophic subset (n-CyFNS) A of X and for any neighborhood V of f[A], there exists a neighborhood U of A such that for any B ⊂ U implies, f[B] ⊂ V.
III. For any n-CyFNS A of X and for any neighborhood V of f [A], there exists a neighborhood U of A such that U ⊂ $f^{-1}[V]$.
IV. For any n-CyFNS A of X and for any neighborhood V of f[A], $f^{-1}[V]$ is a neighborhood of A.

Proof: for equivelency of I) and II),

Suppose f is n-cylindrical fuzzy neutrosophic continuous and A ⊂ X is n-CyFNS and V be the neighborhood of f[A]. Then there exist a neighborhood U of A such that f[U] ⊂ V, by the definition of continuity. If B ⊂ U, then f[B] ⊂ f[U] ⊂ V.

For equivelency of II) and III),

Suppose II) holds. Let A ⊂ X is n-CyFNS and V be the neighborhood of f[A], then there exist a neighborhood U of A such that for any B ⊂ U, then f[B] ⊂ V, then we can write B ⊂ f^{-1}f[B] ⊂ f^{-1}[V]. Since B is arbitrary, III) follows.

For equivelency of III) and IV),

Suppose III) holds ie, for any neighborhood U of A ⊂ X, A is n-CyFNS in X, U ⊂ f^{-1}[V]. By the definition of neighborhood, A⊂ C ⊂ U, C is n-CyFNOS of X.

But A⊂ C ⊂ U ⊂ f^{-1}[V]. Thus A⊂ C ⊂ f^{-1}[V]. This implies IV)

For equivelency of IV) and I),

Suppose IV) holds. ie, A ⊂ C ⊂ f^{-1}[V]; C is n-CyFNOS of X.

Now f[C] ⊂ f(f^{-1}[V]) this shows that f[C] ⊂ f(f^{-1}[V]) ⊂ V. Since C is open, it is a neighborhood of A. Thus f is n-CyFN continuous.

Example 1. Let X = {a, b, c} and family of n-CyFN subsets of X, $\tau_X = \{1cyN, 0cyN, A, B, C, D\}$. Also Y = { p, q, r } with τ_y = { 1_{cyN} , 0_{cyN}, P, Q, R, S} and let f: (X, τ_X) → (Y, τ_y) be a function. Also,

A = {<a; 0.4, 0.5, 0.6>, <b; 0.6, 0.5, 0.3>, <c; 0.7, 0.5, 0.5>},

B = {<a; 0.7, 0.5, 0.6>, <b; 0.7, 0.6, 0.4>, <c; 0.7, 0.6, 0.6>},

C = {<a; 0.7, 0.5, 0.6>, <b; 0.7, 0.6, 0.3>, <c; 0.7, 0.6, 0.5>},

D = {<a; 0.4, 0.5, 0.6>, <b; 0.6, 0.5, 0.4>, <c; 0.7, 0.5, 0.6>},

P = {<p; 0.6, 0.5, 0.3>, <q; 0.4, 0.5, 0.6>, <r; 0.7, 0.5, 0.5>},

Q = {<p; 0.7, 0.6, 0.4>, <q; 0.7, 0.5, 0.6>, <r; 0.7, 0.6, 0.6>},

R = {<p; 0.7, 0.6, 0.3>, <q; 0.7, 0.5, 0.6>, <r; 0.7, 0.6, 0.5>},

S = {<p; 0.6, 0.5, 0.4>, <q; 0.4, 0.5, 0.6>, <r; 0.7, 0.5, 0.6>}.

Clearly (X, τ_X) and (Y, τ_Y) are n-CyFNTS.

If f: $(X, \tau_X) \rightarrow (Y, \tau_y)$ is defined by f(a) = q, f(b) = p and f(c) = f(r), then f is n-CyFN continuous.

Theorem 2. Let (X, τ_X) and (Y, τ_Y) be two n-CyFNTS and let f: X→Y is n-CyFN continuous if and only if for each n-CyFN open subset B of Y, we have $f^{-1}[B]$ is an n-CyFN open subset of X.

Proof: let f: X→Y is n-CyFN continuous, then for any n-CyFNS A of X and V is a neighborhood of f[A], there exists a neighborhood U of A such that f[U] ⊂ V. Let B ⊂ Y be n-CyFN open in Y, then there is a neighborhood V of f[A] such that V ⊂ B, by iv) of *Theorem 1*, $f^{-1}[V]$ is a neighborhood of A. Since V ⊂ B implies $f^{-1}[V] \subset f^{-1}[B]$. Thus $f^{-1}[B]$ is a neighborhood and A is arbitrary, it follows the definition of n-CyFN open subset.

Conversely suppose, A ⊂ X is n-CyFNS and V is a neighborhood of f[A]. Then, there exists an n-CyFN open subset C such that f[A] ⊂ C ⊂ V and $f^{-1}[C]$ is open. Also we can write $A \subset f^{-1}f[A] \subset f^{-1}[C] \subset f^{-1}[V]$. Thus $f^{-1}[V]$ is a neighborhood of A. hence f is n-CyFN continuous.

Definition 20. Let (X, τ_X) and (Y, τ_Y) be two n-CyFNTS and let f: X→Y is an n-CyFN function. Then

I. f is called an n-CyFN open function if f(A) is n-CyFN open in Y for every n-CyFN open set A in X.
II. f is called n-CyFN closed function if f(B) is n-CyFN closed in Y for every n-CyFN closed set B in X.

Example 2. In *Example 1*, f is clearly an open function but not closed.

Theorem 3. Let X be a non-empty set and let (Y, τ_Y) be an n-CyFNTS and f: X→Y be a function. Then, there exists a coarsest n-cylindrical fuzzy neutrosophic topology τ_x on X such that f is n-CyFN continuous.

Proof: clearly,

I) $0_{cyN}, 1_{cyN} \in \tau_X$.
II) and III) are evident. Also from *Theorem 2*, f is n-CyFN continuous.

To prove that τ_X is the coarsest n-CyFNT on X such that the f is n-CyFN continuous.

Suppose $\tau_c \subset \tau_X$ is the coarsest n-CyFNT on X such that f is n-CyFN continuous. If $B \in \tau_X$, then there is V in τ_Y such that $f^{-1}[V] = B$. But f is n-CyFN continuous with respect to τ_c.

Then $B = f^{-1}[V] \in \tau_c$.

Thus $\tau_X \subset \tau_c$ this implies $\tau_c = \tau_X$.

4 | Interior and Closure of n-Cylindrical Fuzzy Neutrosophic Sets

We will now define the interior and closure of a set in n-CyFNTS.

Chang gave the definition of a neighborhood of a fuzzy open set instead of a neighborhood of a point.

Definition 21. Let A and N be two n-cylindrical fuzzy neutrosophic subsets of an n-CyFNTS. N is called neighborhood of A if there exists an n-CyFNOS, O such that $A \subset O \subset N$.

Preposition 2. A ⊂ X is n-cylindrical fuzzy neutrosophic open in $(X, \tau_{\mathscr{C}})$ if and only if it carries a neighborhood of its each subset.

Definition 22. Let (X, τ_X) be an n-CyFNTS and let $P = \{\langle x, \alpha_P(x), \beta_P(x), \gamma_P(x) \rangle \mid x \in X\}$ is an n-CyFNS in X. Then the n-Cylindrical fuzzy neutrosophic interior (int_{CyN}) is defined as the n-CyFN union of all n-CyFN open subsets of X ie, $int_{CyN}(P) = \cup \{A: A \in \tau_X \text{ and } A \subseteq P\}$.

Clearly $int_{CyN}(P)$ is the biggest n-CyFN open set that is contained by P.

Example 3. Let $X = \{x, y\}$ and $\tau_X = \{1_{cyN}, 0_{cyN}, P, Q, R, S\}$, where

P = {<x; 0.4, 0.5, 0.8>, <y; 0.3, 0.5, 0.3>}, Q = {<x; 0.5, 0.5, 0.3>, <y; 0.5, 0.5, 0.9>},

R = {<x; 0.5, 0.5, 0.3>, <y; 0.5, 0.5, 0.3>}, S = {<x; 0.4, 0.5, 0.8>, <y; 0.3, 0.5, 0.9>}.

Clearly (X, τ_X) is an n-CyFNTS.

Let G = {<x; 0.5, 0.5, 0.2 >, <y; 0.5, 0.5, 0.2>}, Clearly $int_{CyN}(G) = R$.

Theorem 4. Let (X, τ_X) be an n-CyFNTS and $G \subseteq X$. G is an n-CyFN open set if and only if $G = int_{CyN}(G)$.

Proof: if G is an n-CyFN open set, then the largest n-CyFN open set contained by G is $int_{CyN}(G)$. Hence $G = int_{CyN}(G)$.

Conversely, $int_{CyN}(G)$ is an n-CyFN open set and, if $G = int_{CyN}(G)$ then G is n-CyFN open set.

Theorem 5. Let (X, τ_X) be an n-CyFNTS and $G, H \subseteq X$. Then,

I. $int_{CyN}(int_{CyN}(G)) = int_{CyN}(G)$.
II. $G \subseteq H \Longrightarrow int_{CyN}(G) \subseteq int_{CyN}(H)$.
III. $int_{CyN}(G) \cap int_{CyN}(H) = int_{CyN}(G \cap H)$.
IV. $int_{CyN}(G) \cup int_{CyN}(H) \subseteq int_{CyN}(G \cup H)$.

Proof:

I. Let $int_{CyN}(G) = A$. Then $A \in \tau_X$ if and only if $A = int_{CyN}(A)$, therefore, $int_{CyN}(G) = int_{CyN}(int_{CyN}(G))$.
II. Let $G \subseteq H$. From the definition of n-CyFN interior, $int_{CyN}(G) \subseteq G$ and $int_{CyN}(H) \subseteq H$. Also $int_{CyN}(H)$ is the largest n-CyFN open set contained by H. Hence $G \subseteq H \Longrightarrow int_{CyN}(G) \subseteq int_{CyN}(H)$.
III. By the definition of n-CyFN interior, $int_{CyN}(G) \subseteq G$ and $int_{CyN}(H) \subseteq H$. Then $int_{CyN}(G) \cap int_{CyN}(H) \subseteq G \cap H$. But $int_{CyN}(G \cap H)$ is the largest open set contained by $G \cap H$, then $int_{CyN}(G) \cap int_{CyN}(H) \subseteq int_{CyN}(G \cap H)$. (1)

On the other hand, $G \cap H \subseteq G$ and $G \cap H \subseteq H$ then $int_{CyN}(G \cap H) \subseteq int_{CyN}(G)$ and $int_{CyN}(G \cap H) \subseteq int_{CyN}(H)$, and $int_{CyN}(G \cap H) \subseteq int_{CyN}(G) \cap int_{CyN}(H)$. (2)

From (1) and (2), the result follows.

IV. We have $int_{CyN}(G) \subseteq G$ and $int_{CyN}(H) \subseteq H$. Then $int_{CyN}(G) \cup int_{CyN}(H) \subseteq (G \cup H)$. But $int_{CyN}(G \cup H)$ is the largest open set contained by $G \cup H$. Hence, $int_{CyN}(G) \cup int_{CyN}(H) \subseteq int_{CyN}(G \cup H)$.

Definition 23. Let (X, τ_X) be an n-CyFNTS and let $P = \{\langle x, \alpha_P(x), \beta_P(x), \gamma_P(x) \rangle \mid x \in X\}$ is an n-CyFNS in X. Then the n-Cylindrical fuzzy neutrosophic closure (cl_{CyN}) of P.

ie, cl_{CyN}(P) = ∩ {B: B is an n-CyFNCS in X and $B \subseteq P$}.

It is to be noted that, cl_{CyN}(P) is the smallest closed set that contains P.

Example 4. From *Example 3*,

The set of closed sets of τ_X is denoted as $\tau_X{}^c = \{\{1_{cyN}, 0_{cyN}, P^c, Q^c, R^c, S^c\}$

P^c = {<x; 0.8, 0.5, 0.4>, <y; 0.3, 0.5, 0.3>}, Q^c = {<x; 0.3, 0.5, 0.5>, <y; 0.9, 0.5, 0.5>},

R^c = {<x; 0.3, 0.5, 0.5>, <y; 0.3, 0.5, 0.5>}, S^c = {<x; 0.8, 0.5, 0.4>, <y; 0.9, 0.5, 0.3>},

Let H = {<x; 0.2, 0.5, 0.5 >, <y; 0.2, 0.5, 0.5>}, Clearly cl_{CyN}(H) =R^c.

Theorem 6. Let (X, τ_X) be an n-CyFNTS and H ⊆X. H is an n-CyFN closed set if and only if H = cl_{CyN}(H).

Proof: the proof is obvious.

Theorem 7. Let (X, τ_X) be an n-CyFNTS and H ⊆X. Then $int_{CyN}(H) \subseteq H \subseteq cl_{CyN}(H)$.

Proof: from the definition of n-CyFN interior, it is clear that int_{CyN}(H) is the largest n-CyFN open set contained by H. Hence $int_{CyN}(H) \subseteq H$.

Also from the definition of n-CyFN closure, cl_{CyN}(H) is the smallest closed set that contains H. Therefore $H \subseteq cl_{CyN}(H)$.

Hence proved.

Theorem 8. Let (X, τ_X) be an n-CyFNTS and G, H ⊆ X. Then,

I. $cl_{CyN}(cl_{CyN}(H)) = cl_{CyN}(H)$.
II. G ⊆H ⟹ $cl_{CyN}(G) \subseteq cl_{CyN}(H)$.
III. $cl_{CyN}(G \cap H) \subseteq cl_{CyN}(G) \cap cl_{CyN}(H)$.
IV. $cl_{CyN}(G) \cup cl_{CyN}(H) = cl_{CyN}(G \cup H)$.
V. $(cl_{CyN}\ H))^c = int_{CyN}(H^c)$.

Proof:

I. Let cl_{CyN}(H) = K. Then K is an n-CyFN closed set. Then K = cl_{CyN}(K) Hence $cl_{CyN}(cl_{CyN}(H)) = cl_{CyN}(H)$.
II. Let G ⊆H. From the definition of n-CyFN closure, G ⊆ cl_{CyN}(G) and H ⊆ cl_{CyN}(H). Also cl_{CyN}(H) is the smallest n-CyFN closed set that contains H. Hence G ⊆ H ⟹ $cl_{CyN}(G) \subseteq cl_{CyN}(H)$.
III. cl_{CyN}(G) and cl_{CyN}(H) are n-CyFN closed sets. So $cl_{CyN}(G) \cap cl_{CyN}(H)$ is an n-CyFN closed set. By the definition of n-CyFN closure, $G \subseteq cl_{CyN}(G)$ and H ⊆ cl_{CyN}(H). Then G∩H⊆ $cl_{CyN}(G) \cap cl_{CyN}(H)$. But cl_{CyN}(G∩H) is the smallest closed set that contains G∩H, then $cl_{CyN}(G \cap H) \subseteq cl_{CyN}(G) \cap cl_{CyN}(H)$.
IV. We have $G \subseteq cl_{CyN}(G)$ and $H \subseteq cl_{CyN}(H)$, then G∪H ⊆ $cl_{CyN}(G) \cup cl_{CyN}(H)$.

But cl_{CyN}(G∪H) is the smallest closed set that contains G∪H. Therefore $cl_{CyN}(G \cup H) \subseteq cl_{CyN}(G) \cup cl_{CyN}(H)$.

Conversely, $G \subseteq cl_{CyN}(G) \subseteq cl_{CyN}(G \cup H)$ and $H \subseteq cl_{CyN}(H) \subseteq cl_{CyN}(G \cup H)$. Then $cl_{CyN}(G) \cup cl_{CyN}(H) \subseteq cl_{CyN}(G \cup H)$.Hence the result holds.

V. If we take the basic definition of n-CyFN closure and n-CyFN interior, we get, $(cl_{CyN}\ H))^c = (\cap G_j)^c$, $H \subseteq G_j$ and $G_j^c \in \tau_X$.

But $(\cap G_j)^c = \cup G_j^c = int_{CyN}(H^c)$.

Theorem 9. Let (X, τ_X) and (Y, τ_Y) be two n-CyFNTS and let f: X→Y is an n-CyFN function. Then

I. f is n-CyFN open function if $f(int_{CyN}(G)) \subseteq int_{CyN}(f\ G))$ for each n-CyFN set G over X.
II. f is n-CyFN closed function if $cl_{CyN}(f(H)) \subseteq f(cl_{CyN}(H))$, for each n-CyFN set H over X.

Proof: straight forward.

Definition 24. Let (X, τ_X) and (Y, τ_Y) be two n-CyFNTS and let f: X→Y is an n-CyFN function. Then f is n-CyFN homeomorphism if:

I. f is a bijection.
II. f is n-CyFN continuous.
III. Inverse of f, f^{-1} is also n-CyFN continuous.

Example 5. Let X = {a, b, c, d}. Now consider the n-CyFN subsets I, J, K, L of X as

I = {<a; 0.4, 0.5, 0.6>, <b; 0.6, 0.5, 0.3>, <c; 0.7, 0.5, 0.5>, <d; 0.7, 0.5, 0.6>},

J = {<a; 0.7, 0.5, 0.6>, <b; 0.6, 0.5, 0.3>, <c; 0.7, 0.5, 0.5>, <d; 0.4, 0.5, 0.6>},

K = {<a; 0.7, 0.5, 0.6>, <b; 0.6, 0.5, 0.3>, <c; 0.7, 0.5, 0.5>, <d; 0.7, 0.5, 0.6>},

L = {<a; 0.4, 0.5, 0.6>, <b; 0.6, 0.5, 0.3>, <c; 0.7, 0.5, 0.5>, <d; 0.4, 0.5, 0.6>}.

Let $\tau_X = \{0_{cyN}, 1_{cyN}, I, J, K, L\}$. Then (X, τ_X) is an n-CyFNTS. Consider θ: X→X defined by $\theta(a) = d$, $\theta(b) = b$, $\theta(c) = c$ and $\theta(d) = a$. Clearly θ is an n-CyFN homeomorphism.

Theorem 10. Let (X, τ_X) and (Y, τ_Y) be two n-CyFNTS and let f: X→Y is an n-CyFN function. Then the following conditions are equivalent.

I. f is f is n-CyFN homeomorphism.
II. f is n-CyFN continuous and n-CyFN open.
III. f is n-CyFN continuous and n-CyFN closed.

Proof: from the definition and properties of n-CyFN continuous function, n-CyFN interior and n-CyFN closure, it follows.

5 | Conclusion

n-Cylindrical fuzzy neutrosophic sets are the latest extension of neutrosophic sets in which I as an independent neutrosophic component. In our previous work we defined the topological space in n-CyFNS context. So far, we have introduced n-CyFN base, n-CyFN subbase and related theorems. Through this paper our aim is to extend the important concepts like continuity, interiors, closure and related theorems to n-cylindrical fuzzy neutrosophic environment. Here, we define the membership functions of an image and its pre image in n-CyFNSs. Using this concept, we then introduced the n-Cylindrical fuzzy neutrosophic continuity of a function defined between two n-CyFNTS. In addition, we present some fundamental results related to this concept, as well as a characterization of the n-Cylindrical fuzzy neutrosophic continuity. Also defined are the n-Cylindrical fuzzy neutrosophic interior and n-Cylindrical fuzzy neutrosophic closure of n-CyFN subsets of n-CyFNTS. We examine some properties based on these concepts. The concept of n-CyFN open function, n-CyFN closed function and n-CyFN homeomorphism are also presented. To that extent, the study sheds more light on the subject, allowing for further investigation in other important concepts of topology like compactness, connectedness, separation axioms etc.

Funding

This research received no external funding.

Conflicts of Interest

The authors declare no conflict of interest.

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