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An Algebraic Analysis on Exploring q-Rung Orthopair Multi-Fuzzy Sets

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Abstract

The q-Rung Orthopair Fuzzy set (qROF-set) environment is a contemporary tool for handling uncertainty and vagueness in decision-making scenarios. In this paper, we delve into the algebraic examination of q-rung orthopair Multi-Fuzzy Sets (MFSs) and explore their operational laws. The novel q-Rung Orthopair Multi-Fuzzy subgroup (qROMF-subgroup) is the extension of Intuitionistic Multi-Fuzzy Subgroup (IMF-subgroup) to encompass the domain of groups. The properties of the proposed fuzzy subgroup are examined in detail, and the paper concludes by defining two additional concepts: qROMF-coset and qROMF-normal subgroup. Finally, we present a comparison of the newly introduced model with existing approaches to validate its superior performance.

Keywords: qROMF-set, qROMF-subgroup, qROMF-coset and qROMF-normal subgroup.

1 | Introduction

Fuzzy sets, Multi-Fuzzy Sets (MFSs), and their generalizations have consistently held significance for mathematicians and researchers dealing with uncertainties. In our daily lives, numerous models and conventional approaches are utilized to manage imprecision and uncertainty. To address this, Zadeh [1] introduced the theory of fuzzy sets as a modified version of crisp sets. In this theory, the degree of belongingness (ζ) of an element is represented as a single value between zero and one. However, in real-world scenarios, it is not always accurate to assume that the degree of non-belongingness (δ) of an element in a fuzzy set is simply $(1 - \zeta)$, as there might be some hesitant degree. To accommodate this, Atanassov [2] proposed a generalization of fuzzy sets called intuitionistic fuzzy set (IF-Set) that includes a degree of hesitation called the hesitation margin, defined as $1 - (\zeta + \delta)$. Pythagorean Fuzzy-Set (PF-Set), which was introduced by Yager [3] is the improvement of IF-Set and is helpful in various dynamic problems. Another important development to address some of the limitations of existing fuzzy set theory is the qROF-set, which was proposed by Yager [4].

Blizard [5] and Yager [6] introduced the idea of multiset which represents a generalization of the classical or standard set, allowing for an unordered collection of objects with the possibility of

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repetition. Multisets have found diverse applications in fields such as mathematics, computer science, economics, and more [7, 8]. Sebastian and Ramakrishnan [9], [10] introduced the concept of Multi Fuzzy Set (MFS), which serves as a broader framework than L-fuzzy set theory [11]. MFS offers an improved approach to representing the quantities of elements present in a set, using varying and/or identical membership values. The concept of MFSs offers a novel approach to representing problems that may be challenging to describe using other extensions of fuzzy set theory. By combining MFS and SS, Yang et al. [12] developed the idea of an MFSS and used it in the context of MCDM. The Intuitionistic Multi-Fuzzy Soft Set (IMF-Set) model and the Hesitant Multi-Fuzzy Soft Set [13] model as an extension of MFSS were introduced by Das and Kar [14] and Asit and Dey [13].

A pioneering work by Rosenfeld [15] initiated the exploration of fuzzy subgroups, prompting numerous mathematicians to investigate traditional group theoretic principles in different fuzzy contexts. The notion of multi-fuzzy subgroups [16] has also been studied extensively in recent years, and has been used in the modeling of complex systems and decision-making scenarios. Muthuraj and Balamurugan [17] conducted research on the concept of multi-fuzzy groups and their level subgroups [17], and they also explored the topic of intuitionistic multi-anti fuzzy subgroups [18]. Additionally, Rasuli [19] introduced the idea of fuzzy multi-groups under t-norms [20]. According to studies, researchers have added a variety of new multi-fuzzy algebraic structures, including, complex multi-fuzzy subgroups [21], multi-fuzzy soft subgroups [22] and more [23]–[26]. To learn about these developments, readers are advised to refer [20], [27]–[32].

The recent extensions of qROF-set theory and MF-set theory have opened up new avenues of research and provided a powerful framework for dealing with uncertain or imprecise data. Thus the combined framework of these two extensions known as qROMF-set is launched recently by Vimala et al. [33] to improve the handling of imprecise and vague information. We illustrate the concept of qROMF-subgroup and establish its features using the most recent development of qROMF-set.

The primary significance of this paper include:

I. The establishment of q-rung orthopair multi-fuzzy subgroup (qROMF-subgroup) as an extension of the Intuitionistic Multi-Fuzzy Subgroup (IMF-subgroup).
II. This extension is valuable because qROMF-sets offer greater flexibility compared to IMF-sets ranging from 1 to the qth power of multi-belongingness and multi-non-belongingness degrees.
III. The investigation of algebraic structures on q-rung orthopair MFSs, includes qROMF subgroups, qROMF-cosets, and qROMF-normal subgroups.

This structure of this study is as follows; Section 2 provides an overview of the preliminary concepts that are required to define qROMF-subgroup. In Section 3, we present the definition of qROMF-subgroup and its properties. This is accomplished by defining qROMFS and introducing its rudimentary operations, such as subset, equality, complement, union and intersection. Furthermore, we define qROMF-coset and qROMF-normal subgroup, which are important concepts in the study of qROMF-subgroup. Section 5, gives a comparative analysis with prior literature and to demonstrate the superior and more effective nature of our proposed work.

2 | Basic Concepts

This part discusses the preliminary concepts needed for the development of preceeding sections. Throughout the discussion, the symbol $\breve{\mathscr{W}}$ will be employed to represent a universal set.

Definition 1 ([1]). The F-set $\breve{F}$ on $\breve{\mathscr{W}}$ is a mapping $\zeta: \breve{\mathscr{W}} \to [0,1]$ where $\zeta(\breve{w})$ represents the belongingness degree of elements in $\breve{\mathscr{W}}$ and it is depicted in the form

$$\breve{\mathscr{F}} = \left\{ \breve{w}: \zeta_{\mathscr{F}}(\breve{w})/\breve{w} \in \breve{\mathscr{W}} \right\}.$$

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Definition 2 ([6]). Let $\breve{\mathscr{W}}$ be the initial universal set. A fuzzy multiset (FM-set) $\breve{\mathscr{A}}$ of a set $\breve{\mathscr{W}}$ is defined by a count membership function $CM_{\breve{\mathscr{A}}}: \breve{\mathscr{W}} \longrightarrow [0,1]$ of which the value is a multi-set of the unit interval $I = [0,1]$.

Definition 3 ([8]). Let $\breve{\mathscr{C}}$ be a group. A FM-set $\breve{\mathscr{A}}$ of $\breve{\mathscr{W}}$ is said to be a fuzzy multigroup of $\breve{\mathscr{C}}$ if

I. $CM_{\breve{\mathscr{A}}}\ \breve{w}, \breve{x}) \geq min\left(CM_{\breve{A}}\ \breve{w}), CM_{\breve{A}}\ \breve{x})\right) \quad for\ all\ \breve{w}, \breve{x} \in \breve{\mathscr{C}}.$
II. $CM_{\breve{\mathscr{A}}}\left(\breve{w}^{-1}\right) = CM_{\breve{A}}\ \breve{w}) \quad for\ all\ \breve{w} \in \breve{\mathscr{C}}.$

Definition 4 ([9], [10]). A MFS $\breve{\mathscr{O}}$ of dimension n in $\breve{\mathscr{W}}$ is a set of ordered strings:

$$\breve{\mathscr{O}} = \left\{\breve{w}/\left\langle \zeta^1_{\breve{O}}(\breve{w}), \zeta^2_{\breve{O}}(\breve{w}), \dots, \zeta^i_{\breve{O}}(\breve{w})\right\rangle : \breve{w} \in \breve{\mathscr{W}}\right\},$$

where n be a positive integer. Then the function $\zeta^i_{\breve{O}} = \left(\zeta^1_{\breve{O}}, \zeta^2_{\breve{O}}, \ldots \zeta^i_{\breve{O}}\right)$ is called the multi-belongingness degree of a MF-set and the collection of all MF-set of dimension n in $\breve{\mathscr{W}}$ is expressed as $M^iFS(\breve{\mathscr{W}})$.

Definition 5 ([14]). An intuitionistic MFS (IMF-set) $\breve{A}$ is of the form

$$\breve{A} = \left\{\breve{w}/\left(\zeta^1_{\breve{A}}(\breve{w}), \delta^1_{\breve{\mathscr{A}}}\breve{w}\right)\right), \left(\zeta^2_{\breve{\mathscr{A}}}(\breve{w}), \delta^2_{\breve{\mathscr{A}}}(\breve{w})\right), \cdots, \left(\zeta^n_{\breve{\mathscr{A}}}(\breve{w}), \delta^n_{\breve{A}}\breve{w}\right)\right) : \breve{w} \in \breve{\mathscr{W}}\right\},$$

where $\zeta^1_{\breve{\mathscr{A}}}(\breve{w}), \delta^1_{\breve{\mathscr{A}}}(\breve{w}): \breve{\mathscr{W}} \rightarrow [0,1]$ are the real valued functions satisfying the condition that $0 \leq \zeta^1_{\breve{\mathscr{A}}}(\breve{w}) + \delta^1_{\breve{\mathscr{A}}}(\breve{w})\big) \leq 1$ for $i = 1,2, \dots, n$ and for each $\breve{w} \in \breve{\mathscr{W}}$. The set of all IMF-set is represented as $IMFS(\breve{\mathscr{W}})$.

Definition 6 ([16]). Let $\breve{\mathscr{O}} \in M^iFS(\breve{\mathscr{W}})$ is called a multi-fuzzy subgroup (MF-subgroup) of a group $\breve{\mathscr{C}}$ if $for\ all\ \breve{w}, \breve{x} \in \breve{\mathscr{C}}.$

I. $\breve{\mathscr{O}}\ \breve{w}\breve{x}) \geq min\left(\breve{\mathscr{O}}\ \breve{w}), \breve{\mathscr{O}}\ \breve{x})\right)$

II. $\breve{\mathscr{O}}\left(w^{-1}\right) = \breve{\mathscr{O}}\ \breve{w}).$

Definition 7 ([18]). A IMF-set $\breve{A} = \left\{\breve{w}/\left(\zeta^1_{\breve{A}}(\breve{w}), \delta^1_{\breve{A}}(\breve{w})\right), \left(\zeta^2_{\breve{\mathscr{A}}}(\breve{w}), \delta^2_{\breve{\mathscr{A}}}(\breve{w})\right), \cdots, \left(\zeta^n_{\breve{A}}(\breve{w}), \delta^n_{\breve{\mathscr{A}}}(\breve{w})\right) : \breve{w} \in \breve{\mathscr{W}}\right\}$ of a group $\breve{\mathscr{C}}$ is said to be a IMF-subgroup of $\check{C}$ if

I. $\zeta_{\breve{\mathscr{R}}}(\breve{w}\breve{x}) \geq min\left(\zeta_{\breve{\mathscr{R}}}\left((\breve{w}), \zeta_{\breve{\mathscr{R}}}(\breve{x})\right)\right.$ and $\delta_{\breve{\mathscr{R}}}(\breve{w}\breve{x}) \leq max\left(\delta_{\breve{\mathscr{R}}}\left((\breve{w}), \delta_{\breve{\mathscr{R}}}(\breve{x})\right)\right. \quad for\ all\ \breve{w}, \breve{x} \in \breve{\mathscr{C}}.$
II. $\zeta_{\breve{R}}\left(\breve{w}^{-1}\right) = \zeta_{\breve{R}}(\breve{w})$ and $\delta_{\breve{\mathscr{R}}}\left(\breve{w}^{-1}\right) = \delta_{\breve{\mathscr{R}}}(\breve{w}), \breve{w} \in \breve{\mathscr{C}}.$

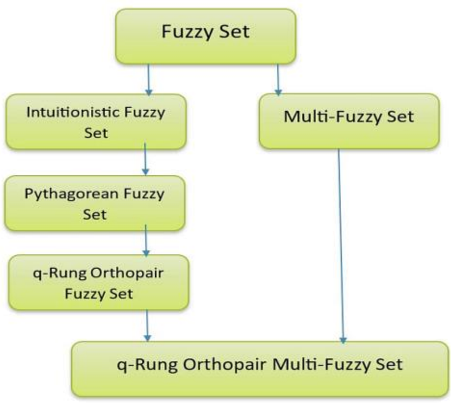

Fig. 1. Hierarchial structure of the q-rung orthopair MFS.

Definition 8 ([4]). A qROF-set $\breve{\mathscr{A}}$ in $\breve{\mathscr{W}}$ is represented as

$$\breve{\mathscr{A}} = \left\{\left\langle \breve{w}, \left(\zeta_{\breve{\mathscr{A}}}(\breve{w}), \delta_{\breve{\mathscr{A}}}(\breve{w})\right)\right\rangle : \breve{w} \in \breve{\mathscr{W}}\right\}.$$

The function $\zeta_{\breve{\mathscr{A}}}, \delta_{\breve{\mathscr{A}}}: \breve{\mathscr{W}} \to [0,1]$ indicates the degree of belongingness and non-belongingness of the element in $\breve{\mathscr{W}}$ with the restriction that $0 \le \zeta_{\breve{\mathscr{A}}}(\breve{w}))^q + \delta_{\breve{\mathscr{A}}}(\breve{w}))^q \le 1$ and the degree of hesitancy is $\pi_{\breve{\mathscr{A}}}(\breve{w}) = \left[1 - \zeta_{\breve{\mathscr{A}}}(\breve{w}))^q + \delta_{\breve{\mathscr{A}}}(\breve{w}))^q)\right]^{\frac{1}{q}}$.

Table 1. IMFS, qROMFS.

IMFS	qROMFS
$0 \le \zeta_k + \delta_k \le 1$	$0 \le (\zeta_k)^q + (\delta_k)^q \le 1$
$\Pi_k = 1 - \zeta_k - \delta_k$	$\Pi_k = \sqrt[q]{1 - (\zeta_k)^q - (\delta_k)^q}$
$\zeta_k + \delta_k + \Pi_k = 1$	$(\zeta_k)^q + (\delta_k)^q + (\Pi_k)^q = 1$

3 | q-Rung Orthopair Multi-Fuzzy Subgroup

This section introduces the operational laws of q-rung orthopair MFS and defines the algebraic structures of qROMF-set such as qROMF-subgroup, qROMF-coset and qROMF-normal subgroup.

Definition 9 ([33]). Let $\breve{\mathscr{R}}$ be a q-rung orthopair MFS (qROMF-set) of dimension n in $\breve{\mathscr{W}}$ is defined as:

$$\breve{\mathscr{R}} = \left\{\left\langle \breve{w}, \left(\zeta^1_{\breve{\mathscr{R}}}(\breve{w}), \delta^1_{\breve{\mathscr{R}}}(\breve{w}), \left(\zeta^2_{\breve{\mathscr{R}}}(\breve{w}), \delta^2_{\breve{\mathscr{R}}}(\breve{w}), \cdots, \left(\zeta^n_{\breve{\mathscr{R}}}(\breve{w}), \delta^n_{\breve{\mathscr{R}}}(\breve{w})\right\rangle : \breve{w} \in \breve{\mathscr{W}}\right\}.$$

The function $\zeta^i_{\breve{\mathscr{R}}}, \delta^i_{\breve{\mathscr{R}}}: \breve{\mathscr{W}} \longrightarrow [0,1]$ *for all* $i = 1,2,..,n$, denotes the degree of multi-belongingness and multinon-belongingness respectively with the restriction that $0 \le \left(\zeta^i_{\breve{\mathscr{R}}}(\breve{w})\right)^q + \left(\delta^i_{\breve{\mathscr{R}}}(\breve{w})\right)^q \le 1$ *for all* $\breve{w} \in \breve{\mathscr{W}}$. The collection of all MF-set of dimension n in $\breve{\mathscr{W}}$ is expressed as $qROM^nFS(\breve{\mathscr{W}})$. For ease, a qROMFN is represented as $\breve{\mathscr{R}} = \left(\zeta^i_{\breve{\mathscr{R}}}, \delta^i_{\breve{\mathscr{R}}}\right)$.

Example 1. Suppose a company's resource and development department intends to conduct interviews for four potential candidates. Let $\breve{\mathscr{W}} = \{\breve{w}_1, \breve{w}_2, \breve{w}_3, \breve{w}_4\}$ be the collection of candidates. The department assesses each candidate based on their academic performance, work experience and age using three-dimensional degree of belongingness, non-belongingness.

$$\begin{aligned}\breve{\mathscr{R}} = \{ \quad & \{\langle \breve{w}_1, (0.8,0.5), (0.3,0.5), (0.4,0.7)\rangle, \langle \breve{w}_2, (0.7,0.3), (0.5,0.4), (0.6,0.2)\rangle,\\ & \langle \breve{w}_3, (0.5,0.3), (0.9,0.5), (0.7,0.3)\rangle, \langle \breve{w}_4, (0.4,0.3), (0.5,0.2), (0.9,0.4)\rangle\}.\end{aligned}$$

Definition 10 ([33]). Let $\breve{\mathscr{R}}, \breve{\mathscr{S}} \in qROM^iFS(\breve{\mathscr{W}})$ over $\breve{\mathscr{W}}$ and n be a positive integer such that $\breve{\mathscr{R}} = \{ \breve{w}/\left(\zeta^i_{\breve{\mathscr{R}}}(\breve{w}), \delta^i_{\breve{\mathscr{R}}}(\breve{w})\right); \breve{w} \in \breve{\mathscr{W}}\}$ and $\breve{\mathscr{S}} = \left\{\breve{w}/\left(\zeta^i_{\breve{\mathscr{S}}}(\breve{w}), \delta^i_{\breve{\mathscr{S}}}(\breve{w})\right); \breve{w} \in \breve{\mathscr{W}}\right\}$ *for all* $i = 1,2,..,n$ respectively.

Then the subset, equality, union, intersection and complement of $\breve{\mathscr{S}}$ and $\breve{\mathscr{T}}$ are defined as:

I. $\breve{\mathscr{R}} \subset \breve{\mathscr{S}}$ iff $\zeta^i_{\breve{\mathscr{R}}}(\breve{w}) \le \zeta^i_{\breve{\mathscr{R}}}(\breve{w}), \delta^i_{\breve{\mathscr{S}}}(\breve{w}) \ge \delta^i_{\breve{\mathscr{S}}}(\breve{w})$ *for all* $\breve{w} \in \breve{\mathscr{W}}$.

II. $\breve{\mathscr{R}} = \breve{\mathscr{S}}$ iff $\zeta^i_{\breve{\mathscr{R}}}(\breve{w}) = \zeta^i_{\breve{\mathscr{S}}}(\breve{w}), \delta^i_{\breve{\mathscr{R}}}(\breve{w}) = \delta^i_{\breve{\mathscr{S}}}(\breve{w})$ *for all* $\breve{w} \in \breve{\mathscr{W}}$.

III. $\breve{\mathscr{R}} \cup \breve{\mathscr{S}}$ iff $\left\{\left\langle \breve{w}, \left(\zeta^i_{\breve{\mathscr{R}}}(\breve{w}) \vee \zeta^i_{\breve{\mathscr{S}}}(\breve{w})\right), \left(\delta^i_{\breve{\mathscr{R}}}(\breve{w}) \wedge \delta^i_{\breve{\mathscr{S}}}(\breve{w})\right)\right\rangle : \breve{w} \in \breve{\mathscr{W}}\right\}$.

IV. $\breve{\mathscr{R}} \cap \breve{\mathscr{S}}$ iff $\left\{\left\langle \breve{w}, \left(\zeta^i_{\breve{\mathscr{R}}}(\breve{w}) \wedge \zeta^i_{\breve{\mathscr{S}}}(\breve{w})\right), \left(\delta^i_{\breve{\mathscr{R}}}(\breve{w}) \vee \delta^i_{\breve{\mathscr{S}}}(\breve{w})\right)\right\rangle : \breve{w} \in \breve{\mathscr{W}}\right\}$.

V. $\breve{\mathscr{R}}^c = \left\{\left\langle \breve{w}, \left(\delta^i_{\breve{\mathscr{R}}}(\breve{w}), \zeta^i_{\breve{\mathscr{R}}}(\breve{w})\right)\right\rangle : \breve{w} \in \breve{\mathscr{W}}\right\}$.

Where $\breve{\mathscr{R}}^c$ denotes the complement of $\breve{\mathscr{R}}$. The symbols $\vee$ and $\wedge$ represent the max and min operators, respectively, in the operations of union and intersection.

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3.1 | Operational Laws of Qromfs

Definition 11. Let $\breve{\mathscr{R}} = \left(\zeta^{i}_{\breve{\mathscr{R}}}, \delta^{i}_{\breve{\mathscr{R}}}\right), \breve{\mathscr{R}}_{\infty} = \left(\zeta^{i}_{\breve{\mathscr{R}}_1}, \delta^{i}_{\mathscr{R}_1}\right), \breve{\mathscr{R}}_2 = \left(\zeta^{i}_{\mathscr{R}_2}, \delta^{i}_{\mathscr{R}_2}\right)$ are any three q-ROMFNs of dimension n over $\mathscr{W}$ and $\Omega > 0$, then

I. $\breve{\mathscr{R}}_1 \cup \breve{\mathscr{R}}_2 = \left\langle \max\left\{\zeta^{i}_{\breve{\mathscr{R}}_1}, \zeta^{i}_{\mathscr{R}_2}\right\}, \min\left\{\delta^{i}_{\breve{\mathscr{R}}_1}, \delta^{i}_{\mathscr{R}_2}\right\}\right\rangle$.

II. $\breve{\mathscr{R}}_1 \cap \breve{\mathscr{R}}_2 = \left\langle \min\left\{\zeta^{i}_{\breve{\mathscr{R}}_1}, \zeta^{i}_{\breve{\mathscr{R}}_2}\right\}, \max\left\{\delta^{i}_{\breve{\mathscr{R}}_1}, \delta^{i}_{\mathscr{R}_2}\right\}\right\rangle$.

III. $\breve{\mathcal{R}}_1 \oplus \breve{\mathcal{R}}_2 = \left\langle \sqrt[q]{\left(\zeta^{i}_{\breve{\mathcal{R}}_1}\right)^q + \left(\zeta^{i}_{\breve{\mathcal{R}}_2}\right)^q - \left(\zeta^{i}_{\breve{\mathcal{R}}_1}\right)^q\left(\zeta^{i}_{\breve{\mathcal{R}}_2}\right)^q},\ \delta^{i}_{\breve{\mathcal{R}}_1}\delta^{i}_{\breve{\mathcal{R}}_2}\right\rangle$.

IV. $\breve{\mathcal{R}}_1 \otimes \breve{\mathcal{R}}_2 = \left\langle \zeta^{i}_{\breve{\mathcal{R}}_1}\zeta^{i}_{\breve{\mathcal{R}}_2}, \sqrt[q]{\left(\delta^{i}_{\breve{\mathcal{R}}_1}\right)^q + \left(\delta^{i}_{\breve{\mathcal{R}}_2}\right)^q - \left(\delta^{i}_{\breve{\mathcal{R}}_1}\right)^q\left(\delta^{i}_{\breve{\mathcal{R}}_2}\right)^q}\right\rangle$.

V. $\Omega\breve{\mathcal{R}} = \left\langle \sqrt[q]{1 - \left(1 - \left(\zeta^{i}_{\breve{\mathcal{R}}}\right)^q\right)^{\Omega}}, \left(\delta^{i}_{\breve{\mathcal{R}}}\right)^{\Omega}\right\rangle$.

VI. $\breve{\mathcal{R}}^{\Omega} = \left\langle \left(\zeta^{i}_{\breve{\mathcal{R}}}\right)^{\Omega}, \sqrt[q]{1 - \left(1 - \left(\delta^{i}_{\breve{\mathcal{R}}}\right)^q\right)^{\Omega}}\right\rangle$.

VII. $\breve{\mathcal{R}}^c = \left(\delta^{i}_{\breve{\mathcal{R}}}, \zeta^{i}_{\breve{\mathcal{R}}}\right)$.

Example 2. Assume that $\breve{\mathscr{R}}_1 = [(0.5,0.9),(0.6,0.8)]$, $\breve{\mathscr{R}}_2 = [(0.4,0.8),(0.3,0.9)]$ are both q $ROMFSs(q \geq 3)$. Then

I. $\breve{\mathscr{R}}_1 \cup \breve{\mathscr{R}}_2 = \langle\ max\{0.5,0.4\}, min\{0.9,0.8\}),\ \ max\{0.6,0.3\}, min\{0.8,0.9\})\rangle = \langle\ 0.5,0.8),\ \ 0.6,0.8)\rangle$.

II. $\breve{\mathscr{R}}_1 \cap \breve{\mathscr{R}}_2 = \langle(min\{0.5,0.4\}, max\{0.9,0.8\}), (min\{0.6,0.3\}, max\{0.8,0.9\})\rangle = \langle(0.4,0.9),(0.3,0.9)\rangle$.

III. $\breve{\mathcal{R}}_1 \oplus \breve{\mathcal{R}}_2 = \left\langle\left(\sqrt[3]{0.5^3 + 0.4^3 - (0.5)^3(0.4)^3}, (0.9)(0.8)\right), \left(\sqrt[3]{0.6^3 + 0.3^3 - (0.6)^3(0.3)^3}, (0.8)(0.9)\right)\right\rangle =$ $(0.57,0.72),(0.62,0.72)\rangle$

IV. $\breve{\mathcal{R}}_1 \otimes \breve{\mathcal{R}}_2 = \left\langle\left((0.5)(0.4), \sqrt[3]{0.9^3 + 0.7^3 - (0.9)^3(0.7)^3}\right), \left((0.6)(0.3), \sqrt[3]{0.8^3 + 0.9^3 - (0.8)^3(0.9)^3}\right)\right\rangle =$ $\langle(0.20,0.94),(0.18,0.87)\rangle$

V. $\Omega\breve{\mathcal{R}} = \left\langle \sqrt[q]{1 - \left(1 - \left(\zeta^{i}_{\breve{\mathcal{R}}}\right)^q\right)^{\Omega}}, \left(\delta^{i}_{\breve{\mathcal{R}}}\right)^{\Omega}\right\rangle$

$$= \left\langle\left(\sqrt[3]{1 - (1 - (0.5)^3)^3}, (0.9)^3\right), \left(\sqrt[3]{1 - (1 - (0.6)^3)^3}, (0.8)^3\right)\right\rangle$$

$$\approx \langle(0.911,0.729),(0.8035,0.5120)\rangle$$

VI. $\breve{\mathcal{R}}^{\Omega} = \left\langle \left(\zeta^{i}_{\breve{\mathcal{R}}}\right)^{\Omega}, \sqrt[q]{1 - \left(1 - \left(\delta^{i}_{\breve{\mathcal{R}}}\right)^q\right)^{\Omega}}\right\rangle$, for $\Omega = 3$

$$= \left\langle\left((0.5)^3, \sqrt[q]{1 - (1 - (0.9))^3)^3}, (0.6)^3, \sqrt[q]{1 - (1 - (0.8)^3)^3}\right\rangle$$

$\approx \langle(0.1250,0.9933),(0.2160,0.9596)\rangle$ for $\Omega = 3$.

VII. $\breve{\mathscr{R}}^c = \left(\delta^{i}_{\breve{\mathscr{R}}}, \zeta^{i}_{\breve{\mathscr{R}}}\right) = \langle(0.9,0.3),(0.8,0.6)\rangle$.

3.2 | qROMF-Subgroup

Definition 12. A qROMF-set $\breve{\mathscr{R}} = \left\{\left\langle \breve{w}, \left(\zeta^{i}_{\breve{\mathscr{R}}}(\breve{w}), \delta^{i}_{\breve{\mathscr{R}}}(\breve{w})\right)\right\rangle : \breve{w} \in \breve{\mathscr{W}}\right\}$ of a group $\breve{\mathscr{C}}$ is said to be a qROMF-subgroup of $\breve{\mathscr{C}}$ if

I. $\zeta^{q}_{\breve{R}}(\breve{w}\breve{x}) \geq min\left(\zeta^{q}_{\breve{R}}\left((\breve{w}), \zeta^{q}_{\breve{R}}(\breve{x})\right)\right.$ and $\delta^{q}_{\breve{R}}(\breve{w}\breve{x}) \leq max\left(\delta^{q}_{\breve{R}}\left((\breve{w}), \delta^{q}_{\breve{R}}(\breve{x})\right)\right.$ $\quad for\ all\ \breve{w}, \breve{x} \in \breve{\mathscr{C}}$.
II. $\zeta^{q}_{\breve{R}}\left(\breve{w}^{-1}\right) = \zeta^{q}_{\breve{R}}(\breve{w})$ and $\delta^{q}_{\breve{R}}\left(\breve{w}^{-1}\right) = \delta^{q}_{\breve{R}}\ \breve{w})$.

Here $\zeta^{q}_{\breve{R}}(\breve{w}) = \left(\zeta^{q}_{\breve{R}}(\breve{w})\right)$ and $\delta^{q}_{\breve{R}}(\breve{w}) = \left(\delta^{q}_{\breve{R}}(\breve{w})\right), q \geq 1$ and $\quad for\ all\ \breve{w} \in \breve{\mathscr{C}}$.

Definition 13. Let $\breve{\mathscr{R}}$ be a qROMF-subgroup of a group $\breve{\mathscr{C}}$. Then for $\eta \in \breve{\mathscr{C}}$, the qROMF-left coset of $\breve{\mathscr{R}}$ is the qROMF-subgroup $\eta\breve{\mathscr{R}} = \left(\eta\zeta^{q}_{\breve{R}}, \eta\delta^{q}_{\breve{R}}\right)$ defined by $\left(\eta\zeta^{q}_{\breve{R}}\right)(\breve{w}) = \zeta^{q}_{\breve{R}}\left(\eta^{-1}\breve{w}\right), \left(\eta\delta^{q}_{\breve{R}}\right)(\breve{w}) = \delta^{q}_{\breve{\mathscr{R}}}\left(\eta^{-1}\breve{w}\right)$ and the qROMF-right coset of $\breve{\mathscr{R}}$ is the qROMF-subgroup $\breve{\mathscr{R}}\eta = \ \zeta_{\breve{R}}\eta, \delta_{\breve{R}}\eta)$ defined by $\left(\zeta^{q}_{\breve{R}}\eta\right)\ \breve{w}) = \zeta^{q}_{\breve{\mathscr{R}}}\left(\breve{w}\eta^{-1}\right), \left(\delta^{q}_{\breve{R}}\eta\right)\ \breve{w}) = \delta^{q}_{\breve{\mathscr{R}}}\left(\breve{w}\eta^{-1}\right) \quad for\ all\ \breve{w} \in \breve{\mathscr{C}}$.

Definition 14. Let $\breve{\mathscr{R}}$ be a qROMF-subgroup of a group $\breve{\mathscr{C}}$. Then $\breve{\mathscr{R}}$ is a qROMF-normal subgroup of a group $\breve{\mathscr{C}}$ if every qROMF-left coset of $\breve{\mathscr{R}}$ is also a qROMF-right coset of $\breve{\mathscr{R}}$ in $\breve{\mathscr{C}}$. Equivalently, $\eta\breve{\mathscr{R}} = \breve{\mathscr{R}}\eta, for\ all\ \eta \in \breve{\mathscr{C}}$.

Proposition 1. Let $\breve{\mathscr{R}}$ be a qROMF-subgroup of a group $\breve{\mathscr{C}}$ and $\breve{e}_{\breve{\mathscr{C}}}$ is the identity element of $\breve{\mathscr{C}}$. Then

I. $\zeta^{q}_{\breve{R}}\left(\breve{e}_{\breve{\mathscr{C}}}\right) \geq \zeta^{q}_{\breve{R}}(\breve{w})$ and $\delta^{q}_{\breve{\mathscr{R}}}\left(\breve{e}_{\breve{\mathscr{C}}}\right) \leq \ \delta^{q}_{\breve{\mathscr{R}}}(\breve{w}$.
II. $\zeta^{q}_{\breve{R}}\left(\breve{w}^{-1}\right) = \zeta^{q}_{\breve{R}}(\breve{w})$ and $\delta^{q}_{\breve{\mathscr{R}}}\left(\breve{w}^{-1}\right) = \delta^{q}_{\breve{\mathscr{R}}}\ \breve{w}) \quad for\ all\ \breve{w} \in \breve{\mathscr{C}}$ respectively.

Proof: Let $\breve{\mathscr{R}} = \ \zeta_{\breve{R}}, \delta_{\breve{R}})$ be a qROMF-subgroup of $\breve{\mathscr{C}}$.

Then by the Definition 10,

I. $\zeta^{q}_{\breve{\mathscr{R}}}\left(\breve{e}_{\breve{\mathscr{C}}}\right) = \zeta^{q}_{\breve{\mathscr{R}}}\left(\breve{w}\breve{w}^{-1}\right) \geq min\left(\zeta^{q}_{\breve{\mathscr{R}}}(\breve{w}), \zeta^{q}_{\breve{\mathscr{R}}}\left(\breve{w}^{-1}\right) = \zeta^{q}_{\breve{\mathscr{R}}}(\breve{w})\right.$.

Also, $\delta^{q}_{\breve{R}}\ \breve{e}_{\breve{C}}) = \delta^{q}_{\breve{R}}\left(\breve{w}^{-1}\right) \leq max\left(\delta^{q^2}_{\breve{R}}(\breve{w}), \delta^{q}_{\breve{R}}\left(\breve{w}^{-1}\right) = \delta^{q}_{\breve{R}}(\breve{w})\right.$.

II. We have $\zeta^{q}_{\breve{\mathscr{R}}}\left(\breve{w}^{-1}\right) \geq \zeta^{q}_{\breve{\mathscr{R}}}(\breve{w})$ and $\delta^{q}_{\breve{\mathscr{R}}}\left(\breve{w}^{-1}\right) \leq \delta^{q}_{\breve{\mathscr{R}}}\ \breve{w}) \quad for\ all\ \breve{w} \in \breve{\mathscr{C}}$.

Substitute $\breve{w}^{-1}$ instead of $\breve{w}$, we get $\zeta^{q}_{\breve{R}}\left(\left((\breve{w})^{-1}\right)^{-1}\right) \geq \zeta^{q}_{\breve{R}}\left(\breve{w}^{-1}\right) \Longrightarrow \zeta^{q}_{\breve{R}}(\breve{w}) \geq \zeta^{q}_{\breve{\mathscr{R}}}\left(\breve{w}^{-1}\right)$.

Again, $\delta^{q}_{\breve{\mathscr{R}}}\left(\left((\breve{w})^{-1}\right)^{-1}\right) \leq \delta^{q}_{\breve{\mathscr{R}}}\left(\breve{w}^{-1}\right) \Longrightarrow \delta^{q}_{\breve{R}}\left(\breve{w}'\right) \geq \delta^{q}_{\breve{\mathscr{R}}}\left(\breve{w}^{-1}\right) \quad for\ all\ \breve{w} \in \breve{\mathscr{C}}$.

Therefore, by combining all the results, we obtain $\zeta^{q}_{\breve{R}}\left(\breve{w}^{-1}\right) = \zeta^{q}_{\breve{R}}(\breve{w})$ and $\delta^{q}_{\breve{R}}\left(\breve{w}^{-1}\right) = \delta^{q}_{\breve{\mathscr{R}}}(\breve{w}) \quad for\ all\ \breve{w} \in \breve{\mathscr{C}}$.

Proposition 2. $\breve{\mathscr{R}}$ is a qROMF-subgroup of a group $\breve{\mathscr{C}}$ if and only if $\breve{\mathscr{R}}^{c}$ is a qROMF-subgroup of a group $\breve{\mathscr{C}}$.

Proof: Suppose $\breve{\mathscr{R}}$ is a qROMF-subgroup of a group $\breve{\mathscr{C}}$.

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Then

I. $\zeta^{q}_{\breve{\mathscr{R}}}(\breve{w}\breve{x}) \geq \min\left(\zeta^{q}_{\breve{\mathscr{R}}}\left((\breve{w}), \zeta^{q}_{\breve{\mathscr{R}}}(\breve{x})\right)\right.$ $\quad for\ all\ \breve{w}, \breve{x} \in \breve{\mathscr{C}}$

$\Longrightarrow 1 - \zeta^{q}_{\breve{R}}(\breve{w}\breve{x}) \geq 1 - min\left(\zeta^{q}_{\breve{R}}\left((\breve{w}), \zeta^{q}_{\breve{R}}(\breve{x})\right)\right.$

$\Longrightarrow \delta^{q}_{\breve{R}}(\breve{w}\breve{x}) \geq max\left(\delta^{q}_{\breve{R}}\left((\breve{w}), \delta^{q}_{\breve{R}}(\breve{x})\right)\right.$.

II. $\zeta^{q}_{\breve{\mathscr{R}}}\left(\breve{w}^{-1}\right) = \zeta^{q}_{\breve{\mathscr{R}}}\ \breve{w})$ $\quad for\ all\ \breve{w} \in \breve{\mathscr{C}}$

$\Longrightarrow 1 - \zeta^{q}_{\breve{\mathscr{R}}}\left(\breve{w}^{-1}\right) = 1 - \zeta^{q}_{\breve{\mathscr{R}}}\ \breve{w})$

$\Longrightarrow \delta^{q}_{\breve{\mathscr{R}}}\left(\breve{w}^{-1}\right) = \delta^{q}_{\breve{\mathscr{R}}}\ \breve{w})$

and $\delta^{q}_{\breve{\mathscr{R}}}\left(\breve{w}^{-1}\right) = \delta^{q}_{\breve{\mathscr{R}}}\ \breve{w})$ $\quad for\ all\ \breve{x} \in \breve{\mathscr{C}}$

$\Longrightarrow 1 - \delta^{q}_{\breve{\mathscr{R}}}\left(\breve{w}^{-1}\right) = 1 - \delta^{q}_{\breve{\mathscr{R}}}(\breve{w})$

$\Longrightarrow \zeta^{q}_{\breve{\mathscr{R}}}\left(\breve{w}^{-1}\right) = \zeta^{q}_{\breve{\mathscr{R}}}\ \breve{w})$.

Hence $\breve{\mathscr{R}}^{c}$ is a qROMF-subgroup of a group $\breve{\mathscr{C}}$.

Proposition 3. Let $\breve{\mathscr{R}}$ be any qROMF-subgroup of a group $\breve{\mathscr{C}}$ and $\breve{e}_{\breve{C}}$ is the identity element of $\breve{\mathscr{C}}$. Then

I. $\zeta^{q}_{\breve{\mathscr{R}}}\left(\breve{w}\breve{x}^{-1}\right) = \zeta^{q}_{\breve{\mathscr{R}}}\ \breve{e}_{\breve{C}}) \Longrightarrow \zeta^{q}_{\breve{\mathscr{R}}}\ \breve{w}) = \zeta^{q}_{\breve{\mathscr{R}}}\ \breve{x})$.

II.$\left(\breve{w}\breve{x}^{-1}\right) = \delta^{q}_{\breve{\mathscr{R}}}\ \breve{e}_{\breve{C}}) \Longrightarrow \delta^{q}_{\breve{R}}\ \breve{w}) = \delta^{q}_{\breve{R}}\ \breve{x})$ $\quad for\ all\ \breve{w}, \breve{x} \in \breve{\mathscr{C}}$ respectively.

Proof: Let $\breve{\mathscr{R}}$ be any qROMF-subgroup of a group $\breve{\mathscr{C}}$ (given).

I. $\zeta^{q}_{\breve{\mathscr{R}}}\ \breve{w}) \geq \zeta^{q}_{\breve{\mathscr{R}}}\ \breve{e}_{\breve{C}})$ $\quad for\ all\ \breve{w}, \breve{x} \in \breve{\mathscr{C}}$

$$\begin{aligned}
\zeta^{q}_{\breve{\mathscr{R}}}(\breve{w}) = \zeta^{q}_{\breve{\mathscr{R}}}\left(\breve{w}\breve{e}_{\breve{\mathscr{C}}}\right) &= \zeta^{q}_{\breve{\mathscr{R}}}\left(\breve{w}\left(\breve{x}^{-1}\breve{x}\right)\right) \\
&= \zeta^{q}_{\breve{\mathscr{R}}}\left(\breve{w}\breve{x}^{-1}\breve{x}\right) \\
&= \zeta^{q}_{\breve{\mathscr{R}}}\left(\left(\breve{w}\breve{x}^{-1}\right)\breve{x}\right) \\
&= \min\left(\zeta^{q}_{\breve{\mathscr{R}}}\left(\breve{w}\breve{x}^{-1}\right), \zeta^{q}_{\breve{\mathscr{R}}}(\breve{x})\right) \\
&= \min\left(\zeta^{q}_{\breve{\mathscr{R}}}\left(\breve{e}_{\breve{\mathscr{C}}}\right), \zeta^{q}_{\breve{\mathscr{R}}}(\breve{x})\right) \\
&= \zeta^{q}_{\breve{\mathscr{R}}}(\breve{x})
\end{aligned}$$

1. Therefore $\zeta^{q}_{\breve{\mathscr{R}}}\ \breve{x}) \geq \zeta^{q}_{\breve{\mathscr{R}}}\ \breve{x})$.

$$\begin{aligned}\zeta^{q}_{\breve{R}}(\breve{x}) &= \zeta^{q}_{\breve{\mathscr{R}}}\left(\breve{x}\breve{e}_{\breve{C}}\right) = \zeta^{q}_{\breve{\mathscr{R}}}\left(\breve{x}\left(\breve{w}^{-1}\breve{w}\right)\right)\\ &= \zeta^{q}_{\breve{R}}\left(\breve{x}\breve{w}^{-1}\breve{w}\right)\\ &= \zeta^{q}_{\breve{R}}\left(\left(\breve{x}\breve{x}^{-1}\right)\breve{w}\right)\\ &= \min\left(\zeta^{q}_{\breve{\mathscr{R}}}\left(\breve{x}\breve{w}^{-1}\right),\zeta^{q}_{\breve{\mathscr{R}}}(\breve{w})\right)\\ &= \min\left(\zeta^{q}_{\breve{\mathscr{R}}}\left(\breve{e}_{\breve{\mathscr{C}}}\right),\zeta^{q}_{\breve{\mathscr{R}}}(\breve{w})\right)\\ &= \zeta^{q}_{\breve{R}}\ \breve{w}).\end{aligned}$$

2. Therefore $\zeta^{q}_{\breve{R}}\ \breve{x}) \geq \zeta^{q}_{\breve{R}}\ \breve{w})$.

From (1) and (2), we obtain $\zeta^{q}_{\breve{\mathscr{R}}}(\breve{w}) = \zeta^{q}_{\breve{R}}(\breve{x})$.

II. $\zeta^{q}_{\breve{\mathscr{R}}}\ \breve{w}) \geq \zeta^{q}_{\breve{\mathscr{R}}}\ \breve{e}_{\breve{C}})\quad for\ all\ \breve{w},\breve{x} \in \breve{\mathscr{C}}$.

$$\begin{aligned}\zeta^{q}_{\breve{\mathscr{R}}}(\breve{w}) = \zeta^{q}_{\breve{\mathscr{R}}}\left(\breve{w}\breve{e}_{\breve{C}}\right) &= \zeta^{q}_{\breve{\mathscr{R}}}\left(\breve{w}\left(\breve{x}^{-1}\breve{x}\right)\right)\\ &= \zeta^{q}_{\breve{\mathscr{R}}}\left(\breve{w}\breve{x}^{-1}\breve{x}\right)\\ &= \zeta^{q}_{\breve{\mathscr{R}}}\left(\left(\breve{w}\breve{x}^{-1}\right)\breve{x}\right)\\ &= \min\left(\zeta^{q}_{\breve{\mathscr{R}}}\left(\breve{w}\breve{x}^{-1}\right),\zeta^{q}_{\breve{\mathscr{R}}}(\breve{x})\right)\\ &= \min\left(\zeta^{q}_{\breve{\mathscr{R}}}\left(\breve{e}_{\breve{\mathscr{C}}}\right),\zeta^{q}_{\breve{\mathscr{R}}}(\breve{x})\right)\\ &= \zeta^{q}_{\breve{\mathscr{R}}}\ \breve{x}).\end{aligned}$$

3. Therefore $\zeta^{q}_{\breve{R}}\ \breve{w}) \geq \zeta^{q}_{\breve{R}}\ \breve{x})$.

$$\begin{aligned}\delta^{q}_{\breve{R}}\ \breve{x}) &= \delta^{q}_{\breve{\mathscr{R}}}\left(\breve{x}\breve{e}_{\breve{\mathscr{C}}}\right) = \delta^{q}_{\breve{\mathscr{R}}}\left(\breve{x}\left(\breve{w}^{-1}\breve{w}\right)\right)\\ &= \delta^{q}_{\breve{R}}\left(\breve{x}\breve{w}^{-1}\breve{w}\right)\\ &= \delta^{q}_{\breve{\mathscr{R}}}\left(\left(\breve{x}\breve{w}^{-1}\right)\breve{w}\right)\\ &= \max\left(\delta^{q}_{\breve{\mathscr{R}}}\left(\breve{x}\breve{w}^{-1}\right),\delta^{q}_{\breve{\mathscr{R}}}(\breve{w})\right)\\ &= \max\left(\delta^{q}_{\breve{R}}\left(\breve{e}_{\breve{C}}\right),\delta^{q}_{\breve{R}}(\breve{w})\right)\\ &= \delta^{q}_{\breve{R}}\ \breve{w}).\end{aligned}$$

4. Therefore $\delta^{q}_{\breve{R}}\ \breve{x}) \leq \delta^{q}_{\breve{R}}\ \breve{w})$.

From (3) and (4), we obtain $\delta^{q}_{\breve{\mathscr{R}}}(\breve{w}) = \delta^{q}_{\breve{R}}(\breve{x})$.

Proposition 4. Let $\breve{\mathscr{R}}$ be any qROMF-subgroup of a group $\breve{\mathscr{C}}$ and $\breve{e}_{\breve{\mathscr{C}}}$ is the identity element of $\breve{\mathscr{C}}$. Then $\breve{\mathscr{R}} = \left\{\breve{w} \in \breve{\mathscr{C}} / \zeta^{q}_{\breve{\mathscr{R}}}(\breve{w}) = \zeta^{q}_{\breve{\mathscr{R}}}\ \breve{e}_{\breve{C}})\ \text{and}\ \delta^{q}_{\breve{\mathscr{R}}}(\breve{w}) = \delta^{q}_{\breve{\mathscr{R}}}\ \breve{e}_{\breve{C}})\right\}$ forms a subgroup of the $\breve{\mathscr{C}}$.

Proof: Here $\breve{\mathscr{R}} = \left\{\breve{w} \in \breve{\mathscr{C}} / \zeta^{q}_{\breve{\mathscr{R}}}(\breve{w}) = \zeta^{q}_{\breve{\mathscr{R}}}\ \breve{e}_{\breve{C}})\ \text{and}\ \delta^{q}_{\breve{\mathscr{R}}}(\breve{w}) = \delta^{q}_{\breve{\mathscr{R}}}\ \breve{e}_{\breve{C}})\right\}$, clearly $\breve{\mathscr{R}}$ is non-empty as $\breve{e}_{\breve{C}} \in \breve{\mathscr{R}}$.

To show that $\breve{\mathscr{R}}$ is a subgroup of $\breve{\mathscr{C}}$, we have to show that $\breve{w}\breve{x}^{-1} \in \breve{\mathscr{R}}$.

Let $\breve{w},\breve{x} \in \breve{\mathscr{R}}$.

Then $\zeta^{q}_{\breve{\mathscr{R}}}(\breve{w}) = \zeta^{q}_{\breve{R}}\ \breve{e}_{\breve{C}}) = \zeta^{q}_{\breve{\mathscr{R}}}(\breve{x}), \delta^{q}_{\breve{R}}(\breve{w}) = \delta^{q}_{\breve{\mathscr{R}}}\left(\breve{e}_{\breve{\mathscr{C}}}\right) = \delta^{q}_{\breve{\mathscr{R}}}(\breve{x})$.

Since $\breve{\mathscr{R}} = \left(\zeta^{q}_{\breve{R}},\delta^{q}_{\breve{R}}\right)$ is a qROMF-subgroup of a group $\breve{\mathscr{C}}$, Then

$$\begin{aligned}\zeta^q_{\breve{\mathscr{R}}}\left(\breve{w}\breve{x}^{-1}\right) \geq \quad & min\left(\zeta^q_{\breve{\mathscr{R}}}\left((\breve{w}), \zeta^q_{\breve{\mathscr{R}}}\left(\breve{x}^{-1}\right)\right)\right. \\ & = min\left(\zeta^q_{\breve{\mathscr{R}}}\left((\breve{w}), \zeta^q_{\breve{\mathscr{R}}}(\breve{x})\right)\right. \\ & = min\left(\zeta^q_{\breve{\mathscr{R}}}\left(\breve{e}_{\breve{\mathscr{C}}}\right), \zeta^q_{\breve{\mathscr{R}}}\left(\breve{e}_{\breve{\mathscr{C}}}\right)\right) \\ & \zeta^q_{\breve{\mathscr{R}}}\left(\breve{e}_{\breve{\mathscr{C}}}\right).\end{aligned}$$

Therefore, $\zeta^q_{\breve{R}}\left(\breve{w}\breve{x}^{-1}\right) \geq \zeta^q_{\breve{R}}\left(\breve{w}\breve{x}^{-1}\right)$.

Similarly, we can show that $\delta^q_{\breve{\mathscr{R}}}\left(\breve{w}\breve{x}^{-1}\right) \leq \delta^q_{\breve{\mathscr{R}}}\left(\breve{w}\breve{x}^{-1}\right)$.

Again by *Proposition 1*, we have $\zeta^q_{\breve{\mathscr{R}}}\left(\breve{e}_{\breve{\mathscr{C}}}\right) \geq \zeta^q_{\breve{\mathscr{R}}}\left(\breve{w}\breve{x}^{-1}\right)$ and $\delta^q_{\breve{\mathscr{R}}}\left(\breve{e}_{\breve{\mathscr{C}}}\right) \leq \delta^q_{\breve{\mathscr{R}}}\left(\breve{w}\breve{x}^{-1}\right)$. Therefore $\zeta^q_{\breve{\mathscr{R}}}\left(\breve{x}\breve{x}^{-1}\right) \geq \zeta^q_{\breve{\mathscr{R}}}\left(\breve{e}_{\breve{\mathscr{C}}}\right)$ and $\delta^q_{\breve{\mathscr{R}}}\left(\breve{w}\breve{x}^{-1}\right) \leq \delta^q_{\breve{\mathscr{R}}}\left(\breve{e}_{\breve{\mathscr{C}}}\right)$.

Hence $\breve{\mathscr{R}}$ is a subgroup of $\breve{\mathscr{C}}$.

4 | Comparative Analysis

We present a comprehensive comparison of the q-rung orthopair MFS with other prominent fuzzy set techniques such as those referenced in [1, 2, 9, 10], as well as MFS techniques mentioned in [29], [7]. When considering decision-making issues, it becomes clear that the multifuzzy set methodology is unable to take the degree of non-belongingness into account. Despite being a generalisation of intuitionistic fuzzy sets and PF-Sets, q-ROFS still has difficulty dealing with decision-making issues including several agents, characteristics, objects, indices, and different types of uncertainty. As a result, our proposed model offers a refined representation of multi-fuzzy information by amalgamating the strengths of both q-ROFS and MFS. This integration provides a more versatile and robust approach to addressing complex decision-making scenarios, bridging the gap between the limitations of existing techniques, as highlighted in *Table 2*.

4.1 | Supriority of the Proposed Model

A q-rung orthopair MFS is a generalization of previous studies such as fuzzy set, MFS [27], q-rung orthopair fuzzy set, and intuitionistic MFS [22] by taking into account a lot more information about an object while the process is being carried out and to deal with the multi-dimensional information in a set. In several disciplines (including decision-making), this generalised form gives novel answers for issues based on fuzzy sets and their expansions.

By incorporating the idea of raising multi-belongingness and multi-non-belongingness to the power of q, we have established a more flexible and effective framework for modelling fuzzy systems and facilitating decision-making under uncertain conditions. This framework is known as the qROMFS (with the multi-belongingness and multi-non-belongingness are real valued functions).

4.2 | Conclusion

In this research, the algebraic structures of qROMF-set and its operational laws are introduced. Furthermore, the exploration extends to encompass the concept of a qROMF-subgroup, delving into its inherent properties to unveil a deeper understanding. Moreover, the study extends its reach to define two additional key constructs, namely the qROMF-coset and the qROMF-normal subgroup, enriching the conceptual landscape with their explicit delineations. This strategic direction holds the promise of advancing the theoretical framework, potentially yielding novel insights and practical applications, thereby contributing to the continuous evolution of this fascinating field of study.

Table 2. Comparison of q-ROMFSs with prominent models.

Approach	Truthiness	Falsity	Loss of Information	Ability to Explain Multi-Dimensional Information with Parameterization on q	Limitations
FS [1]	✓	×	×	×	Incapable of addressing the degree of non-belongingness.
IFS [2]	✓	✓	✓	×	Inability to address issues where $\zeta + \delta > 1$.
PFS [3]	✓	✓	✓	×	Inability to address issues where $\zeta^2 + \delta^2 > 1$.
q-ROFS [4]	✓	✓	✓	×	Inability to address issues where $\zeta^q + \delta^q > 1$.
MFS [9, 10]	✓	×	×	×	Incapable of addressing The degree of multi-non-belongingness.
IMFS [14]	✓	✓	✓	×	Inability to address issues where $\zeta_i + \delta_i > 1$, where i be the dimension.
Proposed approach	✓	✓	✓	✓	Inability to address issues where $(\zeta_i)^q + (\delta_i)^q > 1$.

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