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(α, β)-CUT of Intuitionistic Fuzzy Normed Ideals

Nour Abed Alhaleem[1,*], Abd Ghafur Ahmad[1]

[1] Universiti Kebangsaan Malaysia,UKM Bangi, Selangor, Malaysia; noorb@ymail.com; ghafur@ukm.edu.my.

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Abstract

The main aim of this paper is to introduce the notion of (α, β)-level sets with respect to intuitionistic fuzzy normed ideals. We extend the notion of intuitionistic fuzzy normed ideals to the (α, β)-level subsets where α and β are elements in the interval [0,1]. Several related properties for (α, β)-cut of intuitionistic fuzzy normed ideals in a normed ring (NR) will be studied and proven. Further, for any two normed rings NR, NR' with a mapping $f: NR \rightarrow NR'$, a relation between the intuitionistic fuzzy normed ideal I of NR and the intuitionistic fuzzy normed ideal $f(I)$ (the image of I) of NR' will be obtained with the support of their (α, β)-level subsets.

Keywords: Intuitionistic fuzzy normed ideals, Level sets, (α, β)-cut, Homomorphisms.

1 | Introduction

The conception of a fuzzy set was created as a generalization of a crisp set by Zadeh [1], which led to the foundation of fuzzy set theory. These sets characterize a membership function that assigns each object a grade of membership between 0 and 1. Followed by the studies of fuzzy subgroup by Rosenfeld [2] which utilized the notion of fuzzy sets to groups theory, he specified subgroups and subgroupoids and characterized various related properties. Later, Liu [3] presented the notion of the fuzzy ring and studied fuzzy subrings and fuzzy ideals and proved some fundamental properties. In [4], Atanassov initiated the idea of intuitionistic fuzzy sets as a characterization of fuzzy sets, various properties that were connected to the operations and relations defined over intuitionistic fuzzy sets were proved. Later, various researchers utilized this notion in various sections of mathematics, especially in algebra. Biswas [5] initiated the notion of intuitionistic fuzzy groups and investigated their properties. Followed by a work by Hur et al. [6] who studied intuitionistic fuzzy subgroups and intuitionistic fuzzy subrings and examined some of their characteristics. Also, intuitionistic fuzzy ideals were defined along with the properties of their level subsets. The same year, Banerjee and Basnet [7] studied intuitionistic fuzzy subrings and ideals based on intuitionistic fuzzy sets. The conception of intuitionistic fuzzy normed rings was defined. In the paper [8], they initiated the concepts of intuitionistic fuzzy normed subrings and intuitionistic fuzzy normed ideals. In [9], Basent introduced (α, β)-cut of intuitionistic fuzzy ideals and established some important links among

Corresponding Author: noorb@ymail.com

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intuitionistic fuzzy sets and crisp sets. Later, Sharma [10] introduced the (α,β)-cut of intuitionistic fuzzy groups and studied intuitionistic fuzzy subgroup with the help of their (α,β)-level subsets. In [11], various different types of interval (α,β) cut-set of generalized interval-valued intuitionistic fuzzy sets were investigated.

In this study, (α,β)-cut of intuitionistic fuzzy normed ideals is investigated and various related properties for (α,β)-cut of intuitionistic fuzzy normed ideals are examined and proved. In addition, the relation between the intuitionistic fuzzy normed ideals under epimorphism mapping with the support of their (α,β)-level subsets will be defined.

2 | Preliminaries

In this section, the most significant definitions and results needed for the following section are presented.

Definition 1 ([1]). The fuzzy set (FS) E on a universal X is a set of ordered pairs:

$$E = \{(x, \mu_E(x)): x \in X\},$$

where, $\mu_E(x)$ is the membership function of x in E satisfies $0 \leq \mu_E(x) \leq 1$ for all $x \in X$.

Definition 2 ([4]). An intuitionistic fuzzy set (IFS) A in X is defined as an object of the following form:

$$A = \{(x, \mu_A(x), \gamma_A(x)): x \in X\},$$

where the functions $\mu_A(x): X \to [0,1]$ and $\gamma_A(x): X \to [0,1]$ denotes the degree of membership and the degree of nonmembership, respectively, where $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in X$. An intuitionistic fuzzy set A is written symbolically in the form $A = (\mu_A, \gamma_A)$.

Definition 3 ([4]). Let A and B be two intuitionistic fuzzy sets. The following operations and relations are valid:

I. $A = B$ if $\mu_A(x) = \mu_B(x)$ and $\gamma_A(x) = \gamma_B(x)$ for all $x \in X$.
II. $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$.
III. $A^c = \{(x, \gamma_A(x), \mu_A(x)): x \in X\}$ and denoted by $A^c = (\gamma_A, \mu_A)$.
IV. $A \cap B = \{(x, min\{\mu_A(x), \mu_B(x)\}, max\{\gamma_A(x), \gamma_B(x)\}): x \in X\}$.
V. $A \cup B = \{(x, max\{\mu_A(x), \mu_B(x)\}, min\{\gamma_A(x), \gamma_B(x)\}): x \in X\}$.

Definition 4 ([12]). Let A be an intuitionistic fuzzy set of a universe set X. Then (α,β)-cut of A is a crisp subset $C_{\alpha,\beta}(A)$ of A and defined by

$$C_{\alpha,\beta}(A) = \{x \in X: \mu_A(x) \geq \alpha \text{ and } \gamma_A(x) \leq \beta\},$$

and called the (α,β)-level subset of A, where $\alpha + \beta \leq 1$ and $\alpha, \beta \in [0,1]$.

Proposition 1 ([10]). If A and B are two IFSs of a universe set X, the following holds for all $\alpha, \beta, \theta, \vartheta \in [0,1]$:

I. $C_{\alpha,\beta}(A) \subseteq C_{\theta,\vartheta}(A)$ if $\alpha \geq \theta$ and $\beta \leq \vartheta$.
II. $C_{1-\beta,\beta}(A) \subseteq C_{\alpha,\beta}(A) \subseteq C_{\alpha,1-\alpha}(A)$.
III. $A \subseteq B$ implies $C_{\alpha,\beta}(A) \subseteq C_{\alpha,\beta}(B)$.
IV. $C_{\alpha,\beta}(A \cap B) = C_{\alpha,\beta}(A) \cap C_{\alpha,\beta}(B)$.
V. $C_{\alpha,\beta}(A \cup B) \supseteq C_{\alpha,\beta}(A) \cup C_{\alpha,\beta}(B)$, equality holds if $\alpha + \beta = 1$.
VI. $C_{0,1}(A) = X$.

VII. $C_{\alpha,\beta}(\cup A_k) = \cup C_{\alpha,\beta}(A_k)$ for $k \in K$ and $k < \infty$.

Definition 5 ([13]). Let $*$: $[0; 1] \times [0; 1] \rightarrow [0; 1]$ be a binary operation. Then $*$ is a t-norm if $*$ conciliates the conditions of commutativity, associativity, monotonicity and neutral element 1. We shortly use t-norm and write $x * y$ instead of $*(x, y)$.

Definition 6 ([13]). Let $\diamond$: $[0,1] \times [0,1] \rightarrow [0,1]$ be a binary operation. Then $\diamond$ is a s-norm if $\diamond$ conciliates the conditions of commutativity, associativity, monotonicity and neutral element 0. We shortly use s-norm and write $x \diamond y$ instead of $\diamond(x, y)$.

A t-norm (s-norm) is called continuous if it is continuous as a function, in the usual interval topology on $[0,1]^2$.

Definition 7 ([7]). Let $A = \{(x, \mu_A\ x), \gamma_A\ x)) : x \in X\}$ and $B = \{(x, \mu_B\ x), \gamma_B\ x)) : x \in X\}$ be two intuitionistic fuzzy sets in a ring R.

Then $A + B$ is defined as:

$$A + B = \left\{\left(x, \mu_{A+B}\ x), \gamma_{A+B}\ x)\right) : x \in X\right\},$$

where

$$\mu_{A+B}\ x) = \begin{cases} \mathop{\diamond}\limits_{x=y+z} (\mu_A\ y) * \mu_B\ z)\ ), & \text{if } x = y + z, \\ 0\ , & \text{otherwise.} \end{cases}$$

For all $x, y, z \in X$.

Definition 8 ([7]). Let $A = \{(x, \mu_A\ x), \gamma_A\ x)) : x \in X\}$ and $B = \{(x, \mu_B(x), \gamma_B(x) : x \in X\}$ be two intuitionistic fuzzy sets in a ring R. Then $A \bigcirc B$ is defined:

$$A \bigcirc B = \left\{\left(x, \mu_{A\circledast B}\ x), \gamma_{A\otimes B}\ x)\right) : x \in X\right\},$$

where

$$\mu_{A\circledast B}\ x) = \begin{cases} \mathop{\diamond}\limits_{x=yz} \left(\mu_A(y) * \mu_B\ z)\right), & \text{if } x = yz, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\gamma_{A\otimes B}\ x) = \begin{cases} \mathop{*}\limits_{x=yz} \left(\gamma_A(y) \diamond \gamma_B\ z)\right), & \text{if } x = yz, \\ 1, & \text{otherwise.} \end{cases}$$

For all $x, y, z \in X$.

Definition 9 ([14]). A normed ring (NR) is a ring that possesses a norm $\|\ \|$, that is a non-negative real-valued function $\|\ \| : NR \rightarrow \mathbb{R}$, satisfies the following conditions for all $x, y \in NR$:

I. $\| x \| = 0 \Leftrightarrow x = 0$.
II. $\| x + y \| \leq \| x \| + \| y \|$.
III. $\| xy \| \leq \| x \| \| y \|$.
IV. $\| x \| = \| -x \|$ (Thus, $\| 1_A \| = 1 = \| -1_A \|$ if identity exists).

Definition 10 ([15]). Let $*$ be a continuous t-norm and $\diamond$ be a continuous s-norm. An intuitionistic fuzzy set $S = \{(x, \mu_S(x), \gamma_S(x)): x \in NR\}$ is called an intuitionistic fuzzy normed subring (IFNSR) of the normed ring $(NR, +, .)$ if the following are fulfilled for all $x, y \in NR$:

I. $\mu_S(x - y) \geq \mu_S(x) * \mu_S(y)$.
II. $\mu_S(xy) \geq \mu_S(x) * \mu_S(y)$.
III. $\gamma_S(x - y) \leq \gamma_S(x) \diamond \gamma_S(y)$.
IV. $\gamma_S(xy) \leq \gamma_S(x) \diamond \gamma_S(y)$.

Definition 11 ([16]). An intuitionistic fuzzy subring $I = \{(x, \mu_I(x), \gamma_I(x)): x \in NR\}$ is an intuitionistic fuzzy normed ideal of the normed ring NR if it fulfils the following conditions for all $x, y \in NR$:

I. $\mu_I(x - y) \geq \mu_I(x) * \mu_I(y)$.
II. $\mu_I(xy) \geq \mu_I(x) \diamond \mu_I(y)$.
III. $\gamma_I(x - y) \leq \gamma_I(x) \diamond \gamma_I(y)$.
IV. $\gamma_I(xy) \leq \gamma_I(x) * \gamma_I(y)$.

Throughout the paper, for the intuitionistic fuzzy normed ideal, it will be shortly used IFNI.

Example 1 ([14]). The field of real numbers $\mathbb{R}$ is a normed ring with respect to the absolute value and the field of complex numbers $\mathbb{Q}$ is a normed ring with respect to the modulus. More general examples are the ring of real square matrices with the matrix norm and the ring of real polynomials with a polynomial norm.

Theorem 2 ([16]). If I and J are two intuitionistic fuzzy ideals of NR. Then $I \cap J$ is an IFNI of NR.

3 | (α, β)-Cut of Intuitionistic Fuzzy Normed Ideals

All through this study, $\mathbb{Z}$ is the set of integers, NR is a normed ring and $C_{\alpha,\beta}(A)$ is the (α, β)-level subset.

Definition 12. Let $A = \{(x, \mu_A\ x), \gamma_A\ x)): x \in NR\}$ be an intuitionistic fuzzy set of a normed ring NR. The (α, β)-cut of A is the level subset $C_{\alpha,\beta}\ A)$ and defined as

$$C_{\alpha,\beta}\ A) = \{x \in NR:\ \mu_A(x) \geq \alpha \text{ and } \gamma_A(x) \leq \beta\},$$

where $\alpha + \beta \leq 1$ and $\alpha, \beta \in [0,1]$.

Lemma 1. If I is an IFNI. Then $C_{\alpha,\beta}(I)$ is an ideal of NR if $\mu_I(0) \geq \alpha$ and $\gamma_I(0) \leq \beta$ with $\alpha + \beta \leq 1$.

Proof: Clearly $C_{\alpha,\beta}(I) \neq \phi$ as $0 \in C_{\alpha,\beta}(I)$. Let $x, y \in C_{\alpha,\beta}(I)$. Then $\mu_I(x) \geq \alpha$, $\mu_I(y) \geq \alpha$ and $\gamma_I(x) \leq \beta$, $\gamma_I(y) \leq \beta$. Then, $\mu_I(x - y) \geq \mu_I(x) * \mu_I(y) \geq \alpha$ and $\gamma_I(x - y) \leq \gamma_I(x) \diamond \mu_I(y) \leq \beta$. Thus, $x - y \in C_{\alpha,\beta}(I)$. Then $C_{\alpha,\beta}$ is a subring.

Now, let $r \in NR$, $\mu_I\ rx) \geq \mu_I\ r) \diamond \mu_I\ x) \geq \mu_I\ x) \geq \alpha$, $\mu_I(xr) \geq \mu_I(x) \diamond \mu_I(r) \geq \mu_I(x) \geq \alpha$ and $\gamma_I(rx) \leq \gamma_I(r) * \gamma_I(x) \leq \gamma_I(x) \leq \beta$, $\gamma_I(xr) \leq \gamma_I(x) * \gamma_I(r) \leq \gamma_I(x) \leq \beta$. Then, xr and $rx \in C_{\alpha,\beta}(I)$. Therefore, $C_{\alpha,\beta}(I)$ is an ideal.

Theorem 3. If I is an IFNI in NR, then $C_{\alpha,\beta}(I) \subseteq C_{\theta,\vartheta}(I)$ if $\alpha \geq \theta$ and $\beta \leq \vartheta$.

Proof: Let $x \in C_{\alpha,\beta}(I)$. Then $\mu_I(x) \geq \alpha$ and $\gamma_I(x) \leq \beta$.

As $\alpha \geq \theta$ and $\beta \leq \vartheta$, then $\mu_I(x) \geq \alpha \geq \theta$ and $\gamma_I(x) \leq \beta \leq \vartheta$. Therefore, $x \in C_{\theta,\vartheta}(I)$.

Hence, $C_{\alpha,\beta}(I) \subseteq C_{\theta,\vartheta}(I)$.

Corollary 1. If $\alpha + \beta \leq 1$, then $C_{1-\beta,\beta}(I) \subseteq C_{\alpha,\beta}(I) \subseteq C_{\alpha,1-\alpha}(I)$.

Proof: As $\alpha + \beta \leq 1$, then $1 - \beta \geq \alpha$. Then, by Theorem 3, $C_{1-\beta,\beta}(I) \subseteq C_{\alpha,\beta}(I)$.

Also, $\beta \leq 1 - \alpha$, then $C_{\alpha,\beta}$ $I) \subseteq C_{\alpha,1-\alpha}$ $I)$.

Therefore, $C_{1-\beta,\beta}(I) \subseteq C_{\alpha,\beta}(I) \subseteq C_{\alpha,1-\alpha}(I)$.

Theorem 4. Let $C_{\alpha,\beta}(I)$ be an ideal with $\alpha + \beta \leq 1$ and with $\alpha, \beta \in [0,1]$, then I is an IFNI in NR.

Proof: Let $x, y \in NR$ and $\alpha = \mu_I(x) * \mu_I(y)$ and $\beta = \gamma_I$ $x) \diamond \gamma_I$ $y)$. Then $\mu_I(x) \geq \alpha$, $\gamma_I(x) \leq \beta$ and $\mu_I(y) \geq \alpha$, $\gamma_I(y) \leq \beta$. Then $x, y \in C_{\alpha,\beta}(I)$. Since $C_{\alpha,\beta}(I)$ is an ideal of NR, so $x - y \in C_{\alpha,\beta}$ $I)$. Therefore, $\mu_I(x-y) \geq \alpha = \mu_I(x) * \mu_I(y)$ and $\gamma_I(x-y) \leq \beta = \gamma_I(x) \diamond \gamma_I(y)$. Assume $\lambda = \mu_I(x) \diamond \mu_I(y)$. More precisely, let $\lambda = \mu_I(x)$. Since $\mu_I(x) + \gamma_I(x) \leq 1$, so $\gamma_I(x) \leq 1 - \mu_I(x) \leq 1 - \lambda$. Then $x \in C_{\lambda,1-\lambda}(I)$. Since, $C_{\lambda,1-\lambda}(I)$ is an ideal of NR, so $xy \in C_{\lambda,1-\lambda}(I)$. Thus, $\mu_I(xy) \geq \lambda = \mu_I(x) \diamond \mu_I(y)$.

Similarly, it can be proven that $\gamma_I(xy) \leq \gamma_I(x) * \gamma_I(y)$. Hence I is an IFNI of NR.

Proposition 2. Let I and J be two IFNI's of NR. If $I \subseteq J$, then $C_{\alpha,\beta}(I) \subseteq C_{\alpha,\beta}(J)$ for all $\alpha, \beta \in [0,1]$.

Proof: Let $x \in C_{\alpha,\beta}(I)$, then $\mu_I(x) \geq \alpha$ and $\gamma_I(x) \leq \beta$. As $I \subseteq J$ implies $\mu_J(x) \geq \mu_I(x) \geq \alpha$ and $\gamma_J(x) \leq \gamma_I(x) \leq \beta$. Hence $\mu_J(x) \geq \alpha$ and $\gamma_J(x) \leq \beta$. So $x \in C_{\alpha,\beta}(J)$.

Therefore, $C_{\alpha,\beta}(I) \subseteq C_{\alpha,\beta}(J)$.

Theorem 5. If I and J are two IFNI's of NR. Then, $C_{\alpha,\beta}$ $I \cap J) = C_{\alpha,\beta}(I) \cap C_{\alpha,\beta}(J)$ for all $\alpha, \beta \in [0,1]$.

Proof: Since $I \cap J \subseteq I$ and $I \cap J \subseteq J$. Therefore, $C_{\alpha,\beta}(I \cap J) \subseteq C_{\alpha,\beta}(I)$ and $C_{\alpha,\beta}(I \cap J) \subseteq C_{\alpha,\beta}(J)$. Hence,

$$C_{\alpha,\beta}(I \cap J) \subseteq C_{\alpha,\beta}(I) \cap C_{\alpha,\beta}(J) \quad (1)$$

Also, let $x \in C_{\alpha,\beta}(I) \cap C_{\alpha,\beta}(J) \Rightarrow x \in C_{\alpha,\beta}(I)$ and $x \in C_{\alpha,\beta}(J)$. Then

$\mu_I(x) \geq \alpha$, $\mu_J(x) \geq \alpha$ and $\gamma_I(x) \leq \beta$, $\gamma_J(x) \leq \beta$,

$min\{\mu_I(x), \mu_J(x)\} \geq \alpha$ and $max\{\gamma_I(x), \gamma_J(x)\} \leq \beta$,

$\mu_{I \cap J}(x) \geq \alpha$ and $\gamma_{I \cap J}(x) \leq \beta$.

Therefore, $x \in C_{\alpha,\beta}(I \cap J)$. Thus,

$$C_{\alpha,\beta} \ I) \cap C_{\alpha,\beta} \ J) \subseteq C_{\alpha,\beta} \ I \cap J). \quad (2)$$

By *Eqs. (1)* and *(2)*, $C_{\alpha,\beta}(I) \cap C_{\alpha,\beta}(J) = C_{\alpha,\beta}(I \cap J)$.

Lemma 2. $C_{\alpha,\beta}(\cap I_k | k \in K) = \cap \{C_{\alpha,\beta}(I_k) | k \in K\}$ for $k < \infty$ and for all $\alpha, \beta \in [0,1]$.

Proof: Let $x \in C_{\alpha,\beta} \cap I_k) \Leftrightarrow (\cap \mu_{I_k})(x) \geq \alpha$ and $(\cap \gamma_{I_k})(x) \leq \beta$,

$\Leftrightarrow \mu_{I_k}(x) \geq \alpha$ and γ_{I_k} $x) \leq \beta$,

$\Leftrightarrow x \in C_{\alpha,\beta}\ I_k)$ for all k,

$$\Leftrightarrow x \in \cap\ C_{\alpha,\beta}(I_k).$$

Therefore, $C_{\alpha,\beta} \cap I_k\ ) = \cap \left\{C_{\alpha,\beta}\ I_k)\right\}$.

Theorem 6. If I and J are two IFNI's of NR. Then $C_{\alpha,\beta}(I) \cup C_{\alpha,\beta}(J) \subseteq C_{\alpha,\beta}(I \cup J)$. If $\alpha + \beta = 1$ then $C_{\alpha,\beta}\ I) \cup C_{\alpha,\beta}\ J) = C_{\alpha,\beta}(I \cup J)$.

Proof: Since $I \subseteq I \cup J$ and $J \subseteq I \cup J$. Therefore, $C_{\alpha,\beta}(I) \subseteq C_{\alpha,\beta}(I \cup J)$ and $C_{\alpha,\beta}(J) \subseteq C_{\alpha,\beta}(I \cup J)$. Hence,

$$C_{\alpha,\beta}\ I) \cup C_{\alpha,\beta}\ J) \subseteq C_{\alpha,\beta}\ I \cup J). \quad (3)$$

Now equality holds if $\alpha + \beta = 1$. Prove that $C_{\alpha,\beta}(I \cup J) \subseteq C_{\alpha,\beta}(I) \cup C_{\alpha,\beta}(J)$.

Let $x \in C_{\alpha,\beta}(I \cup J)$, then $\mu_{I \cup J}(x) \geq \alpha$ and $\gamma_{I \cup J}\ x) \leq \beta$. So, $max\{\mu_I(x), \mu_J(x)\} \geq \alpha$ and $min\{\gamma_I(x), \gamma_J(x)\} \leq \beta$.

If $\mu_I\ x) \geq \alpha$, then $\gamma_I\ x) \leq 1 - \mu_I\ x) \leq 1 - \alpha = \beta$. It follows that $x \in C_{\alpha,\beta}\ I) \subseteq C_{\alpha,\beta}\ I) \cup C_{\alpha,\beta}\ J)$. Similarly, if $\mu_J\ x) \geq \alpha$, then $\gamma_J\ x) \leq 1 - \mu_J\ x) \leq 1 - \alpha = \beta$. It follows that $x \in C_{\alpha,\beta}\ J) \subseteq C_{\alpha,\beta}\ I) \cup C_{\alpha,\beta}\ J)$. Therefore, $x \in C_{\alpha,\beta}(I) \cup C_{\alpha,\beta}(J)$. Hence,

$$C_{\alpha,\beta}(I \cup J) \subseteq C_{\alpha,\beta}(I) \cup C_{\alpha,\beta}(J) \quad (4)$$

By *Eqs. (3)* and *(4)*, $C_{\alpha,\beta}(I \cup J) = C_{\alpha,\beta}(I) \cup C_{\alpha,\beta}(J)$.

Remark 1. In *Theorem 6*, the equality may not hold unless $\alpha + \beta = 1$, as shown in the following example:

Let $\mathbb{Z}_5 = \{0,1,2,3,4\}$ and consider the ring $(\mathbb{Z}_5, +_5, \times_5)$. Let I and J be two *IFNI*'s in $\mathbb{Z}_5$ with:

$\mu_I\ 0) = 0.7, \mu_I\ 2) = \mu_I\ 5) = 0.4,\ \ \mu_I\ 1) = \mu_I\ 3) = 0.1,$

$\gamma_I\ 0) = 0.1, \gamma_I\ 2) = \gamma_I\ 5) = 0.3, \gamma_I\ 1) = \gamma_I\ 3) = 0.5,$

$\mu_J\ 0) = 0.8, \mu_J\ 2) = \mu_J\ 5) = 0.3, \mu_J\ 1) = \mu_J\ 3) = 0.25,$

$\gamma_J\ 0) = 0.1, \gamma_J\ 2) = \gamma_J\ 5) = 0.2, \gamma_J\ 1) = \gamma_J\ 3) = 0.4.$

Then, $C_{0.4,0.2}(I) = \{0\}$ and $C_{0.4,0.2}(J) = \{0\}$.

Now,

$\mu_{I \cup J}(0) = 0.8, \mu_{I \cup J}\ 2) = \mu_{I \cup J}(5) = 0.4, \mu_{I \cup J}\ 1) = \mu_{I \cup J}\ 3) = 0.25,$

and

$\gamma_{I \cup J}(0) = 0.1, \gamma_{I \cup J}(2) = \gamma_{I \cup J}(5) = 0.2, \gamma_{I \cup J}(1) = \gamma_{I \cup J}(3) = 0.4.$

Then, $C_{0.4,0.2}(I \cup J) = \{0,2,5\}$. Correspondingly, $C_{0.4,0.2}(I \cup J) \nsubseteq C_{0.4,0.2}(I) \cup C_{0.4,0.2}(J)$.

Therefore, $C_{\alpha,\beta}(I) \cup C_{\alpha,\beta}(J)$ may not equal $C_{\alpha,\beta}\ I \cup J)$ when $\alpha + \beta \neq 1$.

Definition 13. Let I and J be two IFNI's in NR. Then,

$$C_{\alpha,\beta}(I + J) = \{(x, \mu_{I+J}(x) \geq \alpha \text{ and } \gamma_{I+J}(x) \leq \beta) : x \in NR\}.$$

Theorem 7. For any two IFNI's I and J of a ring NR, $C_{\alpha,\beta}(I)+C_{\alpha,\beta}(J)\subseteq C_{\alpha,\beta}(I+J)$ and equality holds if $\alpha+\beta=1$.

Proof: Let $x=y+z\in C_{\alpha,\beta}(I)+C_{\alpha,\beta}(J)$ such that $y\in C_{\alpha,\beta}(I)$ and $z\in C_{\alpha,\beta}(J)$. Then

$$\mu_I(y)\geq\alpha,\ \mu_J(z)\geq\alpha \text{ and } \gamma_I(y)\leq\beta, \gamma_J(z)\leq\beta.$$
$$\mu_I(y)*\mu_J(z)\geq\alpha \text{ and } \gamma_I(y)\diamond\gamma_J(z)\leq\beta.$$
$$\overset{\diamond}{_{x=y+z}}\left(\mu_I(y)*\mu_J(z)\right)\geq\alpha \text{ and } \overset{*}{_{x=y+z}}\left(\gamma_I(y)\diamond\gamma_J(z)\right)\leq\beta.$$
$$\mu_{I+J}(x)\geq\alpha \text{ and } \gamma_{I+J}(x)\leq\beta.$$

Therefore, $x\in C_{\alpha,\beta}(I+J)$. Hence, $C_{\alpha,\beta}(I)+C_{\alpha,\beta}(J)\subseteq C_{\alpha,\beta}(I+J)$.

For the second part, let $\alpha+\beta=1$ and let $x\in C_{\alpha,\beta}(I+J)$. Then, $\mu_{I+J}(x)\geq\alpha$ and $\gamma_{I+J}(x)\leq\beta$. Take, $\mu_{I+J}(x)\geq\alpha\Rightarrow\overset{\diamond}{_{x=y+z}}\left(\mu_I(y)*\mu_J(z)\right)\geq\alpha\Rightarrow\mu_I(w)*\mu_J(v)\geq\alpha$ for some $x=w+v$. Therefore, $\mu_I(w)\geq\alpha$ and $\mu_J(v)\geq\alpha$. Also, $\gamma_I(w)\leq1-\mu_I(w)\leq1-\alpha=\beta$ and $\gamma_J(v)\leq1-\mu_J(v)\leq1-\alpha=\beta$. Thus, $\gamma_I(w)\leq\beta$ and $\gamma_J(v)\leq\beta$. Hence $w\in C_{\alpha,\beta}(I)$, and $v\in C_{\alpha,\beta}(J)$. Then $x=w+v\in C_{\alpha,\beta}(I)+C_{\alpha,\beta}(J)$.

Thus, $C_{\alpha,\beta}(I+J)\subseteq C_{\alpha,\beta}(I)+C_{\alpha,\beta}(J)$. So, the equality follows:

Definition 14. Let I and J be two IFNI's in NR. Then, for $\alpha,\beta\in[0,1]$.

$$C_{\alpha,\beta}(I\circ J)=\{(x,\mu_{I\circledast J}(x)\geq\alpha \text{ and } \gamma_{I\otimes J}(x)\leq\beta): x\in NR\}.$$

Theorem 8. For any IFNI's I and J in NR, $C_{\alpha,\beta}(I)C_{\alpha,\beta}(J)\subseteq C_{\alpha,\beta}(I\circ J)$. Equality holds if $\alpha+\beta=1$.

Proof: By *Definition 5*,

$$\mu_{I\circledast J}(x)=\overset{\diamond}{_{x=yz}}\left(\mu_I(y)*\mu_J(z)\right) \text{ and } \gamma_{I\otimes J}(x)=\overset{*}{_{x=yz}}\left(\gamma_I(y)\diamond\gamma_J(z)\right).$$

Let $x=yz\in C_{\alpha,\beta}(I)C_{\alpha,\beta}(J)$ such that $y\in C_{\alpha,\beta}(I)$ and $z\in C_{\alpha,\beta}(J)$. So, $\mu_I(y)\geq\alpha$, $\gamma_I(y)\leq\beta$ and $\mu_J(z)\geq\alpha$,

$$\begin{aligned}\mu_{I\circledast J}(x)&=\overset{\diamond}{_{x=yz}}\left(\mu_I(y)*\mu_J(z)\right)\\&\geq\mu_I(y)*\mu_J(z)\\&\geq\alpha,\end{aligned}$$

and

$$\begin{aligned}\gamma_{I\otimes J}(x)&=\overset{*}{_{x=yz}}\left(\gamma_I(y)\diamond\gamma_J(z)\right)\\&\leq\gamma_I(y)\diamond\gamma_J(z)\\&\leq\beta.\end{aligned}$$

$\gamma_J(z)\leq\beta$. Hence

Therefore, $x\in C_{\alpha,\beta}(I\circ J)$. Then, $C_{\alpha,\beta}(I)C_{\alpha,\beta}(J)\subseteq C_{\alpha,\beta}(I\circ J)$.

To prove $C_{\alpha,\beta}(I)C_{\alpha,\beta}(J)=C_{\alpha,\beta}(I\circ J)$, let $\alpha+\beta=1$ and $x\in C_{\alpha,\beta}(I\circ J)$. Then, $\mu_{I\circ J}(x)\geq\alpha$ and $\gamma_{I\circ J}(x)\leq\beta$.

Now, $\mu_{I\circledast J}(x)\geq\alpha\Rightarrow\mu_{I\circledast J}(x)=\overset{\diamond}{_{x=yz}}\left(\mu_I(y)*\mu_J(z)\right)\geq\alpha\Rightarrow\mu_I(y)*\mu_J(z)\geq\alpha$. Then $\mu_I(y)\geq\alpha$ and $\mu_J(z)\geq\alpha$. Also, $\gamma_I(y)\leq1-\mu_I(y)\leq1-\alpha=\beta$ and $\gamma_J(z)\leq1-\mu_J(z)\leq1-\alpha=\beta$. Then, $\gamma_I(y)\leq\beta$ and $\gamma_J(z)\leq\beta$. Hence $y\in C_{\alpha,\beta}(I)$ and $z\in C_{\alpha,\beta}(J)$. So, $x\in C_{\alpha,\beta}(I)C_{\alpha,\beta}(J)$. Thus, $C_{\alpha,\beta}(I\circ J)\subseteq C_{\alpha,\beta}(I)C_{\alpha,\beta}(J)$.

Therefore, $C_{\alpha,\beta}(I)C_{\alpha,\beta}(J)=C_{\alpha,\beta}(I\circ J)$.

4 | Homomorphism of Intuitionistic Fuzzy Normed Ideals

The correlation between the intuitionistic fuzzy normed ideal I of NR and that of intuitionistic fuzzy normed ideal $f(I)$ of NR' with the help of their (α, β)-level subsets is obtained.

Definition 15 ([16]). Let NR, NR' be two rings and $f: NR \to NR'$ be a ring homomorphism. Let $A = \{(x, \mu_A(x), \gamma_A(x)): x \in NR\}$ and $B = \{(y, \mu_B(y), \gamma_B(y)): y \in NR'\}$ be intuitionistic fuzzy subsets of NR, NR', respectively. Then, the image of A is denoted by $f(A)$ and defined as $f(A) = \{(y, \mu_{f(A)}(y), \gamma_{f(A)}(y)): y \in NR'\}$ where:

$$\mu_{f(A)}(y) = \begin{cases} \underset{x \in f^{-1}(y)}{\diamond} \mu_A(x), & \text{if } f^{-1}(y) \neq \phi, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\gamma_{f(A)}(y) = \begin{cases} \underset{x \in f^{-1}(y)}{*} \gamma_A(x), & \text{if } f^{-1}(y) \neq \phi, \\ 1, & \text{otherwise.} \end{cases}$$

Also, $f^{-1}(B) = \{(x, \mu_{f^{-1}(B)}(x), \gamma_{f^{-1}(B)}(x)): x \in NR\}$ is called the inverse image (pre-image) of B, where $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ and $\gamma_{f^{-1}(B)}(x) = \gamma_B(f(x))$ for every $x \in NR$.

Theorem 8 ([16]). Let $f: NR \to NR'$ be an epimorphism mapping.

I. If D is an intuitionistic fuzzy subring in NR', then $f^{-1}(D)$ is an intuitionistic fuzzy subring in NR.
II. If S is an intuitionistic fuzzy subring in NR, then $f(S)$ is an intuitionistic fuzzy subring in NR'.

Theorem 9. Let $f: NR \to NR'$ be an epimorphism mapping. Let $I = \{(x, \mu_I(x), \gamma_I(x)): x \in NR\}$ be an IFNI in NR. Then $f(C_{\alpha,\beta}(I)) \subseteq C_{\alpha,\beta}(f(I))$. The equality holds if μ_I has the sup property ($\diamond$) and $\alpha + \beta = 1$.

Proof: Let $y \in f(C_{\alpha,\beta}(I))$. Then there exit an element $x \in C_{\alpha,\beta}(I)$ such that $f(x) = y$, such that $\mu_I(x) \geq \alpha$ and $\gamma_I(x) \leq \beta$. Then,

$$\underset{x \in f^{-1}(y)}{\diamond} \mu_I(x) \geq \alpha \ \text{ and } \ \underset{x \in f^{-1}(y)}{*} \gamma_I(x) \leq \beta. \text{ So, } \mu_{f(I)}(y) \geq \alpha \ \text{ and } \ \gamma_{f(I)}(y) \leq \beta.$$

Which implies that $y \in C_{\alpha,\beta}(f(I))$. Hence, $f(C_{\alpha,\beta}(I)) \subseteq C_{\alpha,\beta}(f(I))$.

For the other way, Let $y \in C_{\alpha,\beta}(f(I))$. Then,

$$\mu_{f(I)}(y) \geq \alpha \ \text{ and } \ \gamma_{f(I)}(y) \leq \beta,$$

$$\underset{x \in f^{-1}(y)}{\diamond} \mu_I(x) \geq \alpha \ \text{ and } \ \underset{x \in f^{-1}(y)}{*} \gamma_I(x) \leq \beta.$$

Since μ_I has the sup property so there exists $z \in f^{-1}(y)$ such that $\mu_I(z) = \underset{x \in f^{-1}(y)}{\diamond} \mu_I(x) \geq \alpha$. Then $\gamma_I(z) \leq 1 - \mu_I(z) \leq 1 - \alpha = \beta$. Thus, $z \in C_{\alpha,\beta}(I)$ and hence $y = f(z) \in f(C_{\alpha,\beta}(I))$. Therefore, $C_{\alpha,\beta}(f(I)) \subseteq f(C_{\alpha,\beta}(I))$.

Hence, $C_{\alpha,\beta}(f(I)) = f(C_{\alpha,\beta}(I))$.

Theorem 10. Define $f: NR \to NR'$ to be a homomorphism mapping. Let $f^{-1}(J) = \{(x, \mu_{f^{-1}(J)}(x), \gamma_{f^{-1}(J)}(x)) : x \in NR\}$ be an IFNI of NR. Then $C_{\alpha,\beta}\big(f^{-1}\ J)\big) = f^{-1}(C_{\alpha,\beta}(J))$.

Proof: Let $x \in C_{\alpha,\beta}\big(f^{-1}\ J)\big) \Leftrightarrow \mu_{f^{-1}\ J)}\ x) \geq \alpha$ and $\gamma_{f^{-1}\ J)}\ x) \leq \beta$

$$\Leftrightarrow \mu_J\big(f\ x)\big) \geq \alpha \text{ and } \gamma_J\big(f\ x)\big) \leq \beta$$

$$\Leftrightarrow f\ x) \in C_{\alpha,\beta}\ J)$$

$$\Leftrightarrow x \in f^{-1}(C_{\alpha,\beta}(J)).$$

Hence, $C_{\alpha,\beta}\big(f^{-1}\ J)\big) = f^{-1}(C_{\alpha,\beta}(J))$.

5 | Conclusion

In this paper, intuitionistic fuzzy ideals, with the help of some properties of their (α,β)-level subsets, were presented and studied. Also, some important links between intuitionistic fuzzy ideals and (α,β)-level subsets were established. Further, the homomorphic image was characterized by using the properties of (α,β)-level sets, such that the relation between the intuitionistic fuzzy normed ideal I of NR and the intuitionistic fuzzy normed ideal $f(I)$ of NR' were generalized with the support of their (α,β)-level subsets. To extend our work, further research can be done to introduce the concept of upper and lower (α,β)-intuitionistic fuzzy normed ideals and to examine various properties of (α,β)-level subsets for intuitionistic fuzzy normed ideals.

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Conflicts of Interest

The authors declare no conflict of interest.

References

[1] Zadeh, L. A. (1965). Fuzzy sets. *Information and control*, *8*(3), 338–353.

[2] Rosenfeld, A. (1971). Fuzzy groups. *Journal of mathematical analysis and applications*, *35*(3), 512–517. DOI:10.1016/0022-247X(71)90199-5

[3] Liu, W. (1982). Fuzzy invariant subgroups and fuzzy ideals. *Fuzzy sets and systems*, *8*(2), 133–139.

[4] Atanassov, K. T. (1986). Intuitionistic fuzzy sets. *Fuzzy sets and systems*, *20*(1), 87–96.

[5] Biswas, R. (1989). Intuitionistic fuzzy subgroups, Mathematical Forum, Vol. *Mathematical forum*, *10*(2), 37–46.

[6] Hur, K., Kang, H. W., & Song, H. K. (2003). Intuitionistic fuzzy subgroups and subrings. *Honam mathematical journal*, *25*(2), 19–41.

[7] Banerjee, B., & Basnet, D. (2003). Intuitionistic fuzzy subrings and ideals. *Journal of fuzzy mathematics*, *11*(1), 139–155.

[8] Abed Alhaleem, N., & Ahmad, A. G. (2020). Intuitionistic fuzzy normed subrings and intuitionistic fuzzy normed ideals. *Mathematics*, *8*(9), 1594.

[9] Basnet, D. K. (2010). (α, β)-cut of intuitionistic fuzzy ideals. *International journal of algebra*, *4*(27), 22–27.

[10] Sharma, P. K. (2011). (α,β) – Cut of Intuitionistic Fuzzy Groups. *International mathematics forum*, *6*(53), 2605–2614.

[11] Adak, A. K., Bhowmik, M., & Pal, M. (2012). Interval cut-set of generalized interval-valued intuitionistic fuzzy sets. *International journal of fuzzy system applications (IJFSA)*, *2*(3), 35–50.

[12] Klir, G., & Yuan, B. (1995). *Fuzzy sets and fuzzy logic* (Vol. 4). Prentice hall New Jersey.

[13] M.M. Gupta, J. Q. (1991). Theory of T-norms and fuzzy inference methods. *Fuzzy sets and systems*, *40*(3), 431–450.

[14] Naimark, M. A. (1959). *Normed Rings*. Groningen.

[15] Alhaleem, N. A., & Ahmad, A. G. Bin. (2021). Intuitionistic anti fuzzy normed ideals. *Notes on intuitionistic fuzzy sets*. https://api.semanticscholar.org/CorpusID:237896038

[16] Alhaleem, N. A., & others. (2021). Intuitionistic fuzzy normed prime and maximal ideals. *AIMS mathematics*, *6*(10), 10565–10581.