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Complex Fuzzy Lie Subalgebras and Complex Fuzzy Ideals Under t-Norms

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Abstract

In this paper, we define the conceps of complex fuzzy Lie subalgebras and complex fuzzy ideals of Lie algebras with respect to t-norms and investigate some of characteristics and relationship between them. Next, we introduce the concepos of quotient subalgebras, intersection, sum and direct product of them and prove some results about them. Finally, we introduce and study the image and the inverse image of them under Lie algebra homomorphisms.

Keywords: Lie algebras, Ideals, Fuzzy set theory, t-norms, Complex fuzzy sets, Intersections, Homomorphisms.

1 | Introduction

Lie algebras were introduced to study the concept of infinitesimal transformations by Marius Sophus Lie in the 1870s and independently discovered by Wilhelm Killing in the 1880s. The name Lie algebra was given by Hermann Weyl in the 1930s. The notion of fuzzy sets was firstly introduced by Zadeh [1]. The fuzzy set theory states that there are propositions with an infinite number of truth values, assuming two extreme values, 1 (totally true), 0 (totally false) and a continuum in between, that justify the term fuzzy. Applications of this theory can be found, for example, in artificial intelligence, computer science, control engineering, decision theory, logic and management science. After the introduction of fuzzy sets by Zadeh [1], there are a number of generalizations of this fundamental concept. The notion of complex fuzzy sets is introduced by Ramot et al. [2] is one among them. Here, the membership values are complex numbers, drawn from the unit disc of the complex plane. For more details on complex fuzzy sets, we refer the readers to [2], [3]. The fuzzy Lie subalgebras and fuzzy Lie ideals are considered by Kim and Lee [4], and by Yehia [5], [6]. The author by using norms, investigated some properties of fuzzy algebraic structures [7]-[23]. By using t-norms, the objective of this article is to present the notion of complex fuzzy Lie subalgebras as $CFST(L)$ and complex fuzzy ideals as $CFIT(L)$ of Lie algebra L such that we characterize them with the level subset of subalgebras and ideasl of Lie subalgebras and prove some fundamental results about them. Next we define the intersection, sum and direct product of them and we prove that if $\mu_1, \mu_2 \in CFST(L)$ and $\vartheta_1, \vartheta_1 \in CFIT(L)$, then $\mu_1 \cap \mu_2$,

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$\mu_1+\mu_2 \in CFST(L)$ and $\vartheta_1 \cap \vartheta_1, \vartheta_1+\vartheta_1 \in CFIT(L)$. Also if $\mu_1 \in CFST(L_1), \mu_2 \in CFST(L_2)$ and $\vartheta_1 \in CFIT(L_1), \vartheta_2 \in CFIT(L_2)$, then $\mu_1 \times \mu_2 \in CFST(L_1 \times L_2)$ and $\vartheta_1 \times \vartheta_1 \in CFIT(L_1 \times L_2)$. Finally, we investigate the image and the inverse image of them under Lie algebra homomorphisms. If $f{:}L \to M$ be an epimorphism of Lie algebras such that $\mu \in CFST(L)$, $\vartheta \in CFIT(L)$ and $\alpha \in CFST(M)$, $\beta \in CF\,IT(M)$, then $f(\mu) \in CFST(M)$, $f(\vartheta) \in CF\,IT(M)$ and $f^{-1}(\alpha) \in CFST(L)$, $f^{-1}(\beta) \in CF\,IT(L)$.

2 | Preliminaries

In this section, we first review some elementary aspects that are necessary for this paper. For more details we refer the readers to [3], [7].

Definition 1. A Lie algebra is a vector space L over a field F (equal to R or C) on which $L \times L \to L$ denoted by $(x,y) \to [x,y]$ is defined satisfying the following axioms:

I. $[x,y]$ is bilinear.
II. $[x,x] = 0$ for all $x \in L$.
III. $[[x,y], z]+[[y,z],x]+[[z,x],y]=0$ (Jacobi identity).

For all $x,y,z \in L$.

In this paper by L will be denoted a Lie algebra. We note that the multiplication in a Lie algebra is not associative, i.e., it is not true in general that $\big[[x,y],z\big]=\big[x,[y,z]\big]$. But it is anti commutative, i.e., $[x,y]=-[y,x]$. A subspace H of L closed under $[,]$ will be called a Lie subalgebra. A subspace I of L with the property $[I,L] \subseteq I$ will be called a Lie ideal of L. Obviously, any Lie ideal is a subalgebra.

Definition 2. Let L_1 and L_2 be Lie algebras over a field F. A linear transformation $f{:}L_1 \to L_2$ is called a Lie homomorphism if $f([x,y])=[f(x), f(y)]$ for all $x, y \in L_1$.

Definition 3. Let X be a nonempty set. A complex fuzzy set A on X is an object having the form $A = \{(x, \mu_A(x)) | x \in X\}$, where μ_A denotes the degree of membership function that assigns each element $x \in X$ a complex number $\mu_A(x)$ lies within the unit circle in the complex plane. We shall assume that is $\mu_A(x)$ will be represented by $r_{A(x)} e^{iw_A(x)}$ where $i = \sqrt{-1}$, and $r: X \to [0,1]$ and $w: X \to [0,2\pi]$. Note that by setting $w(x) = 0$ in the definition above, we return back to the traditional fuzzy subset. Let $\mu_1=r_1e^{w_1}$, and $\mu_2=r_2e^{w_2}$ be two complex numbers lie within the unit circle in the complex plane. By $\mu_1 \le \mu_2$, we mean $r_1 \le r_2$ and $w_1 \le w_2$.

Definition 4. A t-norm T is a function $T{:}[0, 1] \times [0, 1] \to [0, 1]$ having the following four properties:

- (T1) $T(x,1)= x$ (neutral element).
- (T2) $T(x, y) \le T(x, z)$ if $y \le z$ (monotonicity).
- (T3) $T(x, y)= T(y, x)$ (commutativity).
- (T4) $T\big(x,T(y, z)\big)= T(T(x, y), z)$ (associativity).

For all $x, y, z \in [0, 1]$.

It is clear that if $x_1 \ge x_2$ and $y_1 \ge y_2$, then $T(x_1, y_1) \ge T(x_2, y_2)$.

Example 1.

I. Standard intersection t-norm $T_m(x, y)=min\{x, y\}$.

II. Bounded sum t-norm $T_b(x, y)=max\{0, x+y-1\}$.
III. Algebraic product t-norm $T_p(x, y)=xy$.
IV. Drastic t-norm.

$$T_D(x,y)=\begin{cases} y, & \text{if } x=1, \\ x, & \text{if } y=1, \\ 0, & \text{otherwise.} \end{cases}$$

V. Nilpotent minimum t-norm.

$$T_{nM}(x,y)=\begin{cases} \min\{x, y\}, & \text{if } x+y>1, \\ 0, & \text{otherwise.} \end{cases}$$

VI. Hamacher product t-norm.

$$T_{H_0}(x,y)=\begin{cases} 0, & \text{if } x=y=0, \\ \dfrac{xy}{x+y-xy}, & \text{otherwise.} \end{cases}$$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm: $T_D(x,y)\leq T(x, y)\leq T_m(x, y)$ for all $x, y\in[0, 1]$.

Recall that t-norm T will be idempotent if for all $x\in[0,1]$ we have $T(x, x)=x$.

Lemma 1. Let T be a t-norm. Then

$$T(T(x, y), T(w, z))=T(T(x, w), T(y, z)).$$

For all $x, y, w, z\in[0, 1]$.

3 | Main Results

Definition 5. Let L be a Lie subalgebra and E^2 is the unit disc. Define $\mu=re^{iw}:L\rightarrow E^2$ be a complex fuzzy set on L with $r:L\rightarrow[0,1]$ and $w:L\rightarrow[0, 2\pi]$. Then μ is said to be a complex fuzzy subalgebra of L under t-norm T if the following conditions hold:

I. $r(x+y)\geq T(r(x), r(y))$.
II. $r(\alpha x)\geq r(x)$,
III. $r([x, y])\geq T(r(x),r(y))$.
IV. $w(x+y)\geq min\{w(x), w(y)\}$.
V. $w(\alpha x)\geq w(x)$,
VI. $w([x,y])\geq min\{w(x),w(y)\}$.

For all $x,y\in L$ and $\alpha\in F$.

Denote by $CFST(L)$, the set of all complex fuzzy subgroups of L under t-norm T.

Definition 6. Let L be a Lie subalgebra and E^2 is the unit disc. Define $\mu=re^{iw}:L\rightarrow E^2$ be a complex fuzzy set on L with $r:L\rightarrow[0,1]$ and $w:L\rightarrow[0, 2\pi]$. Then μ is said to be a complex fuzzy Lie ideal of L under t-norm T if the following conditions hold:

I. $r(x+y)\geq T(r(x), r(y))$.
II. $r(\alpha x)\geq r(x)$.
III. $r([x,y])\geq r(x)$.
IV. $w(x+y)\geq min\{w(x),w(y)\}$.
V. $w(\alpha x)\geq w(x)$.
VI. $w([x,y])\geq w(x)$.

For all $x, y \in L$ and $a \in F$.

Denote by $CFIT(L)$, the set of all complex fuzzy Lie ideals of L under t-norm T.

Example 2. It is well known that the set $R^3 = \{(x_1,x_2,x_3) : x_1,x_2,x_3 \in R\}$ of all 3-dimensional real vectors forms a Lie algebra over $F = R$ with the usual cross product $\times$: define $\mu : R^3 \to E^2$, where (E^2 is the unit disc), by

$$r(x_1,x_2,x_3)=\begin{cases}1, & \text{if } x_1=x_2=x_3=0,\\ 0.65, & \text{if } x_1=x_2=0 \text{ and } x_3 \neq 0,\\ 0, & \text{otherwise,}\end{cases}$$

and

$$w(x_1,x_2,x_3)=\begin{cases}0.7\pi, & \text{if } x_1=x_2=x_3=0,\\ 0.5\pi, & \text{if } x_2=x_3=0 \text{ and } x_1 \neq 0,\\ 0, & \text{otherwise.}\end{cases}$$

If $T(a, b)=ab$ for all $a,b \in [0,1]$, then $\mu \in CFST(L)$. Let $x=(0,0,1) \in L=R^3$ and $y=(1,0,1) \in L=R^3$ then $[x,y]=(0,1,0)$ and so $r([x,y])=r(0,1,0)=0 \ngeq 0.65=r(x)$ thus $\mu \notin CFIT(L)$.

Lemma 2. Let $\mu \in CFST(L)$ and T be an idempotent t-norm.

I. $\mu(0) \geq \mu(x)$ for all $x \in L$.
II. $\mu(x) = \mu(-x)$ for all $x \in L$.
III. $\mu([x,y]=\mu([y,x])$ for all $x,y \in L$.
IV. $\mu(x- y)= \mu(0)$ implies that $\mu(x)= \mu(y)$ for all $x,y \in L$.

Proof:

I. Let $x \in L$. Then

$$r(0)=r(x + (-x)) \geq T(r(x), r(-x)) \geq T(r(x), r(x))=r(x),$$

and

$$w(0)=w(x+(-x)) \geq \min\{w(x), w(-x)\} \geq \min\{w(x),w(x)\}=w(x).$$

Thus $r(0) \geq r(x)$ and $w(0) \geq w(x)$ and so $\mu(0)= r(0)e^{iw(0)} \geq r(x)e^{iw(x)} =\mu(0)$.

II. Let $x \in L$. As

$$r(-x)= r((-1)x) \geq r(x)=r(-(-x)) \geq r(-x),$$

and

$$w(-x)=w((-1)x) \geq w(x)=w(-(-x)) \geq w(-x).$$

So $r(x)=r(-x)$ and $w(x)=w(-x)$ then $\mu(x)=r(x)e^{iw(x)}=r(-x)e^{iw(-x)}= \mu(-x)$.

III. Let $x,y \in L$. Then $r([x,y])=r(-[y, x])=r([y, x])$ and $w([x,y])=w(-[y,x])= w([y, x])$ which mean that

$$\mu([x,y])=r([x,y])e^{iw([x,y])}=r([y,x])e^{iw([y,x])}=\mu([y,x]).$$

IV. Let $x,y \in L$. Therefore

$$r(y)=r\big(x-(x-y)\big)$$
$$=r\Big(x+\big(-(x-y)\big)\Big)$$
$$\geq T\Big(r(x),r\big(-(x-y)\big)\Big)$$
$$=T\big(r(x),r(x-y)\big)$$
$$=T\big(r(x),r(0)\big)$$
$$\geq T\big(r(x),r(x)\big)$$
$$=r(x)$$
$$=r(x-y+y)$$
$$\geq T\big(r(x-y),r(y)\big)$$
$$=T\big(r(0),r(y)\big)$$
$$\geq T\big(r(y),r(y)\big)$$
$$=r(y).$$

Thus $r(x)=r(y)$. Also

$$w(y)=w(x-(x-y))$$
$$=w(x+(-(x-y)))$$
$$\geq \min\{w(x),w(-(x-y))\}$$
$$=\min\{w(x),w(x-y)\}$$
$$=\min\{w(x),w(0)\}$$
$$\geq \min\{w(x),w(x)\}$$
$$=w(x)$$
$$=w(x-y+y)$$
$$\geq \min\{w(x-y),w(y)\}$$
$$=\min\{w(0),w(y)$$
$$\geq \min\{w(y),w(y)\}$$
$$=w(y).$$

Thus $w(x)=w(y)$. Therefore $\mu(x)=r(x)e^{iw(x)}=r(y)e^{iw(y)}=\mu(y)$.

Proposition 1. Let L be a Lie subalgebra and $\mu:L\to[0,1]$ be a complex fuzzy set on L. If T be idempotent t-norm, then the following statements are equivalent:

I. $\mu=re^{iw}\in CFST(L)$.

II. The set

$$U_\mu^t=\{x\in L:\ \mu(x)=r(x)e^{iw(x)}\geq t\}=\{x\in L:r(x)\geq t;\ w(x)\geq t\},$$

is a subalgebra of L for every $t\in Im(\mu)$.

Proof: Let $\mu=re^{iw}\in CFST(L)$ and $x,y\in U_\mu^t$ and $\alpha\in F$. We prove that $x+y,\alpha x,[x,y]\in U_\mu^t$.

As

$$r(x+y)\geq T(r(x),r(y))\geq T(t,t)=t,$$

and

$$w(x+y)\geq \min\{w(x),w(y)\}\geq w\{t,t\}=t.$$

We will have that $x+y\in U_\mu^t$. Also $r(\alpha x)\geq r(x)\geq t$ and $w(\alpha x)\geq w(x)\geq t$ so $\alpha x\in U_\mu^t$.

Moreover,

$$r([x,y]) \geq T(r(x),r(y)) \geq T(t,t)=t,$$

and

$$w([x,y]) \geq \min\{w(x),w(y)\} \geq w\{t,t\}=t.$$

Mean that $[x,y] \in U_\mu^t$. Therefore U_μ^t will be subalgebra of L for every $t \in Im(\mu)$. Conversely, let U_μ^t be a Lie subalgebra of L for every $t \in Im(\mu)$. Let $x,y \in L$ and $\alpha \in F$. We can say that $\mu(y) \geq \mu(x)=t$ so $r(y) \geq r(x)=t$ and $w(y) \geq w(x)=t$ thus $x,y \in U_\mu^t$ and then $x+y, \alpha x, [x,y] \in U_\mu^t$. Now $r(x+y) \geq t = T(r(x),r(y))$ and $w(x+y) \geq t = \min\{w(x),w(y)\}$. Also $r(\alpha x) \geq t = r(x)$ and $w(\alpha x) \geq t = w(x)$. Next $r([x,y]) \geq t = T(r(x),r(y))$ and $w([x,y]) \geq t = \min\{w(x),w(y)\}$. Therefore $\mu = re^{iw} \in CFST(L)$.

Corollary 1. Let L be a Lie subalgebra and $\mu: L \to [0,1]$ be a complex fuzzy set on L. If T be idempotent t-norm, then the following statements are equivalent:

I. $\mu = re^{iw} \in CFST(L)$.
II. The set

$$U_\mu^t = \{x \in L: \mu(x) = r(x)e^{iw(x)} > t\} = \{x \in L: r(x) > t;\ w(x) > t\},$$

is a subalgebra of L for every $t \in Im(\mu)$.

Proposition 2. Let L be a Lie subalgebra and $\mu: L \to [0,1]$ be a complex fuzzy set on L. If T be idempotent t-norm, then the following statements are equivalent:

I. $\mu = re^{iw} \in CFIT(L)$.
II. The set

$$U_\mu^t = \{x \in L: \mu(x) = r(x)e^{iw(x)} \geq t\} = \{x \in L: r(x) \geq t;\ w(x) \geq t\},$$

is an ideal of L for every $t \in Im(\mu)$.

Proof: The proof is similar to the *Proposition 1*.

Corollary 2. Let L be a Lie subalgebra and $\mu: L \to [0,1]$ be a complex fuzzy set on L. If T be idempotent t-norm, then the following statements are equivalent:

I. $\mu = re^{iw} \in CFST(L)$.
II. The set

$$U_\mu^t = \{x \in L: \mu(x) = r(x)e^{iw(x)} > t\} = \{x \in L: r(x) > t;\ w(x) > t\},$$

is an ideal of L for every $t \in Im(\mu)$.

Definition 7. Let $f: G \to H$ be a mapping and $\mu_G = r_G e^{iw_G}$ and $\mu_H = r_H e^{iw_H}$ be two complex fuzzy sets on G and H respectively. Define $f(\mu_G): H \to [0,1]$ as:

$$f(\mu_G) = f(r_G e^{iw_G}) = f(r_G)e^{if(w_G)}.$$

Such that for all $h \in H$, we define:

$$f(r_G)(h) = \sup\{r_G(g) \mid g \in G: f(g) = h\},$$

and

$$f(w_G)(h)=\sup\{w_G(g) \mid g\in G;\ f(g)=h\}.$$

Also define $f^{-1}(\mu_H):G\to[0,1]$ as:

$$f^{-1}(r_He^{iw_H})= f^{-1}(r_H)e^{if^{-1}(w_H)},$$

Such that for all $g\in G$, we define:

$$f^{-1}\left(r_He^{iw_H}\right)(g)= r_H\left(f(g)\right)e^{iw_H\left(f(g)\right)}.$$

Definition 8. Let $\mu_1=r_1e^{iw_1}$ and $\mu_2=r_2e^{iw_2}$ be two complex fuzzy sets on L. Define the intersection $\mu_1\cap\mu_2$ as:

$$\mu_1\cap\mu_2=r_1e^{iw_1}\cap r_2e^{iw_2}=(r_1\cap r_2)e^{i(w_1\cap w_2)}.$$

Such that $r_1\cap r_2: L\to[0,1]$ and $w_1\cap w_2: L\to[0, 2\pi]$ and for all $x \in L$ define:

$$(r_1\cap r_2)(x)=T(r_1(x),r_2(x)),$$

and

$$(w_1\cap w_2)(x)=\min\{w_1(x),w_1(x)\}.$$

Proposition 3. Let $\mu_1 = r_1e^{iw_1} \in CFST(L)$ and $\mu_2=r_2e^{iw_2}\in CFST(L)$. Then $\mu_1\cap\mu_2\in CFST(L)$.

Proof: Let $x,y\in L$ and $\alpha\in F$. Then

I.

$$(r_1\cap r_2)(x+y)=T(r_1(x + y),r_2(x + y))$$
$$\geq T(T\left(r_1(x),r_1(y)\right),T(r_2(x),r_2(y)))$$
$$=(T\left(r_1(x),r_2(x)\right),T(r_1(y),r_2(y)))$$
$$T((r_1\cap r_2)(x),(r_1\cap r_2)(y)),$$

and

$$(\mathbf{w_1} \cap \mathbf{w_2})(\mathbf{x} + \mathbf{y}) = \min\{w_1(x+y),w_2(x+y)$$
$$\geq\min\{\min\left\{w_1(x),w_1(y)\right\},\min\{w_2(x),w_2(y)\}\}$$
$$= \min\{\min\{w_1(x),w_2(x)\},\min\{w_1(y),w_2(y)\}\}$$
$$=\min\{(w_1\cap w_2)(x), (w_1\cap w_2)(y)\},$$

thus

$$(r_1\cap r_2)(x+y)\geq T((r_1\cap r_2)(x),(r_1\cap r_2)(y)),$$

and

$$w_1\cap w_2)(x+y)\geq\min\{(w_1\cap w_2)(x), (w_1\cap w_2)(y)\}.$$

II.

$$(r_1\cap r_2)([x,y])=T(r_1([x,y]),r_2([x,y]))$$
$$\geq T\left(T\left(r_1(x),r_1(y)\right),T\left(r_2(x),r_2(y)\right)\right)$$
$$= \mathbf{T}(\mathbf{T}\left(\mathbf{r_1}(\mathbf{x}), \mathbf{r_2}(\mathbf{x})\right), \mathbf{T}(\mathbf{r_1}(\mathbf{y}), \mathbf{r_2}(\mathbf{y})))$$
$$= T((r_1\cap r_2)(x),(r_1\cap r_2)(y)),$$

so

$$(r_1\cap r_2)([x,y])\geq T\left((r_1\cap r_2)(x),(r_1\cap r_2)(y)\right).$$

Also

$$(w_1 \cap w_2)([x, y]) = \min\{w_1([x,y]), w_2([x,y])\}$$
$$\geq \min\{\min\{w_1(x), w_2(y)\}, \min\{w_2(x), w_2(y)\}\}$$
$$= \min\{\min\{w_1(x), w_2(x)\}, \min\{w_1(y), w_2(y)\}\}$$
$$\min\{(w_1 \cap w_2)(x), (w_1 \cap w_2)(y)\},$$

thus

$$(w_1 \cap w_2)([x,y]) \geq \min\{(w_1 \cap w_2)(x), (w_1 \cap w_2)(y)\}.$$

III.

$$(r_1 \cap r_2)(\alpha x) = T(r_1(\alpha x), r_2(\alpha x)) \geq T(r_1(x), r_2(x)) = (r_1 \cap r_2)(x),$$

and

$$(w_1 \cap w_2)(\alpha x) = \min\{w_1(\alpha x), w_2(\alpha x)\} \geq \min\{w_1(x), w_2(x)\} = (w_1 \cap w_2)(x).$$

Thus *Cases (1)* to *(3)* give us that $\mu_1 \cap \mu_2 \in CFST(L)$.

Proposition 4. Let $\mu_1 = r_1 e^{iw_1} \in CFIT(L)$ and $\mu_2 = r_2 e^{iw_2} \in CFST(L)$. Then $\mu_1 \cap \mu_2 \in CFIT(L)$.

Proof: It is similar to the proof of *Proposition 3*.

Definition 9. Let $\mu_1 = r_1 e^{iw_1}$ and $\mu_2 = r_2 e^{iw_2}$ be two complex fuzzy sets on L. Define the sum $\mu_1 + \mu_2$ as:

$$\mu_1 + \mu_2 = r_1 e^{iw_1} + r_2 e^{iw_2} = (r_1 + r_2) e^{i(w_1 + w_2)}.$$

Such that $r_1 + r_2 : L \to [0,1]$ and $w_1 + w_2 : L \to [0, 2\pi]$ and for all $x \in L$ define:

$$(r_1 + r_2)(x) = \sup_{x=ab} T\big(r_1(a), r_2(b)\big),$$

and

$$(w_1 + w_2)(x) = \min_{x=ab} \{w_1(a), w_2(b)\}.$$

Proposition 5. Let $\mu_1 = r_1 e^{iw_1} \in CFST(L)$ and $\mu_2 = r_2 e^{iw_2} \in CFST(L)$. Then $\mu_1 + \mu_2 \in CFST(L)$.

Proof: Let $x, y, a, b, c, d \in L$ and $\alpha \in F$.

I.

$$(r_1 + r_2)(x+y) = \sup_{x+y=a+b+c+d} T\big(r_1(a+b), r_2(c+d)\big)$$

$$\geq \sup_{x+y=a+b+c+d} T(T(r_1(a), r_1(b)), T(r_2(c), r_2(d)))$$

$$= \sup_{x+y=a+c+b+d} T(T(r_1(a), r_2(c)), T(r_1(b), r_2(d)))$$

$$= T\Big(\sup_{x=a+c} T\big(r_1(a), r_2(c)\big), \sup_{y=b+d} T(r_1(b), r_2(d)\Big)$$

$$= T((r_1 + r_2)(x), (r_1 + r_2)(y)),$$

and then

$$(r_1 + r_2)(x+y) \geq T((r_1 + r_2)(x), (r_1 + r_2)(y)).$$

Also

$$(w_1+w_2)(x+y)=\min_{x+y=a+b+c+d}\{w_1(a+b),w_2(c+d)\}$$

$$\geq \min_{x+y=a+b+c+d}\{\min\{w_1(a),w_1(b)\},\min\{w_2(c),w_2(d)\}\}$$

$$=\min_{x+y=a+c+b+d}\{\min\{w_1(a),w_2(c)\},\min\{w_1(b),w_2(d)\}\}$$

$$\min\{\min_{x=a+c}\{w_1(a),w_2(c)\},\min_{y=b+d}\{w_1(b),w_2(d)\}\}$$

$$=\min\{(w_1+w_2)(x),(w_1+w_2)(y)\}.$$

Thus

$$(w_1+w_2)(x+y)\geq\min\{(w_1+w_2)(x),(w_1+w_2)(y)\}.$$

II.

$$(r_1+r_2)([x,y])=\sup_{[x,y]=[a,b]+[c,d]} T\big(r_1([a,b]), r_2([c,d])\big)$$

$$\geq \sup_{[x,y]=[a,b]+[c,d]} T(T(r_1(a), r_1(b))T(r_2(c),r_2(d)))$$

$$=\sup_{[x,y]=[a+c,b+d]} T(T(r_1(a), r_2(c)),T(r_1(b),r_2(d)))$$

$$=T(\sup_{x=a+c} T\big(r_1(a), r_2(c)\big), \sup_{y=b+d} T(r_1(b),r_2(d))$$

$$=T((r_1+r_2)(x),(r_1+r_2)(y)),$$

so

$$(r_1+r_2)([x,y]) \geq T((r_1+r_2)(x),(r_1+r_2)(y)).$$

Also

$$(w_1+w_2)([x,y])=\min_{x+y=a+b+c+d}\{w_1([a,b]),w_2([c,d])\}$$

$$\geq \min_{[x,y]=[a,b]+[c,d]}\{\min\{w_1(a),w_1(b)\},\min\{w_2(c),w_2(d)\}\}$$

$$=\min_{[x,y]=[a+c,b+d]}\{\min\{w_1(a),w_2(c)\},\min\{w_1(b),w_2(d)\}\}$$

$$=\min\{\min_{x=a+c}\{w_1(a),w_2(c)\},\min_{y=b+d}\{w_1(b),w_2(d)\}\}$$

$$=\min\{(w_1+w_2)(x),(w_1+w_2)(y)\},$$

Thus

$$(w_1+w_2)([x,y])\geq\min\{(w_1+w_2)(x),(w_1+w_2)(y)\}.$$

III.

$$(r_1+r_2)(\alpha x)=\sup_{\alpha x=\alpha a+\alpha b} T\big(r_1(\alpha a), r_2(\alpha b)\big)$$

$$\geq \sup_{x=a+b} T\big(r_1(a), r_2(b)\big)$$

$$=(r_1+r_2)(x),$$

and

$$(w_1+w_2)(\alpha x)=\min_{\alpha x=\alpha a+\alpha b}\{w_1(\alpha a),w_1(\alpha b)\}$$

$$\geq \min_{x=a+b}\{w_1(a),w_1(b)\}$$

$$=(w_1+w_2)(x),$$

Then from *Cases (1)* to *(3)* we get that $\mu_1+\mu_1 \in CFST(L)$.

Proposition 6. Let $\mu_1=r_1e^{iw_1}\in CFST(L)$ and $\mu_2=r_2e^{iw_2}\in CFST(L)$. Then $\mu_1+\mu_2\in CFST(L)$.

Proof: It is similar to the proof of *Proposition 5*.

Definition 10. Let L be a Lie subalgebra and I be an ideal of L: define $\mu_{\frac{L}{I}}:\frac{L}{I}\to[0,1]$ as:

$$\mu_{\frac{L}{I}}(x+I)=r_{\frac{L}{I}}(x+I)e^{w_{\frac{L}{I}}(x+I)}.$$

Such that $r_{\frac{L}{I}}:\frac{L}{I}\to[0,1]$ with

$$r_{\frac{L}{I}}(x+I)=\begin{cases} T\big(r(x),r(i)\big), & \text{if } x\neq i,\\ 1, & \text{otherwise,}\end{cases}$$

and $w_{\frac{L}{I}}:\frac{L}{I}\to[0,1]$ with

$$w_{\frac{L}{I}}(x+I)=\begin{cases} \min\{w(x),w(i)\}, & \text{if } x\neq i,\\ 1, & \text{otherwise.}\end{cases}$$

Proposition 7. Let μ_L and I be an ideal of L. Then $\mu_{\frac{L}{I}}\in CFST\left(\frac{L}{I}\right)$.

Proof: Let $\alpha\in F$ and $x,y\in L$ such that $x, y\neq i$. Then

I.

$$r_{\frac{L}{I}}((x+I)+(y+I))=r_{\frac{L}{I}}((x+y)+I)$$
$$=T\big(r(x+y),r(i)\big)$$
$$\geq T\Big(T\big(r(x),r(y)\big),T\big(r(i),r(i)\big)\Big)$$
$$= T\Big(T\big(r(x),r(i)\big),T\big(r(y),r(i)\big)\Big)$$
$$=T(r_{\frac{L}{I}}(x+I),r_{\frac{L}{I}}(y+I)).$$

Thus $r_{\frac{L}{I}}((x+I)+(y+I))\geq T\left(r_{\frac{L}{I}}(x+I),r_{\frac{L}{I}}(y+I)\right)$.

II.

$$r_{\frac{L}{I}}\big(\alpha(x+I)\big)=r_{\frac{L}{I}}(\alpha x+I)=T\big(r(\alpha x),r(i)\big)\geq T\big(r(x),r(i)\big)=r_{\frac{L}{I}}(x+I).$$

III.

$$\begin{aligned}
r_{\frac{L}{I}}([x+I,y+I])&=r_{\frac{L}{I}}([x,y]+I)\\
&=T\big(r([x,y]),r(i)\big)\\
&\geq T\Big(T\big(r(x),r(y)\big),T\big(r(i),r(i)\big)\Big)\\
&= T\Big(T\big(r(x),r(i)\big),T\big(r(y),r(i)\big)\Big)\\
&= T\big(r_{\frac{L}{I}}(x+I),r_{\frac{L}{I}}(y+I)\big).
\end{aligned}$$

Then $r_{\frac{L}{I}}([x+I,y+I])\geq T\left(r_{\frac{L}{I}}(x+I),r_{\frac{L}{I}}(y+I)\right)$.

IV.

$$\begin{aligned}
w_{\frac{L}{I}}((x+I)+(y+I))&=r_{\frac{L}{I}}((x+y)+I)\\
&=\min\{w(x+y),w(i)\}\\
&\geq\min\{\min\{w(x),w(y)\},\min\{w(i),w(i)\}\} =\\
&\min\{\min\{w(x),w(i)\},\min\{w(y),w(i)\}\}\\
&=\min\{w_{\frac{L}{I}}(x+I),w_{\frac{L}{I}}(y+I)\}.
\end{aligned}$$

Then $w_{\frac{L}{I}}((x+I)+(y+I))\geq \min\left\{w_{\frac{L}{I}}(x+I),w_{\frac{L}{I}}(y+I)\right\}$.

V.

$$w_{\frac{L}{I}}\big(\alpha(x+I)\big)=w_{\frac{L}{I}}(\alpha x+I)=\min\{w(\alpha x),w(i)\}\geq\min\{w(x),w(i)\}=w_{\frac{L}{I}}(x+I).$$

VI.

$$\begin{aligned}
w_{\frac{L}{I}}([x+I,y+I])&=w_{\frac{L}{I}}([x,y]+I)\\
&=\min\{w([x,y]),w(i)\}\\
&\geq\min\{\min\{w(x),w(y)\},\min\{w(i),w(i)\}\}\\
&=\min\{\min\{w(x),w(i)\},\min\{w(y),w(i)\}\}\\
&=\min\{w_{\frac{L}{I}}(x+I),w_{\frac{L}{I}}(y+I)\}.
\end{aligned}$$

Then $w_{\frac{L}{I}}([x+I,y+I])\geq\min\{w_{\frac{L}{I}}(x+I),w_{\frac{L}{I}}(y+I)\}$.

Therefore *Cases (1)* to *(6)* give us that $\mu_{\frac{L}{I}}\in CFST\left(\frac{L}{I}\right)$.

Definition 11. Let $\mu_1=r_1e^{iw_1}$ and $\mu_2=r_2e^{iw_2}$ be two complex fuzzy sets on L_1 and L_2 respectively. Define the product $\mu_1\times\mu_2$ as:

$$\mu_1\times\mu_2=r_1e^{iw_1}\times r_2e^{iw_2}=(r_1\times r_2)e^{i(w_1\times w_2)},$$

such that $r_1\times r_2 : L_1\times L_2\rightarrow[0,1]$ and $w_1\times w_2: L_1\times L_2\rightarrow[0,2\pi]$ with

$$(r_1\times r_2)(x_1, x_2)= T(r_1(x_1),r_2(x_2)),$$

and

$(w_1 \times w_2)(x_1, x_2)=\min\{w_1(x_1),w_1(x_2)\}$.

For all $(x_1, x_2)\in L_1 \times L_2$.

Proposition 8. Let $\mu_1=r_1e^{iw_1}\in CFST(L_1)$ and $\mu_2=r_2e^{iw_2}\in CFST(L_2)$. Then $\mu_1\times\mu_2\in CFST(L_1\times L_2)$.

Proposition 9. Let $\mu_1=r_1e^{iw_1}\in CFIT(L_1)$ and $\mu_2 = r_2e^{iw_2} \in CFIT(L_2)$. Then $\mu_1\times\mu_2\in CFIT(L_1\times L_2)$.

Proposition 10. Let $\mu_L=r_Le^{iw_L}\in CFST(L)$ and $f:L\to M$ be an epimorphism of Lie algebras. Then $f(\mu_L)\in CFST(M)$.

Proof:

I.

Let $m_1,m_2\in M$ and $l_1,l_2\in L$ such that $m_1=f(l_1)$ and $m_2=f(l_2)$. Then

$$f(r_L)(m_1+m_2) = \sup\{r_L(l_1+l_2) \mid l_1,l_2\in L;\ f(l_1+l_2)=m_1+m_2\}$$
$$\geq \sup\{T(r_L(l_1),r_L(l_2)) \mid l_1,l_2\in L,\ f(l_1) = m_1;\ f(l_2) = m_2\}$$
$$=T(\sup\{r_L(l_1) \mid l_1\in L; f(l_1)= m_1\},\sup\{r_L(l_2) \mid l_2\in L; f(l_2)= m_2\})$$
$$=T(f(r_L)(m_1),f(r_L)(m_2)),$$

and so

$$f(r_L)(m_1+m_2)\geq T(f(r_L)(m_1),f(r_L)(m_2)).$$

Also

$$f(w_L)(m_1+m_2)=\sup\{w_L(l_1+l_2) \mid l_1,l_2\in L;\ f(l_1+l_2)=m_1+m_2\}$$
$$\geq \sup\{\min\{w_L(l_1),w_L(l_2)\} \mid l_1,l_2\in L;\ f(l_1) = m_1;\ f(l_2) = m_2\}$$
$$= \min\{\sup\{w_L(l_1) \mid l_1\in L; f(l_1)= m_1\},\sup\{w_L(l_2) \mid l_2\in L; f(l_2)= m_2\}\}$$
$$=\min\{f(w_L)(m_1),f(w_L)(m_2)\}.$$

Thus

$$f(w_L)(m_1+m_2)\geq \min\{f(w_L)(m_1),f(w_L)(m_2)\}.$$

II. Let $m\in M$, $a \in F$ and $l\in L$ such that $m=f(l)$. Now

$$f(r_L)(\alpha m)=\sup\{r_L(\alpha l) \mid l \in L;\ f(\alpha l) = \alpha m\}$$
$$\geq \sup\{r_L(l) \mid l \in L;\ f(l) = m\}$$
$$=f(r_L)(m),$$

and

$$f(w_L)(\alpha m)=\sup\{w_L(\alpha l) \mid l \in L;\ f(\alpha l)=\alpha m\} \geq \sup\{w_L(l) \mid l \in L;\ f(l)=m\} = f(w_L)(m).$$

III. Let $m_1,m_2\in M$ and $l_1,l_2\in L$ such that $m_1=f(l_1)$ and $m_2=f(l_2)$. As

$$(r_L)([m_1,m_2])=\sup\{r_L([l_1,l_2]) \mid l_1,l_2\in L;\ f([l_1,l_2])=[m_1,m_2]\}$$
$$\geq \sup\{T(r_L(l_1), r_L(l_2)) \mid l_1,l_2\in L;\ ([f(l_1),f(l_2)])=[m_1,m_2]\}$$
$$=T(\sup\{r_L(l_1) \mid l_1\in L;\ f(l_1)=m_1\}, \sup\{r_L(l_2) \mid l_2\in L;\ f(l_2)=m_2\})$$
$$=T(f(r_L)(m_1),f(r_L)(m_2)).$$

Then $f(r_L)([m_1,m_2])\geq T(f(r_L)(m_1),f(r_L)(m_2))$.

Moreover,

$f(w_L)([m_1,m_2])=\sup\{w_L([l_1,l_2]) \mid l_1,l_2\in L;\ f([l_1,l_2]) = [m_1,m_2]\}$
$\geq \sup\{\min\{w_L(l_1), w_L(l_2)\} \mid l_1,l_2\in L;\ ([f(l_1),f(l_2)])=[m_1,m_2]\}$
$= \min\{\sup\{w_L(l_1) \mid l_1\in L;\ f(l_1)=m_1\}, \sup\{w_L(l_2) \mid l_2\in L;\ f(l_2)=m_2\}\}$
$=\min\{f(w_L)(m_1),f(w_L)(m_2)\}.$

Then $f(w_L)([m_1,m_2])\geq \min\{f(w_L)(m_1),f(w_L)(m_2)\}$.

Therefore, *Cases (1)* to *(3)* mean that $f(\mu_L)=f(r_L e^{iw_L})=f(r_L)e^{if(w_L)}\in CFST(M)$.

Proposition 11. Let $\mu_L = r_L e^{iw_L} \in CFIT(L)$ and $f: L \to M$ be an epimorphism of Lie algebras. Then $f(\mu_L)\in CFIT(M)$.

Proof: It is similar to the proof of *Proposition 10*.

Proposition 12. Let $\mu_L = r_L e^{iw_L} \in CFST(M)$ and $f: L \to M$ be an epimorphism of Lie algebras. Then $f^{-1}(\mu_L) \in CFST(L)$.

Proof:

I. Let $l_1,l_2\in L$. Thus

$f^{-1}(r_M)(l_1+l_2)=r_M(f(l_1+l_2))$
$= r_M(f(l_1)+ f(l_2))$
$\geq T(r_M(f(l_1),r_M(f(l_2))$
$=T(f^{-1}(r_M)(l_1),f^{-1}(r_M)(l_2)),$

and

$f^{-1}(w_M)(l_1+l_2)=w_M(f(l_1+l_2))$
$=w_M(f(l_1)+ f(l_2))$
$\geq \min\{w_M(f(l_1),w_M(f(l_2))\}$
$=\min\{f^{-1}(w_M)(l_1),f^{-1}(w_M)(l_2)\},$

and so

$f^{-1}(r_M)(l_1+l_2)\geq T(f^{-1}(r_M)(l_1),f^{-1}(r_M)(l_2))$

and

$f^{-1}(w_M)(l_1+l_2)\geq \min\{f^{-1}(w_M)(l_1),f^{-1}(w_M)(l_2)\}.$

II. Let $l_1, l_2 \in L$. Now

$f^{-1}(r_M)([l_1,l_2])=r_M(f([l_1,l_2]))$
$=r_M([f(l_1)+ f(l_2)])$
$\geq T(r_M(f(l_1),r_M(f(l_2))$
$=T(f^{-1}(r_M)(l_1),f^{-1}(r_M)(l_2)),$

and

$f^{-1}(w_M)([l_1,l_2])=w_M(f([l_1,l_2]))$
$=w_M([f(l_1), f(l_2)])$
$\geq \min\{w_M(f(l_1),w_M(f(l_2))\}$
$=\min\{f^{-1}(w_M)(l_1),f^{-1}(w_M)(l_2)\},$

then

$$f^{-1}(r_M)([l_1,l_2])\geq T\left(f^{-1}(r_M)(l_1),f^{-1}(r_M)(l_2)\right),$$

and

$$f^{-1}(w_M)([l_1,l_2])\geq \min\left\{f^{-1}(w_M)(l_1),f^{-1}(w_M)(l_2)\right\}.$$

III. Let $l\in L$ and $\alpha\in F$. Then

$$f^{-1}(r_M)(\alpha l)=r_M(f(\alpha l)) = r_M(\alpha f(l)) \geq r_M(f(l)) = f^{-1}(r_M)(l),$$

and

$$f^{-1}(w_M)(\alpha l)=w_M\left(f(\alpha l)\right)= w_M\left(\alpha f(l)\right)\geq w_M\left(f(l)\right)= f^{-1}(w_M)(l).$$

Then from *Cases (1)* to *(3)* we get that $f^{-1}(\mu_L)\in$CFST(L).

Proposition 13. Let $\mu_L=r_Le^{iw_L}\in CFIT(M)$ and $f:L\to M$ be an epimorphism of Lie algebras. Then $f^{-1}(\mu_L)\in$CFIT(L).

Proof: It is similar to the proof of *Proposition 12*.

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The author has seen and agreed with the contents of the manuscript and there is no financial interest to report and certifies that the submission is original work and is not under review at any other publication.

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