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Vulnerability Parameters in Picture Fuzzy Soft Graphs and Their Applications to Locate a Diagnosis Center in Cities

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Abstract

The notion of Fuzzy Graphs (FGs) and Intuitionistic Fuzzy Graphs (IFGs) is generalized in the Picture Fuzzy Graph (PFG), which is a more common platform for expressing the degree of positive, negative, and neutral membership functions. Picture Fuzzy Soft Graphs (PFSGs) are powerful mathematical tools for modeling real-world vagueness. The concept of vulnerability parameters of PFSG is provided in this research work by introducing the novel cardinality, domination number, integrity, and Domination Integrity (ĐÏ) of PFSG. Furthermore, a decision-making method for the PFSG has been presented using ĐÏ to suggest an algorithm for determining the ideal location for establishing a city diagnosis center.

Keywords: Picture fuzzy soft graphs, Cardinality, Dominating set, Integrity, Domination integrity.

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1 | Introduction

The Fuzzy Set (FS) [1] and the soft set [2] can be used to model problems involving vagueness and uncertainty. The Intuitionistic Fuzzy Set (IFS) is a generalization of the FS introduced by Atanassov [3]. Cuong and Kreinovich [4] introduced the Picture Fuzzy Set (PFS), which is a straightforward extension of the IFS and defined certain operations on PFS. Thumbakara and George [5] introduced soft graphs. They discussed the characteristics of soft complete graphs, soft trees, and soft graph homomorphisms. Graph theory was introduced by [6]. The Fuzzy Graph (FG) was introduced by [7], Akram and Nawaz [8], who defined fuzzy soft graphs. Atanassov and Shannon [9] defined the Intuitionistic Fuzzy Graph (IFG). The concept of an Intuitionistic Fuzzy Soft Graph (IFSG) was introduced by Shyla and Varkey [10]. Smarandache [11] introduced the neutrosophic set. Broumi et al. [12] introduced single-valued neutrosophic graphs. The Picture Fuzzy Graph (PFG) is an effective technique for dealing with ambiguous concerns in everyday life that an IFG may not be able to address precisely. A PFG is extremely useful for dealing with problems that have several options, such as no, yes, refuse, and abstain. By combining FGs with PFSs, Zuo et al. [13] developed the idea of the FG and introduced it to PFG. Various products and other operations on PFGs are also discussed. Mohamed Ismayil and Asha Bosley [14] introduced the Domination in PFG. Recently, Chellamani et al. [15] introduced Picture Fuzzy Soft Graph (PFSG) with properties and defined some operations.

Domination may be used in many fields, including decision-making theory, computer science, psychology, neurological systems, artificial intelligence, etc. Today, a lot of researchers are attempting to develop new applications for domination in their specialized fields. Dominating sets also give system designers more freedom and let them use membership grades with fewer restrictions. We now provide a crucial application of dominance in an ambiguous environment, which is necessary to comprehend the idea of dominating sets in PFSGs completely. The idea of integrity was first presented by [16]. It acts as a valuable measurement of vulnerability. Integrity assesses the damages as well as how difficult it is to compromise the network. A network is connected to the other nodes by a minimally dominant set of nodes; if this set is eliminated, the network is greatly impacted. This results in both a paralysis of decision-making and reduced communication among the surviving members. When the prevailing sets of nodes are attacked, the damage will be greater. This motivated studying the ĐÏ concept in graphs introduced by [17]. In FGs, Saravanan et al. [18] extended the concept of integrity in FGs. Ganesan et al. [19] introduced strong ĐÏ in graphs and FGs, and Jaikumar et al. [20]] introduced integrity and ĐÏ in neutrosophic soft graphs. This inspired us to introduce integrity and DI in PFSGs. Additionally, certain characteristics of integrity and DI concepts are discussed. The remaining portions of the manuscript are outlined below.

Section 2 contains the fundamental definitions for PFG with certain operations. Section 3 introduces cardinality, lower and upper ĐN of PFSG. Section 4 provides integrity and ĐÏ of PFSG with examples and properties. Section 5 includes an application. Section 6 contains the article's conclusion.

2 | Preliminaries

Brief discussions of fundamental concepts related to the operations of PFS, Picture Fuzzy Soft Set (PFSS), and PFSG are given in this section.

Definition 1 ([4]). A PFS Þ on V is in the form of $Þ = \{\langle\vartheta, \mu_Þ(\vartheta), \eta_Þ(\vartheta), \gamma_Þ(\vartheta)\rangle | \vartheta \in V\}$ where $\mu_Þ(\vartheta) \in [0,1]$ is positive membership degree of ϑ in Þ, $\eta_Þ(\vartheta) \in [0,1]$ is neutral membership degree of ϑ in Þ, $\gamma_Þ(\vartheta) \in [0,1]$ is negative membership degree of ϑ in Þ and $0 \leq \mu_Þ(\vartheta) + \eta_Þ(\vartheta) + \gamma_Þ(\vartheta) \leq 1, \text{for all } \vartheta \in V$. The degree of refusal membership can be determined by $\delta_Þ(\vartheta) = 1 - \langle\mu_Þ(\vartheta) + \eta_Þ(\vartheta) + \gamma_Þ(\vartheta)\rangle \leq 1, \text{for all } \vartheta \in V$.

Definition 2 ([4]). Let Þ be the PFSs of U. Let ε be the parameter sets and $\hat{O} \subseteq \varepsilon$. A couple $(₣, \hat{O})$ is a PFSS over U, where $₣ \colon \hat{O} \to Þ$. Clearly, $\text{for all } ë \in \varepsilon, ₣(ë)$ is defined as a PFS such that $₣(ë) = \{\langle\vartheta, \mu_{₣(ë),}(\vartheta), \eta_{₣(ë),}(\vartheta), \gamma_{₣(ë),}(\vartheta)\rangle | \vartheta \in U\}$.

Definition 3 ([15]). Let ℘* = (V, ε) be a graph. A couple ℘ = ($\tilde{A}_1$, $\tilde{A}_2$) is a PFG on ℘*, where $\tilde{A}_1 = (\mu_{\tilde{A}_1}, \eta_{\tilde{A}_1}, \nu_{\tilde{A}_1})$ is a PFS on V and $\tilde{A}_2 = (\mu_{\tilde{A}_2}, \eta_{\tilde{A}_2}, \nu_{\tilde{A}_2})$ is a PFS on ε ⊆ V × V such that for every arc ϑψ ∈ ε.

I. $\mu_{\tilde{A}_2}(\vartheta, \psi) \leq \wedge(\mu_{\tilde{A}_2}(\vartheta), \mu_{\tilde{A}_2}(\psi))$.

II. $\eta_{\tilde{A}_2}(\vartheta, \psi) \leq \wedge(\eta_{\tilde{A}_2}(\vartheta), \eta_{\tilde{A}_2}(\psi))$.

III. $\nu_{\tilde{A}_2}(\vartheta, \psi) \geq \vee(\nu_{\tilde{A}_2}(\vartheta), \nu_{\tilde{A}_2}(\psi))$.

Definition 4 ([15]). A PFSG ℘ = (℘*, Ғ, K̈, Ô) is a quadrable such that

I. ℘* = (V, ε) is a graph.

II. Ô denotes a non-void parameter set.

III. (Ғ, Ô) and (K̈, Ô) are two PFSS over the vertex V and the edge ε.

IV. (Ғ(o), K̈ (o)), for all o ∈ Ô is a PFSG of ℘*, then it satisfies the following:

- $\mu_{\ddot{K}(o)}(\vartheta\psi) \leq \wedge\, (\mu_{Ғ(o)}(\vartheta), \mu_{Ғ(o)}(\psi))$.
- $\eta_{\ddot{K}(o)}\ (\vartheta\psi) \leq \wedge\, (\eta_{Ғ(o)}(\vartheta), \eta_{Ғ(o)}(\psi))$.
- $\nu_{\ddot{K}(o)}\ (\vartheta\psi) \leq \vee\, (\nu_{Ғ(o)}(\vartheta), \nu_{Ғ(o)}(\psi))$.

Such that $0 \leq \mu_{\ddot{K}(o)}(\vartheta) + \eta_{\ddot{K}(o)}(\vartheta) + \gamma_{\ddot{K}(o)}(\vartheta) \leq 1$, for all o ∈ Ô; $\vartheta, \psi \in V$. The PFSG (Ғ(o), K̈ (o)) is symbolized by **ℤ(ë).**

Definition 5 ([15]). Let $\wp_1 = (\wp_1^*, Ғ_1, \ddot{K}_1, \hat{0}_1)$ and $\wp_2 = (\wp_2^*, Ғ_2, \ddot{K}_2, \hat{0}_2)$ be two PFSG. $\wp_1$ is known as a picture fuzzy soft subgraph (PFSSG) of $\wp_2$ if $\hat{0}_1 \subseteq \hat{0}_2$ and $\mathbb{Z}_1(o)$ is a partial subgraph of $\mathbb{Z}_2(o)$ for all o ∈ Ô.

Definition 6 ([15]). Let ℘ = (℘*, Ғ, K̈, Ô) be PFSG of ℘* = (V, ε). PFSG ℘ is known as regular PFSG if **ℤ(ë)** is a regular PFG for all ë ∈ Ô.

An example of a regular PFSG is given in *Fig. 1*.

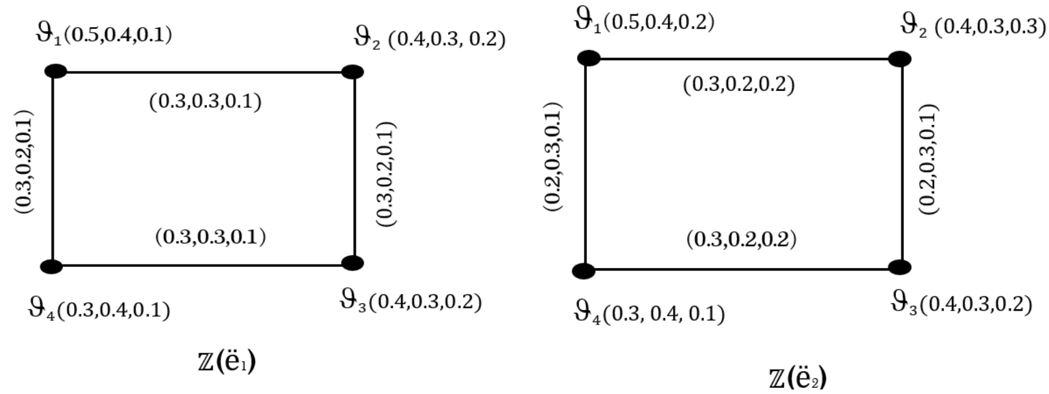

Fig. 1. Regular PFSG.

Definition 7. Let ℘* = (V, ε) be a PFSG. If ℤ (ë) is a strong PFG for all ë ∈ Ô, then ℘ = (℘*, Ғ, K̈, Ô) is a strong PFSG of ℘*, that is

I. $\mu_{\ddot{K}(ë)}(\vartheta, \psi) = \wedge\, (\mu_{Ғ(ë)}(\vartheta), \mu_{Ғ(ë)}(\psi))$.

II. $\eta_{\ddot{K}(ë)}\ (\vartheta, \psi) = \wedge\, (\eta_{Ғ(ë)}(\vartheta), \eta_{Ғ(ë)}(\psi))$.

III. $\nu_{\ddot{K}(ë)}\ (\vartheta, \psi) = \vee\, (\nu_{Ғ(ë)}(\vartheta), \nu_{Ғ(ë)}(\psi))$ for all ϑ ψ ∈ ε, ë ∈ Ô.

Example of a strong PFSG is given in *Fig. 2*.

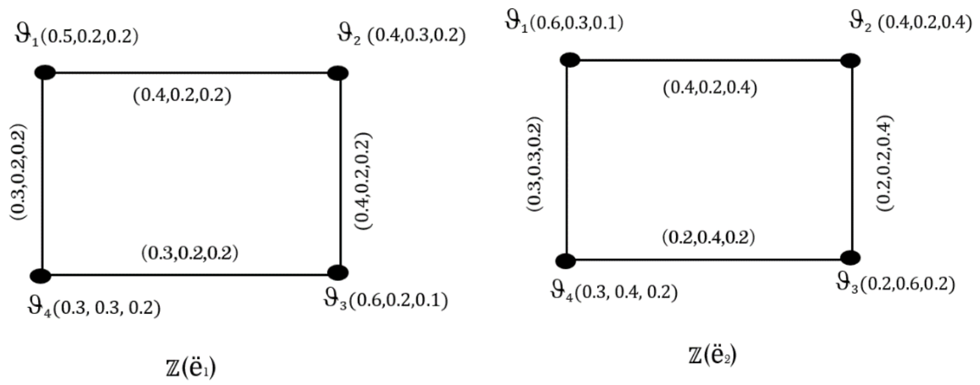

Fig. 2. Strong PFSG.

Definition 8. Let $\wp^* = (V, \varepsilon)$ be a PFSG. If $\mathbb{Z}$ (ë) is a complete PFG, then $\wp = (\wp^*, \text{Ꞙ}, \ddot{K}, \hat{O})$ is complete PFSG of $\wp^*$ i.e.

I. $\mu_{\ddot{K}(\ddot{e})}(\vartheta, \psi) = \wedge\, (\mu_{\text{Ꞙ}(\ddot{e})}(\vartheta), \mu_{\text{Ꞙ}(\ddot{e})}(\psi))$.

II. $\eta_{\ddot{K}(\ddot{e})}\, (\vartheta, \psi) = \wedge\, (\eta_{\text{Ꞙ}(\ddot{e})}(\vartheta), \eta_{\text{Ꞙ}(\ddot{e})}(\psi))$.

III. $\nu_{\ddot{K}(\ddot{e})}\, (\vartheta, \psi) = \vee\, (\nu_{\text{Ꞙ}(\ddot{e})}(\vartheta), \nu_{\text{Ꞙ}(\ddot{e})}(\psi))$ for all $\vartheta, \psi \in V$ and $\ddot{e} \in \hat{O}$

An example of a complete PFSG is given in *Fig. 3*.

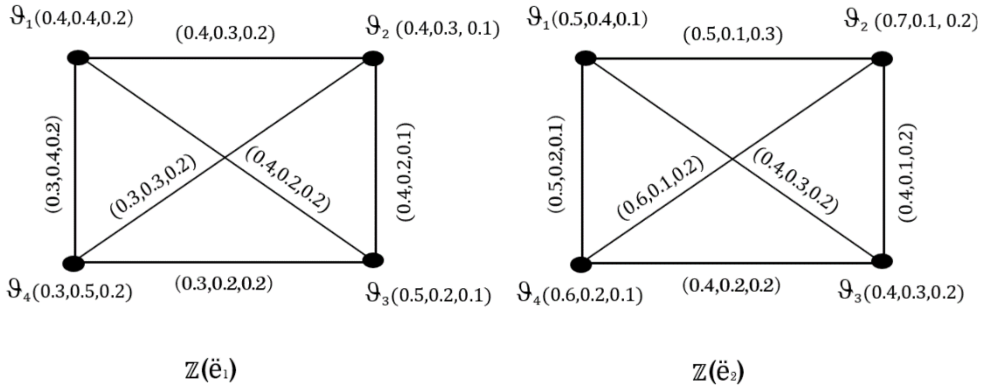

Fig. 3. Complete PFSG.

Definition 9. If $\wp = (V, \varepsilon)$ is a PFSG, then the complement of PFSG $\wp = (\text{Ꞙ}, \ddot{K}, \hat{O})$ is $\wp^C = (\text{Ꞙ}^C, \ddot{K}^C, \hat{O}^C)$ and

I. $\hat{O}^C = \hat{O}$.

II. $\text{Ꞙ}^C(\ddot{e}) = \text{Ꞙ}(\ddot{e})$.

III. $\mu^c_{\ddot{K}(\ddot{e})}\, (\vartheta, \psi) = \wedge\, \{(\mu_{\text{Ꞙ}(\ddot{e})}(\vartheta), \mu_{\text{Ꞙ}(\ddot{e})}(\psi))\} - \mu_{\ddot{K}(\ddot{e})}(\vartheta, \psi)$.

IV. $\eta^c_{\ddot{K}(\ddot{e})}\, (\vartheta, \psi) = \wedge\, \{(\eta_{\text{Ꞙ}(\ddot{e})}(\vartheta), \eta_{\text{Ꞙ}(\ddot{e})}(\psi))\} - \eta_{\ddot{K}(\ddot{e})}(\vartheta, \psi)$.

V. $\nu^c_{\ddot{K}(\ddot{e})}(\vartheta, \psi) = \vee\, \{(\nu_{\text{Ꞙ}(\ddot{e})}(\vartheta), \nu_{\text{Ꞙ}(\ddot{e})}(\psi))\} - \nu_{\ddot{K}(\ddot{e})}(\vartheta, \psi)$.

Definition 10. The order of PFSG $\wp = (V, \varepsilon)$ is determined by

$$\Theta(\wp) = \left(\sum_{\ddot{e}_i \in \hat{O}} \Theta_\mu(\wp), \sum_{\ddot{e}_i \in \hat{O}} \Theta_\eta(\wp), \sum_{\ddot{e}_i \in \hat{O}} \Theta_\nu(\wp) \right),$$

where $\Theta_\mu(\wp) = \sum_{\vartheta \in V}(\mu_{\text{Ꞙ}(\ddot{e}_i)}(\vartheta)), \Theta_\eta(\wp) = \sum_{\vartheta \in V}(\eta_{\text{Ꞙ}(\ddot{e}_i)}(\vartheta)), \Theta_\nu(\wp) = \sum_{\vartheta \in V}(\nu_{\text{Ꞙ}(\ddot{e}_i)}(\vartheta))$.

Definition 11. The size of PFSG $\wp$ = (V, ε) is determined by

$$S(\wp) = \left(\sum_{ë_i \in Ô} S_\mu(\wp), \sum_{ë_i \in Ô} S_\eta(\wp), \sum_{ë_i \in Ô} S_\nu(\wp)\right),$$

where $S_\mu(\wp) = \sum_{\mathfrak{A}\psi\in\varepsilon}\left(\mu_{\ddot{K}(ë)}(\vartheta,\psi)\right)$, $S_\eta(\wp) = \sum_{\mathfrak{A}\psi\in\varepsilon}\left(\eta_{\ddot{K}(ë)}(\vartheta,\psi)\right)$, $S_\nu(\wp) = \sum_{\mathfrak{A}\psi\in\varepsilon}(\nu_{\ddot{K}(ë)}(\vartheta,\psi))$.

3 | Cardinality and Domination Number in PFSG

The concept of the novel cardinality and various domination numbers are specified as follows in this section. This section defines the novel cardinality idea and various domination numbers.

Definition 12. The cardinality of PFSG $\wp$ = (V, ε) is determined by

$$|\wp| = \sum_{ë_i \in Ô}\left|\sum_{\vartheta\in V}\left(\frac{1 + 2\left(\mu_{Ғ(ë_i)}(\vartheta)\right) + \left(\frac{\eta_{Ғ(ë_i)}(\vartheta)}{2}\right) - \nu_{Ғ(ë_i)}(\vartheta)}{3}\right)\right.$$
$$\left. + \sum_{\vartheta\psi\in\varepsilon}\left(\frac{1 + 2\left(\mu_{\ddot{K}(ë)}(\vartheta,\psi)\right) + \left(\frac{\eta_{\ddot{K}(ë)}(\vartheta,\psi)}{2}\right) - \nu_{\ddot{K}(ë)}(\vartheta,\psi)}{3}\right)\right|.$$

Definition 13. The vertex cardinality of PFSG $\wp$ = (V, ε) is determined by

$$|V| = \sum_{ë_i \in Ô}\left|\sum_{\vartheta\in V}\left(\frac{1 + 2\left(\mu_{Ғ(ë_i)}(\vartheta)\right) + \left(\frac{\eta_{Ғ(ë_i)}(\vartheta)}{2}\right) - \nu_{Ғ(ë_i)}(\vartheta)}{3}\right)\right|.$$

Definition 14. The edge cardinality of PFSG $\wp$ = (V, ε) is determined by

$$|\varepsilon| = \sum_{ë_i \in Ô}\left|\sum_{\vartheta\psi\in\varepsilon}\left(\frac{1 + 2\left(\mu_{\ddot{K}(ë)}(\vartheta,\psi)\right) + \left(\frac{\eta_{\ddot{K}(ë)}(\vartheta,\psi)}{2}\right) - \nu_{\ddot{K}(ë)}(\vartheta,\psi)}{3}\right)\right|.$$

Example 1. Consider PFSG shown in *Fig. 2.* The vertex cardinality of the PFSG of $\wp$ which corresponds to $ë_1$ and $ë_2$ is $\mathbb{Z}_{|V|}(ë) = \{\mathbb{Z}_{|V|}(ë_1), \mathbb{Z}_{|V|}(ë_2)\} = \{2.47, 2.28\}$. The edge cardinality of the PFSG of $\wp$ which corresponds to $ë_1$ and $ë_2$ is $\mathbb{Z}_{|\varepsilon|}(ë) = \{\mathbb{Z}_{|\varepsilon|}(ë_1), \mathbb{Z}_{|\varepsilon|}(ë_2)\} = \{2.13, 1.85\}$. The cardinality of the PFSG of $\wp$ which corresponds to $ë_1$ and $ë_2$ is $\mathbb{Z}_{|G|}(ë) = \{\mathbb{Z}_{|G|}(ë_1), \mathbb{Z}_{|G|}(ë_2)\}=\{4.60, 4.13\}$.

Definition 15. Let $\wp = (\wp^*, Ғ, \ddot{K}, Ô)$ be a PFSG, the degree of a vertex $\deg(\vartheta)$ is defined as

$$\deg(\vartheta) = \left(\deg_{\mu_{Ғ(ë)}}(\vartheta), \deg_{\eta_{Ғ(ë)}}(\vartheta), \deg_{\nu_{Ғ(ë)}}(\vartheta)\right),$$

where $\deg_{\mu_{Ғ(ë)}}(\vartheta) = \sum_{ë\in Ô}\left(\sum_{\vartheta\notin\psi\in V}\mu_{\ddot{K}(ë)}(\vartheta,\psi)\right)$, $\deg_{\eta_{Ғ(ë)}}(\vartheta) = \sum_{ë\in Ô}\left(\sum_{\vartheta\notin\psi\in V}\eta_{\ddot{K}(ë)}(\vartheta,\psi)\right)$, $\deg_{\nu_{Ғ(ë)}}(\vartheta) = \sum_{ë\in Ô}\left(\sum_{\vartheta\notin\psi\in V}\nu_{\ddot{K}(ë)}(\vartheta,\psi)\right)$.

Definition 16. Let $\wp = (\wp^*, Ғ, \ddot{K}, Ô)$ be a PFSG, then the total degree of a vertex $\vartheta \in G$ is defined as $Tdeg(\vartheta) = \left(Tdeg_{\mu_{Ғ(ë)}}(\vartheta), Tdeg_{\eta_{Ғ(ë)}}(\vartheta), Tdeg_{\nu_{Ғ(ë)}}(\vartheta)\right)$, Where

$$\mathrm{Tdeg}_{\mu_{₣(\ddot{e})}}(\vartheta)=\sum_{\ddot{e}\in\hat{O}}\left(\sum_{\vartheta\notin\psi\in V}\mu_{\ddot{K}(\ddot{e})}(\vartheta,\psi)+\mu_{₣(\ddot{e})}(\vartheta,\psi)\right),$$

$$\mathrm{Tdeg}_{\eta_{₣(\ddot{e})}}(\vartheta)=\sum_{\ddot{e}\in\hat{O}}\left(\sum_{\vartheta\notin\psi\in V}\eta_{\ddot{K}(\ddot{e})}(\vartheta,\psi)+\eta_{₣(\ddot{e})}(\vartheta,\psi)\right),$$

$$\mathrm{Tdeg}_{\nu_{₣(\ddot{e})}}(\vartheta)=\sum_{\ddot{e}\in\hat{O}}\left(\sum_{\mathfrak{A}\notin\psi\in V}\nu_{\ddot{K}(\ddot{e})}(\vartheta,\psi)+\nu_{₣(\ddot{e})}(\vartheta,\psi)\right).$$

Example 2. Consider a PFSG $\wp * = (V, \varepsilon)$ with $V=\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$ and $\varepsilon=\{\vartheta_1\vartheta_2, \vartheta_2\vartheta_3, \vartheta_3\vartheta_4, \vartheta_1\vartheta_4\}$.

Let (₣, $\hat{O}$) be a PFSS over V with the approximation function ₣: $\hat{O} \to Þ(V)$ described by ₣$(\ddot{e}_1)$ $=\{\vartheta_1(0.5,0.4,0.1), \vartheta_2\ (0.4,0.3, 0.2), \vartheta_3(0.4,0.3,0.2), \vartheta_4(0.3,0.4,0.1)\}$, ₣$(\ddot{e}_2)$ $=\{\vartheta_1(0.5,0.4,0.2), \vartheta_2\ (0.4,0.3,0.3), \vartheta_3(0.4,0.3,0.2), \vartheta_4(0.3, 0.4, 0.1)\}$.

Let ($\ddot{K}$, $\hat{O}$) be a PFSS over ε with $\ddot{K}$: $\hat{O} \to Þ(\varepsilon)$ described by $\ddot{K}(\ddot{e}_1)=\{\vartheta_1\vartheta_2(0.3,0.3,0.1), \vartheta_2\vartheta_3(0.3,0.2,0.1), \vartheta_3\vartheta_4\ (0.3,0.3,0.1), \vartheta_1\vartheta_4\ (0.3,0.2,0.1)\}$, $\ddot{K}(\ddot{e}_2)=\{\vartheta_1\vartheta_2(0.3,0.2,0.2), \vartheta_2\vartheta_3(0.2,0.3,0.1), \vartheta_3\vartheta_4\ (0.3,0.2,0.2), \vartheta_1\vartheta_4\ (0.2,0.3,0.1)\}$.

Clearly, $\mathbb{Z}(\ddot{e}_1) = \{$ ₣$(\ddot{e}_1), \ddot{K}(\ddot{e}_1)\}$ and $\mathbb{Z}(\ddot{e}_2) = \{$ ₣$(\ddot{e}_2), \ddot{K}(\ddot{e}_2)\}$ are the PFSG corresponding to $\ddot{e}_1$ and $\ddot{e}_2$ as shown in *Fig. 1*.

deg $[\mathbb{Z}(\ddot{e}_1)]$ are as follows:

$\deg(\vartheta_1) = \deg(\vartheta_2) = \deg(\vartheta_3) = \deg(\vartheta_4) =(0.6, 0.5, 0.2)$.

deg $[\mathbb{Z}(\ddot{e}_2)]$ are as follows:

$\deg(\vartheta_1) = \deg(\vartheta_2) = \deg(\vartheta_3) = \deg(\vartheta_4) =(0.5, 0.5, 0.3)$.

Thus $\mathbb{Z}(\ddot{e}) = \{\mathbb{Z}(\ddot{e}_1), \mathbb{Z}(\ddot{e}_2)\}$ is a regular PFSG.

Definition 17. The minimum and maximum degree of the PFSG $\wp$ is determined by

I. $\delta(\wp) = \min\{\deg_{\wp}(\vartheta)\ /\vartheta \in V, \ddot{e} \in \hat{O}\}$,

II. $\Delta(\wp) = \max\{\deg_{\wp}(\vartheta)\ /\vartheta \in V, \ddot{e} \in \hat{O}\}$.

Definition 18. A sequence of various vertices $\vartheta_0, \vartheta_1, \vartheta_2,...,\vartheta_m$ that forms a path þ of length m in a PFSG such that $(\mu_{\ddot{K}(\ddot{e})}(\vartheta_{j-1}\vartheta_j)\ \eta_{\ddot{K}(\ddot{e})}(\vartheta_{j-1}\vartheta_j)\ \nu_{\ddot{K}(\ddot{e})}(\vartheta_{j-1}\vartheta_j)) > 0$, j = 1, 2, ..., m.

Definition 19. A path þ of length m in $\wp$ connecting the vertices ϑ and ψ such as p: $\vartheta_0, \vartheta_1, \vartheta_2, ...\ \vartheta_{m-1}, \vartheta_m$ then $\mu_{\ddot{K}(\ddot{e})}(\vartheta,\psi)$, $\eta_{\ddot{K}(\ddot{e})}(\vartheta,\psi)$ and $\nu_{\ddot{K}(\ddot{e})}(\vartheta,\psi)$ are expressed as

I. $\mu^{m}_{\ddot{K}(\ddot{e})}(\vartheta,\psi) = \mu_{\ddot{K}(\ddot{e})}(\vartheta, \vartheta_1) \wedge \mu_{\ddot{K}(\ddot{e})}(\vartheta_1, \vartheta_2) \wedge ... \wedge \mu_{\ddot{K}(\ddot{e})}(\vartheta_{m-1}, \psi)$.

II. $\eta^{m}_{\ddot{K}(\ddot{e})}(\vartheta,\psi) = \eta_{\ddot{K}(\ddot{e})}(\vartheta, \vartheta_1) \wedge \eta_{\ddot{K}(\ddot{e})}(\vartheta_1, \vartheta_2) \wedge ... \wedge \eta_{\ddot{K}(\ddot{e})}(\vartheta_{m-1}, \psi)$.

III. $\nu^{m}_{\ddot{K}(\ddot{e})}(\vartheta,\psi) = \nu_{\ddot{K}(\ddot{e})}(\vartheta, \vartheta_1) \vee \nu_{\ddot{K}(\ddot{e})}(\vartheta_1, \vartheta_2) \vee ... \vee \nu_{\ddot{K}(\ddot{e})}(\vartheta_{m-1}, \psi)$.

Let $(\mu^{\infty}_{\ddot{K}(\ddot{e})}(\vartheta,\psi))$, $\eta^{\infty}_{\ddot{K}(\ddot{e})}(\vartheta,\psi)$, $\nu^{\infty}_{\ddot{K}(\ddot{e})}(\vartheta,\psi)$ be the strength of connectedness between the vertices ϑ and ψ of a PFSG $\wp$. Then $\mu^{\infty}_{\ddot{K}(\ddot{e})}(\vartheta,\psi)$, $\eta^{\infty}_{\ddot{K}(\ddot{e})}(\vartheta,\psi)$, and $\nu^{\infty}_{\ddot{K}(\ddot{e})}(\vartheta,\psi)$ are expressed as

I. $\mu^{\infty}_{\ddot{K}(\ddot{e})}(\vartheta,\psi) = \sup\{\mu^{m}_{\ddot{K}(\ddot{e})}(\vartheta,\psi)\,|\,m = 1, 2, ...\}$.

II. $\eta^{\infty}_{\ddot{K}(\ddot{e})}(\vartheta,\psi) = \sup\{\eta^{m}_{\ddot{K}(\ddot{e})}(\vartheta,\psi) \mid m = 1, 2, \ldots\}$.

III. $\nu^{\infty}_{\ddot{K}(\ddot{e})}(\vartheta,\psi) = \inf\{\nu^{m}_{\ddot{K}(\ddot{e})}(\vartheta,\psi) \mid m = 1, 2, \ldots\}$.

Where the minimum membership value is determined by inf, and the maximum value is determined by sup.

Definition 20. A PFSG $\wp = (V, \varepsilon)$ is connected to PFSG if for every vertex $\vartheta, \psi \in V$, $\mu^{\infty}_{\ddot{K}(\ddot{e})}(\vartheta,\psi) > 0$ or $\eta^{\infty}_{\ddot{K}(\ddot{e})}(\vartheta,\psi) > 0$ or $\nu^{\infty}_{\ddot{K}(\ddot{e})}(\vartheta,\psi) < 1$.

Definition 21. An arc (ϑ,ψ) in a PFSG $\wp = (V, \varepsilon)$ is a strong arc if $\mu_{\ddot{K}(\ddot{e})}(\vartheta,\psi) > \mu^{\infty}_{\ddot{K}(\ddot{e})}(\vartheta,\psi)$, $\eta_{\ddot{K}(\ddot{e})}(\vartheta,\psi) > \eta^{\infty}_{\ddot{K}(\ddot{e})}(\vartheta,\psi)$, $\nu_{\ddot{K}(\ddot{e})}(\vartheta,\psi) > \nu^{\infty}_{\ddot{K}(\ddot{e})}(\vartheta,\psi)$.

Definition 22. Let $\wp = (V, \varepsilon)$ be a PFSG on V if there exists a strong arc between the vertices ϑ and ψ in $\wp$, then ϑ dominates ψ. Clearly,

I. For every $\vartheta, \psi \in V$. Domination is a symmetric relation on V. If ϑ dominates ψ and ψ dominates ϑ.

II. For every $\psi \in V$, the neighborhood of $\vartheta \in V$ that is dominated by ψ.

III. If $\mu_{\ddot{K}(\ddot{e})}(\vartheta,\psi) < \mu^{\infty}_{\ddot{K}(\ddot{e})}(\vartheta,\psi)$, $\eta_{\ddot{K}(\ddot{e})}(\vartheta,\psi) < \eta^{\infty}_{\ddot{K}(\ddot{e})}(\vartheta,\psi)$, $\nu_{\ddot{K}(\ddot{e})}(\vartheta,\psi) < \nu^{\infty}_{\ddot{K}(\ddot{e})}(\vartheta,\psi)$ for all $\vartheta, \psi \in V$, $\ddot{e} \in \hat{O}$ then the Đ-set of $\wp$ is V.

Definition 23. A subset $\mathbb{S}$ of V is a Đ-set in $\ddot{G}$ if for all $\psi \in V - \mathbb{S}$ there exist vertex $\vartheta \in \mathbb{S}$, such that ϑ dominates ψ for all $\vartheta, \psi \in V$ and $\ddot{e} \in \hat{O}$.

Definition 24. If no proper subset of $\mathbb{S}$ is a Đ-set for all $\vartheta, \psi \in V$ and $\ddot{e} \in \hat{O}$, then Đ-set $\mathbb{S}$ of a PFSG $\wp = (V, \varepsilon)$ is Minimal Dominating Set (MĐ-set).

Definition 25. The lowest cardinality amongst all MĐ-set in $\wp$ is known as the lower ĐN of $\wp$ and is represented by $\sum_{\ddot{e} \in \hat{O}}(\mathscr{Y}_p(\wp))$ for all $\ddot{e} \in \hat{O}$ and $\vartheta, \psi \in V$.

Definition 26. The greatest cardinality amongst all MĐ-set in $\wp$ is known as the upper ĐN of $\wp$ and is represented by $\sum_{\ddot{e} \in \hat{O}}(\Gamma_p(\wp))$ for all $\ddot{e} \in \hat{O}$ and $\vartheta, \psi \in V$.

Example 3. Consider the PFSG in *Fig. 1* as an example.

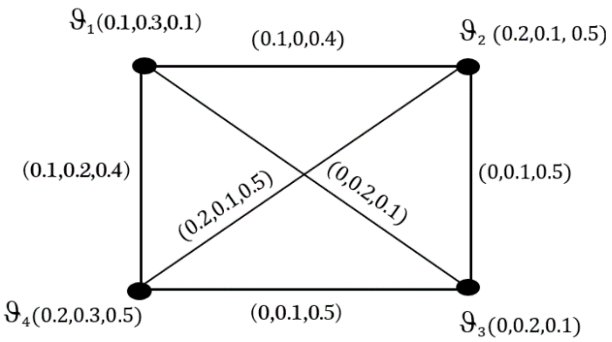

Fig. 4. Domination in PFSG using strong arcs.

The minimum Đ-sets are $\mathbb{S}_1 = \{\vartheta_1, \vartheta_2\}$, $\mathbb{S}_2 = \{\vartheta_2, \vartheta_3\}$, $\mathbb{S}_3 = \{\vartheta_3, \vartheta_4\}$, $\mathbb{S}_4 = \{\vartheta_1, \vartheta_4\}$.

The lowest cardinality amongst all MĐ-set is in $\wp$ is $|\mathbb{S}_2|$.

The lower ĐN of $\wp$ is $\sum_{\ddot{e} \in \hat{O}}(\mathscr{Y}_p(\wp)) = |\mathbb{S}_2| = 0.65$.

The greatest cardinality amongst all MĐ-set is in $\wp$ is $|\mathbb{S}_9|$.

The upper ĐN of $\wp$ is $\sum_{\ddot{e} \in \hat{O}}(\Gamma_p(\wp)) = |\mathbb{S}_4| = 0.77$.

Definition 27. An edge (ϑ, ψ) is an effective edge in a PFSG if $\mu_{\ddot{K}(\ddot{e})}(\vartheta, \psi) = \wedge(\mu_{F(\ddot{e})}(\vartheta), \mu_{F(\ddot{e})}(\psi))$, $\eta_{\ddot{K}(\ddot{e})}(\vartheta, \psi) = \wedge(\eta_{F(\ddot{e})}(\vartheta), \eta_{F(\ddot{e})}(\psi))$ and $\nu_{\ddot{K}(\ddot{e})}(\vartheta, \psi) = \vee(\nu_{F(\ddot{e})}(\vartheta), \nu_{F(\ddot{e})}(\psi))$.

Definition 28. A vertex $\mathfrak{A} \in V$ of PFSG $\wp = (V, \varepsilon)$ is an isolated vertex if $\mu_{\ddot{K}(\ddot{e})}(\vartheta, \psi) = 0$, $\eta_{\ddot{K}(\ddot{e})}(\vartheta, \psi) = 0$, $\nu_{\ddot{K}(\ddot{e})}(\vartheta, \psi) = 0$. As a result, an isolated vertex dominates no other vertex in $\wp$.

Definition 29. If no strong arc exists between the vertices $\vartheta, \psi \in V$ of the PFSG $\wp = (V, \varepsilon)$, then vertices ϑ, ψ are known as independent.

Definition 30. If a subset $\mathbb{S}$ of V in a PFSG is an independent set of $\wp$ then $\mu_{\ddot{K}(\ddot{e})}(\vartheta, \psi) < \mu^{\infty}_{\ddot{K}(\ddot{e})}(\vartheta, \psi\ \eta_{\ddot{K}(\ddot{e})})$, $(\vartheta, \psi) < \eta^{\infty}_{\ddot{K}(\ddot{e})}(\vartheta, \psi)$, $\nu_{\ddot{K}(\ddot{e})}(\vartheta, \psi) < \nu^{\infty}_{\ddot{K}(\ddot{e})}(\vartheta, \psi)$ for all $\vartheta, \psi \in \mathbb{S}$, $\ddot{e} \in \hat{O}$.

Remark 1: If and only if there are no adjacent vertices in $\mathbb{S}$, then it is an independent set of $\wp$.

Definition 31. An independent set $\mathbb{S}$ of a PFSG $\wp = (V, \varepsilon)$ is known as a Maximal Independent Set (MXI-set) if for each $\psi \in V - \mathbb{S}$, then $\mathbb{S} \cup \{\psi\}$ is not independent.

Definition 32. The upper independence number of $\wp$ is the maximal cardinality amongst all MXI-set in $\wp$ and is symbolized by $\sum_{\ddot{e} \in \hat{O}}(I_p(\wp))$ for all $\vartheta, \psi \in V$. The lower independence number of $\wp$ is the minimal cardinality amongst all MXI-set in $\wp$ and is symbolized by $\sum_{\ddot{e} \in \hat{O}}(i_p(\wp))$ for all $\vartheta, \psi \in V$.

4 | Integrity and Domination Integrity in PFSG

In this section, firstly, we proposed integrity and Domination Integrity (ĐÏ) in PFSG with examples and proved some theorems.

Definition 33. The integrity of PFSG $\wp = (\wp^*, F, \ddot{K}, \hat{O})$ is determined as $\ddot{I}(\wp) = \wedge\{|\mathbb{S}| + \mathfrak{M}(\wp - \mathbb{S})\}$, where $|\mathbb{S}| = \sum_{\ddot{e}_i \in \hat{O}} \left| \sum_{\vartheta \in V} \left(\frac{1 + 2\left(\mu_{F(\ddot{e}_i)}(\vartheta)\right) + \left(\frac{\eta_{F(\ddot{e}_i)}(\vartheta)}{2}\right) - \nu_{F(\ddot{e}_i)}(\vartheta)}{3} \right) \right|$ represents the cardinality of $\mathbb{S}$, and $\mathfrak{M}(\wp - \mathbb{S}) = \sum_{\ddot{e}_i \in \hat{O}} \left| \sum_{\vartheta \in V(\wp - \mathbb{S})} \left(\frac{1 + 2\left(\mu_{F(\ddot{e}_i)}(\vartheta)\right) + \left(\frac{\eta_{F(\ddot{e}_i)}(\vartheta)}{2}\right) - \nu_{F(\ddot{e}_i)}(\vartheta)}{3} \right) \right|$ denotes the order of the greatest component of $\wp - \mathbb{S}$.

Definition 34. An integrity set (Ï- set) of PFSG $\wp = (\wp^*, F, \ddot{K}, \hat{O})$ is a subset $\mathbb{S}$ of V in $\wp$ for which $\ddot{I}(\wp) = \wedge\{|\mathbb{S}| + \mathfrak{M}(\wp - \mathbb{S})\}$.

Example 4. Consider *Fig. 3*, amongst all the subsets of $\mathbb{S}$, $\{\vartheta_2\}$ is a Ï - set of $\wp$. The integrity of $\wp$ is $\ddot{I}(\wp) = \{0.32 + 1.10\} = 1.42$.

Definition 35. The ĐÏ of PFSG $\wp = (\wp^*, F, \ddot{K}, \hat{O})$ is determined as $Đ\ddot{I}(\wp) = \wedge\{|\mathbb{S}| + \mathfrak{M}(\wp - \mathbb{S})\}$ and $\mathbb{S}$ is a Đ-set of $\wp$, where $|\mathbb{S}| = \sum_{\ddot{e}_i \in \hat{O}} \left| \sum_{\vartheta \in V} \left(\frac{1 + 2\left(\mu_{F(\ddot{e}_i)}(\vartheta)\right) + \left(\frac{\eta_{F(\ddot{e}_i)}(\vartheta)}{2}\right) - \nu_{F(\ddot{e}_i)}(\vartheta)}{3} \right) \right|$ represents the cardinality of $\mathbb{S}$, and $\mathfrak{M}(\wp - \mathbb{S}) = \sum_{\ddot{e}_i \in \hat{O}} \left| \sum_{\vartheta \in V(\wp - \mathbb{S})} \left(\frac{1 + 2\left(\mu_{F(\ddot{e}_i)}(\vartheta)\right) + \left(\frac{\eta_{F(\ddot{e}_i)}(\vartheta)}{2}\right) - \nu_{F(\ddot{e}_i)}(\vartheta)}{3} \right) \right|$ denotes the order of the largest component of $\wp - \mathbb{S}$.

Definition 36. A ĐÏ-set of $\wp = (\wp^*, F, \ddot{K}, \hat{O})$ is a subset $\mathbb{S}$ of V in $\wp$ and is defined by $Đ\ddot{I}(\wp) = \wedge\{|\mathbb{S}| + \mathfrak{M}(\wp - \mathbb{S})\}$.

Example 5. From the *Example 3*, To find $Đ\ddot{I}(\wp)$ for the following MĐ-sets:

$\mathbb{S}_1 = \{\vartheta_1, \vartheta_2\}, \mathbb{S}_2 = \{\vartheta_2, \vartheta_3\}, \mathbb{S}_3 = \{\vartheta_3, \vartheta_4\}, \mathbb{S}_4 = \{\vartheta_1, \vartheta_4\}$.

From the subsets, $\mathbb{S}_2$ is a ĐÏ set, $Đ\ddot{I}(\wp) = \{0.65 + 0.77\} = 1.4$.

Example 6. In *Fig. 4*, $\{(\vartheta_2, \vartheta_4)\}$ is the Đ-set which corresponds to ë.

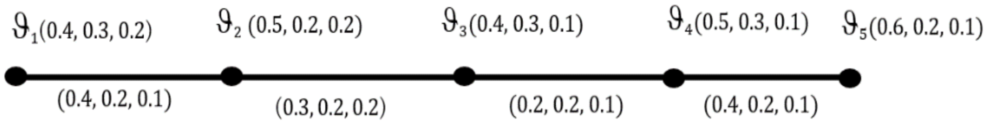

Fig. 5. **$\mathbb{Z}$(ë).**

Table 1 and *Table 2* show the values of integrity and ĐÏ of $\mathbb{Z}$**(**ë**).**

Table 1. Integrity of Z(ë).

$\mathbb{S}$	$\lvert\mathbb{S}\rvert$	$\mathfrak{M}(\wp - \mathbb{S})$	Ï($\wp$)
$\{\vartheta_3\}$	0.62	1.42 for $\{\vartheta_4, \vartheta_5\}$	1.84

Table 2. ĐÏ Integrity of Z(ë).

$\mathbb{S}$	$\lvert\mathbb{S}\rvert$	$\mathfrak{M}(\wp - \mathbb{S})$	ĐÏ ($\wp$)
$\{\vartheta_2, \vartheta_4\}$	1.32	0.73 for $\{\vartheta_5\}$	2.05

In the crisp graph, Ï($\wp$) ≤ ĐÏ($\wp$).

Definition 37. Let $\wp$ = ($\wp$*, Ғ, Ḱ, Ô) be a PFSG. If a subset has at least one end of each strong arc of $\wp$, then the subset $\mathbb{S} \subset V(\wp)$ is known as a vertex covering of $\wp$. A minimal vertex covering $\mathbb{S}$ of $\wp$ is one in which no subset of $\mathbb{S}$ is a vertex covering. The vertex covering number of $\wp$ is the least cardinality of all minimal vertex coverings of $\wp$ and is symbolized by $\Sigma_{ë_i \in Ô}(C_p(\wp))$.

Theorem 1. Let $\wp$ be a PFSG. Then $\Sigma_{ë_i \in Ô}\left(I_p(\wp)\right) + \Sigma_{ë_i \in Ô}\left(C_p(\wp)\right) = |V(\wp)|$.

Proof: Suppose $\mathbb{S}$ is an MXI-set of a PFSG $\wp$ and minimum vertex covering of $\wp$. Thus, $\Sigma_{ë_i \in Ô}\left(I_p(\wp)\right) + \Sigma_{ë_i \in Ô}\left(C_p(\wp)\right) = |V(\wp)|$.

Definition 38. A PFSG $\wp$ is called strong arc PFSG if every arc in $\wp$ is strong.

Theorem 2. Let $\wp$ be strong arc PFSG, Ï($\wp$) ≤ ĐÏ($\wp$) ≤ $|V(\wp)|$. Also Ï($\wp$) ≤ ĐÏ($\wp$) ≤ $|V(\wp)| - \Sigma_{ë_i \in Ô}\left(C_p(\wp)\right) + 1$.

Proof: Each arc in a strong PFSG is a strong arc. Thus, Ï($\wp$) ≤ ĐÏ($\wp$). The induced graph $\wp - \mathbb{S}$ is an independent set if $\mathbb{S}$ is vertex covering in $\wp$. We get independent vertices by the removal of $\mathbb{S}$. That is, $\mathfrak{M}(\wp - \mathbb{S}) = 1$. Hence, Ï($\wp$) ≤ ĐÏ($\wp$) ≤ $|V(\wp)| - \Sigma_{ë_i \in Ô}\left(C_p(\wp)\right) + 1$.

Theorem 3. For any PFSG, $\Sigma_{ë_i \in Ô}\left(d_p(\wp)\right) \leq$ ĐÏ($\wp$).

Proof: The Đ-set $\mathbb{S}$ and $\mathfrak{M}(\wp - \mathbb{S})$ of a PFSG $\wp$ determine its ĐÏ number. Then $\Sigma_{ë_i \in Ô}\left(d_p(\wp)\right) <$ ĐÏ($\wp$). The equality is valid when a PFSG's vertices are all present. Hence $\Sigma_{ë_i \in Ô}\left(d_p(\wp)\right) \leq$ ĐÏ($\wp$).

Theorem 4. For every strong arc PFSG, $\delta s(\wp) + 1 \leq$ Ï($\wp$) ≤ ĐÏ($\wp$).

Proof: Let $\wp$ be a strong PFSG and $\mathbb{S} \subset V(\wp)$. Let $|d_s(\vartheta)| = \delta_s(\wp)$, where $\vartheta \in V(\wp)$ is the minimum strong degree vertex of $\wp$. We get, $\mathfrak{M}(\wp - \mathbb{S}) \geq 1$ by removing the vertices in $\mathbb{S}$ from $\wp$. Hence, $\delta s(\wp) + 1 \leq$ Ï($\wp$).

Theorem 5. Let H = ($\wp_1$*, $Ғ_1$, $Ḱ_1$, $Ô_1$) be a PFSSG of $\wp$ = ($\wp_2$*, $Ғ_2$, $Ḱ_2$, $Ô_2$). Then Ï(H) ≤ Ï($\wp$).

Proof: Since H = ($\wp_1$*, $Ғ_1$, $Ḱ_1$, $Ô_1$) is a PFSSG of $\wp$ = ($\wp_2$*, $Ғ_2$, $Ḱ_2$, $Ô_2$), $|V(H)| \leq |V(\wp)|$, at least one vertex $\mathfrak{A} \in$ H. In comparison to $\wp$'s membership value, it has a lesser membership value. Or else, $|V(\wp)| \leq |V(H)|$). Furthermore, for each PFSSG H, Ï(H) ≤ $|H| < |\wp|$.

Suppose $\ddot{I}(\wp) > \ddot{I}(H)$ for an Ï- set $\mathbb{S}$ of $H \subset \wp$. Then $\mathfrak{M}(H - \mathbb{S}) < \ddot{I}(\wp) - |\mathbb{S}|$. Suppose $\mathbb{S}$ is Ï- set of $\wp$, then $\mathfrak{M}(H - \mathbb{S}) < \mathfrak{M}(\wp - \mathbb{S})$, which is not possible. Suppose $\mathbb{S}$ is not Ï- set of $\wp$, then $\ddot{I}(\wp) - |\mathbb{S}| < \mathfrak{M}(\wp - \mathbb{S})$; this contradicts our result.

Theorem 6. Let $\wp$ be a complete PFSG. Then $\ddot{I}(\wp) = |V(\wp)| = Đ\ddot{I}(\wp)$.

Proof: It is obvious that in a complete PFSG, all of the vertices are adjacent to one another. Once any subset $\mathbb{S}$ of vertices in $\ddot{G}$ have been eliminated, $\mathfrak{M}(\wp - \mathbb{S}) = |V(\wp)| - |\mathbb{S}|$.

Theorem 7. If $\wp$ is a strong PFSG and its complement $\wp^c$, then $\ddot{I}(\wp \cup \wp^c) = |V(\wp)|$.

Proof: Since $\wp$ is a strong PFSG and $\wp^c$ is a PFSG, then $\wp \cup \wp^c$ is a PFSG. Hence $\ddot{I}(\wp \cup \wp^c) = |V(\wp)|$.

Theorem 8. If $\wp_1$ and $\wp_2$ are two connected PFSGs and $\wp = \wp_1 \cup \wp_2$ with $|\wp_1| \geq |\wp_2|$, then $\ddot{I}(\wp) = \wedge \{|\wp_1|, \ddot{I}(\wp_1), |\mathbb{S}| + |V(\wp_2)|, |\mathbb{S}| + \vee \{\mathfrak{M}(\wp_1 - \mathbb{S}), \mathfrak{M}(\wp_2 - \mathbb{S})\}\}$.

Proof: Let $\wp_1$, $\wp_2$ be two connected PFSGs and $\wp = \wp_1 \cup \wp_2$ with $|\wp_1| \geq |\wp_2|$. Assume that $|\wp_1| > |\wp_2|$. Vertices in the Ï- set $\mathbb{S}$ of $\wp$ can be either from $\wp_1$ to $\wp_2$ or both or blank. Since $|\wp_1| \geq |\wp_2|$, $\mathbb{S}$ cannot contain vertices from $\wp_2$ alone.

Theorem 9. Let $\wp_1$ and $\wp_2$ be two connected PFSGs and $\wp = \wp_1 + \wp_2$ with $V_1 \cap V_2 \neq \emptyset$. Then $\ddot{I}(\wp) = \wedge\{\ddot{I}(\wp_1) + |V(\wp_2)|, \ddot{I}(\wp_2) + |V(\wp_1)|\}$.

Proof: Let $\wp_1$ and $\wp_2$ be complete PFSGs. Obviously, $\wp$ is a complete PFSG. Hence, $\ddot{I}(\wp) = \ddot{I}(\wp_1) + \ddot{I}(\wp_2) = \ddot{I}(\wp_1) + |V(\wp_2)| = |V(\wp_1)| + \ddot{I}(\wp_2)$.

If every vertex of $\wp_1$ in the Ï-set of $\wp$, then $\wp_2$ is connected and also a component, we know that each vertex from $\wp_1$ is connected with $\wp_2$ by an edge. Similarly, consider $\wp_2$. Furthermore, other subsets of $V(\wp)$, the remaining vertices of $\wp$ are in $\mathfrak{M}(\wp - \mathbb{S})$. $\ddot{I}(\wp) = \wedge\{\ddot{I}(\wp_1) + |V(\wp_2)|, \ddot{I}(\wp_2) + |V(\wp_1)|\}$.

5 | Application of PFSG Using ĐÏ

This section presents a decision-making method for the PFSG using the previously defined ĐÏ. Then, it suggests an algorithm for determining the ideal location for establishing a city diagnosis center. The cities ϑ_1, ϑ_2, ϑ_3, ϑ_4, ϑ_5, ϑ_6 are calculated as alternatives based on parameters $\ddot{e}_1$ = illness and symptoms, $\ddot{e}_2$ = economic circumstances, $\ddot{e}_3$ = density of population. Let V = $\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5, \vartheta_6\}$ be the universal set of six cities and $\hat{O} = \{\ddot{e}_1, \ddot{e}_2, \ddot{e}_3\}$ represent the parameter set that defines the city's risk. PFSS (Ꞙ, Ô) over V, which describes the impact of the disease in the cities corresponding to the parameters.

Table. 3. PFSS (Ꞙ, Ô) over V.

PFSS (Ꞙ, Ô) Over V	Ꞙ($\ddot{e}_1$)	Ꞙ($\ddot{e}_2$)	Ꞙ($\ddot{e}_3$)	Ꞙ($\ddot{e}$)
ϑ_1	(0.6,0.2,0.1)	(0.8,0.1,0.1)	(0.7,0.2,0.1)	(0.6,0.1,0.1)
ϑ_2	(0.5,0.2,0.3)	(0.7,0.2,0.1)	(0.6,0.2,0.2)	(0.5,0.2,0.3)
ϑ_3	(0.3,0.4,0.3)	(0.5,0.3,0.2)	(0.4,0.4,0.2)	(0.3,0.3,0.3)
ϑ_4	(0.4,0.3,0.2)	(0.4,0.2,0.4)	(0.4,0.3,0.3)	(0.4,0.2,0.4)
ϑ_5	(0.7,0.1,0.1)	(0.3,0.3,0.3)	(0.5,0.2,0.2)	(0.3,0.1,0.3)
ϑ_6	(0.8,0.1,0.1)	(0.6,0.2,0.2)	(0.7,0.2,0.1)	(0.6,0.1,0.2)

($\ddot{K}$, Ô) is a PFSS over $\varepsilon = \{\vartheta_1\vartheta_2, \vartheta_1\vartheta_3, \vartheta_1\vartheta_4, \vartheta_1\vartheta_5, \vartheta_1\vartheta_6, \vartheta_2\vartheta_3, \vartheta_2\vartheta_4, \vartheta_2\vartheta_5, \vartheta_2\vartheta_6, \vartheta_3\vartheta_4, \vartheta_3\vartheta_5, \vartheta_3\vartheta_6, \vartheta_4\vartheta_5, \vartheta_4\vartheta_6, \vartheta_5\}$ describe the membership of the relation between cities and the parameters $\ddot{e}_1$, $\ddot{e}_2$, and $\ddot{e}_3$.

Table 4. PFSS (K̈, Ô) over ε.

PFSS (K̈, Ô) Over ε	K̈ (ë₁)	K̈ (ë₂)	K̈ (ë₃)
$\vartheta_1\vartheta_2$	-	(0.7,0.1,0.1)	(0.5,0.1,0.2)
$\vartheta_1\vartheta_3$	(0.3,0.2,0.2)	-	-
$\vartheta_1\vartheta_4$	(0.4,0.2,0.2)	(0.4,0.1,0.4)	(0.3,0.2,0.2)
$\vartheta_1\vartheta_5$	(0.6,0.1,0.1)	(0.3,0.1,0.3)	-
$\vartheta_1\vartheta_6$	-	(0.6,0.1,0.2)	-
$\vartheta_2\vartheta_3$	-	(0.5,0.2,0.2)	(0.3,0.2,0.1)
$\vartheta_2\vartheta_4$	(0.4,0.2,0.3)	(0.4,0.2,0.3)	(0.3,0.2,0.2)
$\vartheta_2\vartheta_5$	(0.5,0.1,0.3)	(0.3,0.2,0.2)	(0.4,0.1,0.1)
$\vartheta_2\vartheta_6$	(0.5,0.1,0.2)	-	-
$\vartheta_3\vartheta_4$	-	(0.4,0.2,0.4)	-
$\vartheta_3\vartheta_5$	-	(0.3,0.2,0.3)	(0.3,0.2,0.1)
$\vartheta_3\vartheta_6$	(0.3,0.1,0.2)	(0.3,0.1,0.2)	(0.4,0.2,0.1)
$\vartheta_4\vartheta_5$	-	(0.3,0.2,0.4)	(0.3,0.2,0.2)
$\vartheta_4\vartheta_6$	(0.4,0.1,0.2)	(0.4,0.1,0.3)	-
$\vartheta_5\vartheta_6$	(0.7,0.1,0.1)	(0.3,0.1,0.3)	(0.5,0.2,0.2)

The PFSG **ℤ(ë₁)**, **ℤ(ë₂)**, and **ℤ(ë₃)**, which correspond to $ë_1$ = illness and symptoms, $ë_2$ = economic circumstances, $ë_3$ = density of population is expressed by the incidence matrices shown below:

$$\mathbb{Z}(ë_1)=\begin{bmatrix} - & - & (0.3,0.2,0.2) & (0.4,0.2,0.2) & (0.6,0.1,0.1) & - \\ - & - & - & (0.4,0.2,0.3) & (0.5,0.1,0.3) & (0.5,0.1,0.2) \\ (0.3,0.2,0.2) & - & - & - & - & (0.3,0.1,0.2) \\ (0.4,0.2,0.2) & (0.4, 0.2, 0.3) & - & - & - & (0.4,0.1,0.2) \\ (0.6,0.1,0.1) & (0.5, 0.1, 0.3) & - & - & - & (0.7,0.1,0.1) \\ - & (0.5, 0.1, 0.2) & (0.3,0.1,0.2) & (0.4,0.1,0.2) & (0.7,0.1,0.1) & - \end{bmatrix},$$

$$\mathbb{Z}(ë_2)=\begin{bmatrix} - & (0.7,0.1,0.1) & - & (0.4,0.1,0.4) & (0.3,0.1,0.3) & (0.6,0.1,0.2) \\ (0.7, 0.1, 0.1) & - & (0.5,0.2,0.2) & (0.4,0.2,0.3) & (0.3,0.2,0.2) & - \\ - & (0.5,0.2,0.2) & - & (0.4,0.2,0.4) & (0.3,0.2,0.3) & (0.3,0.1,0.2) \\ (0.4, 0.1, 0.4) & (0.4,0.2,0.3) & (0.4,0.2,0.4) & - & (0.3,0.2,0.4) & (0.4,0.1,0.3) \\ (0.3, 0.1, 0.3) & (0.3,0.2,0.2) & (0.3,0.2,0.3) & (0.3,0.2,0.4) & - & (0.3,0.1,0.3) \\ (0.6,0.1,0.2) & - & (0.3,0.1,0.2) & (0.4,0.1,0.3) & (0.3,0.1,0.3) & - \end{bmatrix},$$

$$\mathbb{Z}(ë_3)=\begin{bmatrix} - & (0.5, 0.1, 0.2) & - & (0.3,0.2,0.2) & - & - \\ (0.5, 0.1, 0.2) & - & (0.3,0.2,0.1) & (0.3,0.2,0.2) & (0.4,0.1,0.1) & - \\ - & (0.3,0.2,0.1) & - & - & (0.3,0.2,0.1) & (0.4,0.2,0.1) \\ (0.3, 0.2, 0.2) & (0.3,0.2,0.2) & - & - & (0.3,0.2,0.2) & - \\ - & (0.4,0.1,0.1) & (0.3,0.2,0.1) & (0.3,0.2,0.2) & - & (0.5,0.2,0.2) \\ - & - & (0.4,0.2,0.1) & - & (0.5,0.2,0.2) & - \end{bmatrix}.$$

We get, the resultant PFSG **ℤ(ë)**, where $ë = ë_1 \cap ë_2 \cap ë_3$. The decision matrix of PFSG is

$$\mathbb{Z}(ë)=\begin{bmatrix} - & - & - & (0.3,0.1,0.4) & - & - \\ - & - & - & (0.3,0.2,0.3) & (0.3,0.1,0.3) & - \\ - & - & - & - & - & (0.3,0.1,0.2) \\ (0.3, 0.1, 0.4) & (0.3,0.2,0.3) & - & - & - & - \\ - & (0.3,0.1,0.3) & - & - & - & (0.3,0.1,0.3) \\ - & - & (0.3,0.1,0.2) & - & (0.3,0.1,0.3) & - \end{bmatrix}.$$

The PFSG ℤ(ë $_j$) of PFSG ℘ = (₣, K̈, Ô) which corresponds to ë$_j$ for j = 1, 2, 3 are shown in *Figs. 6-9.*

Table 5. ĐÏ **of** ℤ(ë).

	|𝕊|	𝔐 (ℤ(ë) − 𝕊)	ĐÏ(ℤ(ë))
$\{\vartheta_4, \vartheta_6\}$	1.18	0.72 for $\{\vartheta_1\}$	1.90

The set $\{\vartheta_4, \vartheta_6\}$ is lowest cardinality Đ-set in ℤ(ë) and the corresponding minimum value of ĐÏ (ℤ(ë)) is 1.90 which is shown in *Table 5*. Therefore, cities ϑ_4 and ϑ_6 are the best places to locate a diagnosis center.

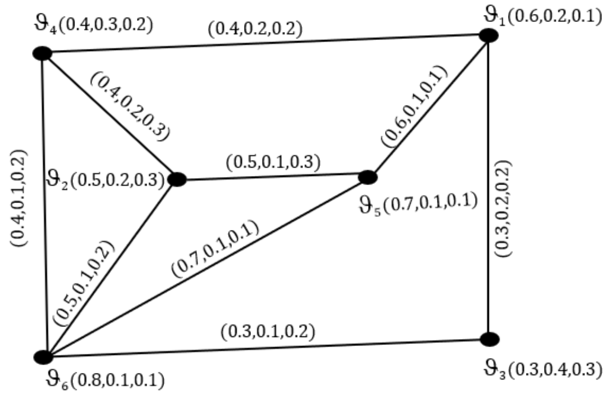

Fig. 6. ℤ(ë$_1$).

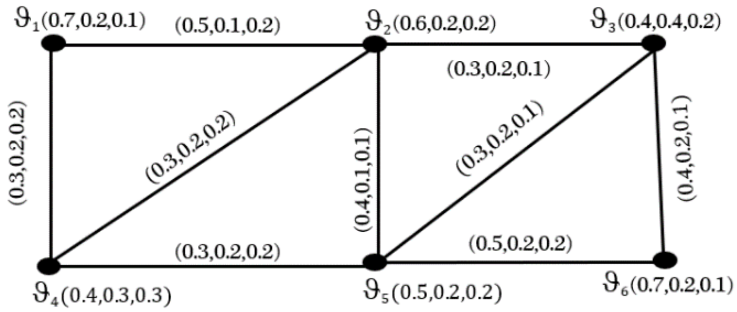

Fig. 7. ℤ(ë$_3$).

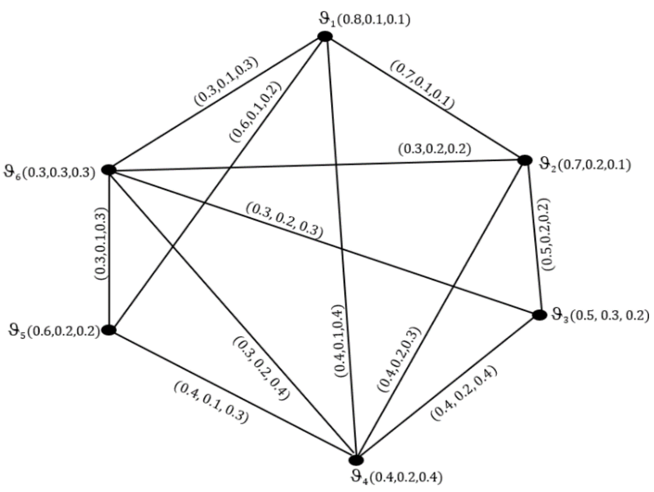

Fig. 8. ℤ(ë$_2$).

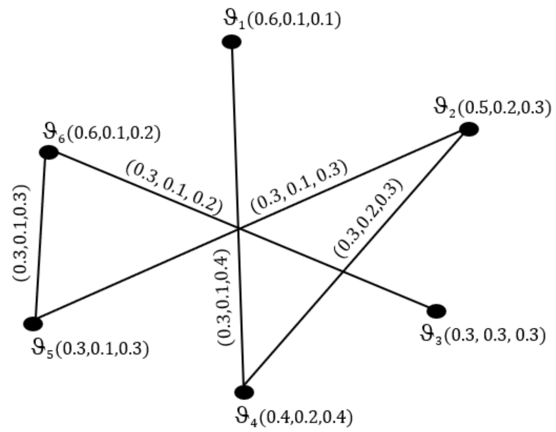

Fig. 9. $\mathbb{Z}$**(ë).**

Our approach is presented as an algorithm applied in the application section.

5.1 | Algorithm

Step 1. Enter the parameter sets.

Step 2. Enter the PFSSs (₣, Ô) and (Ḱ, Ô).

Step 3. Draw the PFSG Ġ = (Ġ*, ₣, Ḱ, Ô).

Step 4. Calculate the resulting PFSG $\mathbb{Z}$**(ë)** = $\bigwedge_j$ ë$_j$ for all j.

Step 5. Construct the PFSG $\mathbb{Z}$**(ë)** using the incidence matrix.

Step 6. Calculate and choose a minimum of ĐÏ($\mathbb{Z}$**(ë)**).

6 | Conclusion

PFSGs can become computationally expensive and complex, especially when dealing with large datasets or intricate structures. The algorithms and methods for analyzing these graphs may require significant computational resources. Picture fuzzy soft graphs can effectively represent uncertainty and imprecision inherent in real-world data. The PFSs allow for a more flexible and nuanced representation of degrees of membership, providing a robust way to model uncertainty. The concept of modeling real-life situations using graph theory is more helpful in structuring the problem. The vague details can be structured more than the normal graph theory using FGs. PFSG is a combination of PFSs and graph theory. This manuscript examines the concepts of novel cardinality lower and upper domination numbers of PFSG and introduces integrity and ĐÏ of PFSGs with examples and some properties. Using this concept, a numerical example is provided to demonstrate the real-world application. The direction of future works of this study may be further focused on single-valued neutrosophic graphs and signed FGs.

Author Contribution

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Conflicts of Interest

The authors declare no conflict of interest.

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