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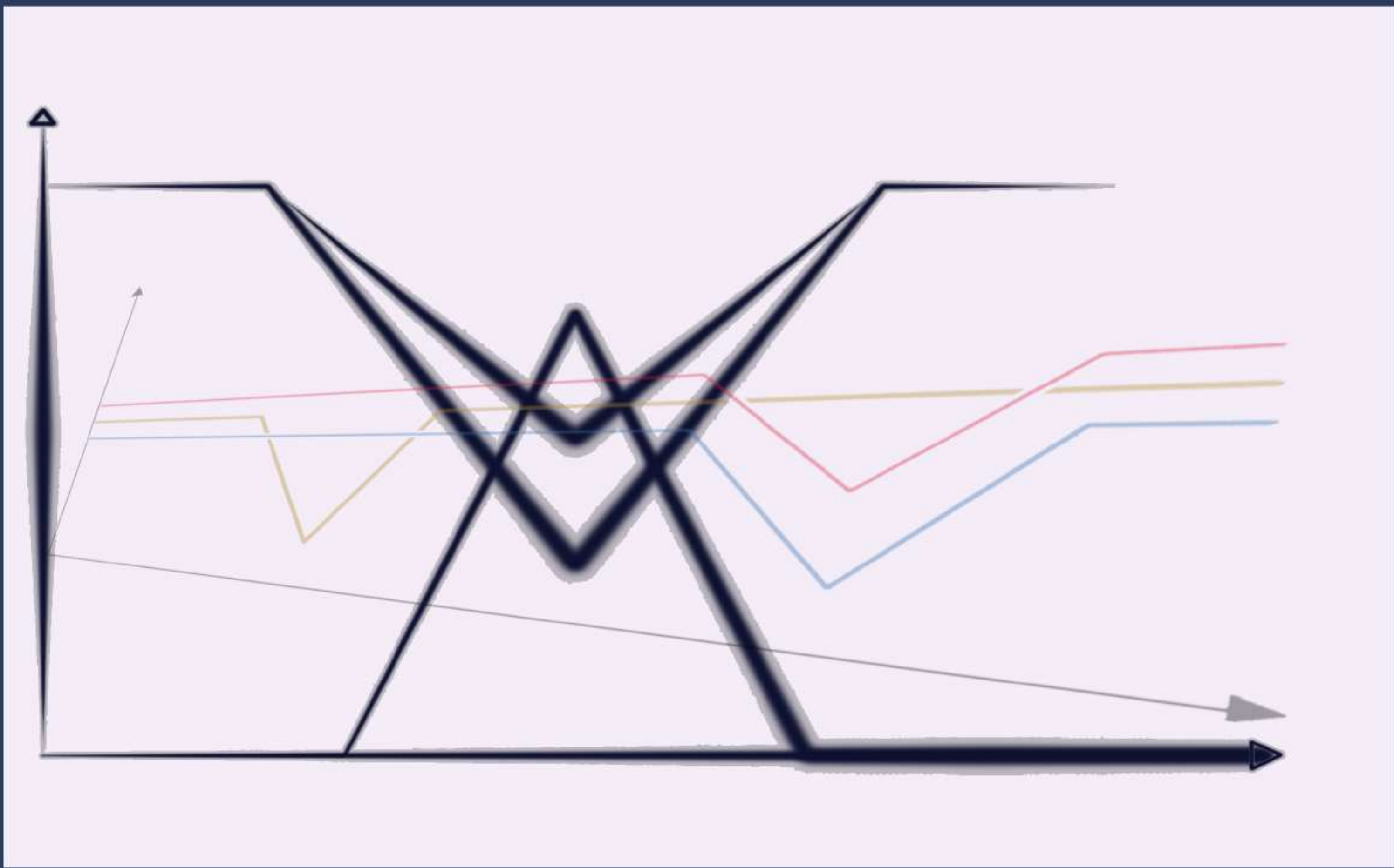

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The Picture Fuzzy Distance Measure in Controlling Network Power Consumption

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Abstract

In order to solve the complex decision-making problems, there are many approaches and systems based on the fuzzy theory were proposed. In 1998, Smarandache [10] introduced the concept of single-valued neutrosophic set as a complete development of fuzzy theory. In this paper, we research on the distance measure between single-valued neutrosophic sets based on the H-max measure of Ngan et al. [8]. The proposed measure is also a distance measure between picture fuzzy sets which was introduced by Cuong [15]. Based on the proposed measure, an Adaptive Neuro Picture Fuzzy Inference System (ANPFIS) is built and applied to the decision making for the link states in interconnection networks. In experimental evaluation on the real datasets taken from the UPV (Universitat Politècnica de València) university, the performance of the proposed model is better than that of the related fuzzy methods.

Keywords: Neutrosophic set, Picture fuzzy set, Distance measure, Decision making, Interconnection network, Power consumption.

1 | Introduction

The fuzzy theory was introduced the first time in 1965 by Zadeh [1]. A fuzzy set is determined by a membership function limited to [0, 1]. Until now, there is a giant research construction of fuzzy theory as well as its application. The fuzzy set is used in pattern recognition, artificial intelligent, decision making, or data mining [2] and [3], and so on. Besides that, the expansion of fuzzy theory is also an interesting topic. The interval-valued fuzzy set [4], the type-2 fuzzy set [5], and the intuitionistic fuzzy set [6] are all developed from the fuzzy set. They replaced the value type or added the other evaluation to the fuzzy set in order to overcome the inadequate simple approach of this traditional fuzzy set. Such as in 1986, the intuitionistic fuzzy set of Atanassov [6] builds up the concept of the non-membership degree. This supplement gives more accurate results in pattern recognition, medical diagnosis and decision making [7]-[9], and so on. In 1998, Smarandache [10] introduced

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neutrosophic set to generalize intuitionistic fuzzy set by three independent components. Until today, many subclasses of neutrosophic sets were studied such as complex neutrosophic sets [11] and [12]. As a particular case of standard neutrosophic sets [13] and [14], the picture fuzzy set introduced in 2013 by Cuong [15], considered as a complete development of the fuzzy theory, allows an element to belong to it with three corresponding degrees where all of these degrees and their sum are limited to [0, 1]. Concerning extended fuzzy set, some recent publications may be mentioned here as in [16]-[20].

As one of the important pieces of set theory, distance measure between the sets is a tool for evaluating different or similar levels between them. Some literature on the application of intuitionistic fuzzy measure from 2012 to present can be found in [7], [21]-[23]. In 2018, Wei introduced the generalized Dice similarity measures for picture fuzzy sets [17]. However, the definition of Wei is without considering the condition related to order relation on picture fuzzy sets. In a decision-making model, a distance measure can be used to compare the similarities between the sets of attributes of the samples and that of the input, such as in predicting dental diseases from images [24]. In this paper, we define the concept of the single-valued neutrosophic distance measure, picture fuzzy distance measure, and represent the specific measure formula. We prove the characteristics of this formula as well as the relation among it and some of the other operators of picture fuzzy sets. The proposed distance measure is inspired by the H-max distance measure of intuitionistic fuzzy sets [8]. Hence, it inherits the advantage of the cross-evaluation in the H-max and moreover it has the completeness of picture fuzzy environment.

The decision-making problems appear in most areas aiming to provide the optimal solution. Saving interconnection network power is always interested, researched and becoming more and more urgent in the current technological era. In 2010, Alonso et al. introduced the power saving mechanism in regular interconnection network [25]. This decision-making model dynamically increases or reduces the number of links based on a thresholds policy. In 2015, they continue to study power consumption control in fat-tree interconnection networks based on the static and dynamic thresholds policies [26]. In general, these threshold policies are rough and hard because they are without any fuzzy approaches, parameter learning and optimizing processes. In 2017, Phan et al. [27] proposed a new method in power consumption estimation of network-on-chip based on fuzzy logic. However, this fuzzy logic system based on Sugeno model [27] is too rudimentary and the parameters here are chosen according to the authors' quantification.

In this paper, aiming to replace the above threshold policy, an Adaptive Neuro Picture Fuzzy Inference System (ANPFIS) based on picture fuzzy distance measure is proposed to make the decisions for the link states in interconnection networks. ANPFIS is a modification and combination between Adaptive Neuro Fuzzy Inference System (ANFIS) [28]-[30], picture fuzzy set, and picture fuzzy distance measure. Hence, ANPFIS operates based on the picture fuzzification and defuzzification processes, the picture fuzzy operators [18] and distance measure, and the learning capability for automatic picture fuzzy rule generation and parameter optimization. In order to evaluate performance, we tested the ANPFIS method on the real datasets of the network traffic history taken from the UPV (Universitat Politècnica de València) university with related methods. The result is that ANPFIS is the most effective algorithm.

The rest of the paper is organized as follows. Section 2 provides some fundamental concepts of the fuzzy, intuitionistic fuzzy, single-valued neutrosophic, and picture fuzzy theories. Section 3 proposes the distance measure of single-valued neutrosophic sets and points out its important properties. Section 4 shows the new decision-making method named Adaptive Neuro Picture Fuzzy Inference System (ANPFIS) and an application of ANPFIS to controlling network power consumption. Section 5 shows the experimental results of ANPFIS and the related methods on real-world datasets. Finally, conclusion is given in Section 6.

2| Preliminary

In this part, some concepts of the theories of fuzzy sets, intuitionistic fuzzy sets, single-valued neutrosophic sets, and picture fuzzy sets are showed.

Let X be a space of points.

Definition 1. [1]. A fuzzy set (FS) A in X,

$$A=\{(x:\mu_A(x))|x\in X\}, \tag{1}$$

is characterized by a membership function, μ_A, with a range in $\lfloor 0,1\rfloor$.

Definition 2. [6]. A Intuitionistic Fuzzy Set (IFS) A in X,

$$A=\{(x:\mu_A(x),\nu_A(x))|x\in X\}, \tag{2}$$

is characterized by a membership function μ_A and a non-membership function ν_A with a range in $\lfloor 0,1\rfloor$ such that $0\leq \mu_A+\nu_A\leq 1$.

Definition 3. [31]. A Single-Valued Neutrosophic Set (SVNS) A in X,

$$A=\{(x:T_A(x),I_A(x),F_A(x))|x\in X\}, \tag{3}$$

is characterized by a truth-membership function T_A, an indeterminacy-membership function I_A, and a false-nonmembership function T_A with a range in $\lfloor 0,1\rfloor$ such that $0\leq T_A+I_A+F_A\leq 3$.

Definition 4. [15]. A Picture Fuzzy Set (PFS) A in X,

$$A=\{(x:\mu_A(x),\eta_A(x),\nu_A(x))|x\in X\}, \tag{4}$$

is characterized by a positive membership function μ_A, a neutral function η_A, and a negative membership function ν_A with a range in $\lfloor 0,1\rfloor$ such that $0\leq \mu_A+\eta_A+\nu_A\leq 1$.

We denote that SVNS(X) is the set of all SVNSs in X and PFS(X) is the set of all PFSs in X. We consider the sets N^* and P^* defined by

$$N^*=\{x=(x_1,x_2,x_3)|0\leq x_1,x_2,x_3\leq 1\}, \tag{5}$$

$$P^*=\{x=(x_1,x_2,x_3)|0\leq x_1,x_2,x_3,x_1+x_2+x_3\leq 1\}. \tag{6}$$

Definition 5. The orders on N^* and P^* are defined as follows

$$x \le y \Leftrightarrow (x_1 < y_1, x_3 \ge y_3) \vee (x_1 = y_1, x_3 > y_3) \vee (x_1 = y_1, x_3 = y_3, x_2 \le y_2),\ \forall x, y \in P^*,\ [19].$$

$$x \ll y \Leftrightarrow x_1 \le y_1, x_2 \le y_2, x_3 \ge y_3, \forall x, y \in N^*.$$

Clearly, on P^*, if $x \ll y$ then $x \le y$.

Remark 1. The lattice $(P^*, \le)$ is a complete lattice [19] but $(P^*, \ll)$ is not. For example, let $x = (0.2, 0.3, 0.5)$ and $y = (0.3, 0, 0.7)$, then there is not any supremum value of x and y on $(P^*, \ll)$.

Otherwise, we have $sup(x, y) = (0.3, 0, 0.5)$ on the lattice $(P^*, \le)$. We denote the units of $(P^*, \le)$ as follows $0_{P^*} = (0, 0, 1)$ and $1_{P^*} = (1, 0, 0)$ [19]. It is easy to see that 0_{P^*} and 1_{P^*} are also the units on $(P^*, \ll)$. Now, some logic operators on PFS(X) are presented.

Definition 6. [19]. A picture fuzzy negation N is a function satisfying

$$N : P^* \to P^*,\ N(0_{P^*}) = 1_{P^*},\ N(1_{P^*}) = 0_{P^*},\ \text{and}\ N(x) \ge N(y) \Leftrightarrow x \le y.$$

Example 1. For every $x \in P^*$, then $N_0(x) = (x_3, 0, x_1)$ and $N_S(x) = (x_3, x_4, x_1)$ are picture fuzzy negations, where $x_4 = 1 - x_1 - x_2 - x_3$.

Remark 2. The operator N_0 also satisfies $N_0(x) \gg N_0(y) \Leftrightarrow x \ll y, \forall x, y \in P^*$.

Now, let $x, y, z \in P^*$ and $I(x) = \{y \in P^* : y = (x_1, y_2, x_3), 0 \le y_2 \le x_2\}$.

Definition 7. [19]. A picture fuzzy t-norm T is a function satisfying

$$T : P^* \times P^* \to P^*,\ T(x, y) = T(y, x),\ T(T(x, y), z) = T(x, T(y, z)),\ T(1_{P^*}, x) \in I(x),\ \text{and}$$
$$T(x, y) \le T(x, z), \forall y \le z.$$

Definition 8. [19]. A picture fuzzy t-conorm S is a function satisfying

$$S : P^* \times P^* \to P^*,\ S(x, y) = S(y, x),\ S(S(x, y), z) = S(x, S(y, z)),\ S(0_{P^*}, x) \in I(x),\ \text{and}$$
$$S(x, y) \le S(x, z), \forall y \le z.$$

Example 2. For all $x, y \in P^*$, the following operators are the picture fuzzy t-norms:

$$T_0(x, y) = (min(x_1, y_1), min(x_2, y_2), max(x_3, y_3)).$$

$$T_1(x,y)=(x_1y_1,x_2y_2,x_3+y_3-x_3y_3).$$

$$T_2(x,y)=(max(0,x_1+y_1-1),max(0,x_2+y_2-1),min(1,x_3+y_3)).$$

$$T_3(x,y)=(max(0,x_1+y_1-1),max(0,x_2+y_2-1),x_3+y_3-x_3y_3).$$

$$T_4(x,y)=(x_1y_1,max(0,x_2+y_2-1),x_3+y_3-x_3y_3).$$

$$T_5(x,y)=(max(0,x_1+y_1-1),x_2y_2,x_3+y_3-x_3y_3).$$

Example 3. For all $x,y\in P^*$, the following operators are the picture fuzzy t-conorms:

$$S_0(x,y)=(max(x_1,y_1),max(0,x_2+y_2-1),min(x_3,y_3)).$$

$$S_1(x,y)=(x_1+y_1-x_1y_1,x_2y_2,x_3y_3).$$

$$S_2(x,y)=(min(1,x_1+y_1),max(0,x_2+y_2-1),max(0,x_3+y_3-1)).$$

$$S_3(x,y)=(x_1+y_1-x_1y_1,max(0,x_2+y_2-1),max(0,x_3+y_3-1)).$$

$$S_4(x,y)=(x_1+y_1-x_1y_1,max(0,x_2+y_2-1),x_3y_3).$$

$$S_5(x,y)=(x_1+y_1-x_1y_1,x_2y_2,max(0,x_3+y_3-1)).$$

Remark 3. For all $x,y,z\in P^*$ and $y\ll z$, the operators $T_{i(i=0,\ldots,5)}$ also satisfy the condition $T(x,y)\ll T(x,z)$. Similarly, $S_{i(i=0,\ldots,5)}$ also satisfy $S(x,y)\ll S(x,z)$.

The logic operators N,T and S on P^* are corresponding to the basic set-theory operators on PFS(X) as follows.

Definition 9. Let N, T and S be the picture fuzzy negation, t-norm and t-conorm, respectively, and $A,B\in$ PFS(X). Then, the complement of A w.r.t N is defined as follows:

$$\overline{A}^{N}=\{(x:N((\mu_A(x),\eta_A(x),\nu_A(x))))|x\in X\}, \tag{7}$$

the intersection of A and B w.r.t T is defined as follows:

$$A\cap_T B=\{(x:T((\mu_A(x),\eta_A(x),\nu_A(x)),(\mu_B(x),\eta_B(x),\nu_B(x))))|x\in X\}, \tag{8}$$

and the union of A and B w.r.t T is defined as follows:

$$A \cup_S B = \left\{ \left(x : S\left(\left(\mu_A(x), \eta_A(x), \nu_A(x) \right), \left(\mu_B(x), \eta_B(x), \nu_B(x) \right) \right) \right) \middle| x \in X \right\}. \tag{9}$$

3| The Single-valued Neutrosophic Distance Measure and the Picture Fuzzy Distance Measure

Recently, Wei has introduced the generalized Dice similarity measures for picture fuzzy sets [17]. However, the definition of Wei is without considering the condition related to order relation on picture fuzzy sets. The new distance measure on picture fuzzy sets is proposed in this section. It is developed from intuitionistic distance measure of Wang et al. [32] and Ngan et al. [8].

Definition 10. A single-valued neutrosophic distance measure d is a function satisfying

$$d : N^* \times N^* \to \left[0, +\infty \right),$$

$$d(x,y) = d(y,x),$$

$$d(x,y) = 0 \Leftrightarrow x = y,$$

If $x \ll y \ll z$ then $d(x,y) \le d(x,z)$ and $d(y,z) \le d(x,z)$.

Definition 11. A picture fuzzy distance measure d is a single-valued neutrosophic distance measure and $d(x,y) \in \left[0,1 \right], \forall x,y \in P^*$.

Definition 12. The measure D_0 is defined as follows

$$D_0(x,y) = \frac{1}{3}\left(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3| + \left| \max\{x_1, y_3\} - \max\{x_3, y_1\} \right| \right), \forall x, y \in P^*. \tag{10}$$

Proposition 1. The measure D_0 is a picture fuzzy distance measure.

Proof. Firstly, we have $\left| \max\{x_1, y_3\} - \max\{x_3, y_1\} \right| \in \left[0,1 \right]$ and

$$|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3| \le \left(|x_1| + |y_1| \right) + \left(|x_2| + |y_2| \right) + \left(|x_3| + |y_3| \right)$$
$$\le \left(x_1 + x_2 + x_3 \right) + \left(y_1 + y_2 + y_3 \right) \le 2.$$

Therefore, $0 \le \frac{1}{3}\left(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3| + \left| \max\{x_1, y_3\} - \max\{x_3, y_1\} \right| \right) \le 1.$

Secondly, we obtain that $D_0(x,y) = D_0(y,x)$ since D_0 has the symmetry property between the arguments.

Thirdly, $D_0(x,y) = 0 \Leftrightarrow |x_1 - y_1| = |x_2 - y_2| = |x_3 - y_3| = \left| \max\{x_1, y_3\} - \max\{x_3, y_1\} \right| = 0 \Leftrightarrow x = y.$

Finally, let $x \ll y \ll z$, then $x_1 \le y_1 \le z_1, x_2 \le y_2 \le z_2, x_3 \ge y_3 \ge z_3$. We obtain that

$$|x_1 - y_1| \le |x_1 - z_1|, |x_2 - y_2| \le |x_2 - z_2|, |x_3 - y_3| \le |x_3 - z_3|.$$

Moreover, $max\{z_1, x_3\} \ge max\{y_1, x_3\} \ge max\{x_1, y_3\} \ge max\{x_1, z_3\}$. Hence, $|max\{x_1, y_3\} - max\{x_3, y_1\}| \le |max\{x_1, z_3\} - max\{x_3, z_1\}|.\lim_{x\to\infty}$. Thus, $D_0(x, y) \le D_0(x, z)$. Similarly, we also have $D_0(y, z) \le D_0(x, z)$.

Remark 4. If d is a picture fuzzy distance measure, then d is a single-valued neutrosophic distance measure. The opposite is not necessarily true. Some picture fuzzy operations were introduced by the group of authors of this paper [18] and [19]. Hence, this research is seen as a complete link to the authors' previous work on picture fuzzy inference systems. An inference system of neutrosophic theory will be developed in another paper as a future work.

Proposition 2. Let $x, y \in P^*$. The measure D_0 satisfies the following properties:

$$D_0(N_0(x), N_S(x)) = \frac{1}{3}x_4.$$

If $x_2 \ge x_4$, then $|D_0(x, N_0(x)) - D_0(x, N_S(x))| = \frac{1}{3}x_4$.

$$|D_0(x, N_0(y)) - D_0(N_0(x), y)| = \frac{1}{3}|x_2 - y_2|.$$

$$|D_0(x, y) - D_0(N_0(x), N_0(y))| = \frac{1}{3}|x_2 - y_2|.$$

If $x_1 + x_3 = y_1 + y_3$, then $D_0(x, y) = D_0(N_S(x), N_S(y))$.

If $x_1 + x_3 = y_1 + y_3$, then $D_0(x, N_S(y)) = D_0(N_S(x), y)$.

$$D_0(x, N_0(x)) = |x_1 - x_3| + \frac{1}{3}x_2.$$

$$D_0(x, N_S(x)) = |x_1 - x_3| + \frac{1}{3}|x_2 - x_4|.$$

$$D_0(x, 1_{P^*}) = \frac{1}{3}(2 - 2x_1 + x_2 + x_3).$$

$$D_0(x, 0_{P^*}) = \frac{1}{3}(2 - 2x_3 + x_1 + x_2).$$

$$D_0(0_{P^*}, 1_{P^*}) = 1.$$

$D_0(x, N_0(x)) = 1$ if and only if $x \in \{0_{P^*}, 1_{P^*}\}$.

$D_0(x, N_S(x)) = 1$ if and only if $x \in \{0_{P^*}, 1_{P^*}\}$.

$D_0(x, N_0(x)) = 0$ if and only if $x_1 = x_3, x_2 = 0$.

$D_0(x, N_S(x)) = 0$ if and only if $x_1 = x_3, x_2 = x_4$.

$$D_0((0,0,0),(0,a,0)) < D_0((0,0,0),(a,0,0)) = D_0((0,0,0),(0,0,a))$$

$$< D_0((a,0,0),(0,0,a)) = D_0((a,0,0),(0,a,0)) = D_0((0,0,a),(0,a,0)), \forall a \in (0,1].$$

Proof. These properties are proved as follows:

We have $D_0(N_0(x), N_S(x)) = D_0((x_3, 0, x_1),(x_3, x_4, x_1))$

$$= \frac{1}{3}\left(|x_3 - x_3| + |0 - x_4| + |x_1 - x_1| + |max\{x_3, x_1\} - max\{x_3, x_1\}|\right) = \frac{1}{3}x_4.$$

We have

$$|D_0(x, N_0(x)) - D_0(x, N_S(x))| = |D_0((x_1, x_2, x_3),(x_3, 0, x_1)) - D_0((x_1, x_2, x_3),(x_3, x_4, x_1))|$$

$$= \left|\frac{1}{3}\left(|x_1 - x_3| + |x_2 - 0| + |x_3 - x_1| + |max\{x_1, x_1\} - max\{x_3, x_3\}|\right)\right.$$

$$\left.-\frac{1}{3}\left(|x_1 - x_3| + |x_2 - x_4| + |x_3 - x_1| + |max\{x_1, x_1\} - max\{x_3, x_3\}|\right)\right|$$

$$= \frac{1}{3}\left|x_2 - |x_2 - x_4|\right| = \frac{1}{3}x_4.$$

We have

$$|D_0(x, N_0(y)) - D_0(N_0(x), y)| = |D_0((x_1, x_2, x_3),(y_3, 0, y_1)) - D_0((x_3, 0, x_1),(y_1, y_2, y_3))|$$

$$= \left|\frac{1}{3}\left(|x_1 - y_3| + |x_2 - 0| + |x_3 - y_1| + |max\{x_1, y_1\} - max\{y_3, x_3\}|\right)\right.$$

$$\left.-\frac{1}{3}\left(|x_3 - y_1| + |0 - y_2| + |x_1 - y_3| + |max\{x_3, y_3\} - max\{y_1, x_1\}|\right)\right| = \frac{1}{3}|x_2 - y_2|.$$

We have

$$|D_0(x, y) - D_0(N_0(x), N_0(y))| = |D_0((x_1, x_2, x_3),(y_1, y_2, y_3)) - D_0((x_3, 0, x_1),(y_3, 0, y_1))|$$

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$$=\left|\frac{1}{3}\left(|x_1-y_1|+|x_2-y_2|+|x_3-y_3|+|max\{x_1,y_3\}-max\{x_3,y_1\}|\right)\right.$$

$$\left.-\frac{1}{3}\left(|x_3-y_3|+|0-0|+|x_1-y_1|+|max\{x_3,y_1\}-max\{y_3,x_1\}|\right)\right|=\frac{1}{3}|x_2-y_2|.$$

We have $D_0(N_S(x),N_S(y))=D_0((x_3,x_4,x_1),(y_3,y_4,y_1))$

$=\frac{1}{3}\left(|x_3-y_3|+|x_4-y_4|+|x_1-y_1|+|max\{x_3,y_1\}-max\{y_3,x_1\}|\right)$. Further, $|x_4-y_4|=|(1-x_1-x_2-x_3)-(1-y_1-y_2-y_3)|=|x_2-y_2|$. Thus, $D_0(N_S(x),N_S(y))=D_0(x,y)$.

We have $D_0(x,N_S(y))=D_0((x_1,x_2,x_3),(y_3,y_4,y_1))$

$=\frac{1}{3}\left(|x_1-y_3|+|x_2-y_4|+|x_3-y_1|+|max\{x_1,y_1\}-max\{y_3,x_3\}|\right)$. In other hand, $D_0(N_S(x),y)=D_0((x_3,x_4,x_1),(y_1,y_2,y_3))$

$=\frac{1}{3}\left(|x_3-y_1|+|x_4-y_2|+|x_1-y_3|+|max\{x_3,y_3\}-max\{y_1,x_1\}|\right)$. Further, $|x_2-y_4|=|x_2-1+y_1+y_2+y_3|=|x_2-1+x_1+y_2+x_3|=|y_2-x_4|$. Thus, $D_0(x,N_S(y))=D_0(N_S(x),y).\lim_{x\to\infty}$

We have $D_0(x,N_0(x))=D_0((x_1,x_2,x_3),(x_3,0,x_1))$

$$=\frac{1}{3}\left(|x_1-x_3|+|x_2-0|+|x_3-x_1|+|max\{x_1,x_1\}-max\{x_3,x_3\}|\right)=|x_1-x_3|+\frac{1}{3}x_2.$$

We have

$D_0(x,N_S(x))=D_0((x_1,x_2,x_3),(x_3,x_4,x_1))$

$$=\frac{1}{3}\left(|x_1-x_3|+|x_2-x_4|+|x_3-x_1|+|max\{x_1,x_1\}-max\{x_3,x_3\}|\right)=|x_1-x_3|+\frac{1}{3}|x_2-x_4|.$$

We have $D_0(x,1_{P^*})=D_0((x_1,x_2,x_3),(1,0,0))$

$$=\frac{1}{3}\left(|x_1-1|+|x_2-0|+|x_3-0|+|max\{x_1,0\}-max\{x_3,1\}|\right)=\frac{1}{3}(2-2x_1+x_2+x_3).$$

We have $D_0(x,0_{P^*})=D_0((x_1,x_2,x_3),(0,0,1))$

$$=\frac{1}{3}\left(|x_1-0|+|x_2-0|+|x_3-1|+|max\{x_1,1\}-max\{x_3,0\}|\right)=\frac{1}{3}(2-2x_3+x_1+x_2).$$

We have $D_0(0_{P^*},1_{P^*})=D_0((1,0,0),(0,0,1))=1.$

Assume that $D_0\left(x, N_0(x)\right) = 1,$ we have $\left|x_1 - x_3\right| + \frac{1}{3}x_2 = 1.$ Since $\left|x_1 - x_3\right| + \frac{1}{3}x_2 \le \left(x_1 + x_3\right) + x_2 \le 1.$

Therefore, $\left|x_1 - x_3\right| + \frac{1}{3}x_2 = \left(x_1 + x_3\right) + x_2 = 1.$ We obtain that $x_2 = 0$ and $\left|x_1 - x_3\right| = 1.$ Thus, $x \in \left\{0_{P^*}, 1_{P^*}\right\}$. Assume that $D_0\left(x, N_S(x)\right) = 1,$ we have $\left|x_1 - x_3\right| + \frac{1}{3}\left|x_2 - x_4\right| = 1.$ Since $\left|x_1 - x_3\right| + \frac{1}{3}\left|x_2 - x_4\right| \le \left(x_1 + x_3\right) + \left(x_2 + x_4\right) = 1.$ We obtain that $x_2 = x_4 = 0$ and $\left|x_1 - x_3\right| = 1.$ Thus, $x \in \left\{0_{P^*}, 1_{P^*}\right\}$. Assume that $D_0\left(x, N_0(x)\right) = 0,$ we have $\left|x_1 - x_3\right| + \frac{1}{3}x_2 = 0.$ Hence, $x_2 = 0$ and $x_1 = x_3$. Assume that $D_0\left(x, N_S(x)\right) = 0,$ we have $\left|x_1 - x_3\right| + \frac{1}{3}\left|x_2 - x_4\right| = 0.$ Hence, $x_2 = x_4$ and $x_1 = x_3$.

We have $D_0\left(\left(0,0,0\right),\left(0,a,0\right)\right) = \frac{a}{3},$

$D_0\left(\left(0,0,0\right),\left(a,0,0\right)\right) = D_0\left(\left(0,0,0\right),\left(0,0,a\right)\right) = \frac{2a}{3},$ and

$D_0\left(\left(a,0,0\right),\left(0,0,a\right)\right) = D_0\left(\left(a,0,0\right),\left(0,a,0\right)\right) = D_0\left(\left(0,0,a\right),\left(0,a,0\right)\right) = a.$

Remark 5. The order "$\ll$" on P^* corresponds to the following order on $\mathrm{PFS}(X)$:

$$A \subseteq_* B \Leftrightarrow \mu_A(x) \le \mu_B(x), \eta_A(x) \le \eta_B(x), \nu_A(x) \ge \nu_B(x), \forall x \in X. \tag{11}$$

Remark 6. The picture fuzzy distance measure on P^* corresponds to the picture fuzzy distance measure on $\mathrm{PFS}(X)$, i.e., for all $A, B \in \mathrm{PFS}\left(X = \left\{x_1, x_2, \ldots, x_m\right\}\right)$, we have the picture fuzzy distance measure D_0 between A and B as follows:

$$\begin{aligned} D_0(A,B) = \frac{1}{3m}\sum_{i=1}^{m}\Big(&\left|\mu_A(x_i) - \mu_B(x_i)\right| + \left|\eta_A(x_i) - \eta_B(x_i)\right| + \left|\nu_A(x_i) - \nu_B(x_i)\right| \\ &+ \left|\max\left\{\mu_A(x_i), \nu_B(x_i)\right\} - \max\left\{\nu_A(x_i), \mu_B(x_i)\right\}\right|\Big). \end{aligned} \tag{12}$$

Proposition 3. Consider the picture fuzzy distance measure D_0 in *Eq. (10)*, the picture fuzzy t-norms $T_{i(i=0,\ldots,5)}$ in *Example 2*, the picture fuzzy t-conorms $S_{i(i=0,\ldots,5)}$ in *Example 3*, and the picture fuzzy negation N_0 in *Example 1*. Let A and B be two picture fuzzy sets on the universe $X = \left\{x_1, x_2, \ldots, x_m\right\}$. Then, we have the following properties:

$$D_0\left(A \cap_{T_2} B, B\right) \ge \max\left\{D_0\left(A \cap_{T_i} B, A \cap_{T_j} B\right), D_0\left(A \cap_{T_k} B, B\right)\right\},$$

$$D_0\left(A \cap_{T_2} B, A\right) \ge \max\left\{D_0\left(A \cap_{T_i} B, A \cap_{T_j} B\right), D_0\left(A \cap_{T_k} B, A\right)\right\},$$

$$D_0\left(\overline{A \cap_{T_2} B}^{N_0}, \overline{A}^{N_0}\right) \ge \max\left\{D_0\left(\overline{A \cap_{T_i} B}^{N_0}, \overline{A \cap_{T_j} B}^{N_0}\right), D_0\left(\overline{A \cap_{T_k} B}^{N_0}, \overline{A}^{N_0}\right)\right\},$$

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$$D_0\left(\overline{A\cap_{T_2} B}^{N_0},\overline{B}^{N_0}\right)\geq max\left\{D_0\left(\overline{A\cap_{T_i} B}^{N_0},\overline{A\cap_{T_j} B}^{N_0}\right),D_0\left(\overline{A\cap_{T_k} B}^{N_0},\overline{B}^{N_0}\right)\right\},$$

$\forall (i,j)\in\{(x,y)/x,y=0,...,5\}\setminus\{(4,5)\}$ and $k=0,1,3,4,5.$

$$D_0\left(A\cup_{S_5} B,A\right)\geq max\left\{D_0\left(A\cup_{S_i} B,A\cup_{S_j} B\right),D_0\left(A\cup_{S_k} B,A\right)\right\},$$

$$D_0\left(A\cup_{S_5} B,B\right)\geq max\left\{D_0\left(A\cup_{S_i} B,A\cup_{S_j} B\right),D_0\left(A\cup_{S_k} B,B\right)\right\},$$

$$D_0\left(\overline{A\cup_{S_5} B}^{N_0},\overline{A}^{N_0}\right)\geq max\left\{D_0\left(\overline{A\cup_{S_i} B}^{N_0},\overline{A\cup_{S_j} B}^{N_0}\right),D_0\left(\overline{A\cup_{S_k} B}^{N_0},\overline{A}^{N_0}\right)\right\},$$

$$D_0\left(\overline{A\cup_{S_5} B}^{N_0},\overline{B}^{N_0}\right)\geq max\left\{D_0\left(\overline{A\cup_{S_i} B}^{N_0},\overline{A\cup_{S_j} B}^{N_0}\right),D_0\left(\overline{A\cup_{S_k} B}^{N_0},\overline{B}^{N_0}\right)\right\},$$

$\forall (i,j)\in\{(x,y)/x,y=0,1,3,4,5\}\setminus\{(1,3)\}$ and $k=0,1,3,4.$

$$D_0\left(A\cup_{S_2} B,A\right)\geq max\left\{D_0\left(A\cup_{S_i} B,A\cup_{S_j} B\right),D_0\left(A\cup_{S_k} B,A\right)\right\},$$

$$D_0\left(A\cup_{S_2} B,B\right)\geq max\left\{D_0\left(A\cup_{S_i} B,A\cup_{S_j} B\right),D_0\left(A\cup_{S_k} B,B\right)\right\},$$

$$D_0\left(\overline{A\cup_{S_2} B}^{N_0},\overline{A}^{N_0}\right)\geq max\left\{D_0\left(\overline{A\cup_{S_i} B}^{N_0},\overline{A\cup_{S_j} B}^{N_0}\right),D_0\left(\overline{A\cup_{S_k} B}^{N_0},\overline{A}^{N_0}\right)\right\},$$

$$D_0\left(\overline{A\cup_{S_2} B}^{N_0},\overline{B}^{N_0}\right)\geq max\left\{D_0\left(\overline{A\cup_{S_i} B}^{N_0},\overline{A\cup_{S_j} B}^{N_0}\right),D_0\left(\overline{A\cup_{S_k} B}^{N_0},\overline{B}^{N_0}\right)\right\},$$

$\forall i,j=0,2,3,4,$ and $k=0,3,4.$

$D_0\left(A\cap_{T_i} B,A\cup_{S_j} B\right)\geq D_0\left(A\cap_{T_0} B,A\cup_{S_0} B\right)$ and

$$D_0\left(\overline{A\cap_{T_i} B}^{N_0},\overline{A\cup_{S_j} B}^{N_0}\right)\geq D_0\left(\overline{A\cap_{T_0} B}^{N_0},\overline{A\cup_{S_0} B}^{N_0}\right),\forall i,j=0,...,5.$$

Proof. These properties are proved as follows. Firstly, we see that for all $x,y\in\left[0,1\right]$,

$max(0,x+y-1)\leq xy\leq min(x,y)$ and $min(1,x+y)\geq x+y-xy\geq max(0,x+y-1)$. Hence, $\left(max(0,x_1+y_1-1),max(0,x_2+y_2-1),min(1,x_3+y_3)\right)\ll\left(x_1y_1,x_2y_2,x_3+y_3-x_3y_3\right)$. This means $T_2\ll T_1$. Similarly, we obtain that $T_2\ll T_3\ll T_4\ll T_1\ll T_0$ and $T_2\ll T_3\ll T_5\ll T_1\ll T_0$. Hence,

$$A\cap_{T_2} B\subseteq_* A\cap_{T_3} B\subseteq_* A\cap_{T_4} B\subseteq_* A\cap_{T_1} B\subseteq_* A\cap_{T_0} B\subseteq_* A,$$

$$A\cap_{T_2} B\subseteq_* A\cap_{T_3} B\subseteq_* A\cap_{T_5} B\subseteq_* A\cap_{T_1} B\subseteq_* A\cap_{T_0} B\subseteq_* A,$$

$$A\cap_{T_2} B\subseteq_* A\cap_{T_3} B\subseteq_* A\cap_{T_4} B\subseteq_* A\cap_{T_1} B\subseteq_* A\cap_{T_0} B\subseteq_* B, \text{ and}$$

$$A\cap_{T_2} B\subseteq_* A\cap_{T_3} B\subseteq_* A\cap_{T_5} B\subseteq_* A\cap_{T_1} B\subseteq_* A\cap_{T_0} B\subseteq_* B.$$

Since D_0 is the picture fuzzy distance measure, thus

$$D_0\left(A\cap_{T_2} B,A\right)\geq max\left\{D_0\left(A\cap_{T_i} B,A\cap_{T_j} B\right),D_0\left(A\cap_{T_k} B,A\right)\right\} \text{ and}$$

$$D_0\left(A\cap_{T_2} B,B\right)\geq max\left\{D_0\left(A\cap_{T_i} B,A\cap_{T_j} B\right),D_0\left(A\cap_{T_k} B,B\right)\right\},$$

$\forall\left(i,j\right)\in\left\{\left(x,y\right)/x,y=0,...,5\right\}\setminus\left\{\left(4,5\right)\right\}$ and $k=0,1,3,4,5.$ Furthermore, we have

$$\overline{A}^{N_0}=\left\{\left(x:N_0\left(\left(\mu_A\left(x\right),\eta_A\left(x\right),\nu_A\left(x\right)\right)\right)\right)\middle|x\in X\right\}=\left\{\left(x:\left(\nu_A\left(x\right),0,\mu_A\left(x\right)\right)\right)\middle|x\in X\right\},\otimes.$$

It is easy to prove the following lemma: If $A\subseteq_* B,$ then $\overline{B}^{N_0}\subseteq_*\overline{A}^{N_0}.$ Thus,

$$D_0\left(\overline{A\cap_{T_2} B}^{N_0},\overline{A}^{N_0}\right)\geq max\left\{D_0\left(\overline{A\cap_{T_i} B}^{N_0},\overline{A\cap_{T_j} B}^{N_0}\right),D_0\left(\overline{A\cap_{T_k} B}^{N_0},\overline{A}^{N_0}\right)\right\} \text{ and}$$

$$D_0\left(\overline{A\cap_{T_2} B}^{N_0},\overline{B}^{N_0}\right)\geq max\left\{D_0\left(\overline{A\cap_{T_i} B}^{N_0},\overline{A\cap_{T_j} B}^{N_0}\right),D_0\left(\overline{A\cap_{T_k} B}^{N_0},\overline{B}^{N_0}\right)\right\},$$

$\forall\left(i,j\right)\in\left\{\left(x,y\right)/x,y=0,...,5\right\}\setminus\left\{\left(4,5\right)\right\}$ and $k=0,1,3,4,5.$

Secondly, we have $S_0\ll S_4\ll S_3\ll S_5,$ $S_0\ll S_4\ll S_3\ll S_2,$ and $S_0\ll S_4\ll S_1\ll S_5.$ Hence,

$$A\subseteq_* A\cup_{S_0} B\subseteq_* A\cup_{S_4} B\subseteq_* A\cup_{S_3} B\subseteq_* A\cup_{S_5} B,$$

$$B\subseteq_* A\cup_{S_0} B\subseteq_* A\cup_{S_4} B\subseteq_* A\cup_{S_3} B\subseteq_* A\cup_{S_5} B,$$

$$A\subseteq_* A\cup_{S_0} B\subseteq_* A\cup_{S_4} B\subseteq_* A\cup_{S_1} B\subseteq_* A\cup_{S_5} B, \text{ and}$$

$$B\subseteq_* A\cup_{S_0} B\subseteq_* A\cup_{S_4} B\subseteq_* A\cup_{S_1} B\subseteq_* A\cup_{S_5} B.$$

Therefore $D_0\left(A\cup_{S_5} B,A\right)\geq max\left\{D_0\left(A\cup_{S_i} B,A\cup_{S_j} B\right),D_0\left(A\cup_{S_k} B,A\right)\right\},$

$$D_0\left(A\cup_{S_5} B,B\right)\geq max\left\{D_0\left(A\cup_{S_i} B,A\cup_{S_j} B\right),D_0\left(A\cup_{S_k} B,B\right)\right\},$$

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$$D_0\left(\overline{A\cup_{S_5}B}^{N_0},\overline{A}^{N_0}\right)\geq max\left\{D_0\left(\overline{A\cup_{S_i}B}^{N_0},\overline{A\cup_{S_j}B}^{N_0}\right),D_0\left(\overline{A\cup_{S_k}B}^{N_0},\overline{A}^{N_0}\right)\right\},\text{ and}$$

$$D_0\left(\overline{A\cup_{S_5}B}^{N_0},\overline{B}^{N_0}\right)\geq max\left\{D_0\left(\overline{A\cup_{S_i}B}^{N_0},\overline{A\cup_{S_j}B}^{N_0}\right),D_0\left(\overline{A\cup_{S_k}B}^{N_0},\overline{B}^{N_0}\right)\right\},$$

$\forall(i,j)\in\{(x,y)/x,y=0,1,3,4,5\}\setminus\{(1,3)\}$ and $k=0,1,3,4.$

Now, we have $A\subseteq_* A\cup_{S_0}B\subseteq_* A\cup_{S_4}B\subseteq_* A\cup_{S_3}B\subseteq_* A\cup_{S_2}B$ and

$B\subseteq_* A\cup_{S_0}B\subseteq_* A\cup_{S_4}B\subseteq_* A\cup_{S_3}B\subseteq_* A\cup_{S_2}B.$

Thus, for all $i,j=0,2,3,4,$ and $k=0,3,4,$ we have

$$D_0\left(A\cup_{S_2}B,A\right)\geq max\left\{D_0\left(A\cup_{S_i}B,A\cup_{S_j}B\right),D_0\left(A\cup_{S_k}B,A\right)\right\},$$

$$D_0\left(A\cup_{S_2}B,B\right)\geq max\left\{D_0\left(A\cup_{S_i}B,A\cup_{S_j}B\right),D_0\left(A\cup_{S_k}B,B\right)\right\},$$

$$D_0\left(\overline{A\cup_{S_2}B}^{N_0},\overline{A}^{N_0}\right)\geq max\left\{D_0\left(\overline{A\cup_{S_i}B}^{N_0},\overline{A\cup_{S_j}B}^{N_0}\right),D_0\left(\overline{A\cup_{S_k}B}^{N_0},\overline{A}^{N_0}\right)\right\},\text{ and}$$

$$D_0\left(\overline{A\cup_{S_2}B}^{N_0},\overline{B}^{N_0}\right)\geq max\left\{D_0\left(\overline{A\cup_{S_i}B}^{N_0},\overline{A\cup_{S_j}B}^{N_0}\right),D_0\left(\overline{A\cup_{S_k}B}^{N_0},\overline{B}^{N_0}\right)\right\}.$$

Finally, we see that $A\cap_{T_2}B\subseteq_* A\cap_{T_0}B\subseteq_* A\subseteq_* A\cup_{S_0}B\subseteq_* A\cup_{S_5}B.$

Thus, we obtain that $D_0\left(A\cap_{T_2}B,A\cup_{S_5}B\right)\geq D_0\left(A\cap_{T_0}B,A\cup_{S_0}B\right)$ and the remaining inequalities of *Proposition 3*.

4| An Application of the Picture Fuzzy Distance Measure for Controlling Network Power Consumption

4.1| The Problem and the Solution

The interconnection network is important in the parallel computer systems. Saving interconnection network power is always interested, researched and becoming more and more urgent in the current technological era. In order to achieve high performance, the architectural design of the interconnection network requires an effective power saving mechanism. The aim of this mechanism is to reduce the network latency (the average latency of a message) and the percentages between the number of links that are kept switched on by the saving mechanism and the total number of links [25]. As a simplified way of understanding, this is a matter of optimizing the number of links opened in a networking system. This is a decision-making problem for the trunk link state.

In 2010, Alonso et al. introduced the power saving mechanism in regular interconnection network [25]. This model dynamically increases or reduces the number of links that compose a trunk link. This is done by measuring network traffic and dynamically turning these individual links on or off based on a u_{on}/u_{off} threshold policy with keeping at least one operational link (see *Fig. 1* and *Fig. 2*).

The two parameters u_{on} and u_{off} are designed based on different requirements of mechanism aggressiveness (controlled by the value $u_{avg} = (u_{on} + u_{off})/2$) and mechanism responsiveness (controlled by the difference $u_{on} - u_{off}$).

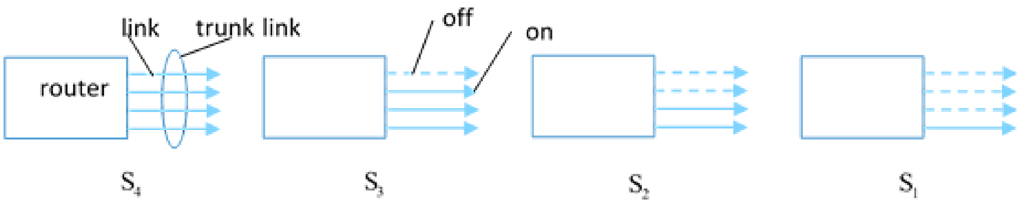

Fig. 1. Four trunk link states.

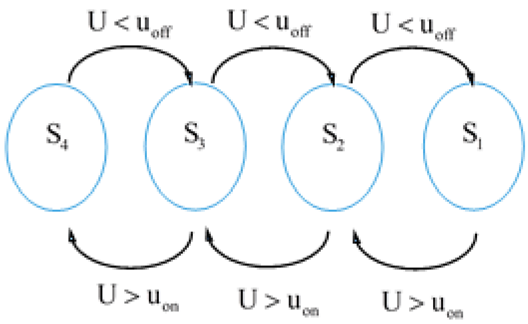

Fig. 2. The operational mechanism of switches.

In order to avoid the possibility of cyclic state transitions that makes the system become unstable, the following restrictions hold in the selection u_{on} and u_{off}:

$$0 < u_{off} \le \frac{u_{on}}{2} \le \frac{U_{max}}{2}. \tag{13}$$

Thus, the different values of u_{off} and u_{on} that satisfy *Eq. (13)* are stiffly chosen in order to achieve different goals of responsiveness and aggressiveness for the power saving mechanism. In 2015, they continue to study and modify power consumption control in fat-tree interconnection networks based on the static and dynamic thresholds policies [26]. In general, this threshold policy is hard because it is without any fuzzy approaches, parameter learning and optimizing processes.

In 2017, Phan et al. [27] proposed a new method in power consumption estimation of network-on-chip based on fuzzy logic [27]. However, this fuzzy logic system based on Sugeno model is too rudimentary and the parameters here are chosen according to the authors' quantification. In this paper, aiming to replace the above threshold policy in decision making problem for the trunk link state, we propose a higher-level fuzzy system based on the proposed single-valued neutrosophic distance measure in Section 3.

4.2| The Adaptive Neuro Picture Fuzzy Inference System (ANPFIS)

In this subsection, an ANPFIS based on picture fuzzy distance measure is introduced to decision making problems. ANPFIS is a modification and combination between ANFIS [28], picture fuzzy set, and picture fuzzy distance measure. Hence, ANPFIS operates based on the picture fuzzification and defuzzification processes, the picture fuzzy operators and distance measure, and the learning capability for automatic picture fuzzy rule generation and parameter optimization. The model is showed as in the *Fig. 3*.

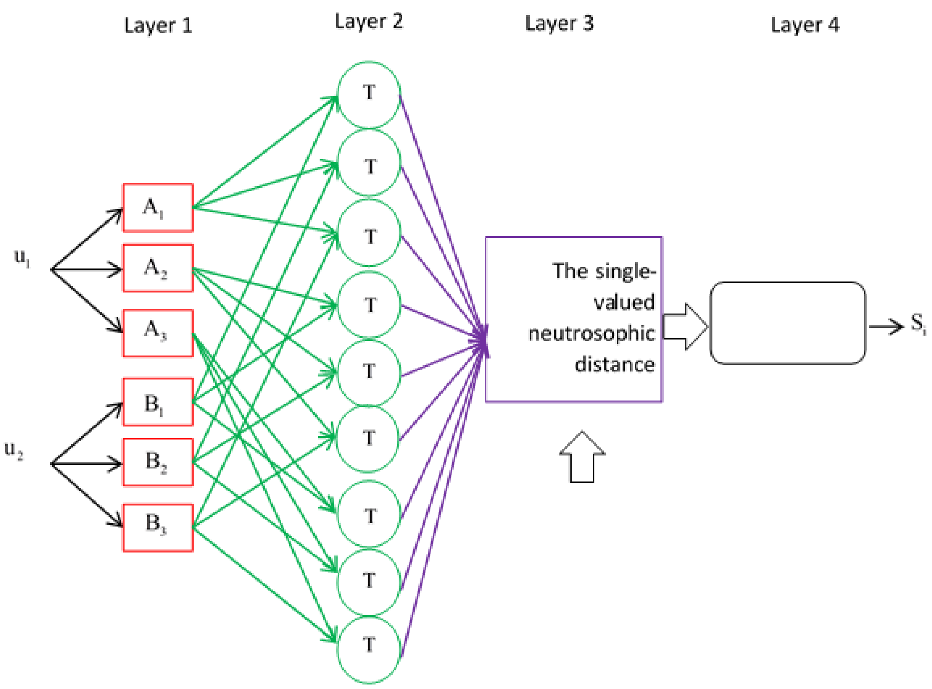

Fig. 3. The proposed ANPFIS decision making model.

The model has the inputs are number values and the output $S_i, i \in \{1,...,n\}$ is the chosen solution. ANPFIS includes four layers as follows:

Layer 1-Picture Fuzzification. Each input value is connected to three neuros O_i^1 , in other words is fuzzified by three corresponding picture fuzzy sets named "High", "Medium", and "Low". We use the Picture Fuzzy Gaussian Function (PFGF): the PFGF is specified by two parameters. The Gaussian function is defined by a central value m and width $k > 0$. The smaller the k, the narrower the curve is. Picture fuzzy Gaussian positive membership, neutral, and negative membership functions are defined as follows

$$\mu(x) = exp\left(-\frac{(x-m)^2}{2k^2}\right),$$

$\nu(x) = c_1(1-\mu(x)), (c_1 \in [0,1]),$ and

$\eta(x) = c_2(1-\mu(x)-\nu(x)), (c_2 \in [0,1]),$ where the parameters $m\otimes$ and $k \triangleright$ are trained.

Layer 2-Automatic Picture Fuzzy Rules. The picture fuzzy t-norm T (see. *Definition 7* and *Example 2*) is used in this step in order to establish the IF-THEN picture fuzzy rules, i.e., the links between the neuros O_i^1 of Layer 1 and the neuros O_k^2 of Layer 2 as follows

"If O_i^1 is x and O_j^1 is y then O_k^2 is $T(x,y)$."

For examples $T(x,y) = T_1^1(x,y)$, where [18]

$$T_1^{\lambda}(x,y)=\left(\frac{x_1y_1}{\lambda_1+(1-\lambda_1)(x_1+y_1-x_1y_1)},\frac{x_2y_2}{\lambda_2+(1-\lambda_2)(x_2+y_2-x_2y_2)},\left(x_3^{\lambda_3}+y_3^{\lambda_3}-x_3^{\lambda_3}y_3^{\lambda_3}\right)^{\frac{1}{\lambda_3}}\right),$$ here

$x,y\in P^*$, and the parameters $\lambda_1,\lambda_2,\lambda_3\in[1,+\infty)$ are trained.

Layer 3 – Calculate the difference to the samples. The difference between the input I and the sample K is calculated by the proposed picture fuzzy distance measure D_0 in *Eq. (10)* as follows

$$\begin{aligned}D_0(I,K)=\frac{1}{3m}\sum_{i=1}^{m}\omega_i\cdot\Big\{&\left|\mu_1(x_i)-\mu_2(x_i)\right|+\left|\eta_1(x_i)-\eta_2(x_i)\right|+\left|\nu_1(x_i)-\nu_2(x_i)\right|\\&+\left|\max\{\mu_1(x_i),\nu_2(x_i)\}-\max\{\mu_2(x_i),\nu_1(x_i)\}\right|\Big\},\end{aligned}$$

where, m is the number of attribute neuro values and $\omega_{i(i=1,\ldots,m)}$ are the trained weights.

Layer 4-Picture Defuzzification. In this final step, we point out the minimum difference value in all values received from Layer 3, $Min D_0(I,K)=D_0(I,K_t)$.

Then, the output value of the ANPFIS is the solution S_t which is corresponding to the sample K_t.

4.3| Application of the ANPFIS algorithm in Controlling Network Power Consumption

In this part, we present the installation of ANPFIS algorithm in the trunk link state Controller of interconnection network.

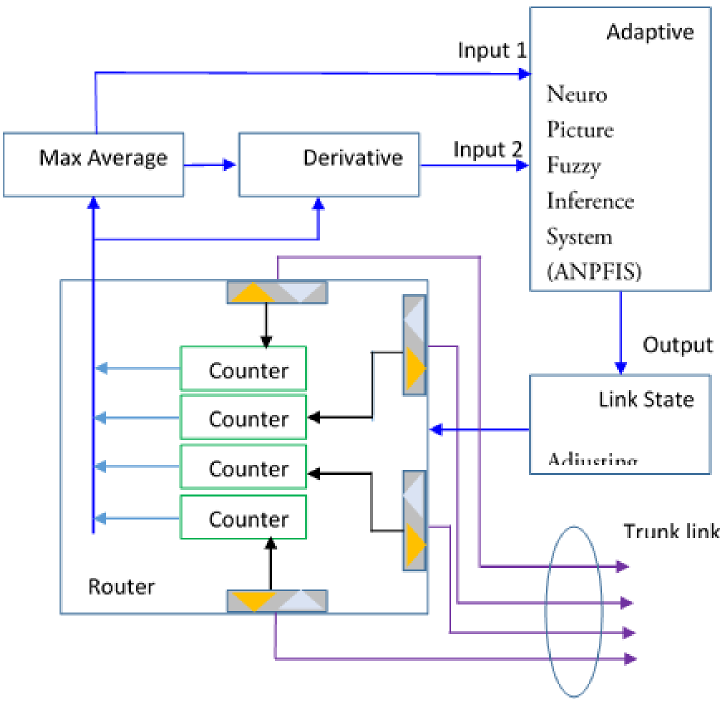

Fig. 4. The architecture of the trunk link state controller based on ANPFIS.

Fig. 4 describes the architecture of the trunk link state Controller based on ANPFIS. This Controller is developed from the previous architecture which is proposed by Phan et al. in 2017 for network-on-chip [27]. For details, each router input port will be equipped with a traffic counter. These counters count the data flits passing through the router in certain clock cycles based on the corresponding response signals from the router. The flits through the router is counted in a slot of time. When the counting

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finish, the traffic through the corresponding port will be calculated [27]. Each port of the router is connected with a Counter, then there are four average values of the traffic.

The Max Average (MA) block receives the values of traffic from the counters which are connected with the routers ports. It compares these values and chose the maximum value for Input 1 of the ANPFIS.

The Derivative (DER) block calculates the derivative of traffics obtained from the counters. This value is defined as an absolute value of the traffics change in a unit of time. This value is determined according to the maximum traffic value decided by MA block. After that, the DER gives it to the Input 2 of the ANPFIS for further processes.

The value domain of Input 1 and Input 2 is from 0 to the maximum bandwidth value. They are normed into [0, 1] by Min Max normalization.

Through the ANPFIS block, the received Output is the trunk link state $S_i, i \in \{1,2,3,4\}$. The received new state are adjusted by the Link State Adjusting block.

5| Experiments on Real-World Datasets

5.1| Experimental Environments

In order to evaluate performance, we test the ANPFIS method on the real datasets of the network traffic history taken from the UPV (Universitat Politècnica de València) university with related methods. The descriptions of the experimental dataset are presented in *Table 1*.

Table 1. The descriptions of the experimental dataset.

No. elements (checking-cycles)	16.571	
No. attributes	2 (MA, DER)	
The normalized value domain of attributes	MA [0,1]	DER [-1,1]
No. classes (No. link states)	4	

We compare the ANPFIS method against the methods of Hung (M2012) [21], Junjun et al. (M2013) [22], Maheshwari et al. (M2016) [23], Ngan et al. (H-max) [8], and ANFIS [28] in the Matlab 2015a programming language. The Mean Squared Error (MSE) degrees of these methods are given out to compare their performance.

5.2| The Quality

The MSE degree of the ANPFIS method are less than those of other methods. The specific values are expressed in *Table 2*.

Table 2. The performance of the methods.

Method	M2012 [21]	M2013 [22]	M2016 [23]	H-max [8]	ANFIS [28]	ANPFIS
MSE	0.2009	0.2606	0.2768	0.1259	0.2006	0.0089

Fig. 5 clearly show the difference between the performance values of six considered algorithms. In *Fig. 5*, the blue columns illustrate the MSE values of the methods. It can be seen that the columns of the other methods are higher than that of the ANPFIS method. That means the accuracy of the proposed method is better than that of the related methods on the considered dataset.

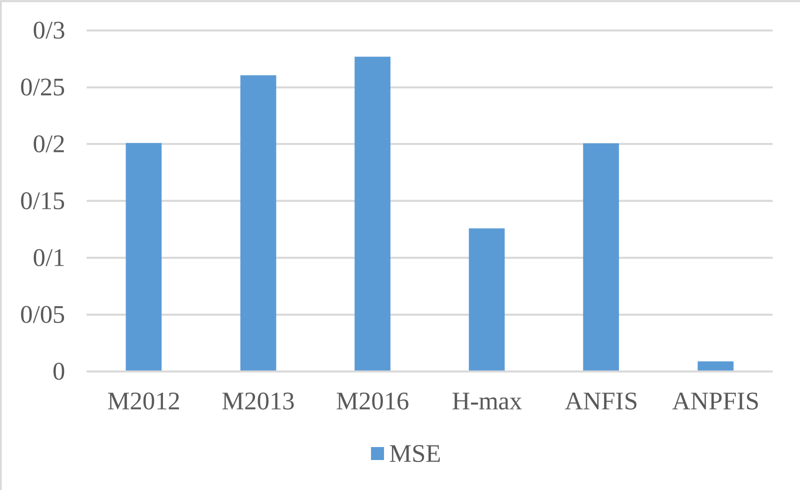

Fig. 5. The MSE values of 6 methods.

6 | Conclusion

The neutrosophic theory increasingly attracts researchers and is applied in many fields. In this paper, a new single-valued neutrosophic distance measure is proposed. It is also a distance measure between picture fuzzy sets and is a development of the H-max measure which was introduced by Ngan et al. [8]. Further, an Adaptive NPFIS based on the proposed measure is shown and applied to the decision making for the link states in interconnection networks. The proposed model is tested on the real datasets taken from the UPV university. The MSE value of the proposed methods is less than that of other methods.

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References

[1] Zadeh, L. A. (1965). Fuzzy sets. *Information and control*, *8*(3), 338-353.

[2] Khorshidi, H. A., & Nikfalazar, S. (2017). An improved similarity measure for generalized fuzzy numbers and its application to fuzzy risk analysis. *Applied soft computing*, *52*, 478-486.

[3] Liang, D., & Xu, Z. (2017). The new extension of TOPSIS method for multiple criteria decision making with hesitant Pythagorean fuzzy sets. *Applied soft computing*, *60*, 167-179.

[4] Zadeh, L. A. (1975). The concept of a linguistic variable and its application to approximate reasoning—I. *Information sciences*, *8*(3), 199-249.

[5] Coupland, S., & John, R. (2008). New geometric inference techniques for type-2 fuzzy sets. *International journal of approximate reasoning*, *49*(1), 198-211.

[6] Atanassov, K. T. (1986). Intuitionistic fuzzy sets. *Fuzzy sets Syst.*, *20*(1), 87–96.

[7] Chen, S. M., Cheng, S. H., & Lan, T. C. (2016). A novel similarity measure between intuitionistic fuzzy sets based on the centroid points of transformed fuzzy numbers with applications to pattern recognition. *Information sciences*, *343-344*, 15-40. DOI: 10.1016/j.ins.2016.01.040

[8] Ngan, R. T., Cuong, B. C., & Ali, M. (2018). H-max distance measure of intuitionistic fuzzy sets in decision making. *Applied soft computing*, *69*, 393-425.

[9] Chen, S. M., & Li, T. S. (2013). Evaluating students' answerscripts based on interval-valued intuitionistic fuzzy sets. *Information sciences*, *235*, 308-322. DOI: 10.1016/j.ins.2012.12.031

[10] Smarandache, F. (1999). *A unifying field in logics. Neutrosophy: Neutrosophic probability, set and logic, rehoboth*. American Research Press, Rehoboth.

[11] Ali, M., & Smarandache, F. (2017). Complex neutrosophic set. *Neural computing and applications*, *28*(7), 1817-1834.

[12] Dat, L. Q., Thong, N. T., Ali, M., Smarandache, F., Abdel-Basset, M., & Long, H. V. (2019). Linguistic approaches to interval complex neutrosophic sets in decision making. *IEEE access*, *7*, 38902-38917.

[13] Smarandache, F. (2016). *Degree of dependence and independence of the (sub) components of fuzzy set and neutrosophic set.* Infinite Study.

[14] Smarandache, F. (2019). *Neutrosophic set is a generalization of intuitionistic fuzzy set, inconsistent intuitionistic fuzzy set (picture fuzzy set, ternary fuzzy set), pythagorean fuzzy set (atanassov's intuitionistic fuzzy set of second type), q-rung orthopair fuzzy set, spherical fuzzy set, and n-hyperspherical fuzzy set, while neutrosophication is a generalization of regret theory, grey system theory, and three-ways decision (revisited)*. Infinite Study.

[15] Cuong, B. C., & Kreinovich, V. (2013, December). Picture fuzzy sets-a new concept for computational intelligence problems. *2013 third world congress on information and communication technologies (WICT 2013)* (pp. 1-6). IEEE.

[16] Thong, P. H. (2015). A new approach to multi-variable fuzzy forecasting using picture fuzzy clustering and picture fuzzy rule interpolation method. In *Knowledge and systems engineering* (pp. 679-690). Springer, Cham. DOI: 10.1007/978-3-319-11680-8_54

[17] Wei, G., & Gao, H. (2018). The generalized Dice similarity measures for picture fuzzy sets and their applications. *Informatica*, *29*(1), 107-124.

[18] Cong, C. B., Ngan, R. T., & Long, L. B. (2017). Some new de morgan picture operator triples in picture fuzzy logic. *Journal of computer sscience and cybernetics*, *33*(2), 143-164.

[19] Cuong, B. C., Kreinovitch, V., & Ngan, R. T. (2016, October). A classification of representable t-norm operators for picture fuzzy sets. *Eighth international conference on knowledge and systems engineering (KSE)* (pp. 19-24). IEEE.

[20] Khalil, A. M., Li, S. G., Garg, H., Li, H., & Ma, S. (2019). New operations on interval-valued picture fuzzy set, interval-valued picture fuzzy soft set and their applications. *IEEE access*, *7*, 51236-51253.

[21] Hung, K. C. (2012). Medical pattern recognition: applying an improved intuitionistic fuzzy cross-entropy approach. *Advances in fuzzy systems*, 863549. 1-6. DOI: 10.1155/2012/863549

[22] Mao, J., Yao, D., & Wang, C. (2013). A novel cross-entropy and entropy measures of IFSs and their applications. *Knowledge-based systems*, *48*, 37-45.

[23] Maheshwari, S., & Srivastava, A. (2016). Study on divergence measures for intuitionistic fuzzy sets and its application in medical diagnosis. *Journal of applied analysis and computation*, *6*(3), 772-789.

[24] Ngan, R. T., Cuong, B. C., & Tuan, T. M. (2018, August). Medical diagnosis from images with intuitionistic fuzzy distance measures. *International joint conference on rough sets* (pp. 479-490). Springer, Cham.

[25] Alonso, M., Coll, S., Martínez, J. M., Santonja, V., López, P., & Duato, J. (2010). Power saving in regular interconnection networks. *Parallel computing*, *36*(12), 696-712.

[26] Alonso, M., Coll, S., Martínez, J. M., Santonja, V., & López, P. (2015). Power consumption management in fat-tree interconnection networks. *Parallel computing*, *48*, 59-80.

[27] Phan, H. P., Tran, X. T., & Yoneda, T. (2017, May). Power consumption estimation using VNOC2. 0 simulator for a fuzzy-logic based low power Network-on-Chip. *2017 IEEE international conference on IC design and technology (ICICDT)* (pp. 1-4). IEEE.

[28] Çaydaş, U., Hasçalık, A., & Ekici, S. (2009). An adaptive neuro-fuzzy inference system (ANFIS) model for wire-EDM. *Expert systems with applications*, *36*(3), 6135-6139.

[29] Ali, M., Deo, R. C., Downs, N. J., & Maraseni, T. (2018). An ensemble-ANFIS based uncertainty assessment model for forecasting multi-scalar standardized precipitation index. *Atmospheric research*, *207*, 155-180.

[30] Karaboga, D., & Kaya, E. (2019). Adaptive network based fuzzy inference system (ANFIS) training approaches: a comprehensive survey. *Artificial intelligence review*, *52*(4), 2263-2293.

[31] Ali, M., Deo, R. C., Downs, N. J., & Maraseni, T. (2018). An ensemble-ANFIS based uncertainty assessment model for forecasting multi-scalar standardized precipitation index. *Atmospheric research*, *207*, 155-180.

[32] Wang, W., & Xin, X. (2005). Distance measure between intuitionistic fuzzy sets. *Pattern recognition letters*, *26*(13), 2063-2069.

Appendix

Source code and datasets of this paper can be found at this link, https://sourceforge.net/projects/pfdm-datasets-code/.

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Some Remarks on Neutro-Fine Topology

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Abstract

The neutro-fine topological space is a space that contains a combination of neutrosophic and fine sets. In this study, the various types of open sets such as generalized open and semi-open sets are defined in such space. The concept of interior and closure on semi-open sets are defined and some of their basic properties are stated. These definitions extend the concept to generalized semi-open sets. Moreover, the minimal and maximal open sets are defined and some of their properties are studied in this space. As well as, discussed the complement of all these sets as its closed sets. The basic properties of the union and intersection of these open sets are stated in some theorems. Only a few sets satisfy this postulates, and others are disproved as shown in the counterexamples. The converse of some theorems is proved in probable examples.

Keywords: Neutro-Fine-Generalized open sets, Neutro-Fine-Semi open sets, Neutro-Fine-Semi interior, Neutro-Fine-Semi closure, Neutro-Fine-Generalized semi open sets, Neutro-Fine minimal open set, Neutro-Fine maximal open sets.

1 | Introduction

The classical set theory developed by Zadeh [40] was termed as a Fuzzy Set (FS), whose elements amuse ambiguous features of true and false membership functions. The FS theory applied in the boundless area of a domain, while Atanassov [39] extended this theory as an Intuitionistic Fuzzy Set (IFS) theory. Later, Smarandache [21] explored a set that contains one more membership function called indeterminacy along with truth and falsity degrees as elements of the Neutrosophic Set (NS). Also, he generalized the NS on IFS [22] and recently proposed his work on attributes valued set, Plithogenic Set (PS) [23]. Nowadays, this set made an outstanding impact on many applications [1]-[4], [11]-[15], [16], [18], [19] and play a vital role in Decision Making (DM) problems [10], [17], [20] and Multi-Criteria DM (MCDM) problems [5], [9].

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Topology is a study of flexible objects under frequent damages without splitting. In recent times, Topological Space (TS) is performing a lead character in the enormous branch of applied sciences and numerous categories of mathematics. The topological structure developed on NS as a generalization of IFTS which was originated by Salama & Alblowi [33], [34], named as Neutrosophic Topological Space (NTS). Few typical sets, open sets, and other TS explored [7], [24], [27], [28], [29], [31], and extended to bi-topological space [6] on such TS.

The most general class of sets which contains few open sets termed as Fine-Open Sets (FOSs), by Powar & Rajak [35], and investigated the special case of generalized TS, called Fine- Topological Space (FTS). Many researchers studied this concept on some sets like FS [26], [30], and others [25], [32]. Recently, this concept extends as Neutro-Fine Topological Space (NFTS) [8], which was introduced by Chinnadurai and Sindhu. The concept of minimal open (closed) and maximal open (closed) sets were exhibited by few researchers [36]- [38].

The aspiration of this paper is to instigate the collection of open sets such as generalized open and semi-open sets defined on NFTS. The concept of interior and closure on neutro-fine-semi open sets are defined and some of their basic properties are stated. These definitions extend the concept to generalized semi-open sets. Moreover, the minimal and maximal open sets are defined and some of their properties are studied in this space. Simultaneously, discussed the complement of all these sets as its closed sets. The basic properties of the union and intersection of these sets are stated in some theorems. Only a few sets satisfy this postulates, and others are disproved as shown in the counterexamples. The converse of some theorems is proved in probable examples.

The layout of this proposal is as follows. In Portion 2, essential definitions of NFTS are recollected. In Portion 3, some type of generalized open sets are defined on NFTS and investigated its properties with illustrative examples. In Portion 4, some more open sets like neutro-fine minimal open sets and neutro-fine maximal open sets are explored via perfect examples. In the end, Portion 6 conveyed the conclusions with some future works.

2| Preliminaries

In this portion, we remind a few major descriptions connected to NFTS.

Definition 1. [8]. Let W be a set of universe and $w_i \in W$ where $i \in I$. Let R be a NS over W. Then the subset of NS R with respect to w_i (sub-NS R_{w_i}) and w_i, w_j (sub-NS R_{w_i,w_j}) are denoted as $\varsigma_R(w_i)$ and $\varsigma_R(w_i, w_j)$, and defined as

$$\varsigma_R(w_i)=\left\{\begin{array}{l}\left\langle w_i, T_R(w_i), I_R(w_i), F_R(w_i)\right\rangle, \left\langle \begin{array}{l} w_{i,j}, max\left(T_R(w_i), T_R(w_j)\right), max\left(I_R(w_i), I_R(w_j)\right), \\ min\left(F_R(w_i), F_R(w_j)\right)\end{array}\right\rangle, \\ \left\langle w_k, T_R(0_n), I_R(0_n), F_R(0_n)\right\rangle, \left\langle w_{k,l}, T_R(0_n), I_R(0_n), F_R(0_n)\right\rangle\end{array}\right\}$$

where $i \in I$, $j \in I-\{i\}$, $k,l \in I-\{i,j\}$ and $k \neq l$ and

$$\varsigma_R(w_i,w_j)=\left\{\begin{array}{l}\left\langle w_i,T_R(w_i),I_R(w_i),F_R(w_i)\right\rangle,\left\langle w_j,T_R(w_j),I_R(w_j),F_R(w_j)\right\rangle,\left\langle w_k,T_R(0_n),I_R(0_n),F_R(0_n)\right\rangle,\\ \left\langle w_{i,j},max\left(T_R(w_i),T_R(w_j)\right),max\left(I_R(w_i),I_R(w_j)\right),min\left(F_R(w_i),F_R(w_j)\right)\right\rangle,\\ \left\langle w_{i,k},max\left(T_R(w_i),T_R(w_k)\right),max\left(I_R(w_i),I_R(w_k)\right),min\left(F_R(w_i),F_R(w_k)\right)\right\rangle,\\ \left\langle w_{j,k},max\left(T_R(w_j),T_R(w_k)\right),max\left(I_R(w_j),I_R(w_k)\right),min\left(F_R(w_j),F_R(w_k)\right)\right\rangle\end{array}\right\}$$

where $i,j,k\in I$ and $i\neq j\neq k$, respectively.

Definition 2. [8]. Let W be a set of universe and $w\in W$. Let R be a NS over *W* and *V* be any proper non- empty subset of *W*. Then $\varsigma_R(V)$ is said to be neutro-fine set (NFS) over *W*.

Definition 3. [8]. Let NFS(W) be the family of all NFSs over W. Then the fine collection of $\varsigma_R(V)$ is denoted as ${}^f\varsigma_W$ and defined over the NT (W, τ_n) as ${}^f\varsigma_W=\left\{0_{nf},1_{nf},\bigcup\varsigma_R(V)\right\}$.

Thus the triplet $\left(W,\tau_n,{}^f\varsigma_W\right)$ is said to be a NFTS over (W, τ_n). The elements belong to ${}^f\varsigma_W$ are said to be neutro-fine open sets (NFOSs) over (W, τ_n) and the complement of NFOSs are said to be neutro-fine closed sets (NFCSs) over (W, τ_n) and denote the collection by ${}^F\varsigma_W$.

Definition 4. [8]. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W, τ_n). Let $\varsigma_R(V)$ be a NFS over W. Then the neutro-fine interior of $\varsigma_R(V)$ is denoted as $Int_{nf}\left(\varsigma_R(V)\right)$ and is defined as the union of all NFOSs contained in $\varsigma_R(V)$.

Clearly, $Int_{nf}\left(\varsigma_R(V)\right)$ is the largest NFOS contained in $\varsigma_R(V)$.

Definition 5. [8]. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W, τ_n). Let $\varsigma_R(V)$ be a NFS over W. Then the neutro-fine closure of $\varsigma_R(V)$ is denoted as $Cl_{nf}\left(\varsigma_R(V)\right)$ and is defined as the intersection of all NFCSs containing $\varsigma_R(V)$.

Clearly, $Cl_{nf}\left(\varsigma_R(V)\right)$ is the smallest NFCS containing $\varsigma_R(V)$.

Definition 6. [8]. Let NF(W) be the family of all NFs over the universe W and $w\in W$. Then NFS $w^{\langle\alpha,\beta,\chi\rangle}$ is said to be a neutro-fine point (NFP), for $0\leq\alpha,\beta,\gamma\leq 1$ and is defined as follows:

$$w^{\langle\alpha,\beta,\gamma\rangle}(v)=\begin{cases}(\alpha,\beta,\gamma), & \text{if } w=v\\ (0,0,1), & \text{if } w\neq v\end{cases}.$$

Every NFS is the union of its NFPs.

Definition 7. [8]. Let $\left(W, \tau_n, {}^f\varsigma_W\right)$ be a NFTS over (W, τ_n). Let $\varsigma_R(V)$ be a NFS over W. Then $\varsigma_R(V)$ is said to be a neutro-fine neighborhood of the NFP $w^{\langle \alpha, \beta, \chi \rangle} \in \varsigma_R(V)$, if there exists a NFOS $\varsigma_R(U)$ such that $w^{\langle \alpha, \beta, \chi \rangle} \in \varsigma_R(U) \subseteq \varsigma_R(V)$.

Proposition 1. [8]. Let $\left(W, \tau_n, {}^f\varsigma_W\right)$ be a NFTS. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be two NFSs over W. Then,

$Int_{nf}\left(0_{nf}\right) = 0_{nf}$ and $Int_{nf}\left(1_{nf}\right) = 1_{nf}$:

$\varsigma_R(V_1)$ is NFOS $\Rightarrow Int_{nf}\left(\varsigma_R(V_1)\right) = \varsigma_R(V_1)$;

$Int_{nf}\left(\varsigma_R(V_1)\right) \subseteq \varsigma_R(V_1)$;

$\varsigma_R(V_1) \subseteq \varsigma_R(V_2) \Rightarrow Int_{nf}\left(\varsigma_R(V_1)\right) \subseteq Int_{nf}\left(\varsigma_R(V_2)\right)$;

$Int_{nf}\left(Int_{nf}\left(\varsigma_R(V_1)\right)\right) = Int_{nf}\left(\varsigma_R(V_1)\right)$;

$Int_{nf}\left(\varsigma_R(V_1) \cap \varsigma_R(V_2)\right) = Int_{nf}\left(\varsigma_R(V_1)\right) \cap Int_{nf}\left(\varsigma_R(V_2)\right)$;

$Int_{nf}\left(\varsigma_R(V_1) \cup \varsigma_R(V_2)\right) \subseteq Int_{nf}\left(\varsigma_R(V_1)\right) \cup Int_{nf}\left(\varsigma_R(V_2)\right)$;

$Int_{nf}\left(\varsigma_R(V_1)'\right) = \left[Cl_{nf}\left(\varsigma_R(V_1)\right)\right]'$.

Proof. Straightforward.

Proposition 2. [8]. Let $\left(W, \tau_n, {}^f\varsigma_W\right)$ be a NFTS. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be two NFSs over W. Then,

$Cl_{nf}\left(0_{nf}\right) = 0_{nf}$ and $Cl_{nf}\left(1_{nf}\right) = 1_{nf}$;

$\varsigma_R(V_1)$ is NFCS $\Rightarrow Cl_{nf}\left(\varsigma_R(V_1)\right) = \varsigma_R(V_1)$;

$Cl_{nf}\left(\varsigma_R(V_1)\right) \supseteq \varsigma_R(V_1)$;

$\varsigma_R(V_1) \subseteq \varsigma_R(V_2) \Rightarrow Cl_{nf}\left(\varsigma_R(V_1)\right) \subseteq Cl_{nf}\left(\varsigma_R(V_2)\right)$;

$Cl_{nf}\left(Cl_{nf}\left(\varsigma_R(V_1)\right)\right) = Cl_{nf}\left(\varsigma_R(V_1)\right)$;

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$$Cl_{nf}\left(\varsigma_R(V_1)\cup\varsigma_R(V_2)\right)=Cl_{nf}\left(\varsigma_R(V_1)\right)\cup Cl_{nf}\left(\varsigma_R(V_2)\right);$$

$$Cl_{nf}\left(\varsigma_R(V_1)\cap\varsigma_R(V_2)\right)\subseteq Cl_{nf}\left(\varsigma_R(V_1)\right)\cap Cl_{nf}\left(\varsigma_R(V_2)\right);$$

$$Cl_{nf}\left(\varsigma_R(V_1)'\right)=\left[Int_{nf}\left(\varsigma_R(V_1)\right)\right]'.$$

Proof. Straightforward.

3| Some Form of Generalized Open Sets in NFTS

In this portion, some open types of generalized open sets on NFTS are defined and probable results are carried by some major expressive examples. This portion is splitted into 3 sub-portions which states neutro-fine-generalized, neutro-fine-semi, and neutro-fine-generalized semi-open sets on NFTS.

3.1| Neutro-Fine-Generalized Open Sets

Let $\varsigma_R(V)$ be a NFS over W of a NFTS $\left(W,\tau_n,{}^f\varsigma_W\right)$. Then $\varsigma_R(V_1)$ is said to be a neutro-fine-generalized closed set (nf -GCS) if $Cl_{nf}\left(\varsigma_R(V)\right)\subseteq\varsigma_R(U)$ whenever $\varsigma_R(V)\subseteq\varsigma_R(U)$ and $\varsigma_R(U)$ is NFOS. The complement of nf-GCS is said to be neutro-fine-generalized open set (nf-GOS).

Theorem 1. Every NFCS is a nf-GCS in NFTS $\left(W,\tau_n,{}^f\varsigma_W\right)$.

Proof. Let $\varsigma_R(V)$ be a NFCS on $\left(W,\tau_n,{}^f\varsigma_W\right)$. Let $\varsigma_R(V)\subseteq\varsigma_R(U)$, where $\varsigma_R(U)$ is NFOS in $\left(W,\tau_n,{}^f\varsigma_W\right)$. Since $\varsigma_R(V)$ is a NFCS, $\varsigma_R(V)=Cl_{nf}\left(\varsigma_R(V)\right)$ $\Rightarrow Cl_{nf}\left(\varsigma_R(V)\right)\subseteq\varsigma_R(V)$. Thus $Cl_{nf}\left(\varsigma_R(V)\right)\subseteq\varsigma_R(V)\subseteq\varsigma_R(U)$. Hence $\varsigma_R(V)$ is a nf-GCS in NFTS $\left(W,\tau_n,{}^f\varsigma_W\right)$.

Remark 1. The converse of the above theorem is not true as shown in the following example.

Example 1. Let $W=\left\{w_1,w_2,w_3\right\}$ and $\tau_n=\left\{0_n,1_n,R,S,T,U\right\}$ where R, S, T and U are NSs over W and are defined as follows

$$R=\left\{\left\langle w_1,.2,.4,.7\right\rangle,\left\langle w_2,.6,.3,.1\right\rangle,\left\langle w_3,.4,.5,.6\right\rangle\right\},$$

$$S=\left\{\left\langle w_1,.9,.3,.6\right\rangle,\left\langle w_2,.6,.5,.4\right\rangle,\left\langle w_3,.7,.8,.1\right\rangle\right\},$$

$$T=\left\{\left\langle w_1,.9,.4,.6\right\rangle,\left\langle w_2,.6,.5,.1\right\rangle,\left\langle w_3,.7,.8,.1\right\rangle\right\}$$ and

$$U=\left\{\left\langle w_1,.2,.3,.7\right\rangle,\left\langle w_2,.6,.3,.4\right\rangle,\left\langle w_3,.4,.5,.6\right\rangle\right\}.$$

Thus (W,τ_n) is a NTS over W.

Then NFOSs over (W,τ_n) are ${}^{f}\varsigma_W=\left\{0_n,1_n,\varsigma_R(w_1),\varsigma_R(w_3),\varsigma_S(w_2,w_3)\right\}$, where

$$\varsigma_R(w_1)=\left\{\left\langle w_1,.2,.4,.7\right\rangle,\left\langle w_2,0,0,1\right\rangle,\left\langle w_3,0,0,1\right\rangle,\left\langle w_{1,2},.6,.4,.1\right\rangle,\left\langle w_{1,3},.4,.5,.6\right\rangle,\left\langle w_{2,3},0,0,1\right\rangle\right\},$$

$$\varsigma_R(w_3)=\left\{\left\langle w_1,0,0,1\right\rangle,\left\langle w_2,0,0,1\right\rangle,\left\langle w_3,.4,.5,.6\right\rangle,\left\langle w_{1,2},0,0,1\right\rangle,\left\langle w_{1,3},.4,.5,.6\right\rangle,\left\langle w_{2,3},.6,.5,.1\right\rangle\right\},$$

$$\varsigma_S(w_2,w_3)=\left\{\left\langle w_1,0,0,1\right\rangle,\left\langle w_2,.6,.5,.4\right\rangle,\left\langle w_3,.7,.8,.1\right\rangle,\left\langle w_{1,2},.9,.5,.4\right\rangle,\left\langle w_{1,3},.9,.8,.1\right\rangle,\left\langle w_{2,3},.7,.8,.1\right\rangle\right\}$$

and NFCSs over (W,τ_n) are ${}^{F}\varsigma_W=\left\{0_n,1_n,\varsigma_R(w_1)',\varsigma_R(w_3)'\ \varsigma_S(w_2,w_3)'\right\}$, where

$$\varsigma_R(w_1)'=\left\{\left\langle w_1,.7,.6,.2\right\rangle,\left\langle w_2,1,1,0\right\rangle,\left\langle w_3,1,1,0\right\rangle,\left\langle w_{1,2},.1,.6,.6\right\rangle,\left\langle w_{1,3},.6,.5,.4\right\rangle,\left\langle w_{2,3},1,1,0\right\rangle\right\},$$

$$\varsigma_R(w_3)'=\left\{\left\langle w_1,1,1,0\right\rangle,\left\langle w_2,1,1,0\right\rangle,\left\langle w_3,.6,.5,.4\right\rangle,\left\langle w_{1,2},1,1,0\right\rangle,\left\langle w_{1,3},.6,.5,.4\right\rangle,\left\langle w_{2,3},.1,.5,.6\right\rangle\right\},$$

$$\varsigma_S(w_2,w_3)'=\left\{\left\langle w_1,1,1,0\right\rangle,\left\langle w_2,.4,.5,.6\right\rangle,\left\langle w_3,.1,.2,.7\right\rangle,\left\langle w_{1,2},.4,.5,.9\right\rangle,\left\langle w_{1,3},.1,.2,.9\right\rangle,\left\langle w_{2,3},.1,.2,.7\right\rangle\right\}.$$

Thus $\left(W,\tau_n,{}^{f}\varsigma_W\right)$ is a NFTS over (W,τ_n). Here nf-GCS$=\left\{\varsigma_S(w_2),\varsigma_S(w_3),\varsigma_S(w_{2,3})\right\}$. Thus $\varsigma_S(w_2)$ is nf-GCS but not NFCS.

Theorem 2. If $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ are nf-GCSs over $\left(W,\tau_n,{}^{f}\varsigma_W\right)$, then $\varsigma_R(V_1)U\varsigma_R(V_2)$ is also a nf-GCS over $\left(W,\tau_n,{}^{f}\varsigma_W\right)$.

Proof. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be nf-GCSs over $\left(W,\tau_n,{}^{f}\varsigma_W\right)$. Then $Cl_{nf}\left(\varsigma_R(V_1)\right)\subseteq\varsigma_R(U)$ whenever $\varsigma_R(V_1)\subseteq\varsigma_R(U)$ and $\varsigma_R(U)$ is NFOS and $Cl_{nf}\left(\varsigma_R(V)\right)\subseteq\varsigma_R(U)$ whenever

$\varsigma_R(V)\subseteq\varsigma_R(U)$ and $\varsigma_R(U)$ is NFOS. Since $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ are subsets of $\varsigma_R(U)$, $\varsigma_R(V_1)\cup\varsigma_R(V_2)$ are subsets of $\varsigma_R(U)$ and $\varsigma_R(U)$ is NFOS. Then by *Proposition 2*, $Cl_{nf}\left(\varsigma_R(V_1)\cup\varsigma_R(V_2)\right)=Cl_{nf}\left(\varsigma_R(V_1)\right)\cup Cl_{nf}\left(\varsigma_R(V_2)\right)$. Thus $Cl_{nf}\left(\varsigma_R(V_1)\cup\varsigma_R(V_2)\right)\subseteq\varsigma_R(U)$. Hence $\varsigma_R(V_1)\cup\varsigma_R(V_2)$ is a nf-GCSs over $\left(W,\tau_n,{}^f\varsigma_W\right)$.

Remark 2. The intersection of two nf-GCSs need not be a tnf-GCS as shown in the following example.

Example 2. Consider the *Example 1*. Here nf-GCS$=\left\{\varsigma_S(w_2),\varsigma_S(w_3),\varsigma_S(w_{2,3})\right\}$. Then

$$\varsigma_S(w_2)\cap\varsigma_S(w_3)=\left\{\left\langle w_1,0,0,1\right\rangle,\left\langle w_2,0,0,1\right\rangle,\left\langle w_3,0,0,1\right\rangle,\left\langle w_{1,2},0,0,1\right\rangle,\left\langle w_{1,3},0,0,1\right\rangle,\left\langle w_{2,3},.7,.8,.1\right\rangle\right\},$$

is not a nf-GCS.

Theorem 3. If $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ are nf-GCSs over $\left(W,\tau_n,{}^f\varsigma_W\right)$, then

$$Cl_{nf}\left(\varsigma_R(V_1)\cap\varsigma_R(V_2)\right)\subseteq Cl_{nf}\left(\varsigma_R(V_1)\right)\cap Cl_{nf}\left(\varsigma_R(V_2)\right).$$

Proof. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be nf-GCSs over $\left(W,\tau_n,{}^f\varsigma_W\right)$. Then $Cl_{nf}\left(\varsigma_R(V_1)\right)\subseteq\varsigma_R(U)$ whenever $\varsigma_R(V_1)\subseteq\varsigma_R(U)$ and $\varsigma_R(U)$ is NFOS and $Cl_{nf}\left(\varsigma_R(V)\right)\subseteq\varsigma_R(U)$ whenever $\varsigma_R(V)\subseteq\varsigma_R(U)$ and $\varsigma_R(U)$ is NFOS.

Since $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ are subsets of $\varsigma_R(U)$, $\varsigma_R(V_1)\cap\varsigma_R(V_2)$ are subsets of $\varsigma_R(U)$ and $\varsigma_R(U)$ is NFOS.

Since $\varsigma_R(V_1)\cap\varsigma_R(V_2)\subseteq\varsigma_R(V_1)$ and $\varsigma_R(V_1)\cap\varsigma_R(V_2)\subseteq\varsigma_R(V_1)$, $Cl_{nf}\left(\varsigma_R(V_1)\cap\varsigma_R(V_2)\right)\subseteq Cl_{nf}\left(\varsigma_R(V_1)\right)$ and $Cl_{nf}\left(\varsigma_R(V_1)\cap\varsigma_R(V_2)\right)\subseteq Cl_{nf}\left(\varsigma_R(V_2)\right)$, by *Proposition 2*. Thus
$Cl_{nf}\left(\varsigma_R(V_1)\cap\varsigma_R(V_2)\right)\subseteq Cl_{nf}\left(\varsigma_R(V_1)\right)\cap Cl_{nf}\left(\varsigma_R(V_2)\right)$.

3.2| Neutro-Fine-Semi Open Sets

Definition 8. Let $\varsigma_R(V)$ be a NFS over W of a NFTS $\left(W,\tau_n,{}^f\varsigma_W\right)$. Then $\varsigma_R(V)$ is said to be a neutro-fine-semi closed set (nf-SCS) if $Int_{nf}\left(Cl_{nf}\left(\varsigma_R(V)\right)\right)\subseteq\varsigma_R(V)$.

The complement of nf-SCS is said to be neutro-fine-semi open set (nf-SOS), i.e., $\varsigma_R(V)\subseteq Cl_{nf}\left(Int_{nf}\left(\varsigma_R(V)\right)\right)$.

Theorem 4. Let $\left(W, \tau_n, {}^f\varsigma_W\right)$ be a NFTS and $\varsigma_R(V)$ be a NFS over W. Then $\varsigma_R(V)$ is nf-SCS if and only if $\varsigma_R(V)'$ is nf-SOS.

Proof. Let $\varsigma_R(V)$ be a nf-SCS. Then $Int_{nf}\left(Cl_{nf}\left(\varsigma_R(V)\right)\right) \subseteq \varsigma_R(V)$.

Taking complement on both sides,

$$\varsigma_R(V)' \subseteq \left[Int_{nf}\left(Cl_{nf}\left(\varsigma_R(V)\right)\right)\right]' = Cl_{nf}\left(Cl_{nf}\left(\varsigma_R(V)\right)\right)'.$$

By using *Proposition 1*, $\varsigma_R(V)' = Cl_{nf}\left(Int_{nf}\left(\varsigma_R(V)'\right)\right)$. Thus $\varsigma_R(V)'$ is a nf-SOS. Conversely, assume that $\varsigma_R(V)'$ is a nf-SOS. Then $\varsigma_R(V)' = Cl_{nf}\left(Int_{nf}\left(\varsigma_R(V)'\right)\right)$.

Taking complement on both sides,

$$\varsigma_R(V) \supseteq \left[Cl_{nf}\left(Int_{nf}\left(\varsigma_R(V)'\right)\right)\right]' = Int_{nf}\left(Int_{nf}\left(\varsigma_R(V)'\right)\right)', \text{ by } \textit{Proposition 1}.$$

By Proposition 2, $\varsigma_R(V) \supseteq Int_{nf}\left(Cl_{nf}\left(\varsigma_R(V)\right)\right)$. Thus $\varsigma_R(V)$ is a nf-SCS.

Theorem 5. If $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ are nf-SCSs over NFTS $\left(W, \tau_n, {}^f\varsigma_W\right)$, then $\varsigma_R(V_1) \cap \varsigma_R(V_2)$ is also a nf-SCS in $\left(W, \tau_n, {}^f\varsigma_W\right)$.

Proof. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be nf-SCSs over $\left(W, \tau_n, {}^f\varsigma_W\right)$. Then $Int_{nf}\left(Cl_{nf}\left(\varsigma_R(V_1)\right)\right) \subseteq \varsigma_R(V_1)$ and $Int_{nf}\left(Cl_{nf}\left(\varsigma_R(V_1)\right)\right) \subseteq \varsigma_R(V_1)$. Thus $\varsigma_R(V_1) \cap \varsigma_R(V_2) \supseteq Int_{nf}\left(Cl_{nf}\left(\varsigma_R(V_1)\right)\right) \cap Int_{nf}\left(Cl_{nf}\left(\varsigma_R(V_2)\right)\right)$

$$= Int_{nf}\left(Cl_{nf}\left(\varsigma_R(V_1)\right) \cap Cl_{nf}\left(\varsigma_R(V_2)\right)\right) \supseteq Int_{nf}\left(Cl_{nf}\left(\varsigma_R(V_1) \cap \varsigma_R(V_2)\right)\right), \text{ by } \textit{Propositions 1} \text{ and } \textit{2}.$$

Hence $\varsigma_R(V_1) \cap \varsigma_R(V_2)$ is a nf-SCSs in $\left(W, \tau_n, {}^f\varsigma_W\right)$.

Theorem 6. If $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ are nf-SOSs over NFTS $\left(W, \tau_n, {}^f\varsigma_W\right)$, then $\varsigma_R(V_1) \cup \varsigma_R(V_2)$ is also a nf-SOS in $\left(W, \tau_n, {}^f\varsigma_W\right)$.

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Proof. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be nf-SCSs over $\left(W,\tau_n,{}^f\varsigma_W\right)$. Then $\varsigma_R(V_1)\subseteq Cl_{nf}\left(Int_{nf}\left(\varsigma_R(V_1)\right)\right)$ and $\varsigma_R(V_2)\subseteq Cl_{nf}\left(Int_{nf}\left(\varsigma_R(V_2)\right)\right)$. Thus $\varsigma_R(V_1)\cup\varsigma_R(V_2)\subseteq Cl_{nf}\left(Int_{nf}\left(\varsigma_R(V_1)\right)\right)\cup Cl_{nf}\left(Int_{nf}\left(\varsigma_R(V_2)\right)\right)$

$=Cl_{nf}\left(Int_{nf}\left(\varsigma_R(V_1)\right)\cup Int_{nf}\left(\varsigma_R(V_2)\right)\right)\supseteq Cl_{nf}\left(Int_{nf}\left(\varsigma_R(V_1)\cup\varsigma_R(V_2)\right)\right)$, by *Propositions 1* and *2*.

Theorem 7. Every NFCS is a nf-SCS in NFTS $\left(W,\tau_n,{}^f\varsigma_W\right)$.

Proof. Let $\varsigma_R(V)$ be a NFCS on $\left(W,\tau_n,{}^f\varsigma_W\right)\frac{1}{2}$. Then $\varsigma_R(V)=Cl_{nf}\left(\varsigma_R(V)\right)$. Thus

$Int_{nf}\left(Cl_{nf}\left(\varsigma_R(V)\right)\right)\subseteq Cl_{nf}\left(\varsigma_R(V)\right)\Rightarrow Int_{nf}\left(Cl_{nf}\left(\varsigma_R(V)\right)\right)\subseteq\varsigma_R(V)$. Hence $\varsigma_R(V)$ is a nf-SCS in $\left(W,\tau_n,{}^f\varsigma_W\right)$.

Definition 9. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n). Let $\varsigma_R(V)$ be a NFS over W. Then the neutro-fine-semi interior of $\varsigma_R(V)$ is denoted as $S^*Int_{nf}\left(\varsigma_R(V)\right)$ and is defined as the union of all nf-SOSs contained in $\varsigma_R(V)$. Clearly, $S^*Int_{nf}\left(\varsigma_R(V)\right)$ is the largest nf-SOS contained in $\varsigma_R(V)$.

Definition 10. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n). Let $\varsigma_R(V)$ be a NFS over W. Then the neutro-fine-semi closure of $\varsigma_R(V)$ is denoted as $S^*Cl_{nf}\left(\varsigma_R(V)\right)$ and is defined as the intersection of all nf-SCSs containing $\varsigma_R(V)$. Clearly, $S^*Cl_{nf}\left(\varsigma_R(V)\right)$ is the smallest nf-SCS containing $\varsigma_R(V)$.

Proposition 3. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be two NFSs over W. Then,

$S^*Int_{nf}\left(\varsigma_R(V_1)\right)\subseteq\varsigma_R(V_1)$;

$\varsigma_R(V_1)$ is nf-SOS $\Rightarrow S^*Int_{nf}\left(\varsigma_R(V_1)\right)=\varsigma_R(V_1)$;

$S^*Int_{nf}\left(S^*Int_{nf}\left(\varsigma_R(V_1)\right)\right)=S^*Int_{nf}\left(\varsigma_R(V_1)\right)$;

$\varsigma_R(V_1)\subseteq\varsigma_R(V_2)\Rightarrow S^*Int_{nf}\left(\varsigma_R(V_1)\right)\subseteq S^*Int_{nf}\left(\varsigma_R(V_2)\right)$.

Proof. Straightforward.

Proposition 4. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be two NFSs over W. Then,

$\varsigma_R(V_1) \subseteq S^* Cl_{nf}\left(\varsigma_R(V_1)\right)$;

$\varsigma_R(V_1)$ is nf-SCS $\Rightarrow S^* Cl_{nf}\left(\varsigma_R(V_1)\right) = \varsigma_R(V_1)$;

$S^* Cl_{nf}\left(S^* Cl_{nf}\left(\varsigma_R(V_1)\right)\right) = S^* Cl_{nf}\left(\varsigma_R(V_1)\right)$;

$\varsigma_R(V_1) \subseteq \varsigma_R(V_2) \Rightarrow S^* Cl_{nf}\left(\varsigma_R(V_1)\right) \subseteq S^* Cl_{nf}\left(\varsigma_R(V_2)\right)$.

Proof. Straightforward.

Proposition 5. Let $\left(W, \tau_n, {}^f\varsigma_W\right)$ be a NFTS. Let $\varsigma_R(V)$ be any NFS over W. Then,

$Int_{nf}\left(\varsigma_R(V)\right) \subseteq S^* Int_{nf}\left(\varsigma_R(V)\right) \subseteq \varsigma_R(V)$;

$\varsigma_R(V) \subseteq S^* Cl_{nf}\left(\varsigma_R(V)\right) \subseteq Cl_{nf}\left(\varsigma_R(V)\right)$;

$S^* Cl_{nf}\left(\varsigma_R(V)'\right) = \left[S^* Int_{nf}\left(\varsigma_R(V)\right)\right]'$;

$S^* Int_{nf}\left(\varsigma_R(V)'\right) = \left[S^* Cl_{nf}\left(\varsigma_R(V)\right)\right]'$.

Proof. Straightforward.

Proposition 6. Let $\left(W, \tau_n, {}^f\varsigma_W\right)$ be a NFTS. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be two NFSs over W. Then,

$S^* Int_{nf}\left(\varsigma_R(V_1) \cap \varsigma_R(V_2)\right) = S^* Int_{nf}\left(\varsigma_R(V_1)\right) \cap S^* Int_{nf}\left(\varsigma_R(V_2)\right)$;

$S^* Int_{nf}\left(\varsigma_R(V_1) \cup \varsigma_R(V_2)\right) \supseteq S^* Int_{nf}\left(\varsigma_R(V_1)\right) \cup S^* Int_{nf}\left(\varsigma_R(V_2)\right)$.

Proof. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be two NFSs over W.

Since $\varsigma_R(V_1) \cap \varsigma_R(V_2) \subseteq \varsigma_R(V_1)$ and $\varsigma_R(V_1) \cap \varsigma_R(V_2) \subseteq \varsigma_R(V_2)$, by using *Proposition 3*, $S^* Int_{nf}\left(\varsigma_R(V_1) \cap \varsigma_R(V_2)\right) \subseteq S^* Int_{nf}\left(\varsigma_R(V_1)\right)$ and $S^* Int_{nf}\left(\varsigma_R(V_1) \cap \varsigma_R(V_2)\right) \subseteq S^* Int_{nf}\left(\varsigma_R(V_2)\right)$.

This implies that,

$$S^* Int_{nf}\left(\varsigma_R(V_1) \cap \varsigma_R(V_2)\right) \subseteq S^* Int_{nf}\left(\varsigma_R(V_1)\right) \cap S^* Int_{nf}\left(\varsigma_R(V_2)\right), \tag{1}$$

By using *Proposition 3*,

$S^* Int_{nf}\left(\varsigma_R(V_1)\right) \subseteq \varsigma_R(V_1)$ and $S^* Int_{nf}\left(\varsigma_R(V_2)\right) \subseteq \varsigma_R(V_2)$

$\Rightarrow S^* Int_{nf}\left(\varsigma_R(V_1)\right) \cap S^* Int_{nf}\left(\varsigma_R(V_2)\right) \subseteq \varsigma_R(V_1) \cap \varsigma_R(V_2)$.

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By using *Proposition 3*,

$$S^*Int_{nf}\left[S^*Int_{nf}\left(\varsigma_R(V_1)\right)\cap S^*Int_{nf}\left(\varsigma_R(V_2)\right)\right]\subseteq S^*Int_{nf}\left(\varsigma_R(V_1)\cap\varsigma_R(V_2)\right).$$

By *Eq. (1)*,

$$S^*Int_{nf}\left(S^*Int_{nf}\left(\varsigma_R(V_1)\right)\right)\cap S^*Int_{nf}\left(S^*Int_{nf}\left(\varsigma_R(V_2)\right)\right)\subseteq S^*Int_{nf}\left(\varsigma_R(V_1)\cap\varsigma_R(V_2)\right).$$

By using *Proposition 3*,

$$S^*Int_{nf}\left(\varsigma_R(V_1)\right)\cap S^*Int_{nf}\left(\varsigma_R(V_2)\right)\subseteq S^*Int_{nf}\left(\varsigma_R(V_1)\cap\varsigma_R(V_2)\right), \tag{2}$$

Hence from *Eqs. (1)-(2)*,

$$S^*Int_{nf}\left(\varsigma_R(V_1)\cap\varsigma_R(V_2)\right)=S^*Int_{nf}\left(\varsigma_R(V_1)\right)\cap S^*Int_{nf}\left(\varsigma_R(V_2)\right).$$

Since $\varsigma_R(V_1)\subseteq\varsigma_R(V_1)\cup\varsigma_R(V_2)$ and $\varsigma_R(V_1)\subseteq\varsigma_R(V_1)\cup\varsigma_R(V_2)$, by using *Proposition 3*,

$S^*Int_{nf}\left(\varsigma_R(V_1)\right)\subseteq S^*Int_{nf}\left(\varsigma_R(V_1)\cup\varsigma_R(V_2)\right)$ and $S^*Int_{nf}\left(\varsigma_R(V_2)\right)\subseteq S^*Int_{nf}\left(\varsigma_R(V_1)\cup\varsigma_R(V_2)\right)$. Hence $S^*Int_{nf}\left(\varsigma_R(V_1)\cup\varsigma_R(V_2)\right)\supseteq S^*Int_{nf}\left(\varsigma_R(V_1)\right)\cup S^*Int_{nf}\left(\varsigma_R(V_2)\right)$.

Proposition 7. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be two NFSs over *W*. Then,

$$S^*Cl_{nf}\left(\varsigma_R(V_1)\cup\varsigma_R(V_2)\right)=S^*Cl_{nf}\left(\varsigma_R(V_1)\right)\cup S^*Cl_{nf}\left(\varsigma_R(V_2)\right);$$

$$S^*Cl_{nf}\left(\varsigma_R(V_1)\cap\varsigma_R(V_2)\right)\subseteq S^*Cl_{nf}\left(\varsigma_R(V_1)\right)\cap S^*Cl_{nf}\left(\varsigma_R(V_2)\right).$$

Proof. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be two NFSs over *W*.

(i) Since $S^*Cl_{nf}\left(\varsigma_R(V_1)\cup\varsigma_R(V_2)\right)=S^*Cl_{nf}\left(\left(\varsigma_R(V_1)\cup\varsigma_R(V_2)\right)'\right)'$, by using *Proposition 5*, $S^*Cl_{nf}\left(\varsigma_R(V_1)\cup\varsigma_R(V_2)\right)=\left[S^*Int_{nf}\left(\left(\varsigma_R(V_1)\cup\varsigma_R(V_2)\right)'\right)\right]'=\left[S^*Int_{nf}\left(\left(\varsigma_R(V_1)'\right)\cap\left(\varsigma_R(V_2)'\right)\right)\right]'$.

Again by using *Proposition 5*, $S^*Cl_{nf}\left(\varsigma_R(V_1)\cup\varsigma_R(V_2)\right)=\left[S^*Int_{nf}\left(\varsigma_R(V_1)'\right)\cap S^*Int_{nf}\left(\varsigma_R(V_2)'\right)\right]'$

$$=\left(S^*Int_{nf}\left(\varsigma_R(V_1)'\right)\right)'\cup\left(S^*Int_{nf}\left(\varsigma_R(V_2)'\right)\right)'.$$

By using *Proposition 5*, $S^*Cl_{nf}\left(\varsigma_R(V_1)\cup\varsigma_R(V_2)\right)=S^*Cl_{nf}\left(\left(\varsigma_R(V_1)'\right)\right)'\cup S^*Cl_{nf}\left(\left(\varsigma_R(V_2)'\right)\right)'$

$$=S^*Cl_{nf}\left(\varsigma_R(V_1)\right)\cup S^*Cl_{nf}\left(\varsigma_R(V_2)\right).$$

Hence $S^{*}Cl_{nf}\left(\varsigma_R(V_1)\cup\varsigma_R(V_2)\right)=S^{*}Cl_{nf}\left(\varsigma_R(V_1)\right)\cup S^{*}Cl_{nf}\left(\varsigma_R(V_2)\right)$.

(ii) Since $\varsigma_R(V_1)\cap\varsigma_R(V_2)\subseteq\varsigma_R(V_1)$ and $\varsigma_R(V_1)\cap\varsigma_R(V_2)\subseteq\varsigma_R(V_2)$, by using *Proposition 4*, $S^{*}Cl_{nf}\left(\varsigma_R(V_1)\cap\varsigma_R(V_2)\right)\subseteq S^{*}Cl_{nf}\left(\varsigma_R(V_1)\right)$ and $S^{*}Cl_{nf}\left(\varsigma_R(V_1)\cap\varsigma_R(V_2)\right)\subseteq S^{*}Cl_{nf}\left(\varsigma_R(V_2)\right)$.

Hence $S^{*}Cl_{nf}\left(\varsigma_R(V_1)\cap\varsigma_R(V_2)\right)\subseteq S^{*}Cl_{nf}\left(\varsigma_R(V_1)\right)\cap S^{*}Cl_{nf}\left(\varsigma_R(V_2)\right)$.

3.3| Neutro-Fine-Generalized Semi Open Sets

Definition 11. Let $\varsigma_R(V)$ be a NFS over W of a NFTS $\left(W,\tau_n,{}^{f}\varsigma_W\right)$. Then $\varsigma_R(V)$ is said to be a neutro-fine-generalized semi closed set (nf -GSCS) if $S^{*}Cl_{nf}\left(\varsigma_R(V)\right)\subseteq\varsigma_R(U)$ whenever $\varsigma_R(V)\subseteq\varsigma_R(U)$ and $\varsigma_R(U)$ is NFOS.

The complement of nf-GSCS is said to be neutro-fine-generalized semi open set (nf-GSOS), i.e., $\varsigma_R(U)\subseteq S^{*}Int_{nf}\left(\varsigma_R(V)\right)$ whenever $\varsigma_R(U)\subseteq\varsigma_R(V)$ and $\varsigma_R(U)$ is NFCS.

Example 3. Consider *Example 1*. Thus nf-SCS$=\left\{\varsigma_R(w_1,w_3),\varsigma_S(w_1)\right\}$, nf-SOS $=\left\{\varsigma_R(w_1,w_3)',\varsigma_S(w_1)'\right\}$ where

$\varsigma_R(w_1,w_3)'=\left\{\left\langle w_1,.7,.6,.2\right\rangle,\left\langle w_2,1,1,0\right\rangle,\left\langle w_3,.6,.5,.4\right\rangle,\left\langle w_{1,2},.1,.6,.6\right\rangle,\left\langle w_{1,3},.6,.5,.4\right\rangle,\left\langle w_{2,3},.1,.5,.6\right\rangle\right\}$,

$\varsigma_S(w_1)'=\left\{\left\langle w_1,.6,.7,.9\right\rangle,\left\langle w_2,.1,1,0\right\rangle,\left\langle w_3,1,1,0\right\rangle,\left\langle w_{1,2},.4,.5,.9\right\rangle,\left\langle w_{1,3},.1,.2,.9\right\rangle,\left\langle w_{2,3},1,1,0\right\rangle\right\}$ and

nf-GSCS$=\left\{\varsigma_S(w_2)\right\}$.

Theorem 8. Every NFCS is a nf-GSCS in NFTS $\left(W,\tau_n,{}^{f}\varsigma_W\right)$.

Proof. Let $\varsigma_R(V)$ be a NFCS in NFTS $\left(W,\tau_n,{}^{f}\varsigma_W\right)$.

Let $\varsigma_R(V)\subseteq\varsigma_R(U)$, where $\varsigma_R(U)$ is NFOS in $\left(W,\tau_n,{}^{f}\varsigma_W\right)$. Since $\varsigma_R(V)$ is a NFCS, $\varsigma_R(V)=Cl_{nf}\left(\varsigma_R(V)\right)$, by *Proposition 2*. Also, by *Proposition 5*, $S^{*}Cl_{nf}\left(\varsigma_R(V)\right)\subseteq Cl_{nf}\left(\varsigma_R(V)\right)$. Thus $S^{*}Cl_{nf}\left(\varsigma_R(V)\right)\subseteq Cl_{nf}\left(\varsigma_R(V)\right)=\varsigma_R(V)\subseteq\varsigma_R(U)$. Hence $\varsigma_R(V)$ is a nf-GSCS in NFTS $\left(W,\tau_n,{}^{f}\varsigma_W\right)$.

Theorem 9. If $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ are nf-GSCSs over NFTS $\left(W,\tau_n,{}^f\varsigma_W\right)$, then $\varsigma_R(V_1)\cap\varsigma_R(V_2)$ is also a nf-GSCS in $\left(W,\tau_n,{}^f\varsigma_W\right)$.

Proof. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be nf-GSCSs over $\left(W,\tau_n,{}^f\varsigma_W\right)$.

If $\varsigma_R(V_1)\cap\varsigma_R(V_2)\subseteq\varsigma_R(U)$ and $\varsigma_R(U)$ is a NFOS, then $\varsigma_R(V_1)\subseteq\varsigma_R(U)$ and $\varsigma_R(V_1)\subseteq\varsigma_R(U)$.

Since $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ are nf–GSCSs, $S^*Cl_{nf}\left(\varsigma_R(V_1)\right)\subseteq\varsigma_R(U)$ and $S^*Cl_{nf}\left(\varsigma_R(V_2)\right)\subseteq\varsigma_R(U)$.

Thus $S^*Cl_{nf}\left(\varsigma_R(V_1)\right)\cap S^*Cl_{nf}\left(\varsigma_R(V_2)\right)\subseteq\varsigma_R(U)$.

By *Proposition 7*, $S^*Cl_{nf}\left(\varsigma_R(V_1)\cap\varsigma_R(V_2)\right)\subseteq S^*Cl_{nf}\left(\varsigma_R(V_1)\right)\cap S^*Cl_{nf}\left(\varsigma_R(V_2)\right)\subseteq\varsigma_R(U)$. This implies that, $S^*Cl_{nf}\left(\varsigma_R(V_1)\cap\varsigma_R(V_2)\right)\subseteq\varsigma_R(U)$. Thus $S^*Cl_{nf}\left(\varsigma_R(V_1)\cap\varsigma_R(V_2)\right)\subseteq\varsigma_R(U)$, $\varsigma_R(V_1)\cap\varsigma_R(V_2)\subseteq\varsigma_R(U)$ and $\varsigma_R(U)$ is a NFOS.

Hence $\varsigma_R(V_1)\cap\varsigma_R(V_2)$ is a nf-GSCS over $\left(W,\tau_n,{}^f\varsigma_W\right)$.

Theorem 10. Every NFOS is a nf-GSOS in NFTS $\left(W,\tau_n,{}^f\varsigma_W\right)$.

Proof. Let $\varsigma_R(V)$ be a NFOS in NFTS $\left(W,\tau_n,{}^f\varsigma_W\right)$. Let $\varsigma_R(U)\subseteq\varsigma_R(V)$, where $\varsigma_R(U)$ is NFCS in $\left(W,\tau_n,{}^f\varsigma_W\right)$.

Since $\varsigma_R(V)$ is a NFOS, $\varsigma_R(V)=Int_{nf}\left(\varsigma_R(V)\right)$, by *Proposition 1*.

Also, by *Proposition 5*, $Int_{nf}\left(\varsigma_R(V)\right)\subseteq S^*Int_{nf}\left(\varsigma_R(V)\right)\subseteq\varsigma_R(V)$. Thus $\varsigma_R(V)=S^*Int_{nf}\left(\varsigma_R(V)\right)$ $\Rightarrow\varsigma_R(U)\subseteq\varsigma_R(V)=S^*Int_{nf}\left(\varsigma_R(V)\right)$.

Hence $\varsigma_R(V)$ is a nf-GSCS in NFTS $\left(W,\tau_n,{}^f\varsigma_W\right)$.

Theorem 11. If $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ are nf-GSOSs over NFTS $\left(W,\tau_n,{}^f\varsigma_W\right)$, then $\varsigma_R(V_1)\cup\varsigma_R(V_2)$ is also a nf-GSOS in $\left(W,\tau_n,{}^f\varsigma_W\right)$.

Proof. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be nf-GSOSs over $\left(W, \tau_n, {}^f\varsigma_W\right)$.

If $\varsigma_R(U) \subseteq \varsigma_R(V_1) \cup \varsigma_R(V_2)$ and $\varsigma_R(U)$ is a NFCS, then $\varsigma_R(U) \subseteq \varsigma_R(V_1)$ and $\varsigma_R(U) \subseteq \varsigma_R(V_2)$.

Since $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ are nf–GSOSs, $\varsigma_R(U) \subseteq S^* Int_{nf}\left(\varsigma_R(V_1)\right)$ and $\varsigma_R(U) \subseteq S^* Int_{nf}\left(\varsigma_R(V_2)\right)$.

Thus $\varsigma_R(U) \subseteq S^* Int_{nf}\left(\varsigma_R(V_1)\right) \cup S^* Int_{nf}\left(\varsigma_R(V_1)\right)$.

By *Proposition 6*, $\varsigma_R(U) \subseteq S^* Int_{nf}\left(\varsigma_R(V_1)\right) \cup S^* Int_{nf}\left(\varsigma_R(V_1)\right) \subseteq S^* Int_{nf}\left(\varsigma_R(V_1) \cup \varsigma_R(V_2)\right)$.

This implies that, $\varsigma_R(U) \subseteq S^* Int_{nf}\left(\varsigma_R(V_1) \cup \varsigma_R(V_2)\right)$.

Thus $\varsigma_R(U) \subseteq S^* Int_{nf}\left(\varsigma_R(V_1) \cup \varsigma_R(V_2)\right)$, $\varsigma_R(U) \subseteq S^* Int_{nf}\left(\varsigma_R(V_1) \cup \varsigma_R(V_2)\right)$ and $\varsigma_R(U)$ is a NFCS.

Hence $\varsigma_R(V_1) \cup \varsigma_R(V_2)$ is a nf-GSOS in $\left(W, \tau_n, {}^f\varsigma_W\right)$.

4| Neutro-Fine Minimal and Maximal Open Sets

In this portion, the minimal and maximal open sets on NFTS are defined and probable results are carried by some major expressive examples.

Definition 12. Let $\varsigma_R(V)$ be a proper non-empty NFOS of a NFTS $\left(W, \tau_n, {}^f\varsigma_W\right)$. Then $\varsigma_R(V)$ is said to be a neutro-fine minimal open set (min_{nf}-OS) if any NFOS which is contained in $\varsigma_R(V)$ is 0_{nf} or $\varsigma_R(V)$. The complement of min_{nf}-OS is said to be neutro-fine minimal closed set (min_{nf}-CS).

Definition 13. Let $\varsigma_R(V)$ be a proper non-empty NFOS of a NFTS $\left(W, \tau_n, {}^f\varsigma_W\right)$. Then $\varsigma_R(V)$ is said to be a neutro-fine maximal open set (max_{nf}-OS) if any NFOS which is contained in $\varsigma_R(V)$ is 1_{nf} or $\varsigma_R(V)$. The complement of max_{nf}-OS is said to be neutro-fine maximal closed set (max_{nf}-CS).

Example 4. Let $W = \left\{ w_1, w_2, w_3 \right\}$ and $\tau_n = \left\{ 0_n, 1_n, R, S \right\}$ where R and *S* are NSs over *W* and are defined as follows

$$R = \left\{ \left\langle w_1, .1, .2, .8 \right\rangle, \left\langle w_2, .4, .7, .3 \right\rangle, \left\langle w_3, .6, .5, .2 \right\rangle \right\} \text{ and}$$

$$S = \left\{ \left\langle w_1, .6, .5, .3 \right\rangle, \left\langle w_2, .9, .8, .1 \right\rangle, \left\langle w_3, .7, .6, .1 \right\rangle \right\}.$$

Thus (W, τ_n) is a NTS over W. Then ${}^f\varsigma_W = \left\{ 0_n, 1_n, \varsigma_R(w_1), \varsigma_R(w_2, w_3), \varsigma_S(w_2) \right\}$, where

$\varsigma_R(w_1) = \left\{ \langle w_1, .1, .2, .8 \rangle, \langle w_2, 0, 0, 1 \rangle, \langle w_3, 0, 0, 1 \rangle, \langle w_{1,2}, .4, .7, .3 \rangle, \langle w_{1,3}, .6, .5, .2 \rangle, \langle w_{2,3}, 0, 0, 1 \rangle \right\}$,

$\varsigma_R(w_2, w_3) = \left\{ \langle w_1, 0, 0, 1 \rangle, \langle w_2, .4, .7, .3 \rangle, \langle w_3, .6, .5, .2 \rangle, \langle w_{1,2}, .4, .7, .3 \rangle, \langle w_{1,3}, .6, .5, .2 \rangle, \langle w_{2,3}, .6, .7, .2 \rangle \right\}$,

$\varsigma_S(w_2) = \left\{ \langle w_1, 0, 0, 1 \rangle, \langle w_2, .9, .8, .1 \rangle, \langle w_3, 0, 0, 1 \rangle, \langle w_{1,2}, .9, .8, .1 \rangle, \langle w_{1,3}, 0, 0, 1 \rangle, \langle w_{2,3}, .9, .8, .1 \rangle \right\}$ are NFOSs over (W, τ_n).

Hence $\left(W, \tau_n, {}^f\varsigma_W \right)$ is a NFTS over (W, τ_n). Thus min_{nf}-OS $= \left\{ 0_n, \varsigma_R(w_1), \varsigma_S(w_2) \right\}$, min_{nf}-CS $= \left\{ 1_n, \varsigma_R(w_1)', \varsigma_S(w_2)' \right\}$, $\max_{nf}$-OS $= \left\{ 0_n, \varsigma_R(w_2, w_3) \right\}$ and max_{nf}-CS $= \left\{ 1_n, \varsigma_R(w_2, w_3)' \right\}$.

Example 5. Consider *Example 1*. Here min_{nf}-OS $= \left\{ 0_n, \varsigma_R(w_1), \varsigma_R(w_3) \right\}$, min_{nf}-CS $= \left\{ 1_n, \varsigma_R(w_1)', \varsigma_R(w_3)' \right\}$,

$\max_{nf}$-OS $= \left\{ 0_n, \varsigma_S(w_2, w_3) \right\}$ and max_{nf}-CS $= \left\{ 1_n, \varsigma_S(w_2, w_3)' \right\}$.

Lemma 1. Let $\left(W, \tau_n, {}^f\varsigma_W \right)$ be a NFTS over (W, τ_n).

If $\varsigma_R(U)$ is a min_{nf}-OS and $\varsigma_R(W)$ is NFOS, then $\varsigma_R(U) \cap \varsigma_R(W) = 0_{nf}$ or $\varsigma_R(U) \subseteq \varsigma_R(W)$.

If $\varsigma_R(U)$ and $\varsigma_R(V)$ are min_{nf}-OSs, then $\varsigma_R(U) \cap \varsigma_R(V) = 0_{nf}$ or $\varsigma_R(U) = \varsigma_R(V)$.

Proof. Let $\varsigma_R(W)$ be a NFOS such that $\varsigma_R(U) \cap \varsigma_R(W) \neq 0_{nf}$.

Since $\varsigma_R(U)$ is a min_{nf}-OS and $\varsigma_R(U) \cap \varsigma_R(W) \subseteq \varsigma_R(U)$, then $\varsigma_R(U) \cap \varsigma_R(W) = \varsigma_R(U)$. Hence $\varsigma_R(U) \subseteq \varsigma_R(W)$.

If $\varsigma_R(U) \cap \varsigma_R(W) \neq 0_{nf}$, then $\varsigma_R(U) \subseteq \varsigma_R(V)$ and $\varsigma_R(V) \subseteq \varsigma_R(U)$, by (i). Hence $\varsigma_R(U) = \varsigma_R(V)$.

Proposition 7. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n). Let $\varsigma_R(U)$ be a min_{nf}-OS. If $w^{\langle\alpha,\beta,\gamma\rangle}$ is a NFP of $\varsigma_R(U)$, then $\varsigma_R(U)\subseteq\varsigma_R(W)$ for any neutro-fine neighborhood $\varsigma_R(W)$ of $w^{\langle\alpha,\beta,\gamma\rangle}$.

Proof. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n).

Let $\varsigma_R(W)$ be a neutro-fine neighborhood of $w^{\langle\alpha,\beta,\gamma\rangle}$ such that $\varsigma_R(U)\not\subset\varsigma_R(W)$. Then $\varsigma_R(U)\cap\varsigma_R(W)$ is a NFOS such that $\varsigma_R(U)\cap\varsigma_R(W)\not\subset\varsigma_R(U)$ and $\varsigma_R(U)\cap\varsigma_R(W)\neq 0_{nf}$.

This contradicts our assumption that $\varsigma_R(U)$ is a min_{nf}-OS. Hence proved.

Proposition 8. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n). Let $\varsigma_R(U)$ be a min_{nf}-OS. Then

$\varsigma_R(U)=\cap\left\{\varsigma_R(W):\varsigma_R(W) \text{ is a neutro-fine neighborhood of } w^{\langle\alpha,\beta,\gamma\rangle}\right\}$, for any NFP $w^{\langle\alpha,\beta,\gamma\rangle}$ of $\varsigma_R(U)$.

Proof. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n). Let $\varsigma_R(U)$ be a min_{nf}-OS.

Since $\varsigma_R(U)$ is a neutro-fine neighborhood of $w^{\langle\alpha,\beta,\gamma\rangle}$, by *Proposition 7*, then

$\varsigma_R(U)\subseteq\cap\left\{\varsigma_R(W):\varsigma_R(W) \text{ is a neutro-fine neighborhood of } w^{\langle\alpha,\beta,\gamma\rangle}\right\}\subseteq\varsigma_R(U)$. Thus $\varsigma_R(U)=\cap\left\{\varsigma_R(W):\varsigma_R(W) \text{ is a neutro-fine neighborhood of } w^{\langle\alpha,\beta,\gamma\rangle}\right\}$.

Proposition 9. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n). Let $\varsigma_R(U)$ be a non-empty NFOS. Then the following conditions are equivalent:

$\varsigma_R(U)$ is a min_{nf}-OS.

$\varsigma_R(U)\subseteq Cl_{nf}\left(\varsigma_R(V)\right)$ for any NFS $\varsigma_R(V)$ of $\varsigma_R(U)$.

$Cl_{nf}\left(\varsigma_R(U)\right)\subseteq Cl_{nf}\left(\varsigma_R(V)\right)$ for any NFS $\varsigma_R(V)$ of $\varsigma_R(U)$.

Proof. (1)$\Rightarrow$(2). Let $\varsigma_R(V)$ be any NFS of $\varsigma_R(U)$.

By *Proposition 7*, for any NFP $w^{\langle\alpha,\beta,\gamma\rangle}$ of $\varsigma_R(U)$ and any neutro-fine neighborhood $\varsigma_R(W)$ of $w^{\langle\alpha,\beta,\gamma\rangle}$, then $\varsigma_R(V)=\left(\varsigma_R(U)\cap\varsigma_R(V)\right)\subseteq\left(\varsigma_R(W)\cap\varsigma_R(V)\right)$. Thus $\varsigma_R(W)\cap\varsigma_R(V)\neq 0_{nf}$, and hence $\varsigma_R(U)\cap\varsigma_R(W)\neq 0_{nf}$ is a NFP of $Cl_{nf}\left(\varsigma_R(V)\right)$. Therefore $\varsigma_R(U)\subseteq Cl_{nf}\left(\varsigma_R(V)\right)$.

(2)$\Rightarrow$(3). Since $\varsigma_R(V)$ is any NFS of $\varsigma_R(U)$, then $\varsigma_R(U)\subseteq Cl_{nf}\left(\varsigma_R(V)\right)$.

Thus by (2), $Cl_{nf}\left(\varsigma_R(U)\right)\subseteq Cl_{nf}\left(Cl_{nf}\left(\varsigma_R(V)\right)\right)=Cl_{nf}\left(\varsigma_R(V)\right)$. Hence $Cl_{nf}\left(\varsigma_R(U)\right)\subseteq Cl_{nf}\left(\varsigma_R(V)\right)$ for any NFS $\varsigma_R(V)$ of $\varsigma_R(U)$.

(3)$\Rightarrow$(1). Suppose that $\varsigma_R(U)$ is not a min_{nf}-OS.

Then there exists a NFS $\varsigma_R(V)$ such that $\varsigma_R(V)\not\subset\varsigma_R(U)$. Then there exists a NFP $w^{\langle\alpha,\beta,\gamma\rangle}\in\varsigma_R(U)$ such that $w^{\langle\alpha,\beta,\gamma\rangle}\notin\varsigma_R(V)$. This implies that, $w^{\langle\alpha,\beta,\gamma\rangle}$ is a NFS. Then it is clear that $Cl_{nf}\left(w^{\langle\alpha,\beta,\gamma\rangle}\right)\subseteq\varsigma_R(V)'$

$\Rightarrow Cl_{nf}\left(w^{\langle\alpha,\beta,\gamma\rangle}\right)\neq Cl_{nf}\left(\varsigma_R(U)\right)$.

Hence the proof.

Lemma 2. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n).

If $\varsigma_R(U)$ is a max_{nf}-OS and $\varsigma_R(W)$ is NFOS, then $\varsigma_R(U)\cup\varsigma_R(W)=1_{nf}$ or $\varsigma_R(W)\subseteq\varsigma_R(U)$.

If $\varsigma_R(U)$ and $\varsigma_R(V)$ are max_{nf}-OSs, then $\varsigma_R(U)\cup\varsigma_R(V)=1_{nf}$ or $\varsigma_R(U)=\varsigma_R(V)$.

Proof. (i) Let $\varsigma_R(W)$ be a NFOS such that $\varsigma_R(U)\cup\varsigma_R(W)\neq 1_{nf}$.

Since $\varsigma_R(U)$ is a max_{nf}-OS and $\varsigma_R(U)\subseteq\varsigma_R(U)\cup\varsigma_R(W)$, then $\varsigma_R(U)\cup\varsigma_R(W)=\varsigma_R(U)$. Hence $\varsigma_R(W)\subseteq\varsigma_R(U)$.

If $\varsigma_R(U)\cup\varsigma_R(W)\neq 1_{nf}$, then $\varsigma_R(U)\subseteq\varsigma_R(V)$ and $\varsigma_R(V)\subseteq\varsigma_R(U)$, by (i). Hence $\varsigma_R(U)=\varsigma_R(V)$.

Proposition 10. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n). Let $\varsigma_R(U)$ be a max_{nf}-OS. If $w^{\langle\alpha,\beta,\gamma\rangle}$ is a NFP of $\varsigma_R(U)$, then for any neutro-fine neighborhood $\varsigma_R(W)$ of $w^{\langle\alpha,\beta,\gamma\rangle}$, $\varsigma_R(U)\cup\varsigma_R(W)=1_{nf}$ or $\varsigma_R(W)\subseteq\varsigma_R(U)$.

Proof. Follows from the *Lemma 2*.

Proposition 11. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n). Let $\varsigma_R(U)$ be a max_{nf}-OS. Then $\varsigma_R(U)=\bigcup\left\{\varsigma_R(W):\varsigma_R(W)\text{is a neutro-fine neighborhood of } w^{\langle\alpha,\beta,\gamma\rangle} \text{ such that } \varsigma_R(U)\bigcup\varsigma_R(W)\neq 1_{nf}\right\}$.

Proof. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n). Let $\varsigma_R(U)$ be a max_{nf}-OS.

Since $\varsigma_R(U)$ is a neutro-fine neighborhood of $w^{\langle\alpha,\beta,\gamma\rangle}$, by *Proposition 10*, then $\varsigma_R(U)\subseteq\bigcup\left\{\varsigma_R(W):\varsigma_R(W)\text{is a neutro-fine neighborhood of } w^{\langle\alpha,\beta,\gamma\rangle} \text{ such that } \varsigma_R(U)\bigcup\varsigma_R(W)\neq 1_{nf}\right\}$ $\subseteq\varsigma_R(U)$. Hence the result.

Theorem 12. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n). Let $\varsigma_R(U_1)$, $\varsigma_R(U_2)$ and $\varsigma_R(U_3)$ be max_{nf}-OSs such that $\varsigma_R(U_1)\neq\varsigma_R(U_2)$. If $\left(\varsigma_R(U_1)\bigcap\varsigma_R(U_2)\right)\subseteq\varsigma_R(U_3)$, then $\varsigma_R(U_1)=\varsigma_R(U_3)$ or $\varsigma_R(U_2)=\varsigma_R(U_3)$.

Proof. Let $\varsigma_R(U_1)$, $\varsigma_R(U_2)$ and $\varsigma_R(U_3)$ be max_{nf}-OSs such that $\varsigma_R(U_1)\neq\varsigma_R(U_2)$. Then

$$\left(\varsigma_R(U_1)\bigcap\varsigma_R(U_3)\right)=\varsigma_R(U_1)\bigcap\left(\varsigma_R(U_3)\bigcap 1_{nf}\right)$$

$$=\varsigma_R(U_1)\bigcap\left(\varsigma_R(U_3)\bigcap\left(\varsigma_R(U_1)\bigcup\varsigma_R(U_2)\right)\right) \text{ (by } \textit{Lemma 2}\text{)}$$

$$=\varsigma_R(U_1)\bigcap\left(\left(\varsigma_R(U_3)\bigcap\varsigma_R(U_1)\right)\bigcup\left(\varsigma_R(U_3)\bigcap\varsigma_R(U_2)\right)\right)$$

$$=\left(\varsigma_R(U_1)\bigcap\varsigma_R(U_3)\right)\bigcup\left(\varsigma_R(U_3)\bigcap\varsigma_R(U_1)\bigcap\varsigma_R(U_2)\right)$$

$$=\left(\varsigma_R(U_1)\bigcap\varsigma_R(U_3)\right)\bigcup\left(\varsigma_R(U_1)\bigcap\varsigma_R(U_2)\right) \text{ (since } \left(\varsigma_R(U_1)\bigcap\varsigma_R(U_2)\right)\subseteq\varsigma_R(U_3)\text{)}$$

$$=\varsigma_R(U_1)\bigcap\left(\varsigma_R(U_3)\bigcup\varsigma_R(U_2)\right).$$

If $\varsigma_R(U_3)\neq\varsigma_R(U_2)$, then $\left(\varsigma_R(U_3)\bigcup\varsigma_R(U_2)\right)=1_{nf}$.

Thus $\left(\varsigma_R(U_1)\bigcap\varsigma_R(U_3)\right)=\varsigma_R(U_1)$ implies $\varsigma_R(U_1)\subseteq\varsigma_R(U_3)$. Since $\varsigma_R(U_1)$ and $\varsigma_R(U_3)$ are max_{nf}-OSs, then hence $\varsigma_R(U_1)=\varsigma_R(U_3)$.

Theorem 13. Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS over (W,τ_n). Let $\varsigma_R(U_1)$, $\varsigma_R(U_2)$ and $\varsigma_R(U_3)$ be , max_{nf}-OSs, which are different from each other. Then $\left(\varsigma_R(U_1)\bigcap\varsigma_R(U_2)\right)\not\subset\left(\varsigma_R(U_1)\bigcap\varsigma_R(U_3)\right)$.

Proof. Let $\varsigma_R(U_1)$, $\varsigma_R(U_2)$ and $\varsigma_R(U_3)$ be , max_{nf} -OSs.

Suppose assume that $\left(\varsigma_R(U_1)\cap\varsigma_R(U_2)\right)\subseteq\left(\varsigma_R(U_1)\cap\varsigma_R(U_3)\right)$. Then

$$\left(\varsigma_R(U_1)\cap\varsigma_R(U_2)\right)\cup\left(\varsigma_R(U_2)\cap\varsigma_R(U_3)\right)\subseteq\left(\varsigma_R(U_1)\cap\varsigma_R(U_3)\right)\cup\left(\varsigma_R(U_2)\cap\varsigma_R(U_3)\right).$$

Thus $\varsigma_R(U_2)\cap\left(\varsigma_R(U_1)\cup\varsigma_R(U_3)\right)\subseteq\left(\varsigma_R(U_1)\cup\varsigma_R(U_2)\right)\cap\varsigma_R(U_3)$.

Since $\varsigma_R(U_1)\cup\varsigma_R(U_3)=1_{nf}=\varsigma_R(U_1)\cup\varsigma_R(U_2)$, then $\varsigma_R(U_2)\subseteq\varsigma_R(U_3)$.

This implies that $\varsigma_R(U_2)=\varsigma_R(U_3)$, which contradicts our assumption. Hence proved.

5| Conclusion

The main objective of this paper is to define some collection of open sets such as neutro-fine-generalized open and neutro-fine-semi open sets on NFTS and analyzed its basic properties with perfect examples. The notion of interior and closure on semi-open sets are described and specified certain properties. These definitions provide the idea of generalized semi-open sets on NFTS. Also, the neutro-fine-minimal and neutro-fine-maximal open sets are defined and some of their properties are studied in this space. Likewise, discussed the complement of all these sets as its closed sets. The basic properties of the union and intersection of these sets are stated in some theorems. Only a few sets satisfy this postulates, and others are disproved as shown in the counterexamples. The converse of some theorems is proved in probable examples. Consequently, the future researchers can extend this NFTS to some special types of sets, whereas soft sets, rough sets, crisp sets, cubic sets, etc., Also, the application part can widen on MCDM problems.

References

[1] Smarandache, F. (2020). Generalizations and alternatives of classical algebraic Structures to NeutroAlgebraic structures and antialgebraic structures. *Journal of fuzzy extension & applications*, *1*(2), 85-87.

[2] Kumar Das, S. (2020). Application of transportation problem under pentagonal Neutrosophic environment. *Journal of fuzzy extension and applications*, *1*(1), 27-41.

[3] Abdel-Basset, M., Mohamed, M., & Smarandache, F. (2020). Comment on" a novel method for solving the fully neutrosophic linear programming problems: suggested modifications". *Neutrosophic sets and systems*, *31*(1), 305-309.

[4] Smarandache, F. (2020). Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra. *Neutrosophic sets and systems*, 33, 290-296.

[5] Chinnadurai, V., Smarandache, F., & Bobin, A. (2020). Multi-aspect decision-making process in equity investment using neutrosophic soft matrices. *Neutrosophic sets and systems*, *31*, 224-241.

[6] Chinnadurai, V., & Sindhu, M. P. (2020). A novel approach for pairwise separation axioms on bi-soft topology using neutrosophic sets and an output validation in real life application. *Neutrosophic sets and systems*, *35*, 435-463.

[7] Chinnadurai, V., & Sindhu, M. P. (2020). A novel approach: neutro-spot topology and its supra topology with separation axioms and computing the impact on COVID-19. *Neutrosophic sets and systems* (Submitted).

[8] Chinnadurai, V., & Sindhu, M. P. (2020). An introduction to neutro-fine topology with separation axioms and decision making. *International journal of neutrosophic science*, *12*(1), 13-28.

[9] Riaz, M., Naeem, K., Zareef, i., & afzal, d. (2020). Neutrosophic n-soft sets with topsis method for multiple Attribute Decision Making. *Neutrosophic sets and systems*, *32*(1), 146-170.

[10] Guleria, A., Srivastava, S., & Bajaj, R. K. (2019). On parametric divergence measure of neutrosophic sets with its application in decision-making models. *Neutrosophic sets and systems*, *29*(1), 101-120.

[11] Yasser, I., Twakol, A., Abd El-Khalek, A. A., Samrah, A., & Salama, A. A. (2020). COVID-X: novel health-fog framework based on neutrosophic classifier for confrontation Covid-19. *Neutrosophic sets and systems*, *35*(1), 1-21.

[12] Nabeeh, N. A., Abdel-Monem, A., & Abdelmouty, A. (2019). A Hybrid Approach of Neutrosophic with MULTIMOORA in Application of Personnel Selection. *Neutrosophic sets and systems*, *30*(1), 1-21.

[13] Nogueira, Y. E. M., Ojeda, Y. E. A., Rivera, D. N., León, A. M., & Nogueira, D. M. (2019). Design and application of a questionnaire for the development of the knowledge management audit using neutrosophic iadov technique. *Neutrosophic sets and systems*, *30*(1), 70-87.

[14] Altinirmak, S., Gul, Y., Okoth, B. O., & Karamasa, C. (2018). Performance evaluation of mutual funds via single valued neutrosophic set (svns) perspective: a case study in turkey. *Neutrosophic sets and systems*, *23*(1), 110-125.

[15] Smarandache, F. (2018). Plithogenic Set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited. *Neutrosophic sets and systems*, *21*(1), 153-166. https://digitalrepository.unm.edu/nss_journal/vol21/iss1/16

[16] Abdel-Basset, M., El-hoseny, M., Gamal, A., & Smarandache, F. (2019). A novel model for evaluation Hospital medical care systems based on plithogenic sets. *Artificial intelligence in medicine*, *100*, 101710.

[17] Abdel-Basset, M., Mohamed, R., Zaied, A. E. N. H., & Smarandache, F. (2019). A hybrid plithogenic decision-making approach with quality function deployment for selecting supply chain sustainability metrics. *Symmetry*, *11*(7), 903.

[18] Abdel-Basset, M., Nabeeh, N. A., El-Ghareeb, H. A., & Aboelfetouh, A. (2019). Utilising neutrosophic theory to solve transition difficulties of IoT-based enterprises. *Enterprise information systems*, 1-21.

[19] Abdel-Baset, M., Chang, V., & Gamal, A. (2019). Evaluation of the green supply chain management practices: A novel neutrosophic approach. *Computers in industry*, *108*, 210-220.

[20] Abdel-Basset, M., Mohamed, M., & Smarandache, F. (2018). An extension of neutrosophic AHP–SWOT analysis for strategic planning and decision-making. *Symmetry*, *10*(4), 116.

[21] Smarandache, F. (1998). *Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis*. American Research Press, Rehoboth.

[22] Smarandache, F. (2005). Neutrosophic set-a generalization of the intuitionistic fuzzy set. *International journal of pure and applied mathematics*, *24*(3), 287-297

[23] Smarandache, F. (2017). Plithogeny, plithogenic set, logic, probability, and statistics. *71st annual gaseous electronics conference, american physical society*. https://arxiv.org/abs/1808.03948

[24] Chinnadurai, V., & Sindhu, M. P. (2020). Generalization of level sets in neutrosophic soft sets and points: a new approach. *Tathapi*, *19*(50), 54-81.

[25] Kalaiselvi, S., & Sindhu, M. P. (2016). $\tau_1\tau_2$-fb-open sets in fine-bitopological spaces. *International journal of multidisciplinary research and modern education*, *2*(2), 435-441.

[26] Nandhini, R., & Amsaveni, D. (2020). Fine fuzzy sp closed sets in fine fuzzy topological space. *International journal of engineering and advanced technology*, *9*(3), 1306-1313.

[27] Pushpalatha, A., & Nandhini, T. (2019). *Generalized closed sets via neutrosophic topological spaces*. Malaya journal of mathermatik, *7*(1), 50-54.

[28] Iswarya, P., & Bageerathi, K. (2019). A study on neutrosophic generalized semi-closed sets in neutrosophic topological spaces. *Journal of emerging technologies and innovative research*, *6*(2), 452-457.

[29] Ozturk, T. Y., Aras, C. G., & Bayramov, S. (2019). A new approach to operations on neutrosophic soft sets and to neutrosophic soft topological spaces. *Communications in mathematics and applications, 10*(3), 481-493.

[30] Rowthri, M., & Amudhambigai, B. (2017). A view on fuzzy fine topological group structures spaces. *International journal of computational and applied mathematics*, *12*(1), 412-422.

[31] Iswarya, P., & Bageerathi, K. (2016). On neutrosophic semi-open sets in neutrosophic topological spaces. Infinite Study. *International journal of mathematics trends and technology*, *37*(3), 214-223.

[32] Rajak, K. (2015). $f\tau^*g^*$ semi-closed sets in fine-topological spaces. *International journal of mathamatical sciences and applica*tions, *5*(2), 329-331.

[33] Salama, A. A., & Alblowi, S. A. (2012). Neutrosophic set and neutrosophic topological spaces. *IOSR journal of mathematics*, *3*(4), 31-35.

[34] Salama, A. A., & Alblowi, S. A. (2012). Generalized neutrosophic set and generalized neutrosophic topological spaces. Infinite Study. *Journal on computer science and engineering*, 2(7), 12-23.

[35] Powar, P. L., & Rajak, K. (2012). Fine-irresolute mappings. *Journal of advance studies in topology*, *3*(4), 125-139.

[36] Nakaoko, F., & Oda, N. (2003). On minimal closed sets. *Proceeding of topological spaces theory and its applications*, 5, 19-21.

[37] Nakaoka, F., & Oda, N. (2003). Some properties of maximal open sets. *International journal of mathematics and mathematical sciences*, *21*, 1331-1340.

[38] Nakaoka, F., & Oda, N. (2001). Some applications of minimal open sets. *International journal of mathematics and mathematical sciences*, 27(8), 471-476.

[39] Atanassov, K. T. (1986). Intuitionistic fuzzy sets. F*uzzy sets and systems,* 20, 87-96.

[40] Zadeh, L. A. (1965). Fuzzy sets. *Information and control*, 8, 338-353.

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An Overview of Portfolio Optimization Using Fuzzy Data Envelopment Analysis Models

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Abstract

A combination of projects, assets, programs, and other components put together in a set is called a portfolio. Arranging these components helps to facilitate the efficient management of the set and subsequently leads to achieving the strategic goals. Generally, the components of the portfolio are quantifiable and measurable which makes it possible for management to manage, prioritize, and measure different portfolios. In recent years, the portfolio in various sectors of economics, management, industry, and especially project management has been widely applied and numerous researches have been done based on mathematical models to choose the best portfolio. Among the various mathematical models, the application of data envelopment analysis models due to the unique features as well as the capability of ranking and evaluating performances has been taken by some researchers into account. In this regard, several articles have been written on selecting the best portfolio in various fields, including selecting the best stocks portfolio, selecting the best projects, portfolio of manufactured products, portfolio of patents, selecting the portfolio of assets and liabilities, etc. After presenting the Markowitz mean-variance model for portfolio optimization, these pieces of research have witnessed significant changes. Moreover, after the presentation of the fuzzy set theory by Professor Lotfizadeh, despite the ambiguities in the selection of multiple portfolios, a wide range of applications in portfolio optimization was created by combining mathematical models of portfolio optimization.

Keywords: Portfolio optimization, Data envelopment analysis, Fuzzy sets, Intuitionistic fuzzy sets, Markowitz model.

1 | Introduction

Nowadays, economic and financial issues of choosing the best investment portfolio for all individuals and legal entities are of special importance and have wide aspects.

Investing means turning financial resources into one or more financial assets to achieve acceptable returns over a period of time and ultimately create wealth for the investor. Investing in monetary and capital markets is one of the most significant pillars of wealth creation and transfer in an economy [1].

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Because of the prosperity of the stock market as well as the desire to invest and the influx of people's capital into this market, choosing the best investment portfolio by achieving the highest return along with the lowest risk, which means choosing the best stock portfolio, are some of the main concerns of the society. Since several factors are involved in choosing the best portfolio, including the financial structure of companies, products, sales and profitability, asset volume, political, commercial, psychological risks, etc., the selection of the right portfolio can be complicated. As a result, using traditional methods in today's complex situation will not much work. The use of mathematical models, due to their flexibility and the involvement of various quantitative and qualitative factors in the process of figuring out the best portfolio, by integrating facts such as ambiguities and doubts helps to make the answers obtained from the models closer to the facts and makes the range of multiple choices predictable. Among the various mathematical models, data envelopment analysis as a widely used technique can be effective in achieving the best portfolio. Especially when the ambiguities are applied in the model, the results of fuzzy data envelopment analysis in portfolio optimization will be much more accurate and consistent with the facts.

In the current work, first, the basic definitions of the concepts are concisely presented and then an overview of some research done in the field of portfolio optimization using fuzzy data envelopment analysis has been made.

2| Definitions

Portfolio. It is the combination of assets or plans the investor or management acquires to achieve the desired return in a given period of time, by accepting the corresponding risks, and by converting some other financial assets. This concept, especially in stock market investing, means choosing and buying a portfolio of various stocks or diverse securities aiming to increase wealth.

Data Envelopment Analysis. It is a non-parametric method in the field of operations research whose task is to evaluate the performance or efficiency of units/portfolios. This method, by considering various inputs and outputs, evaluates performances of multiple units/portfolios and identifies efficient and inefficient units/portfolios. This model was first proposed by Farrell in 1957 [10] and since then, the model has made extensive advances in various sciences and a lot of research has been done in evaluating the performance of different units as well as ranking them [2] and [4].

Fuzzy. It means vague and indefinite in word. It is first introduced by Zadeh in 1965 leading to fundamental changes in the theory of classical mathematics [5]. This concept is based on logic and human decisions so that this logic can be considered as an extension of Aristotelian logic. Unlike the classical logic, which is in the state of right and wrong (zeros and ones), fuzzy logic is flexible, and the correctness or incorrectness of any logical statement is determined by the degree of membership (numerical between zero and one) [5].

In other words, instead of the characteristic function, the membership function is employed as follows:

$$\mathrm{f_A}: \mathrm{U} \to [0,1].$$

In which $f_A(u) \in [0,1]$ refers to the level or degree of belonging of u to A.

Intuitionistic Fuzzy. Although fuzzy sets can plot and investigate the ambiguity, they cannot discuss and plot all the ambiguities that occur in real life. To this end, in the cases with insufficient information, a developed fuzzy theory called intuitive fuzzy theory is used.

An intuitive fuzzy subset of or an intuitive fuzzy subset in U is a set such as A in which to each member in $u\epsilon U$, two degrees are assigned including membership degree and non-membership degree. In other words, for each set of A, the membership function is defined as $f_A: U \to [0,1] * \;[0,1]$, so that

$$F_A(u) = \left(\mu_A(u), \nu_A(u)\right), \quad 0 \leq \mu_A(u) \leq 1.$$

In the intuitive fuzzy sets, $\pi_A = 1 - \mu_A(u) - \nu_A(u)$ refers to the degree of doubt or ambiguity for each member $u \epsilon U$.

In all of the above-mentioned definitions, $\mu_A(u)$ and $\nu_A(u)$ are both fuzzy sets. On the other hand, the values of membership in intuitive fuzzy sets can be considered as $L = \{(x, y)\epsilon[0,1]^2 : 0 \leq x + y \leq 1\}$. Therefore, intuitive fuzzy sets are also called two-dimensional fuzzy sets [6] and [7].

3| Markowitz Model

Markowitz was the first person who introduced a model capable of introducing a suitable criterion for portfolio selection by considering both risks and returns together. The Markowitz mean-variance model is the most popular selection approach for the stock portfolio. This model is based on the following statements:

- *Investors are basically risk-averse and have the expected increased utility.*
- *Each investment option can be infinitely divisible.*
- *Investors select their portfolio based on the average and variance of expected returns.*
- *Investors have a one-period time horizon and this is the same for all investors.*
- *Investors prefer a higher return on a certain level of risk and, conversely, lower risk on a certain level of return.*

A commodity investment portfolio is a portfolio that has the highest return at a certain level of risk or has the lowest risk at a certain level of return [8].

4| Data Envelopment Analysis

Data envelopment analysis is a non-parametric method that is used to calculate the efficiency of homogeneous units, based on the inputs and outputs of the units, and finally the division of all units into efficient and inefficient units. This model was initially proposed by Farrell in 1957 [10] and then developed by Charnes et al. in 1978 [9] through which optimal solutions are sought by dividing a linear combination of outputs by a linear combination of inputs [9] and [10].

The CCR model of data envelopment analysis is expressed as follows:

$$\begin{aligned}
& f_k = \underset{u,v}{\mathrm{Max}} \frac{\sum_{n=1}^{N} (xo)_{nk} v_{nk}}{\sum_{m=1}^{M} (xi)_{mk} u_{mk}}, \\
& \text{s.t.} \\
& \frac{\sum_{n=1}^{N} (xo)_{nk} v_{nk}}{\sum_{m=1}^{M} (xi)_{mk} u_{mk}} \leq 1 \quad j = 1, \ldots, J, \\
& v_{nk}, u_{mk} \geq 0 \quad n = 1, \ldots, N, m = 1, \ldots, M.
\end{aligned}$$

Since data envelopment analysis models are considered as one of the multi-criteria decision-making methods, to regard multiple indicators and criteria in selecting the best options, various types of data envelopment analysis models can be utilized. One of the applications of data envelopment analysis models is for portfolio optimization. In this regard, several types of research have been done to evaluate the efficiency and ranking of stocks as well as to identify the effective variables [10].

Due to the increasing significance of investing in the stock market, choosing the optimal portfolio is one of the most important concerns of investors for which several methods of stock portfolio selection and investment have been presented so far.

Despite the income and profit that the investor can earn from the formation of her portfolio, the issue of reducing investment risk is of particular significance. Nowadays, risk overshadows all aspects of human life and always plays an important role in all matters and decisions of individuals.

Risk conventionally is a kind of danger that is said to happen due to uncertainty about the occurrence of an accident in the future, and the higher this uncertainty is, the higher the risk [1]. There are two viewpoints to risk definition:

First Viewpoint. The risk as any possible fluctuations of economic returns in the future.

Second Viewpoint. The risk as negative fluctuations of economic returns in the future.

In all financial markets, it is the principle that investors are always risk-averse. Therefore, risk will always be present along with return as the two main pillars in portfolio selection. In 1952, Markowit [8] for the first time, proposing the mean-variance model, proved that to form a stock portfolio, one could always minimize the risk by considering a certain level of return. Markowitz's model is known as the first mathematical model of stock portfolio optimization [2].

Before Markowitz introduced his model, it was traditionally believed that increasing diversity in the stock portfolio reduced portfolio risk, but they were unable to measure this risk. Markowitz considered the expected return per share as the average share return in previous periods and the risk per share as the variance of the return per share in previous periods. He showed that the average stock portfolio weight is equal to the stock returns, but the stock portfolio risk is not equal to the average stock weight risk. In order to calculate the expected return per share (E (R)), the shareholder must obtain the probable return on the securities (R) as well as the probability of the expected return (Pr) assuming that the sum of the probabilities is equal to one. In this case, the expected return is as follows:

$$E(R) = \sum_{i=1}^{m} R_i Pr_i.$$

In the above formula, m represents the number of potential returns per share.

Assuming the above, the stock portfolio returns $E(R_p)$ will be equal to the weighted return of each share.

$$E\left(R_p\right) = \sum_{i=1}^{m} w_i\, E(R_i),$$

$$\sum w_i = 1.$$

Where w_i is the weight of the stock portfolio for the ith share.

In the Markowitz model, the risk per share is considered equal to the return variance (VAR (R)) or its second root, the standard deviation (SD (R)) in previous periods [3].

$$VAR(R) = \sigma^2 = \sum_{r=1}^{m} (R_i - E(R))^2 Pr_i,$$

$$\mathrm{VAR(R)} = \sigma^2 = \sum_{r=1}^{m} (R_i - E(R))^2 Pr_i.$$

Accordingly, stock portfolio risk based on Markowitz model is shown below [2]:

$$V_\Omega = \mathrm{Min} \sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{\substack{l=1 \\ l \neq i}}^{n} w_i w_l \psi_{il}.$$

Where w_i and $w_l$are the weight of each share, σ_i^2 is the variance of stock returns, $\psi_{il}$is the covariance of double returns of shares and $V_\Omega$is the variance of stock returns.

Finally, the Markowitz model with two objective functions was defined as maximizing stock portfolio return and minimizing stock portfolio risk as follows:

$$\mathrm{Max}\ \ f_1(x) = \sum_{i=1}^{m} w_i \bar{r}_i ,$$

$$\mathrm{Min}\ \ f_2(x) = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{\substack{l=1 \\ l \neq i}}^{n} w_i w_l \psi_{il},$$

$$\mathrm{s.t:}$$

$$\sum w_i = 1,$$

$$w_i \geq 0.$$

In financial terms, the set of Pareto optimal solutions for the portfolio selection problem is called the efficient set or the efficient boundary. In other words, for a set of assets, a set of portfolios that have the least risk for a given return is called the efficient frontier. The efficient boundary is a non-descending function that shows the best interaction between risk and return. Markowitz used the concepts of mean returns, variance, and covariance to represent the efficient boundary. This model is usually called the *EV* model, in which *E* represents the mean and *V* represents the variance [11].

Tavana et al. [3], using the Markowitz model and using 7 macro criteria and 19 indicators (8 indicators as input and 11 indicators as output), examined various stock exchange industries and chose the best industries from among them to be in the stock portfolio. After calculating the variables, he entered them into the DEA model and calculated the Relative Financial Strength Indices (RFSI) of each company using the Anderson-Peterson model. After calculating the RFSI index, companies with financial strength greater than 0.9 in the BCC-O and CCR-I models as suitable investment options were selected. After selecting the companies, the weight of each share was determined using the Markowitz model (as shown below) and solving the model through genetic technique.

$$\mathrm{Max} \quad \sum_{i=1}^{N} \mu_i x_i - \sum_{i=1}^{N} L(y_i) - \lambda \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij} x_i x_j ,$$

$$\mathrm{s.t.}$$

$$\sum_{i=1}^{N} x_i \leq C^0,$$

$$y_i = |x_i - x_i^0| \quad i = 1, \dots, N,$$

$$x_i \geq 0 \quad i = 1, \dots, N.$$

Sharifighazvini et al. [12] used the Beasley index in a research to obtain a suitable mathematical model with internal market realities to extract the weights of participation of companies' stocks or the fund manager in order to obtain the variance-covariance matrix between firm returns, using the Markowitz model. In the Beasley benchmark, the average and standard deviation of each company's weekly returns and correlation coefficients between 225 Nikkei-listed companies are collected. Therefore, in order to obtain the variance-covariance matrix between firm returns, the relationship between the standard deviation of the yield of each pair of firms and the correlation coefficient between them should be used.

$$\rho_{ab} = \frac{\mathrm{cov}(a,b)}{\sqrt{\mathrm{var}(a).\mathrm{var}(b)}} \quad \Rightarrow \quad \mathrm{cov}(a,b) = \rho_{ab}\,.\sqrt{\mathrm{var}(a).\mathrm{var}(b)}.$$

Thus, the average weekly return and the variance-covariance matrix are obtained for 225 Nikkei-listed companies in the model. In addition, because the index provided by Beasley does not specify the name of each company, information about the price and number of shares of each company is generated as a random number from the corresponding ranges in 30 selected industries of the Iranian stock market. Also in the research, in addition to reducing risk and increasing returns, minimizing portfolio costs has been added to the Markowitz model as a goal function.

$$\begin{aligned}
&\mathrm{Max} \quad \sum_{i=1}^{N} w_i r_i\,,\\
&\mathrm{Min} \quad \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij},\\
&\mathrm{Min} \quad \sum_{i=1}^{N} z_i\,,\\
&\mathrm{s.t.}\\
&w_i = \frac{c_i x_i}{\sum_{i=1}^{N} c_i x_i} \quad i = 1, \dots, N\,,\\
&\sum_{i=1}^{N} w_i \le 1\,,\\
&\sum_{i=1}^{N} v_i \le 1\,,\\
&v_i w_i \le 0.15 \qquad i = 1, \dots, N,\\
&(1 - v_i) w_i \le 0.1 \qquad i = 1, \dots, N,\\
&x_i \le 0.05 S_i \qquad i = 1, \dots, N,\\
&\sum_{i \epsilon [j]} w_i \le 0.3 \qquad j = 1, \dots, t,\\
&\sum_{i=1}^{N} Z_i \le k\,,\\
&x_i \epsilon \{\mathbb{Z}\}\,, v_i, w_i \epsilon \{0,1\}, w_i \ge 0.
\end{aligned}$$

Where x_i is the number of selected stocks of type i, and $v_i$is a zero-one variable that specifies which company's stocks are greater than 15% of the portfolio value.

w_i is the ratio of type i shares in the portfolio and $z_i$is a zero and one variable to indicate the participation or non-participation of type i shares in the portfolio. Model parameters are also:

r_i: Average stock return i.

N: Variety of stocks from which the portfolio is selected.

σ_{ij}: Covariance between stocks i and j.

c_i : Stock price i.

s_i: Number of shares of the company i.

t: Number of industry.

k: Maximum portfolio variety.

In order to optimize the stock portfolio, Ahmadi et al. [13] used a combination of data envelopment analysis and heuristic factor analysis methods. In the presented research, each company is considered as a decision-making unit and input and output indicators are defined for each company. Since each of the indicators shows different dimensions of the companies' performance, it divided the output indicators into the input indicators so that the target indicators become a single and comparable indicator called performance. Finally, in order to reduce the dimensions of the problem and eliminate the correlation between the data, exploratory factor analysis was used.

Factor analysis is a set of different mathematical and statistical techniques that aim to simplify complex data sets. The main question is the answer to the question of whether a set of variables can be described in terms of the number of indicators or fewer factors than the variables and what attribute or feature each of the indicators (factors) represents. Factor analysis is used for correlation between variables.

Due to the fact that the model obtained in this research is an integer programming type, it cannot be solved by mathematical methods. Therefore, to solve the obtained model, the method of genetic algorithm and simulated annealing has been used.

5| Fuzzy Markowitz Model

Mashayekhi and Omrani [14] presented a new multi-objective model for portfolio selection including cross-performance data envelopment analysis and Markowitz mean-variance model in addition to presenting the risk and performance of the portfolio. To take the uncertainty into account, they considered the return on assets as trapezoidal fuzzy numbers and finally solved the model by employing the second type of genetic algorithm –NSGAII. The basic model presented in the current research is the mean-variance cross-sectional performance model of fuzzy Markowitz data envelopment analysis.

Suppose $\tilde{A} = (a, b, \alpha, \beta)$ is a trapezoidal fuzzy number. The cross-sectional performance model of fuzzy Markowitz MV is expressed as:

$$\text{Max } E\left(\sum_{i=1}^{N} \tilde{R}_i w_i\right) = \sum_{i=1}^{N} \frac{1}{2}\left[a_i + b_i + \frac{1}{3}(\beta_i - \alpha_i)\right] w_i ,$$

$$\text{Min } \sigma^2\left(\sum_{i=1}^{N} \tilde{R}_i w_i\right) = (\sum_{i=1}^{N} \frac{1}{2}[b_i - a_i + \frac{1}{3}(\alpha_i + \beta_i)] w_i)^2 + \frac{1}{72}[\sum_{i=1}^{N} (\alpha_i + \beta_i) w_i]^2 ,$$

$$\text{Max } \sum_{i=1}^{N} w_i \bar{e}_i,$$

$$\text{s.t.}$$

$$\sum_{i=1}^{N} z_i \leq h,$$
$$l_i z_i \leq w_i \leq u_i z_i \quad i = 1, \dots, N,$$
$$\sum_{i=1}^{N} w_i = 1 \ ,$$
$$w_i \geq 0 \quad i = 1, \dots, N.$$

Chen et al. [16] illustrated a comprehensive model for selecting a multi-objective portfolio in a fuzzy environment by combining the semi-variance mean model and the cross-sectional data envelopment analysis model. Then, the proposed model was re-formulated with the Sharp ratio by considering resource constraints as well as other constraints. The sharp factor model is expressed as single-factor and multi-factor models. In the single-factor model, it is assumed that the returns of all securities are correlated with each other for only one reason as a common factor to which all securities react with varying degrees of intensity. This common factor is usually considered as the market basket. Due to the fact that in the return study, the positive deviation from the average as a profit is more than expected and this issue is considered as a positive criterion, so to achieve a more consistent result with reality, semi-variance models are used, where only the negative deviation from the expected return is minimized. These models are called the mean-half variance model of portfolio optimization.

Besides the half-variance deviation, there are some downside risks. To assess the Portfolio Performance (PE), Chen and Guy [15] considered three types of data envelopment analysis approaches based on fuzzy portfolio evaluation models and based on the size of different risk scales, namely the probable variance, probable semivariance, and probable semi-absolute deviation.

6| Conclusion and Suggestion

In today's world, due to the variety of choices in each field, reviewing and selecting the best options in each field is of particular importance and this issue is much more significant for everyone to choose the best investment portfolio. To select the best investment portfolio in the general sense and the best stock portfolio, in particular, there are various criteria and indicators. Concentrating on the financial structure of companies, sales, profitability, business environment, various business, political risks, etc. can be effective in choosing a portfolio. Nowadays, according to the development of mathematical models, the use of conceptual mathematical models helps investors to be better informed about the returns and risk of different stock portfolios based on the patterns and principles of mathematical models and let them choose the best type of investment according to their standards as well as their investment policies. Due to the unique features of data envelopment analysis, it has attracted the interest of many researchers among a wide variety of mathematical models to study and select the best portfolios. Despite the ambiguous nature of the data, combining this analysis and Fuzzy concepts leads to coming closer to reality in the models such as the Markowitz model. This paper provides an overview of some of the research conducted in optimizing investment portfolios using data envelopment analysis models, Markowitz model and the concept of fuzzy data. With the advancement of fuzzy concepts and finding new definitions of fuzzy concepts that bring issues and models closer to the realities of society, the presented models can be transformed into a variety of new fuzzy concepts and further studied. One of the new definitions and developments of the fuzzy concept is the description and concept of intuitive fuzzy in which the non-membership function is considered in addition to the membership function in the fuzzy concept and interfered in the relevant calculations. Given that based on field research, very little research has been done in the field of portfolio optimization using the Markowitz model and intuitive fuzzy concepts, more appropriate choices can be suggested to investors for future research by developing portfolio fuzzy optimization models to models with intuitive fuzzy data.

References

[1] Heybati, F., & Haddadzadeh, R. (2008). Portfolio optimization using markowitz's mean-semi variance method on tehran stock exchange. *Future studies management*, *1*(19), 39-55. (**In Persian**). https://jmfr.srbiau.ac.ir/article_5714.html

[2] Rotela Junior, P., Rocha, L. C. S., Aquila, G., Balestrassi, P. P., Peruchi, R. S., & Lacerda, L. S. (2017). Entropic data envelopment analysis: a diversification approach for portfolio optimization. *Entropy*, *19*(9), 352.

[3] Tavana, M., Keramatpour, M., Santos-Arteaga, F. J., & Ghorbaniane, E. (2015). A fuzzy hybrid project portfolio selection method using data envelopment analysis, TOPSIS and integer programming. *Expert systems with applications*, *42*(22), 8432-8444.

[4] Banker, R. D., Charnes, A., & Cooper, W. W. (1984). Some models for estimating technical and scale inefficiencies in data envelopment analysis. *Management science*, *30*(9), 1078-1092.

[5] Zadeh, L. A. (1965). Fuzzy sets. *Information and control*, *8*(3), 338-353.

[6] Atanassov, K. (2016). Intuitionistic fuzzy sets. *International Journal Bioautomation*, *20*, 1.

[7] Edalatpanah, S. A. (2019). A data envelopment analysis model with triangular intuitionistic fuzzy numbers. *International journal of data envelopment analysis*, *7*(4), 47-58.

[8] Markowitz, H. M. (1978). *Portfolio selection*. Wily, New York.

[9] Charnes, A., Cooper, W. W., & Rhodes, E. (1978). Measuring the efficiency of decision making units. *European journal of operational research*, *2*(6), 429-444.

[10] Farrell, M. J. (1957). The measurement of productive efficiency. *Journal of the royal statistical society: series A (General)*, *120*(3), 253-281.

[11] Edalatpanah, S. A. (2018). Neutrosophic perspective on DEA. *Journal of applied research on industrial engineering*, *5*(4), 339-345.

[12] Sharifighazvini, M. R., Ghezavati, V. R., Raissi, S., & Makui, A. (2018). Integration of a new mcdm approach based on the dea, fanp with monlp for efficiency-risk assessment to optimize project portfolio by branch and bound: a real case-study. *Economic computation & economic cybernetics studies & research*, *52*(1).

[13] Ahmadi, M., Osman, M. H. M., & Aghdam, M. M. (2020). Integrated exploratory factor analysis and Data Envelopment Analysis to evaluate balanced ambidexterity fostering innovation in manufacturing SMEs. *Asia pacific management review*, *25*(3), 142-155.

[14] Mashayekhi, Z., & Omrani, H. (2016). An integrated multi-objective Markowitz–DEA cross-efficiency model with fuzzy returns for portfolio selection problem. *Applied soft computing*, *38*, 1-9.

[15] Chen, W., Gai, Y., & Gupta, P. (2018). Efficiency evaluation of fuzzy portfolio in different risk measures via DEA. *Annals of operations research*, *269*(1-2), 103-127.

[16] Chen, W., Li, S. S., Zhang, J., & Mehlawat, M. K. (2020). A comprehensive model for fuzzy multi-objective portfolio selection based on DEA cross-efficiency model. *Soft computing*, *24*(4), 2515-2526.

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Assessment and Linear Programming under Fuzzy Conditions

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Abstract

The present work focuses on two directions. First, a new fuzzy method using triangular/trapezoidal fuzzy numbers as tools is developed for evaluating a group's mean performance, when qualitative grades instead of numerical scores are used for assessing its members' individual performance. Second, a new technique is applied for solving Linear Programming problems with fuzzy coefficients. Examples are presented on student and basket-ball player assessment and on real life problems involving Linear Programming under fuzzy conditions to illustrate the applicability of our results in practice. A discussion follows on the perspectives of future research on the subject and the article closes with the general conclusions.

Keywords: Fuzzy set (FS), Fuzzy number (FN), Triangular FN (TFN), Trapezoidal FN (TpFN), Center of gravity (COG) defuzzification technique, Fuzzy linear programming (FLP).

1 | Introduction

Despite to the initial reserve of the classical mathematicians, fuzzy mathematics and Fuzzy Logic (FL) have found nowadays many and important applications to almost all sectors of human activity (e.g. [1], Chapter 6, [2] Chapters 4-8, [3], etc.). Due to its nature of characterizing the ambiguous real life situations with multiple values, FL offers among others rich resources for assessment purposes, which are more realistic than those of the classical logic [4]-[6].

Fuzzy Numbers (FNs), which are a special form of Fuzzy Sets (FS) on the set of real numbers, play an important role in fuzzy mathematics analogous to the role played by the ordinary numbers in the traditional mathematics. The simplest forms of FNs are the Triangular FNs (TFNs) and the Trapezoidal FNs (TpFNs).

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In the present work we study applications of TFNs and TpFNs to assessment processes and to Linear Programming (LP) under fuzzy conditions. The rest of the paper is formulated as follows: Section 2 contains all the information about FS, FNs and LP which is necessary for the understanding of its contents. Section 3 is divided in two parts. In the first part an assessment method is developed using TFNs/TpFNs as tools, which enables the calculation of the mean performance of a group of uniform objects (individuals, computer systems, etc.) with respect to a common activity performed under fuzzy conditions. In the second part a method is developed for solving LP problems with fuzzy data (Fuzzy LP). In Section 4 examples are presented illustrating the applicability of both methods to real world situations. The assessment outcomes are validated with the parallel use of the GPA index, while the solution of the FLP problems is reduced to the solution of ordinary LP problems by ranking the corresponding fuzzy coefficients. The article closes with a brief discussion for the perspectives of future research on those topics and the final conclusions that are presented in Section 5.

2| Background

2.1| Fuzzy Sets and Logic

For general facts on FS and FL we refer to [2] and for more details to [1]. The FL approach for a problem's solution involves the following steps:

Fuzzification of the problem's data by representing them with properly defined FSs.

Evaluation of the fuzzy data by applying principles and methods of FL in order to express the problem's solution in the form of a unique FS.

Defuzzification of the problem's solution in order to "translate" it in the natural language for use with the original real-life problem.

One of the most popular defuzzification methods is the Center of Gravity (COG) technique. When using it, the fuzzy outcomes of the problem's solution are represented by the coordinates of the COG of the membership function graph of the FS involved in the solution [7].

2.2| Fuzzy Numbers

It is recalled that a FN is defined as follows:

Definition 1. A FN is a FS A on the set R of real numbers with membership function

m_A: R →[0, 1], such that:

A is normal, i.e. there exists x in R such that $m_A(x) = 1$,

A is convex, i.e. all its a-cuts $A^a = \{x \in U: m_A(x) \geq a\}$, a in [0, 1], are closed real intervals, and

Its membership function $y = m_A(x)$ is a piecewise continuous function.

Remark 1. (Arithmetic operations on FNs): One can define the four basic arithmetic operations on FNS in the following two, equivalent to each other, ways:

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With the help of their a-cuts and the Representation-Decomposition Theorem of Ralesscou - Negoita ([8], Theorem 2.1, p.16) for FS. In this way the fuzzy arithmetic is turned to the well known arithmetic of the closed real intervals.

By applying the Zadeh's extension principle ([1], Section 1.4, p.20), which provides the means for any function f mapping a crisp set X to a crisp set Y to be generalized so that to map fuzzy subsets of X to fuzzy subsets of Y.

In practice the above two general methods of the fuzzy arithmetic, requiring laborious calculations, are rarely used in applications, where the utilization of simpler forms of FNs is preferred. For general facts on FNs we refer to [9].

2.3| Triangular Fuzzy Numbers (TFNs)

A TFN (*a, b, c*), with *a, b, c* in *R* represents mathematically the fuzzy statement "the value of *b* lies in the interval [*a, c*]". The membership function of (*a, b, c*) is zero outside the interval [*a, c*], while its graph in [*a, c*] consists of two straight line segments forming a triangle with the OX axis (*Fig.1*).

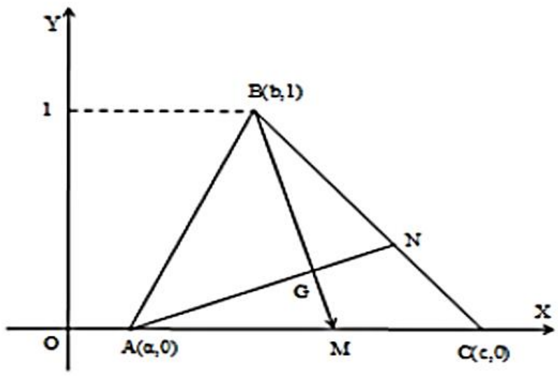

Fig. 1. Graph and COG of the TFN (a, b, c).

Therefore the analytic definition of a TFN is given as follows:

Definition 2. Let *a, b* and *c* be real numbers with $a < b < c$. Then the *TFN* (*a*, *b*, *c*) is a FN with membership function:

$$y = m(x) = \begin{cases} \frac{x-a}{b-a}, & x \in [a,b] \\ \frac{c-x}{c-b}, & x \in [b,c] \\ 0, & x < a \text{ or } x > c \end{cases}$$

Proposition 1. (Defuzzification of a TFN). The coordinates (*X*, *Y*) of the COG of the graph of the TFN (*a*, *b*, *c*) are calculated by the formulas $X = \frac{a+b+c}{3}$, $Y = \frac{1}{3}$.

Proof. The graph of the TFN (*a*, *b*, *c*) is the triangle ABC of *Fig.1*, with A (*a*, 0), B(*b*, 1) and C (*c*, 0). Then, the COG, say G, of ABC is the intersection point of its medians AN and BM. The proof of the proposition is easily obtained by calculating the equations of AN and BM and by solving the linear system of those two equations.

Remark 2. (Arithmetic Operations on TFNs**)** It can be shown [9] that the two general methods of defining arithmetic operations on FNs mentioned in *Remark 2* lead to the following simple rules for the addition and subtraction of TFNs:

Let A = (a, b, c) and B = (a_1, b_1, c_1) be two TFNs. Then:

The sum A + B = ($a+a_1, b+b_1, c+c_1$).

The difference A - B = A + (-B) = ($a-c_1, b-b_1, c-a_1$), where –B = ($-c_1, -b_1, -a_1$) is defined to be the opposite of.

In other words, the opposite of a TFN, as well as the sum and the difference of two TFNs are always TFNs. On the contrary, the product and the quotient of two TFNs, although they are FNs, they are not always TFNs, unless if a, b, c, a_1, b_1, c_1 are in R^+ [9].

The following two scalar operations can be also be defined:

k + A= (k+a, k+b, k+c), k$\in$R.

kA = (ka, kb, kc), if k>0 and kA = (kc, kb, ka), if k<0.

2.4 | Trapezoidal Fuzzy Numbers (TpFNs)

A TpFN (a, b, c, d) with a, b, c, d in R represents the fuzzy statement approximately in the interval $[b, c]$. Its membership function $y=m(x)$ is zero outside the interval [a, d], while its graph in this interval [a, d] is the union of three straight line segments forming a trapezoid with the X-axis (see *Fig. 2*),

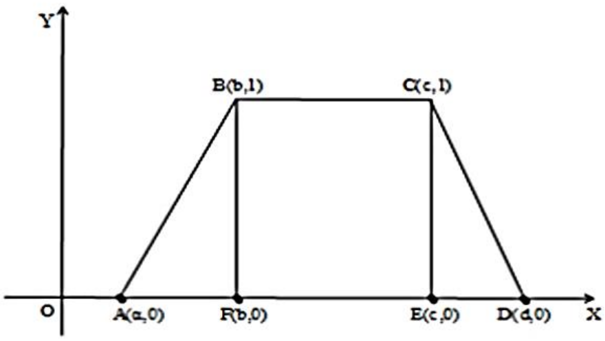

Fig. 2. Graph of the TpFN (a, b, c, d).

Therefore, the analytic definition of a TpFN is given as follows:

Definition 3. Let $a < b < c < d$ be given real numbers. Then the TpFN (a, b, c, d) is the FN with membership function:

$$y = m(x) = \begin{cases} \dfrac{x-a}{b-a}, & x \in [a,b] \\ x = 1, & x \in [b,c] \\ \dfrac{d-x}{d-c}, & x \in [c,d] \\ 0, & x < a \text{ and } x > d \end{cases}$$

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Remark 3. It is easy to observe that the TpFNs are generalizations of TFNs. In fact, the TFN (*a, b, d*) can be considered as a special case of the TpFN (*a, b, c, d*) with *b=c.*

The TFNs and the TpFNs are special cases of the $LR - FNs$ of Dubois and Prade [10]. Generalizing the definitions of TFNs and TpFNs one can define n-agonal FNs of the form $(a_1, a_2, \ldots, a_n)$ for any integer n, $n \geq 3$; e.g. see Section 2 of [11] for the definition of the hexagonal FNs.

It can be shown [9] that the addition and subtraction of two TpFNs are performed in the same way that it was mentioned in *Remark 2* for TFNs. Also, the two scalar operations that have been defined in *Remark 2* for TFNs hold also for TpFNs.

The following two propositions provide two alternative ways for defuzzifying a given TpFN:

Proposition 2. (GOG of a TpFN)**:** The coordinates (*X, Y*) of the COG of the graph of the TpFN (*a, b, c, d*) are calculated by the formulas $X = \frac{c^2+d^2-a^2-b^2+dc-ba}{3(c+d-a-b)}$, $Y = \frac{2c+d-a-2b}{3(c+d-a-b)}$.

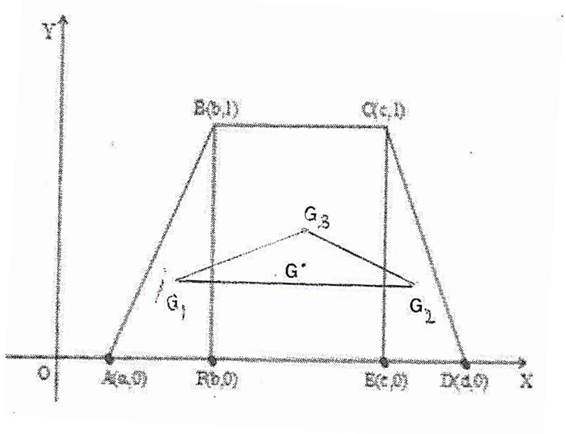
Proof. We divide the trapezoid forming the graph of the TpFN (*a, b, c, d*) in three parts, two triangles and one rectangle (*Fig. 2*). The coordinates of the three vertices of the triangle ABE are (*a,* 0), (*b,* 1) and (*b,* 0) respectively, therefore by *Proposition 4* the COG of this triangle is the point C_1 $(\frac{a+2b}{3}, \frac{1}{3})$.

Similarly one finds that the COG of the triangle FCD is the point C_2 $(\frac{d+2c}{3}, \frac{1}{3})$. Also, it is easy to check that the COG of the rectangle BCFE, being the intersection point of its diagonals, is the point C_3 $(\frac{b+c}{2}, \frac{1}{2})$. Further, the areas of the two triangles are equal to $S_1 = \frac{b-a}{2}$ and $S_2 = \frac{d-c}{2}$ respectively, while the area of the rectangle is equal to $S_3 = c - b$.

Therefore, the coordinates of the COG of the trapezoid, being the resultant of the COGs C_i (x_i, yi), for i=1, 2, 3, are calculated by the formulas $X = \frac{1}{S}\sum_{i=1}^{3} S_i x_i$, $Y = \frac{1}{S}\sum_{i=1}^{3} S_i y_i$ (1), where $S = S_1 + S_2 + S_3 = \frac{c+d-b-a}{2}$ is the area of the trapezoid [12].

The proof is completed by replacing the above found values of S, S_i, x_i and y_i, i = 1, 2, 3, in formulas (1) and by performing the corresponding operations.

Proposition 3. (GOG of the GOGs of a TpFN)**:** Consider the graph of the TpFN (*a, b, c, d*) (*Fig. 3*). Let G_1 and G_2 be the COGs of the rectangular triangles AEB and CFD and let G_3 be the COG of the rectangle



BEFC respectively. Then $G_1G_2G_3$ is always a triangle, whose COG has coordinates $X = \frac{2(a+d)+7(b+c)}{18}$, $Y = \frac{7}{18}$.

Fig. 3. The GOG of the GOGs of the TpFN (a, b, c, d).

Proof. By *Proposition 4* one finds that $G_1\ (\frac{a+2b}{3}, \frac{1}{3})$ and $G_2\ (\frac{d+2c}{3}, \frac{1}{3})$. Further, it is easy to check that the GOG G_3 of the rectangle BCFD, being the intersection of its diagonals, has coordinates $(\frac{b+c}{2}, \frac{1}{2})$.

The y – coordinates of all points of the straight line defined by the line segment G_1G_2 are equal to $\frac{1}{3}$, therefore the point G_3, having y – coordinate equal to $\frac{1}{2}$, does not belong to this line. Hence, by *Proposition 6*, the COG G' of the triangle $G_1G_2G_3$ has coordinates $X = (\frac{a+2b}{3} + \frac{d+2c}{3} + \frac{b+c}{2}) : 3 = \frac{2(a+d)+7(b+c)}{18}$ and $Y = (\frac{1}{3} + \frac{1}{3} + \frac{1}{2}) : 3 = \frac{7}{18}$.

Remark 4. Since the COGs G_1, G_2 and G_3 are the balancing points of the triangles AEB and CFD and of the rectangle BEFC respectively, the COG G' of the triangle $G_1G_2G_3$, being the balancing point of the triangle formed by those COGs, can be considered instead of the COG G of the trapezoid ABCD for defuzzifying the TpFN (*a, b, c, d*). The advantage of the choice of G' instead of G is that the formulas calculating the coordinates of G' (*Proposition 3*) are simpler than those calculating the COG G of the trapezoid ABCD (*Proposition 2*).

An important problem of the fuzzy arithmetic is the ordering of FNs, i.e. the process of determining whether a given FN is larger or smaller than another one. This problem can be solved through the introduction of a ranking function, say R, which maps each FN on the real line, where a natural order exists. Several ranking methods have been proposed until today, like the lexicographic screening [13], the use of an area between the centroid and original points [14], the subinterval average method [11], etc.

Here, under the light of *Propositions (1)* and *(3)* and of *Remark 4*, we define the ranking functions for TFNs and TpFNs as follows:

Definition 11. Let *A* be a FN. Then:

If *A* is a TFN of the form *A* {*a, b, c*}, we define $R(A) = \frac{a+b+c}{3}$.

If *A* is a TpFN of the form *A* {α, *b, c, d*}, we define $R(A) = \frac{2(a+d)+7(b+c)}{18}$.

2.5 | Linear Programming

It is well known that Linear Programming (LP) is a technique for the optimization (maximization or minimization) of a linear objective function subject to linear equality and inequality constraints. The feasible region of a LP problem is a convex polytope, which is a generalization of the three-dimensional polyhedron in the n-dimensional real space R^n, where n is an integer, $n \geq 2$.

A LP algorithm determines a point of the LP polytope, where the objective function takes its optimal value, if such a point exists. In 1947, Dantzig invented the SIMPLEX algorithm [15] that has efficiently tackled the LP problem in most cases. Further, in 1948 Dantzig, adopting a conjecture of John von Neuman, who worked on an equivalent problem in Game Theory, provided a formal proof of the theory of Duality [16]. According to the above theory every LP problem has a dual problem the optimal solution of which, if there exists, provides an optimal solution of the original problem. For general facts about the SIMPLEX algorithm we refer to Chapters 3 and 4 of [17].

LP, apart from mathematics, is widely used nowadays in business and economics, in several engineering problems, etc. Many practical problems of operations research can be expressed as LP problems. However, in large and complex systems, like the socio-economic, the biological ones, etc. ., it is often very difficult to solve satisfactorily the LP problems with the standard theory, since the necessary data cannot be easily determined precisely and therefore estimates of them are used in practice. The reason for this is that such kind of systems usually involve many different and constantly changing factors the relationships among which are indeterminate, making their operation mechanisms to be not clear. In order to obtain good results in such cases one may apply techniques FLP, e.g. see [18] and [19], etc.

3| Main Results

3.1| Assessment under Fuzzy Conditions

Assume that one wants to evaluate the mean performance of a group of uniform objects (individuals, computer systems, etc.) participating in a common activity. When the individual performance of the group's members is assessed by using numerical grades (scores), the traditional method for evaluating the group's mean performance is the calculation of the mean value of those scores. However, cases appear frequently in practice, where the individual performance is assessed by using linguistic instead of numerical grades. For example, this frequently happens for student assessment, usually for reasons of more elasticity that reduces the student pressure created by the existence of the strict numerical scores.

A standard method for such kind of assessment is the use of the linguistic expressions (labels) A = excellent, B = very good, C = good, D = fair and F = unsatisfactory (failed). In certain cases some insert the label E = less than satisfactory between D and F, while others use labels like B^+, B^-, etc., for a more strict assessment. It becomes evident that such kind of assessment involves a degree of fuzziness caused by the existence of the linguistic labels, which are less accurate than the numerical scores. Obviously, in the linguistic assessment the calculation of the mean value of the group's members' grades is not possible.

An alternative method for assessing a group's overall performance in such cases is the calculation of the Grade Point Average (GPA) index ([2], Chapter 6, p.125). The GPA index is a weighted mean calculated by the formula

$$GPA = \frac{0n_F + 1n_D + 2n_C + 3n_B + 4n_A}{n}. \tag{1}$$

In the above formula n denotes the total number of the group's members and n_A, n_B, n_C, n_D and n_F denote the numbers of the group's members that demonstrated excellent, very good, good, fair and unsatisfactory performance respectively. In case of the ideal performance ($n_A = n$) formula (1) gives that GPA = 4, whereas in case of the worst performance ($n_F = n$) it gives that $GPA = 4$. Therefore, we have in general that $0 \leq GPA \leq 4$, which means that values of GPA greater than 2 could be considered as indicating a more than satisfactory performance. However, since in *Eq. (1)* greater coefficients (weights) are assigned to the higher scores, it becomes evident that the GPA index reflects actually not the mean, but the group's quality performance.

Here a method will be developed for an approximate evaluation in such cases of the group's mean performance that uses TFNs or TpFNs as tools. For this, we need the following definition:

Definition 5. Let $A_i = (a_{1i}, a_{2i}, a_{3i}, a_{4i})$, i = 1, 2,…, n be TFNs/TpFNs (see *Remark 3*), where n is a nonnegative integer, $n \geq 2$. Then the mean value of the A_i's is defined to be the TFN /TpFN,

$$A = \frac{1}{n}(A_1 + A_2 + \ldots + A_n). \quad (2)$$

In case of utilizing TFNs as tools the steps of the new assessment method are the following:

Assign a scale of numerical scores from 1 to 100 to the linguistic grades A, B, C, D and F as follows: A (85–100), B (75–84), C (60–74), D (50–59) and F (0–49)[1].

For simplifying the notation use the same letters to represent the above grades by the TFNs.

A = (85, 92.5, 100), B = (75, 79.5, 84), C (60, 67, 74), D (50, 54.5, 59) and F (0, 24.5, 49), respectively, where the middle entry of each of them is equal to the mean value of its other two entries.

Evaluate the individual performance of all the group's members using the above qualitative grades. This enables one to assign a TFN A, B, C, D or F to each member. Then the mean value M of all those TFNs is equal to the TFN

$$M(a, b, c) = \frac{1}{n}(n_A A + n_B B + n_C C + n_D D + n_F F).$$

Use the TFN M (a, b, c) for evaluating the group's mean performance. It is straightforward to check that the three entries of the TFN M are equal to

$$a = \frac{85n_A + 75n_B + 60n_C + 50n_D + 0n_F}{n} \quad b = \frac{92.5n_A + 79.5n_B + 67n_C + 54.5n_D + 24.5n_F}{n}$$ and

$$c - \frac{100n_A + 84n_B + 74n_C + 59n_D + 49n_F}{n}.$$ Then, by *Proposition 6* one gets that

$$X(M) = \frac{a+b+c}{3} = \frac{92.5n_A + 79.5n_B + 67n_C + 54.5n_D + 24.5n_F}{n} = \frac{a+c}{2} = b \ (3).$$

Observe that, in the extreme (hypothetical) case where the lowest possible score has been assigned to each member of the group (i.e. the score 85 to nA members, the score 75 to nB members, etc.) the mean value of all those scores is equal to a. On the contrary, if the greatest score has been assigned to each member, then the mean value of all scores is equal to c. Therefore the value of X(M), being equal to the mean value of a and c, provides a reliable approximation of the group's mean performance.

[1]The scores assigned to the linguistic grades are not standard and may differ from case to case. For instance, in a more rigorous assessment one could take A(90-100), B (80-89), C(70-79), D (60-69), F(<60), etc.

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In cases where multiple referees of the group's performance exist one could utilize TpFNs instead of TFNs for evaluating the group's mean performance. In that case a different TpFN is assigned to each member of the group representing its individual performance, while the other steps of the method remain unchanged (see *Example 16*).

3.2| Fuzzy Linear Programming

The general form of a FLP problem is the following: Maximize (or minimize) the linear expression

$F = A_1x_1 + A_2x_2 + \ldots + A_nx_n$ subject to constraints of the form $x_j \geq 0$,

$A_{i1}x_1 + A_{i2}x_2 + \ldots + A_{in}x_n \leq (\geq) B_i$, where $i = 1, 2, \ldots, m$, $j = 1, 2,,,, n$ and A_j, A_{ij}, B_i are FNs. Here a new method will be proposed for solving FLP problems. We start with the following definition:

Definition 5. The Degree of Fuzziness (DoF) of a n-agonal FN $A = (a_1, \ldots, a_n)$ is defined to be the real number $D = a_n - a_1$. We write then $DoF(A) = D$.

The following two propositions are needed for developing the new method for solving FLP problems:

Proposition 4. Let A be a TFN with $DoF(A) = D$ and $R(A) = R$. Then A can be written in the form

A = (α, 3R-2α-D, α + D), where a is a real number such that $R - \frac{2D}{3} < a < R - \frac{D}{3}$.

Proof. Let A (a, b, c) be the given TFN, with a, b, c real numbers such that $a<b<c$. Then, since

$DoF(A) = c - a = D$, is $c = a + D$. Therefore, $R(A) = \frac{a+b+c}{3} = \frac{2a+b+D}{3} = R$, which gives that

$b = 3R-2a-D$. Consequently we have that $a < 3R-2a-D < a + D$. The left side of the last inequality implies that $3a < 3R-D$, or $a < R-\frac{D}{3}$. Also its right side implies that $-3a < 2D-3R$, or $a > R-\frac{2D}{3}$, which completes the proof.

Proposition 5. Let A be a TpFN with $DoF(A) = D$ and $R(A) = R$. Then A can be written in the form

$A = (a, b, c, a + D)$, where a, b and c are real numbers such that $a < b \leq c < a + D$ and

$b + c = \frac{18R-4a-2D}{7}$.

Proof. Let A (a, b, c, d) be the given TFN, with a, b, c, d real numbers such that $a < b \leq c < d$. Since

$D(A) = d - a = D$, it is $d = a + D$. Also, by *Definition 11* we have that $R = \frac{2(2a+D)+7(b+c)}{18}$ wherefrom one gets the expression of $b + c$ in the required form.

The proposed in this work method for solving a Fuzzy LP problem involves the following steps:

Ranking of the FNs A_j, A_{ij} and B_i.

Solution of the obtained in the previous step ordinary LP problem with the standard theory.

Conversion of the values of the decision variables in the optimal solution to FNs with the desired DoF.

The last step is not compulsory, but it is useful in problems of vague structure, where a fuzzy expression of their solution is often preferable than the crisp one (see *Examples 17* and *18*).

4| Applications

4.1| Examples of Assessment Problems

Example 1. *Table 1* depicts the performance of students of two Departments, say D_1 and D_2, of the School of Management and Economics of the Graduate Technological Educational Institute (T. E. I.) of Western Greece in their common progress exam for the course "Mathematics for Economists I" in terms of the linguistic grades A, B, C, D and F:

Table 1. Student performance in terms of the linguistic grades.

Grade	D_1	D_2
A	60	60
B	40	90
C	20	45
D	30	45
F	20	15
Total	170	255

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It is asked to evaluate the two Departments overall quality and mean performance.

Quality performance (GPA index): Replacing the data of *Table 1* to *Formula (1)* and making the corresponding calculations one finds the same value $GPA = \frac{43}{17} \approx 2.53$ for the two Departments that indicates a more than satisfactory quality performance.

Mean performance (using TFNs): According to the assessment method developed in the previous section it becomes clear that Table 1 gives rise to 170 TFNs representing the individual performance of the students of D_1 and 255 TFNs representing the individual performance of the students of D_2. Applying equation (2) it is straightforward to check that the mean values of the above TFNs are:

$D_1 = \frac{1}{170}.\ (60A+40B+20C+30D+20F) \approx (63.53,\ 73.5,\ 83.47)$, and $D_2 = \frac{1}{255}.\ (60A+90B+45C+45D+15F) \approx (65.88, 72.71, 79.53)$.

Therefore, *Eq. (3)* gives that $X(D_1) = 73.5$ and $X(D_2) = 72.71$. Consequently, both departments demonstrated a good (C) mean performance, with the performance of D_1 being slightly better.

Example 2. The individual performance of the five players of a basket-ball team who started a game was assessed by six different athletic journalists using a scale from 0 to 100 as follows: P_1 (player 1)**:** 43, 48, 49, 49, 50, 52, P_2**:** 81, 83. 85, 88, 91, 95, P_3: 76, 82, 89, 95, 95, 98, P_4**:** 86, 86, 87, 87, 87, 88 and P_5**:** 35, 40, 44, 52, 59, 62. It is asked to assess the mean performance of the five players and their overall quality performance by using the linguistic grades A, B, C, D and F. Also, for reasons of comparison, it is asked to approximate their mean performance in two ways, by using TFNs and TpFNs.

Mean performance: Adding the $5 * 6 = 30$ in total scores assigned by the journalists to the five players and dividing the corresponding sum by 30 one finds that the mean value of those scores is approximately equal to 72.07. Therefore the mean performance of the five players can be characterized as good (C).

Quality performance: A simple observation of the given data shows that 14 of the 30 in total scores correspond to the linguistic grade A, four to B, one to C, four to D and seven to F. Replacing those values to *Formula (1)* one finds that the GPA index is approximately equal to 2.47. Therefore, the five players' overall quality performance can be characterized as more than satisfactory.

Using TFNS: Forming the TFNs A, B, C, D and F and observing the $5*6 = 30$ in total player scores it becomes clear that 14 of them correspond to the TFN A, four to B, one to C, four to D and seven to F. The mean value of the above TFNs (*Definition 11*) is equal to $M = \frac{1}{30}(14A + 4B + C + 4D + 7F) \approx (58.33, 68.98, 79.63)$.

Therefore the mean performance of the five players is approximated by $X(M) = 68.98$ (good).

Using TpFNs: A TpFN (denoted, for simplicity, by the same letter) is assigned to each basket-ball player as follows: $P_1 = (0, 43, 52, 59)$, $P_2 = (75, 81, 95, 100)$, $P_3 = (75, 76, 98, 100)$, $P_4 = (85, 86, 88, 100)$ and $P_5 = (0, 35, 62, 74)$. Each of the above TpFNs describes numerically the individual performance of the corresponding player in the form (a, b, c, d), where a and d are the lower and higher scores respectively corresponding to his performance, while c and b are the lower and higher scores respectively assigned to the corresponding player by the athletic journalists. For example, the performance of the player P_1 was characterized by the journalists from unsatisfactory (scores 43, 48, 49, 49) to fair (scores 50, 52), which means that $a = 0$ (lower score for F) and $d = 59$ (lower score for D), etc.

The mean value of the TpFNs P_i, i =1, 2, 3, 4, 5 (*Definition 13*), is equal to $P = \frac{1}{5}\sum_{i=1}^{5} P_i = (47, 64.2, 79, 86.6)$.

Therefore, under the light of *Remark 10* and *Proposition 9* one finds that $X(P) = \frac{2(47+86.6)+7(64.2+79)}{18} \approx 70.53$. This outcome shows that the five players demonstrated a good (C) mean performance.

The outcomes obtained from the application of the assessment methods used in *Examples (15)* and *(16)* are depicted in *Tables (2)-(3)* below.

Table 2. The outcomes of Example 1.

Method	D_1	D_2	Performance
GPA index	2.53	2.53	More than satisfactory
TFNs	73.5	72.68	Good (C)

Table 3. The outcomes of Example 2.

Method		Performance
Mean value	72.07	Good (C)
GPA index	2.47	More than satisfactory
TFNs	68.98	Good (C)
TpFNs	70.53	Good (C)

Observing those Tables one can see that the fuzzy outcomes (TFNs/TpFNs) are more or less compatible to the crisp ones (mean value/GPA index). This provides a strong indication that the fuzzy assessment method developed in this work "behaves" well. The appearing, relatively small, numerical differences are due to the different "philosophy" of the methods used (mean and quality performance, bi-valued and fuzzy logic).

The approximation of the player mean performance (70.53) obtained in *Example 2* using TpFNs is better (nearer to the accurate mean value 72.07 of the numerical scores) than that obtained by using TFNs (68.98). This is explained by the fact that the TpFNs, due to the way of their construction, describe more accurately than the TFNs each player's individual performance. However, it is not always easy in practice to use TpFNs instead of TFNs. Another advantage of using TpFNs as assessment tools is that, in contrast to TFNs, they make possible the comparison of the individual performance of any two members of the group, even among those whose performance has been characterized by the same qualitative grade. At any case, the fuzzy approximation of a group's mean performance is useful only when no numerical scores are given assessing the idividual performance of its members.

4.2| Examples of FLP Problems

Example 3. In a furniture factory it has been estimated that the construction of a set of tables needs 2 - 3 working hours (w. h.) for assembling, 2.5 - 3.5 w. h. for elaboration (plane, etc.) and 0.75 - 1.25 w. h. for polishing, while the construction of a set of desks needs 0.8 - 1.2, 2 - 4 and 1.5 - 2.5 w. h. respectively for each of the above procedures. According to the factory's existing number of workers, at most 20 w. h. per day can be spent for the assembling, at most 30 w. h. for the elaboration and at most 18 w. h. for the polishing of the tables and desks. If the profit from the sale of a set of tables is between 2.7 and 3.3 hundred euros and of a set of desks is between 3.8 and 4.2 hundred euros[1], find how many sets of tables and desks should be constructed daily to maximize the factory's total profit. Express the problem's optimal solution with TFNs of DoF equal to 1.

Solution. Let x_1 and x_2 be the sets of tables and desks to be constructed daily. Then, using TFNs, the problem can be mathematically formulated as follows[2]:

Maximize $F = (2.7, 3, 3.3)x_1 + (3.8, 4, 4.2)x_2$ *subject to* $x_1, x_2 \geq 0$ and

$(2, 2.5, 3)x_1 + (0.8, 1, 1.2]x_2 \leq (19, 20, 21)$,

$(2.5, 3, 3.5)x_1 + (2, 3, 4)x_2 \leq (29, 30, 31)$,

$(0.75, 1, 1.25)x_1 + (1.5, 2, 2.5)x_2 \leq (15, 16, 17)$.

The ranking of the TFNs involved leads to the following LP maximization problem of canonical form:

Maximize $f(x_1, x_2) = 3x_1 + 4x_2$ *subject to* $x_1, x_2 \geq 0$ and $2.5x_1 + x_2 \leq 20$, $3x_1 + 3x_2 \leq 30$, and $x_1 + 2x_2 \leq 16$.

[1]The profit is changing depending upon the price of the wood, the salaries of the workers, etc.

[2] The mathematical formulation of the problem using TFNs is not unique. Here we have taken $b = \frac{a+c}{2}$ for all the TFNs involved, but this is not compulsory. The change of the values of the above TFNs, changes of course the ordinary LP problem obtained by ranking them, but the change of its optimal solution is relatively small.

Adding the slack variables s_1, s_2, s_3 for converting the last three inequalities to equations one forms the problem's first SIMPLEX matrix, which corresponds to the feasible solution *f(0, 0) = 0*, as follows:

$$\begin{bmatrix} x_1 & x_2 & s_1 & s_2 & s_3 & | & \text{Const.} \\ - & - & - & - & - & - & - \\ 2.5 & 1 & 1 & 0 & 0 & | & 20 = s_1 \\ 3 & 3 & 0 & 1 & 0 & | & 30 = s_2 \\ 1 & 2 & 0 & 0 & 1 & | & 16 = s_3 \\ - & - & - & - & - & | & - \\ -3 & -4 & 0 & 0 & 0 & | & 0 = f(0,0) \end{bmatrix}.$$

Denote by L_1, L_2, L_3, L_4 the rows of the above matrix, the fourth one being the *net evaluation row*. Since -4 is the smaller (negative) number of the net evaluation row and $\frac{16}{2} < \frac{30}{3} < \frac{20}{1}$, the *pivot element* 2 lies in the intersection of the third row and second column Therefore, applying the linear transformations $L_3 \rightarrow \frac{1}{2} L_3 = L'_3$ and $L_1 \rightarrow L_1 - L'_3$, $L_2 \rightarrow L_2 - 3L'_3$, $L_4 \rightarrow L_4 + 4L'_3$, one obtains the second SIMPLEX matrix, which corresponds to the feasible solution $f(0, 8) = 32$ and is the following:

$$\begin{bmatrix} x_1 & x_2 & s_1 & s_2 & s_3 & | & \text{Const.} \\ - & - & - & - & - & - & - \\ 2 & 0 & 1 & 0 & -\frac{1}{2} & | & 12 = s_1 \\ \frac{3}{2} & 0 & 0 & 1 & -\frac{3}{2} & | & 6 = s_2 \\ \frac{1}{2} & 1 & 0 & 0 & \frac{1}{2} & | & 8 = x_2 \\ - & - & - & - & - & | & - \\ -1 & 0 & 0 & 0 & 0 & | & 32 = f(0,8) \end{bmatrix}.$$

In this matrix the pivot element $\frac{3}{2}$ lies in the intersection of the second row and first column, therefore working as above one obtains the third SIMPLEX matrix, which is:

$$\begin{bmatrix} x_1 & x_2 & s_1 & s_2 & s_3 & | & \text{Const.} \\ - & - & - & - & - & - & - \\ 0 & 0 & 1 & -\frac{4}{3} & -\frac{3}{2} & | & 4 = s_1 \\ 1 & 0 & 0 & \frac{2}{3} & -1 & | & 4 = x_1 \\ 0 & 1 & 0 & -\frac{1}{3} & 1 & | & 6 = x_2 \\ - & - & - & - & - & | & - \\ 0 & 0 & 0 & \frac{2}{3} & 1 & | & 36 = f(4,6) \end{bmatrix}.$$

Since there is no negative index in the net evaluation row, this is the last SIMPLEX matrix. Therefore *f (4, 6) = 36* is the optimal solution maximizing the objective function. Further, since both the decision variables x_1 and x_2 are basic variables, i.e. they participate in the optimal solution, the above solution is unique.

Converting, with the help of *Proposition 15*, the values of the decision variables in the above solution to TFNs with DoF equal to 1, one finds that $x_1 = (a, 11\text{-}2a, a+1]$ with $\frac{10}{3} < a < \frac{11}{3}$ and $x_2 = (a, 17\text{-}2a, a+1)$ with. $\frac{16}{3} < a < \frac{17}{3}$. Therefore a fuzzy expression of the optimal solution states that the factory's maximal profit corresponds to a daily production between a and $\alpha+1$ groups of tables with $3.33 < a < 3.67$ and between a and $a+1$ groups of desks with $5.33 < a < 5.67$.

However, taking for example $\alpha = 3.5$ and $a = 5.5$ and considering the extreme in this case values of the daily construction of 4.5 groups of tables and 6.5 groups of desks, one finds that they are needed 33 in total w. h. for elaboration, whereas the maximum available w. h. are only 30. In other words, a fuzzy expression of the solution does not guarantee that all the values of the decision variables within the boundaries of the corresponding TFNs are feasible solutions.

Example 4. Three kinds of food, say F_1, F_2 and F_3, are used in a poultry farm for feeding the chickens, their cost varying between 38 - 42, 17 - 23 and 55 - 65 cents per kilo respectively. The food F_1 contains between 1.5 - 2.5 units of iron and 4 - 6 units of vitamins per kilo, F_2 contains 3.2 - 4.8, 0.6 – 1.4 and F_3 contains 1.7 – 2.3, 0.8 – 1.2 units per kilo respectively. It has been decided that the chickens must receive at least 24 units of iron and 8 units of vitamins per day. How must one mix the three foods so that to minimize the cost of the food? Express the problem's solution with TpFNs of DoF equal to 2.

Solution. Let x_1, x_2 and x_3 be the quantities of kilos to be mixed for each of the foods F_1, F_2 and F_3 respectively. Then, using TpFNs the problem's mathematical model could be formulated as follows[1]:

Minimize $F = (38, 39, 41, 42)\, x_1 + (17, 18, 22, 23)\, x_2 + (55, 56, 64, 65]\, x_3$ subject to $x_1, x_2, x_3 \geq 0$ and

$(1.5, 1.8, 2.2, 2.5)\, x_1 + (3.2, 3.5, 4.5, 4.8)\, x_2 + (1.7, 1.9, 2.1, 2.3]\, x_3 \geq [22, 23, 25, 26]$

$[4, 4.5, 5.5, 6]\, x_1 + [0.6, 0.8, 1.2, 1.4]\, x_2 + [0.8, 0.9, 1.1, 1.2] x_3 \geq (6, 7, 9, 1\ 0).$

The ranking of the TpFNs by *Definition 13* leads to the following LP minimization problem of canonical form:

Minimize $f(x_1, x_2, x_2) = 40x_1 + 20x_2 + 60x_3$ subject to $x_1, x_2, x_3 \geq 0$ and $2x_1 + 4x_2 + 2x_3 \geq 24$, $5x_1 + x_2 + x_3 \geq 8$.

The dual of the above problem is: the following:

Maximize $g(z_1, z_2) = 24z_1 + 8z_2$ subject to $t\ z_1, z_2 \geq 0$, $2z_1 + 5z_2 \leq 40$, $4z_1 + z_2 \leq 20$, $2z_1 + z_2 \leq 60$

Working similarly with *Example 3* it is straightforward to check that the last SIMPLEX matrix of the dual problem is the following:

[1]The problem's mathematical formulation using TpFNs is not unique, but the change of its optimal solution in each case is relatively small.

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$$\left[\begin{array}{cccccc|c} z_1 & z_2 & s_1 & s_2 & s_3 & & \text{Const.} \\ \hline 0 & 1 & \frac{2}{9} & \frac{1}{9} & 0 & & \frac{20}{3}=z_2 \\ 1 & 0 & -\frac{1}{18} & \frac{5}{18} & 0 & & \frac{10}{3}=z_1 \\ 0 & 0 & -\frac{1}{9} & -\frac{4}{9} & 1 & & \frac{140}{3}=s_3 \\ \hline 0 & 0 & \frac{4}{9} & \frac{52}{9} & 0 & & \frac{400}{3}=g(\frac{10}{3},\frac{20}{3}) \end{array}\right].$$

Therefore the solution of the original minimization problem is $f_{min} = f(\frac{4}{9}, \frac{52}{9}, 0) = \frac{400}{3}$.

In other words, the minimal cost of the chicken food is $\frac{400}{3} \approx 133$ cents and will be succeeded by mixing $\frac{4}{9} \approx 0.44$ kilos from food F_1 and $\frac{52}{9} \approx 5.77$ kilos from food F_2.

Converting the values of the decision variables in the above solution to TpFNs with DoF equal to 2 one finds with the help of *Proposition 16* that *x1, x2, x3* must be of the form *(a, b, c, a + 2)* with

$a < b \leq c < a + 2$, $b + c = \frac{18R-4a-4}{7}$ and $R = \frac{4}{9}$ *or* $R = \frac{52}{9}$ or $R = 0$ respectively.

For $R = \frac{4}{9}$ one finds that $b + c = \frac{4-4a}{7}$. Therefore $b < \frac{4-4a}{7}$ - b or $b < \frac{2-2a}{7}$ which gives that

$\alpha < \frac{2-2a}{7}$ or $\alpha < \frac{2}{9}$. Taking for example $\alpha = \frac{1}{9}$, one finds that $b < \frac{2-\frac{2}{9}}{7} = \frac{16}{63}$. Therefore, taking for example $b = \frac{15}{63}$, we obtain that $c = \frac{4-\frac{4}{9}}{7} - \frac{15}{63} = \frac{17}{63}$. Therefore $x_1 = (\frac{7}{63}, \frac{15}{63}, \frac{17}{63}, \frac{133}{63})$.

Working similarly for $R = \frac{52}{9}$ and $R = 0$ one obtains that $x_2 = (\frac{196}{63}, \frac{340}{63}, \frac{362}{63}, \frac{488}{63})$

and $x_3 = (-\frac{21}{63}, -\frac{15}{63}, -\frac{9}{63}, \frac{60}{63})$, respectively.

Therefore, since a TpFN (a, b, c, d) expresses mathematically the fuzzy statement that the interval [b, c] lies within the interval [a, d], a fuzzy expression of the problem's optimal solution states that the minimal cost of the chickens' food will be succeeded by mixing between $\frac{15}{63} \approx 0.24$, $\frac{17}{63} \approx 0.27$, between $\frac{340}{63} \approx 5.4$, $\frac{362}{63} \approx 5.75$ and between $-\frac{15}{63} \approx -0.24$, $-\frac{17}{63} \approx -0.27$ kilos from each one of the foods F_1, F_2 and F_3 respectively. The values of x_3 are not feasible and must be replaced by 0, whereas the values of x_1 and x_2 must be checked as we did in *Example 3*.

5| Discussion and Conclusions

The target of the present paper was two-folded. First, a combination of TFNs / TpFNs and of the COG defuzzification technique was used for assessment purposes. Examples were presented on student and basket-ball player assessment and the new fuzzy method was validated by comparing its outcomes with those of traditional assessment methods (calculation of the mean value of scores and of the GPA index). The advantage of this method is that it can be used for evaluating a group's mean performance when qualitative grades are used instead of numerical scores for assessing the individual performance of its members.

Second, a new technique was developed for solving Fuzzy LP problems by ranking the FNs involved and by solving the ordinary LP problem obtained in this way with the standard theory. Real-life examples were presented to illustrate the applicability of the new technique in practice. In LP problems with a vague structure a fuzzy expression of their solution is often preferable than a crisp one. This was attempted in the present work by converting the values of the decision variables in the optimal solution of the obtained ordinary LP problem to FNs with the desired DoF. The smaller the value of the chosen DoF, the more creditable is the fuzzy expression of the problem's optimal solution.

The new assessment method that has been developed in this work has a general character. This means that, apart for student and athlete assessment, it could be utilized for assessing a great variety of other human or machine (e.g. Case – Based Reasoning or Decision – Making systems) activities. This is an important direction for future research. Also a technique similar to that applied here for solving FLP problems could be used for solving systems of equations with fuzzy coefficients, as well as for solving LP problems and systems of equations with grey coefficients [20].

References

[1] Klir, G. J. & Folger, T. A. (1988). *Fuzzy sets, Uncertainty and information*. Prentice-Hall, London.

[2] Voskoglou, M. Gr. (2017). *Finite markov chain and fuzzy logic assessment models: emerging research and opportunities*. Createspace independent publishing platform, Amazon, Columbia, SC, USA.

[3] Voskoglou, M. G. (2019). *An essential guide to fuzzy systems*. Nova science publishers, New York, USA.

[4] Voskoglou, M. G. (2011). Measuring students modeling capacities: a fuzzy approach. *Iranian journal of fuzzy systems*, *8*(3), 23-33.

[5] Voskoglou, M. Gr. (2012). A study on fuzzy systems. *American journal of computational and applied mathematics*, 2(5), 232-240.

[6] Voskoglou, M. G. (2019). Methods for assessing human–machine performance under fuzzy conditions. *Mathematics*, *7*(3), 230.

[7] Van Broekhoven, E., & De Baets, B. (2006). Fast and accurate center of gravity defuzzification of fuzzy system outputs defined on trapezoidal fuzzy partitions. *Fuzzy sets and systems*, *157*(7), 904-918.

[8] Sakawa, M. (2013). *Fuzzy sets and interactive multiobjective optimization*. Springer science & business media. Plenum press, NY and London.

[9] Kaufman, A., & Gupta, M. M. (1991). *Introduction to fuzzy arithmetic*. New York: Van Nostrand Reinhold Company.

[10] Dubois, D. J. (1980). *Fuzzy sets and systems: theory and applications* (Vol. 144). Academic press, New York.

[11] Dinagar, D. S., & Kamalanathan, S. (2017). Solving fuzzy linear programming problem using new ranking procedures of fuzzy numbers. *International journal of applications of fuzzy sets and artificial intelligence*, *7*, 281-292.

[12] Wikipedia. (2014). *Center of mass: A system of particles.* Retrieved October 10, 2014, from http://en.wikipedia.org/wiki/Center_of_mass#A_system_of_particles

[13] Wang, M. L., Wang, H. F., & Lin, C. L. (2005). Ranking fuzzy number based on lexicographic screening procedure. *International journal of information technology & decision making*, 4(04), 663-678.

[14] Wang, Y. J., & Lee, H. S. (2008). The revised method of ranking fuzzy numbers with an area between the centroid and original points. *Computers & mathematics with applications*, *55*(9), 2033-2042.

[15] Dantzig, G. B. (1951). Maximization of a linear function of variables subject to linear inequalities. *Activity analysis of production and allocation*, *13*, 339-347.

[16] Dantzig, G.B. (1951). Maximization of a linear function of variables subject to linear inequalities. In T. C. Koopmans (Eds.), *Activity analysis of production and allocation* (pp. 339–347). Wiley& Chapman - Hall, New York, London.

[17] Dantzig, G. B. (1983). Reminiscences about the origins of linear programming. In *Mathematical programming the state of the art* (pp. 78-86). Springer, Berlin, Heidelberg.

[18] Taha, H. A. (1967). *Operations research – an introduction, Second Edition*. Collier Macmillan, N. Y., London.

[19] Tanaka, H., & Asai, K. (1984). Fuzzy linear programming problems with fuzzy numbers. *Fuzzy sets and systems*, *13*(1), 1-10.

[20] Verdegay, J. L. (1984). A dual approach to solve the fuzzy linear programming problem. *Fuzzy sets and systems*, *14*(2), 131-141.

[21] Voskoglou, M. G. (2018). Solving linear programming problems with grey data. *Oriental journal of physical sciences*, *3*(1), 17-23.

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Using Interval Type-2 Fuzzy Logic to Analyze Igbo Emotion Words

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Abstract

Several attempts had been made to analyze emotion words in the fields of linguistics, psychology and sociology; with the advent of computers, the analyses of these words have taken a different dimension. Unfortunately, limited attempts have so far been made to using Interval Type-2 Fuzzy Logic (IT2FL) to analyze these words in native languages. This study used IT2FL to analyze Igbo emotion words. IT2F sets are computed using the interval approach method which is divided into two parts: the data part and the fuzzy set part. The data part preprocessed data and its statistics computed for the interval that survived the preprocessing stages while the fuzzy set part determined the nature of the footprint of uncertainty; the IT2F set mathematical models for each emotion characteristics of each emotion word is also computed. The data used in this work was collected from fifteen subjects who were asked to enter an interval for each of the emotion characteristics: Valence, Activation and Dominance on an interval survey of the thirty Igbo emotion words. With this, the words are being analyzed and can be used for the purposes of translation between vocabularies in consideration to context.

Keywords: Affective computing, Valence, Activation, Dominance, Vocabularies

1 | Introduction

Words are vital in our description and understanding of emotions and means different things to different people based on different instances and some are uncertain [1]. Hence, there is a need for a model that can capture the uncertainties of these words. Emotions are feelings or involuntary physiological response of a person to a situation, words or things. Emotion is defined using two approaches: the classical approach and the use of multidimensional space. The classical approach uses fixed number of emotion classes such as {positive, non-positive}, {negative, non-negative}, and {angry, non-angry} in describing emotion-related states.

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However, this approach lacks the ability to handle many types of real-life emotions while the second approach represents emotion using each point in the multidimensional space and the dimensions that are mostly used are valence, activation, and dominance [2] and [3]. Valence represents negative to positive axis, activation represents calm to excited axis while dominance represents weak to strong axis in the three dimensional space [4]. However, the two approaches have failed to represent the uncertainty a person has about the emotion words as reflected in real-world and are somewhat vague and not precisely defined sets.

Therefore, application of fuzzy set model is well suitable because of its ability to cope with uncertainties [5] and [6] and can represent each emotion dimension using intervals rather than fixed points [7].

The Type -1 Fuzzy Set (T1-FS) is a generalization of traditional classical sets in which a concept can possess a certain degree of truth, where the truth value may range between completely true and completely false. However, T1-FS lacks the ability to adequately represent or directly handle data uncertainty because its membership function is crisp in nature [8]. The optimal design of fuzzy systems enables making decisions based on a structure built from the knowledge of experts and guided by membership functions and fuzzy rules [28]. "The membership functions of type-2 fuzzy sets are (n+2)-dimensional, while membership functions of type-1 fuzzy sets are only (n+1)-dimensional (assuming that the universe of discourse has n dimensions). Thus, type-2 fuzzy sets allow more degrees of freedom in representing uncertainty" [32]. Type-2 Fuzzy Set (T2-FS) which is an extension of T1-FS was used in [9] to address the inabilities of T1-FLS. Interval Type-2 Fuzzy Logic (IT2FL) is a simplified version of the general T2FLS that uses intervals to handle uncertainty in the membership function. The structure of T2-FIS is similar to T1-FLS but with additional component called the type reducer modified to accommodate T2F set. Type Reducer is used to reduce the output of the T2 inference engine to type-1 before defuzzification. An IT2FL can better model intrapersonal and interpersonal uncertainties, which are intrinsic to natural language, because the membership grade of an IT2 Fuzzy set is an interval instead of a crisp number as in a T1 FS [31].

Though, many works have been done in the field of emotion words, [8], [10]- [18], however, attempts to analyze emotion words in Igbo non-English languages have not been impacted in any way. This study uses IT2FL to analyze Igbo Emotion Words. The term "Igbo" in this context is used to describe the language spoken primarily by Igbo ethnic group in Nigeria. The Igbo belong to the Sudanic linguistic group of the Kwa division according to [19] and [20]. This wide presence of the Igbos is the basis for selecting the emotions words in Igbo language for analysis using IT2FL.

This study used IT2FL to analyze Igbo emotion words because of its ability to adequately handle emotion word uncertainties described by its Footprint Of Uncertainty (FOU), which is the uncertainty about the union of all the primary MFs. Uncertainty is a characteristic of information, which may be incomplete, inaccurate, undefined, inconsistent. Primary Membership Functions with where both the standard deviation and the uncertain are popular FOUs for a Gaussian because of their parsimony and differentiability [29]. IT2F sets are computed using the interval approach method which is divided into two parts: the data part and the fuzzy set part. The data part preprocessed data and its statistics computed for the interval that survived the preprocessing stages while the fuzzy set part determined the nature of the footprint of uncertainty; the IT2F set mathematical models for each emotion characteristics of each emotion word is also computed. The data used in this work was collected from 15 (fifteen) subjects who were asked to enter an interval for each of the emotion characteristics: Valence, Activation and Dominance on an interval survey of the 30 (thirty) Igbo emotion words.

IT2F sets are computed using the interval approach method which is divided into two parts: the data part and the fuzzy set part. The data part preprocessed data and its statistics computed for the interval that survived the preprocessing stages while the fuzzy set part determined the nature of the footprint of uncertainty; the IT2F set mathematical models for each emotion characteristics of each emotion word is also computed. The data used in this work was collected from 15 (fifteen) subjects who were asked to enter an interval for each of the emotion characteristics: Valence, Activation and Dominance on an interval survey of the 30 (thirty) Igbo emotion words. This paper is organized as follows: In the following section, the Emotion Space, Emotion Vocabularies and Variables, the Igbos and Emotions, T2FLS and Sets, and the IT2FLS are described. The research methodology is presented, the results and discussion are described and the conclusions are drawn.

1.1| Emotion Space, Emotion Vocabularies and Variables

The emotion space E can be considered as a set of all possible emotions and it is represented using variables on the Cartesian product space of valence, activation and dominance scales. An emotional variable ε represents an arbitrary region in the emotion space i.e. $\varepsilon \subset E$. An emotional vocabulary in *Eq. (1)*,

$$V = (W_V, eval_V), \tag{1}$$

is a set of words W_v and a function $eval_v$ that maps words of W_v to their corresponding region in the emotional space, $eval_v : W_v \to E$. Thus, an emotional vocabulary can be seen as a dictionary for looking up the meaning of an emotion word. Words in an emotional vocabulary can be seen as constant emotional variables.

1.2| Igbos and Emotions

Emotions affect every human including the Igbo's and the Igbo emotions have become even more complex over time hence when the Igbos speak of their emotions, the analysis, interpretation and translation to other languages is highly needed. For instance, an Igbo businessman may say "iwe ne enwe m" which means "I am angry". The emotion word in his statement is "iwe" translated as "anger" in English could either be with regards to a business situation or a person. Assuming it is a person that is an individual who annoyed him, it would not change his disposition about his business hence at almost the same time the same businessman could be heard saying "oba go" meaning "it has entered" which insinuates happiness of some sought.

1.3| Type-2 Fuzzy Logic System and Sets

T2FLS is the generalized standard type-1 in order to accommodate and handle more uncertainties in the MF [21]. According to [22] "words mean different things to different people". T2FLS is based on the T2F sets where the MF has multiple values for a crisp input of x, making the need for the creation of a 3-dimensional MF for all x $\mathcal{E}$ X. A characteristic feature of T2FS is the FOU, which is the union of all primary memberships and upper MFs and a lower MFs that are the bounds for the FOU of a T2Fs. Uncertainty in relations and uncertainty in values of the variables are majorly the types of uncertainty considered in developing systems since the high overlapping of the MFS, defining precise values for linguistic variables is not possible [30].

Given a T2Fs $\tilde{A}$, the representation of $\tilde{A}$ is as shown *Eq. (2)*.

$$\tilde{A} = \left\{ \left((x, u), \mu_{\tilde{A}}(x, u) \right) | x \in X, u \in J_x \in [0,1] \right\}. \tag{2}$$

Where $\mu_{\tilde{A}}(x, u)$ is the type-2 fuzzy MF in which $0 \leq \mu_{\tilde{A}}(x, u) \leq 1$

1.4| Interval Type-2 Fuzzy Logic System (IT2FLS)

As a result of the computational complexity of using a general T2FS, IT2Fs is mostly used as a special case an express as, when all $\mu_{\tilde{A}}(x, u) = 1$, then $\tilde{A}$ can rightly be described as an IT2FL as seen in *Eq. (3)* and in *Fig. 1*. The interval type-2 membership function is always equal to 1.

$$\tilde{A} = \int_{x \in X} \int_{u \in \int x} \frac{1}{(x,u)}, \int x \in [0,1]. \quad (3)$$

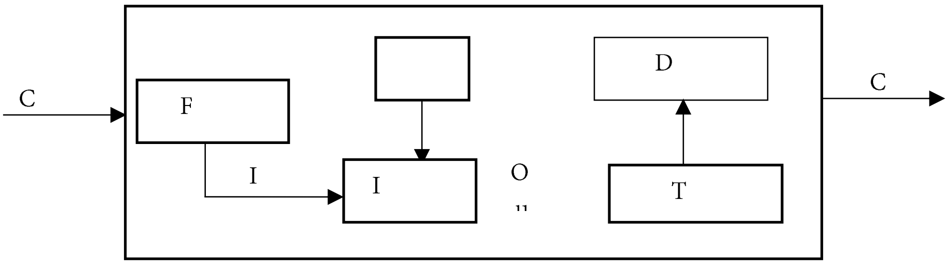

Fig. 1. Typical structure of interval type-2 fuzzy logic system [23].

IT2FLS consists of the fuzzifier, rule base, Inference Engine (IE), Type Reducer (TR). Fuzzification module maps the crisp input to a T2Fs using Gaussian MF. Inference Engine module evaluates the rules in the knowledge base against T2Fs from Fuzzification, to produce another T2Fs. TR uses Karnik-Mendel algorithm to reduce an IT2Fs to T1Fs. Defuzzification module maps the fuzzy set produced by TR to a crisp output using center of gravity defuzzification method. Fuzzy knowledge base stores rules generated from experts' knowledge used by the IE.

2| Research Methodology

The paper employs interval approach where data and fuzzy set parts are considered.

2.1| Interval Approach (IA)

The IA is a method for estimating MFs where the subject does not need to be knowledgeable about fuzzy sets and it has a simple and unambiguous mapping from data to FoU. Feilong and Mendel [24] used the IA to capture the strong points of two previous methods: the Person-MF Approach and the End-point Approach. Using the IA, the data collected from different subjects is subjected to probability distribution. The mean and standard deviation of the distribution are then mapped into the parameters of a T1 MF which are then transformed into T2 MF from which the IT2 MF is derived. The IA is divided into two parts namely the Data Part and the Fuzzy Set Part (FSP) as seen in *Figs. (2)* and *(3)*, respectively.

2.1.1| The Data Part (DP)

The DP of the IA consists of data collection, data preprocessing and probability distribution assignment parts as shown in *Fig. 2*. The DP highlights the valence layer and this is repeated for each word in the vocabulary. In *Fig. 2*, the following steps are carried out to achieve an input to the FSP.

Data Collection. Here, interval survey is performed to collect human intuition about fuzzy predicate.

Data Preprocessing. Pre-processing for n interval data $\left[a^{(i)}, b^{(i)}\right]$, $i=1,\ldots,n$, are performed consisting of 3 stages:

Bad Data Processing. At this stage, results from the survey that do not fall within the given range are removed. If interval end-points given by the respondents satisfy,

$$\left.\begin{array}{l} 0 \le a^i \le 10 \\ 0 \le b^i \le 10 \\ b_i \ge a_i \end{array}\right\} \forall\ i = 1, \cdots, n, \tag{4}$$

then accept the interval; else reject it. After this stage what remains is $n' \le n$ intervals

Outlier Processing. At this stage, a Box and Whisker test is used to eliminate outliers. Outliers are points which do not satisfy

$$\begin{array}{l} a^{(i)} \in [Q_a(0.25) - 1.5IQR_a, Q_a(0.75) + 1.5IQR_a] \\ b^{(i)} \in [Q_b(0.25) - 1.5IQR_b, Q_b(0.75) + 1.5IQR_b] \end{array}. \tag{5}$$

$Q_a(p)$ and $IQR_a = Q_a(0.75) - Q_a(0.25)$ are the p quartile and inter-quartile range for the left end-points and $Q_b(p)$ and $IQR_b = Q_b(0.75) - Q_b(0.25)$ are the p quartile and inter-quartile range for the right end-points, respectively.

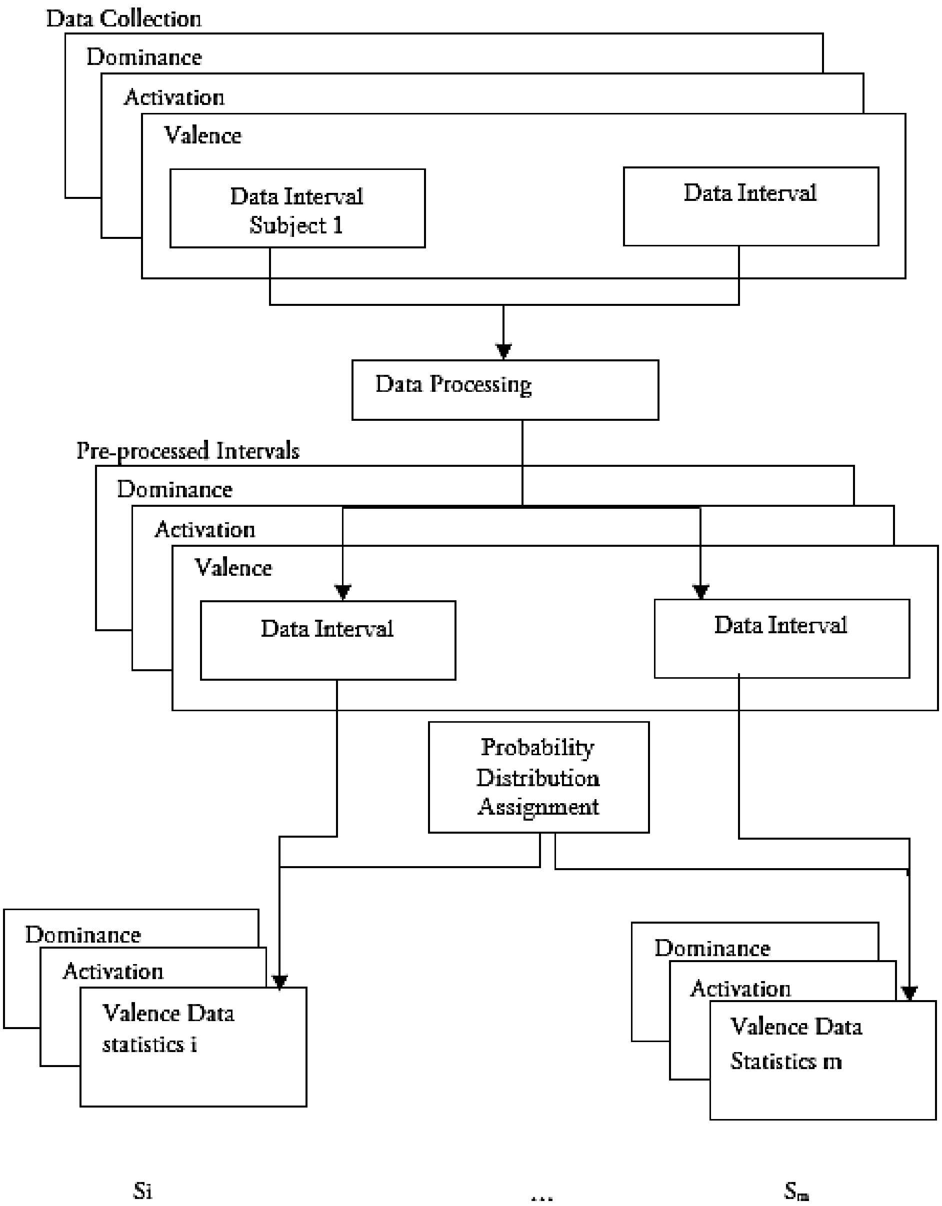

Fig. 2. The data part of the interval approach [25].

Tolerance-Limit Processing. Here, tolerance-limit test is processed using *Eqs. (6)* and *(7)* and if the interval passes, it is accepted else, it is rejected. Then the data intervals are reduced to $m'' \le m'$.

$$a^{(l)} \in [m_l - k\sigma_l, m_l + k\sigma_l,]. \tag{6}$$

$$b^{(r)} \in [m_r - k\sigma_r, m_r + k\sigma_r,]. \tag{7}$$

k is determined by confidence analysis, such that with a $100(1-\gamma)\%$ confidence the given limits contain at least the proportion$1-\alpha$ measurements.

After the data preprocessing, we are left with m data intervals such that $1 \leq m \leq n$.

Assign Probability Distribution to the Interval Data. Here, a probability distribution is assigned to each subject's data interval. The distribution can be uniform, triangular, normal, etc. but for the purpose of this work, the uniform distribution is used.

Compute Data Statistics for the Interval Data. The data statistics $m_l, \sigma_l, m_r and \sigma_r,$ which are the sample means and standard deviation of the left- and right-end points respectively, are computed based on interval data that remains. The last two steps are often merged since they work hand in hand.

The probability distribution $S_i = (m_i, \sigma_i)$ is assigned to the remaining intervals, then the statistics are calculated using the formula for random variables with random distribution stated as

$$m = (a + b)/2 \text{ and} \tag{8}$$

$$\sigma = (b - a)/\sqrt{12}. \tag{9}$$

Then the probability distribution, S_i, is evaluated in *Eq. (10)* and forms the input to the fuzzy set part

$$S_i = \left(m_Y^i, \sigma_Y^i\right) \forall \ i = 1, \ldots, m \tag{10}$$

The S_i is then input to the next stage, the fuzzy set part.

2.1.2| The Fuzzy Set Part (FSP)

The fuzzy set part constructs the interval type-2 fuzzy sets as shown in *Fig. 3*. It takes the result of Data part, i.e. the S_i as input from which it creates the IT2Fs. The Layers denote individual fuzzy sets for valence, activation and dominance. This framework is repeated for each word in the vocabulary.

In *Fig. 3*, the fuzzy set part of IT2FL process is divided into nine (9) steps. Step 1, establishes FS uncertainty measures while Step 2 selects the T1FS model. In Step 3, the uncertainty measures for the selected models are computed. The uncertainty measures on the symmetrical interior triangle compute the mean and Standard Deviation (SD) derived from the mean and SD of the triangular and uniform distributions. In step 4, the S_i from the data part is recollected. The mean and standard SD from the interval model is equated with the corresponding parameters from the previous step. In Step 5, the nature of the FoU is established using the parameters of the models associated with each interval to classify whether an interval should be mapped to an interior MF, or a left or right shoulder MF. The input interval is mapped to a shoulder MF if the parameters show that the distribution is out of the range of the scales. In Step 6, the T1Fs is computed with the intervals and the decision is made in the previous step. Since the MFs derived in this step are based on statistics and not the raw intervals themselves, another preprocessing stage is carried out to delete inadmissible T1FS is with range outside of the limit of the scale of the variable of interest. In Step 7, the fuzzy sets derived are the subject-specific T1FSs. The aggregate is taken to compute the IT2FSs that contain all the subject-specific T1FS in its FOU. This aggregation can be said to be a type-2 union of T1FSs, where the embedded T1FSs describe the FOU of IT2FSs. This includes Steps 8 and 9. After these steps, the IT2FSs word model is derived. The study analyzes the 30 emotion words collected from a wide range of psychological domains based on their MF values of the three characteristics: Valence, Activation, and Dominance.

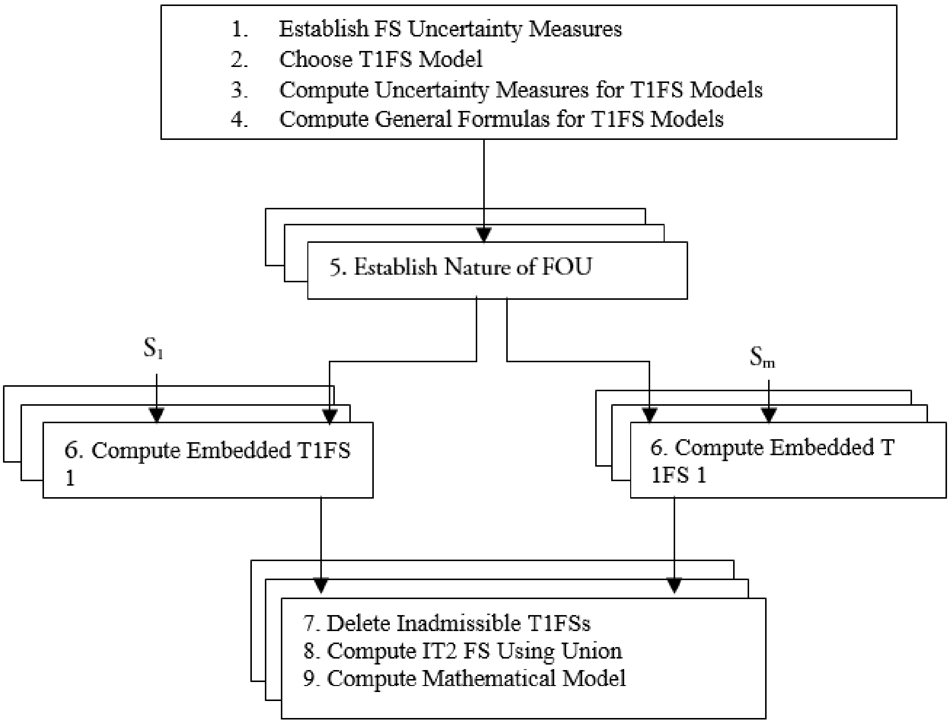

Fig. 3. Fuzzy set part of the interval approach [25].

3| System Design

The architecture for analyzing emotion words from Igbo using IT2FL presented in *Fig. 4*. The architecture is made up of the Knowledge Engine, the Knowledge Base and the User Interface. The Knowledge base further comprises the Database Model, the Interval Surveys and the IT2FL Model. The IT2FL model is composed of the Interval Approach Data Part and the Interval Approach Fuzzy Set Part. The knowledge engine stores all the variables required for the system, the knowledge base stores values for the variables defined in the knowledge engine. The Database stores an organized data to be used in the system as collected from the subjects. Interval survey is where the data are collected in a range of intervals which defines a subject's view of each emotion word given with regards to the emotion characteristics defined in the knowledge engine. The IT2FL model is used to represent words using IT2FSs. The IT2FL comprises Interval Approach which is made up of data and fuzzy set parts. The data part processes the data and makes it ready for use in estimating the MF of each emotion characteristics in each emotion word. The fuzzy set part evaluates the MF representing each emotion characteristics of each word.

3.1| The Knowledge Engine for Analyzing Igbo Emotion Words

For the purpose of this paper, the variables used include the emotional characteristics (Valence, Activation and Dominance). Valence defines an emotion word on the basis of its positivity or negativity. Activation defines an emotion word on the basis of its calmness or excitement. Dominance defines an emotion word on the basis of its submissiveness or aggressiveness. The Igbo emotion vocabulary used in this work is contained in the knowledge engine. The emotion vocabulary used in this research work is the Igbo Emotions vocabulary of 30 words which include: Iwe, Obi Uto, Onuma, Ujo, Ntukwasi Obi, Obi Ojo, Onu, Anuri, Egwu, Ihunanya, Mgbagwoju Anya, Mwute, Ikpe Mara, Enyo, Ihere, Iwe Oku, Ekworo, Anyaufu, Anya Ukwu, Obi Ike, Nrugide, Akwa Uta, Kpebisiri Ike, Kenchekwube, Keechiche, Mkpako, Nwayoo, Obi Abuo, Obi Mgbawa, Ara.

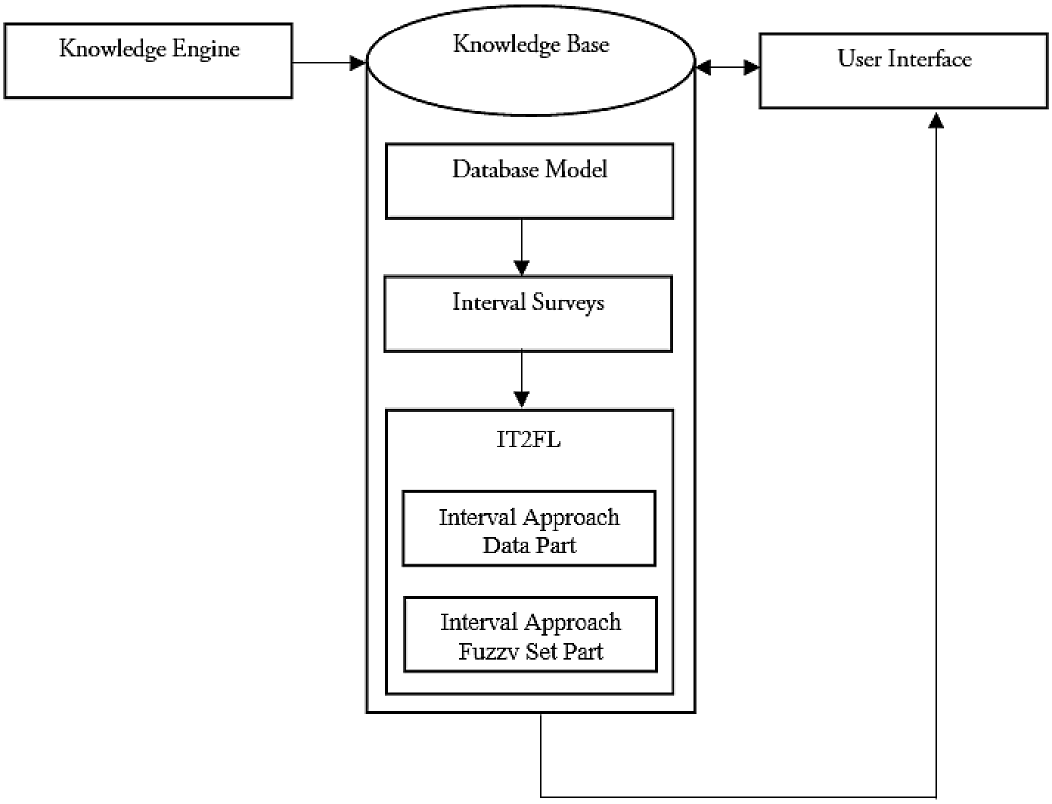

Fig. 4. Architecture for analyzing Igbo emotion words using interval IT2FL.

3.2| The Database Model

The IT2F inference system uses the data as organized in the database. The database schema is created to hold the collected data, data statistics and the membership function values, respectively.

3.3| The Interval Survey

An interval survey is carried out using questionnaires which are given to 15 native speakers of Igbo. The questionnaire began by giving the users instructions then sequentially providing the words to the users. The reason for the interval survey is to collect human intuition about fuzzy predicate which is emotion in this case and provision is given for the subject to give an interval within the range 0-10 that defines the emotion characteristics.

3.4| The IT2FL Model to Analyze Igbo Emotion Words

The IT2FL model for analyzing Igbo emotion words using Interval Approach. The interval approach is used because it takes on the strength of the interval endpoints and the person membership function approaches. Based on *Figs. (2)* and *(3),* the data and fuzzy set parts of Interval Approach for analyzing Igbo emotion words using IT2FL are evaluated.

Table 1. The Igbo emotion words and the english equivalence.

Igbo	English
Iwe	Anger, Annoyance
Obi Uto	Happiness, Delighted
Onuma	Wrath, Great Anger
Ujo	Fear, Shock
Ntukwasi Obi	Trust
Obi Ojo	Wickedness, Bitter
Onu	Joy
Anuri	Gladness
Egwu	Great Fear, Dread
Ihunaya	Love
Mgbagwoju Anya	Confused
Mwute	Regret
Ikpe Mara	Guilty
Enyo	Suspicious
Ihere	Shame
Iwe Oku	Hot-Temper
Ekworo	Jealousy
Anyaufu	Envy
Anya Ukwu	Greed, Discontentment
Obi Ike	Confidence, Courage
Nrugide	Persuasion
AkwaUta	Regret, Condemned
Kpebisiri Ike	Determination, Strong-willed
Kenchekwube	Hopeful
Keechiche	Worry
Mpako	Arrogance, Pride
Nwayoo	Gentle, Calm
Obi Abuo	Doubt, Unsteady
Obi Mgbawa	Heartbroken
Ara	Mad, Disturbed

3.4.1| Interval approach data part for analyzing Igbo emotion words using IT2FL

Using *Fig. 2*, in the data collection part, data are collected using the interval surveys from 10 subjects who are native speakers of the Igbo Language. The 30 emotion words are randomly ordered and presented to the respondents. Each was asked to provide the end-points on an interval for each word on the scale of 0-10. Data processing is performed in estimating the MF of each emotion characteristic in each emotion word and then calculates the statistics of the data intervals. Validation of the data intervals is done starting at 'n' intervals at the data preprocessing part. Bad data processing is performed on the collected dataset and the intervals that do not satisfy the condition in *Eq. (4)* is removed. After bad processing, outlier processing is carried out on the remaining dataset and the intervals that do not satisfy the box and whisker test in *Eq. (5)* are eliminated. Tolerance –limit processing is performed on the remaining intervals and the data intervals are accepted if they satisfy *Eqs. (6)* and *(7)* otherwise, they are rejected.

The constant 'k' is used as estimated by Mendel and Liu [26] and [27] as shown in *Table 2* for 10 intervals at 0.95 confidence limit since we have an average of ten intervals per characteristic.

Table 2. Tolerance factor k for a number of collected data (m'), a proportion of the data (1 - α), and a confidence level 1- γ.

m'	1-γ = 0.95		γ = 0.99	
	1 − α		1 − α	
	0.9	0.95	0.9	0.95
10	2.839	3.379	3.582	4.265
15	2.48	2.954	2.945	3.507
20	2.31	2.752	2.659	3.168
30	2.14	2.549	2.358	2.841
50	1.996	2.379	2.162	2.576
100	1.874	2.233	1.977	2.355
1000	1.709	2.036	1.736	2.718
∞	1.645	1.96	1.645	1.96

Reasonable data intervals are evaluated for the overlapped data intervals and only reasonable data intervals are kept. At this point, the intervals remain the same, i.e. $m = m''$. After the data has been validated, the mean and SD of the data intervals are computed in *Eqs. (8)* and *(9)* on the assumption that the data intervals are uniformly distributed and the probability distribution, S_i, computed in *Eq. (10)* and then becomes an input to the fuzzy set part.

3.4.2| Interval approach fuzzy set part for analyzing Igbo emotion words using

The Interval Approach Fuzzy Set Part for Analyzing Igbo emotion words using IT2FL is used to evaluate the MF based on *Fig. 3*. It consists of nine steps. Step 1 selects a T1FS and computes the mean and SD of the data intervals using symmetrical triangle interior T1FS, left-shoulder T1FS and the right-shoulder T1FS only. In Step 2, the mean and standard deviation are calculated to establish FS uncertainty measures using *Eqs. (11)* and *(12)*.

$$m_A = \frac{\int_{aMF}^{bMF} x\mu_A(x)dx}{\int_{aMF}^{bMF} \mu_A(x)dx}. \tag{11}$$

$$\sigma_A = \left[\frac{\int_{aMF}^{bMF}(x-m_A)^2\mu_A(x)dx}{\int_{aMF}^{bMF}\mu_A(x)dx}\right]^{1/2}. \tag{12}$$

Obviously, $\mu_A(x)/\int_{aMF}^{bMF}\mu_A(x)dx$ is the probability distribution of x, where$x \in [a_{MF}, b_{MF}]$, then m_A and σ_A are the same as the mean and standard deviation used in probability. In Step 3, the Uncertainty Measures are computed for T1FS by calculating the mean and SD for symmetric Triangle Interior (TI) and the Left-Shoulder (LS) and Right-Shoulder (RS) T1MFs using *Eqs. (13) – (15)*, respectively.

$$\text{TI: TI: } m_{MF} = (a_{MF} + b_{MF})/2; \sigma_{MF} = (b_{MF} - a_{MF})/2\sqrt{6}. \tag{13}$$

$$\text{LS: } m_{MF} = (2a_{MF} + b_{MF})/3; \sigma_{MF} = \left[\frac{1}{6}\left[(a_{MF} + b_{MF})^2 + 2a_{MF}^2\right] - m_{MF}^2\right]^{1/2}. \tag{14}$$

$$\text{RS: } m_{MF} = (2a_{MF} + b_{MF})/3; \sigma_{MF} = \left[\frac{1}{6}\left[(a'_{MF} + b'_{MF})^2 + 2a'^2_{MF}\right] - m'^2_{MF}\right]^{1/2}. \tag{15}$$

Where, $a\prime_{MF} = M - b_{MF}$; $b\prime_{MF} = M - a_{MF}$; $m\prime_{MF} = M - m_{MF}$. In Step 4, we compute for the parameters of T1FS models by equating the mean and SD of a T1FS to the mean and SD of the data intervals i.e. $m^i_{MF} = m^i_Y$ and $\sigma^i_{MF} = \sigma^i_Y$ to have *Eqs. (16) – (18)*.

$$\text{IT: } a_{MF} = (a+b)/2 - \sqrt{2}(b-a)/2; b_{MF} = (a+b)/2 + \sqrt{2}(b-a)/2. \quad (16)$$

$$\text{LS: } a_{MF} = (a+b)/2 - (b-a)/\sqrt{6}; b_{MF} = (a+b)/2 + \sqrt{6}(b-a)/3. \quad (17)$$

$$\text{RS:} a_{MF} = M - (a'+b')/2 - (b'-a')/\sqrt{6};\ b_{MF} = M - (a'+b')/2 + \sqrt{6}(b'-a')/3. \quad (18)$$

Where, $a\prime = M - b$ and $b\prime = M - a$. In Step 5, the nature of the FOU is established by mapping the 'm' data intervals into an IT, LS or a RS FOUs using a scale of [0, 10] if and only if (i), (ii) or (iii).

$$\left.\begin{matrix} a^i_{MF} \geq 0 \\ b^i_{MF} \leq 10 \\ b^i_{MF} \geq a^i_{MF} \\ \text{(i)} \end{matrix}\right\} \quad \left.\begin{matrix} a^i_{MF} \geq M \\ b^i_{MF} \leq 10 \\ b^i_{MF} \geq a^i_{MF} \\ \text{(ii)} \end{matrix}\right\} \quad \left.\begin{matrix} a^i_{MF} \geq 0 \\ b^i_{MF} \leq M \\ b^i_{MF} \geq a^i_{MF} \\ \text{(iii)} \end{matrix}\right\} \forall \ i = 1, \dots, m \quad (19)$$

From the mapping, we achieve, 12 TI FOUs, 35 LS FOUs and 23 RS FOUs respectively. In Step 6, embedded T1FSs is computed where the actual mapping to a T1FS is applied to the remaining 'm' data intervals for each word using *Eq. (20)*.

$$A^i = \left(a^i, b^i\right) \rightarrow \left(a^i_{MF}, b^i_{MF}\right), i = 1, \dots, m. \quad (20)$$

In Step 7, we delete inadmissible T1FSs for all data intervals for any T1FS that do not satisfy (*19(i)*) – (*19(iii)*) and m reduces to m*. Step 8 computes an IT2FS using the representation theorem for an IT2FS $\tilde{A}$ in *Eq. (21)*,

$$\tilde{A} = \bigcup_{i=1}^{m*} A^i. \quad (21)$$

Where, A^i is just the computed ith embedded T1FS. In Step 9, the mathematical model for FOU ($\tilde{A}$) is computed by first approximating the parameters of UMF ($\tilde{A}$) and the LMF($\tilde{A}$) for each of the FOU models as shown in *Eqs. (22) – (27)*, respectively.

$$\left.\begin{matrix} \underline{a}_{MF} \equiv \min\limits_{i=1,\dots m*} \left\{a^i_{MF}\right\} \\ \bar{a}_{MF} \equiv \max\limits_{i=1,\dots m*} \left\{a^i_{MF}\right\} \end{matrix}\right\}. \quad (21)$$

$$\left.\begin{matrix} \underline{b}_{MF} \equiv \min\limits_{i=1,\dots m*} \left\{b^i_{MF}\right\} \\ \bar{b}_{MF} \equiv \max\limits_{i=1,\dots m*} \left\{b^i_{MF}\right\} \end{matrix}\right\}. \quad (22)$$

$$C^i_{MF} = \frac{a^i_{MF} + b^i_{MF}}{2}. \quad (23)$$

$$\left.\begin{matrix} \underline{C}_{MF} \equiv \min\limits_{i=1,\dots m*} \left\{C^i_{MF}\right\} \\ \bar{C}_{MF} \equiv \max\limits_{i=1,\dots m*} \left\{C^i_{MF}\right\} \end{matrix}\right\}. \quad (24)$$

$$p = \frac{\underline{b}_{MF}\left(\bar{C}_{MF} - \bar{a}_{MF}\right) + \bar{a}_{MF}\left(\underline{b}_{MF} - \underline{C}_{MF}\right)}{\left(\bar{C}_{MF} - \bar{a}_{MF}\right) + \left(\underline{b}_{MF} - \underline{C}_{MF}\right)}. \quad (25)$$

$$\mu_p = \frac{\underline{b}_{MF} - p}{\left(\underline{b}_{MF} - \underline{C}_{MF}\right)}. \quad (26)$$

The mathematical model of the UMF and the LMF are shown in *Table 3* for TI, L-S and R-S FOU.

	Upper MF	Lower MF
Triangle (Interior) FOU	$(\underline{a}_{MF}, 0)$, $(\underline{C}_{MF}, 1)$, $(\overline{C}_{MF}, 1)$, $(\overline{b}_{MF}, 0)$	$(\underline{a}_{MF}, 0)$, $(\overline{a}_{MF}, 0)$, (p, μ_p), $(\underline{b}_{MF}, 0)$, $(\overline{b}_{MF}, 0)$
Left-Shoulder FOU	$(0, 1)$, $(\overline{a}_{MF}, 1)$, $(\overline{b}_{MF}, 0)$	$(0, 1)$, $(\underline{a}_{MF}, 1)$, $(\underline{b}_{MF}, 0)$, $(\overline{b}_{MF}, 0)$
Right-Shoulder FOU	$(\underline{a}_{MF}, 0)$, $(\underline{b}_{MF}, 1)$, $(M, 1)$	$(\underline{a}_{MF}, 0)$, $(\overline{a}_{MF}, 0)$, $(\overline{b}_{MF}, 1)$, $(M, 1)$

4| Model Experiment

This paper uses IT2FL to analyze Igbo emotion words. The IT2F sets are computed using the interval approach method which comprises data part and fuzzy set part. In order to illustrate the methodology proposed in this paper, we conduct some experiments for Igbo emotion words described in this work.

4.1| The Data Part

Table 4. shows the data intervals collected from the subjects. Bad data are processed and sample presented in *Table 5*. Outlier processing is performed in *Table 5* and the sample result is presented in *Table 6*. The tolerance limit processing in *Table 6* is shown in *Table 7*. Sample result of reasonable interval processing is seen in *Table 8*. The data statistics is computed for all the surviving *m* data intervals and sample are shown in *Table 9*.

Table 4. Parts of the data intervals collected from the subjects.

		S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	S12	S13	S14	S15
	Val	8-10	0-8	3-6	7-9	5-8	5-7	0-5	7-10	6-8	6-9	5-8	8-10	7-10	4-5	4-8
	Act	3-5	1-6	4-7	5-8	7-9	6-7	7-10	0-4	6-10	7-10	7-10	8-10	0-5	9-10	5-9
Iwe	Dom	7-10	2-10	8-10	| | |	7-10	8-10	2-9	6-8	8-10	8-10	8-10	8-10	5-8	8-10	6-9
	Val	9-10	5-6	4-7	7-10	5-8	6-8	5-10	1-4	0-5	0-4	3-5	1-3	5-8	4-5	3-8
	Act	9-10	7-9	5-8	5-8	7-10	8-10	1-10	9-10	8-10	5-9	8-10	8-10	6-9	7-10	5-9
Obi Uto	Dom	0-3	2-6	6-9	6-9	8-10	4-6	8-10	3-5	0-5	4-8	5-8	1-3	8-10	3-5	4-10
	Val	7-9	3-9	8-10	9-10	5-8	4-7	2-7	7-10	9-10	8-10	8-10	8-10	7-10	5-8	4-8
	Act	0-3	2-6	5-8	4-9	9-10	4-6	7-9	5-8	9-10	6-10	3-5	8-10	0-6	7-10	3-7
Onuma	Dom	8-10	3-9	7-10	8-9	5-8	6-9	2-6	10	9-10	7-10	4-6	8-10	5-8	7-10	5-9
	Val	0-3	4-10	2-4	1-3	5-8	6-9	8-10	1-3	8-10	5-8	7-9	8-10	8-10	4-6	2-10
	Act	0-4	2-6	3-5	5-8	7-10	3-6	4-5	0-2	0-5	2-7	0-5	1-3	5-7	7-8	3-9
Ujo	Dom	4-5	3-9	6-8	5-9	7-10	0-4	8-9	0-2	0-5	1-4	3-6	1-3	0-5	8-10	5-10
	Val	7-10	5-7	1-3	6-8	6-10	0-4	3-10	8-10	0-5	1-5	0-3	1-3	7-10	4-7	1-9
	Act	2-3	4-8	4-6	6-10	6-8	4-7	1-9	7-9	0-5	3-6	3-5	1-3	7-10	7-8	2-8
Ntukwasi Obi	Dom	1-3	5-6	8-10	4-8	6-9	4-7	4-9	6-7	0-5	5-8	3-5	1-3	8-10	8-10	5-7
	Val	5-8	0-10	2-5	6-10	5-10	6-10	2-4	5-8	8-10	8-10	8-10	8-10	6-10	4-5	2-8
Obi Ojo	Act	0-2	2-8	1-4	5-10	6-10	5-7	4-8	7-9	0-2	6-10	3-5	5-6	5-9	7-8	3-6

Table 5. Sample result of bad data processing.

		Valence	Activation	Dominance
1	IWE	15	15	14
2	OBI UTO	15	15	15
3	ONUMA	15	15	14
4	UJO	15	15	15
5	NTUKWASI OBI	15	15	15

Table 6. Sample result of outlier processing.

		Valence	Activation	Dominance
1	IWE	9	9	2
2	OBI UTO	12	5	11
3	ONUMA	5	10	5
4	UJO	15	13	12
5	NTUKWASI OBI	15	11	12

Table 7. Sample result of tolerance-limit processing.

		Valence	Activation	Dominance
1	IWE	9	9	2
2	OBI UTO	12	5	11
3	ONUMA	5	10	5
4	UJO	15	13	12
5	NTUKWASI OBI	15	11	12

Table 8. Sample result of reasonable interval processing.

		Valence	Activation	Dominance
1	IWE	9	9	2
2	OBI UTO	12	5	11
3	ONUMA	5	10	5
4	UJO	15	13	12
5	NTUKWASI OBI	15	11	12

4.2| The Fuzzy Set Part

In the fuzzy set part, only the symmetrical triangle interior T1FS, left-shoulder T1FS and the right-shoulder T1FS are used. The mean and SD are the same as in the data part as shown *Table 9*. The uncertainty measure for the chosen T1FS models are computed as is shown in parts in *Table 10*. The nature of FOU for each emotion characteristic for each word is determined as shown in *Table 11*. The embedded T1FSs, aMF and bMF of each interval are calculated as shown in *Table 12* where M=5. The LMF and the UMF of the FOU ($\tilde{A}$) for the five experimental intervals are calculated and summarized in *Table 13*.

Table 9. The sample mean and SD computed for all the surviving m data intervals.

			Mean						Standard Deviation					
			S1	S2	S3	S4	…	S15	S1	S2	S3	S4	…	S15
		Val	0	4	4.5	0	…	6	0	2.309401	0.866025	0	…	1.154701
1	Iwe	Act	4	3.5	5.5	6.5	…	7	0.57735	1.443376	0.866025	0.866025	…	1.154701
		Dom	0	0	0	0	…	0	0	0	0	0	…	0
							…						…	
							…						…	
		Val	0	5.5	5.5	0	…	5.5	0	0.288675	0.866025	0	…	1.443376
2	Obi Uto	Act	0	0	6.5	6.5	…	7	0	0	0.866025	0.866025	…	1.154701
		Dom	1.5	4	7.5	7.5	…	0	0.866025	1.154701	0.866025	0.866025	…	0
							…						…	
							…						…	
		Val	0	0	0	0	…	6	0	0	0	0	…	1.154701
3	Onuma	Act	1.5	4	6.5	6.5	…	5	0.866025	1.154701	0.866025	1.443376	…	1.154701
		Dom	0	0	0	0	…	7	0	0	0	0	…	1.154701
							…						…	
							…						…	
		Val	1.5	7	3	2	…	6	0.866025	1.732051	0.57735	0.57735	…	2.309401
4	Ujo	Act	2	4	4	6.5	…	0	1.154701	1.154701	0.57735	0.866025	…	0
		Dom	4.5	6	7	7	…	0	0.288675	1.732051	0.57735	1.154701	…	0
							…						…	
							…						…	
		Val	8.5	6	2	7	…	5	0.866025	0.57735	0.57735	0.57735	…	2.309401
5	Ntukwasi Obi	Act	2.5	6	5	0	…	5	0.288675	1.154701	0.57735	0	…	1.732051
		Dom	2	5.5	0	6	…	6	0.57735	0.288675	0	1.154701	…	0.57735

Table 10. Sample result of the uncertainty measures for T1FS models.

	m_{MF}	σ_{MF}
Interior	5	2.041241
Left	3.333333	3.535887
Right	3.333333	2.022657

Table 11. Sample result of the nature of the FOU for each emotion characteristics.

			Interior	Left-Shoulder	Right-Shoulder
		Val		1	
1	Iwe	Act		1	
		Dom	1		
		Val		1	
2	Obi Uto	Act	1		
		Dom			1
		Val	1		
3	Onuma	Act		1	
		Dom	1		

Table 11. (Continued).

			Interior	Left-Shoulder	Right-Shoulder
4	Ujo	Val	1		
		Act	1		
		Dom	1		
5	Ntukwasi Obi	Val	1		
		Act	1		
		Dom	1		

Table 12. Embedded T1FS for each emotion characteristics.

			amf	Bmf
1	Iwe	Val	6.183503	8.632993
		Act	7.183503	9.632993
		Dom	5.585786	8.414214
2	Obi Uto	Val	6.183503	8.632993
		Act	5.37868	9.62132
		Dom	1.275255	3.724745
3	Onuma	Val	4.37868	8.62132
		Act	7.183503	9.632993
		Dom	4.171573	9.828427
4	Ujo	Val	7.585786	10.41421
		Act	6.792893	8.207107
		Dom	7.792893	9.207107
5	Ntukwasi Obi	Val	7.585786	10.41421
		Act	6.792893	8.207107
		Dom	5.37868	9.62132

Table 13. Computed LMF and the UMF of the FOU ($\tilde{A}$).

S/No	Word		UMF	LMF
1	Iwe	Val	(6,8)	(0,5,8)
		Act	(7,9)	(0,4,9)
		Dom	(5,6.5,7,8)	(5,6,6.8,8)
2	Obi Uto	Val	(6,8)	(0,3,8)
		Act	(5,6.5,7.5,9)	(5,6,7,8)
		Dom	(0,3,10)	(0,6,9,10)
3	Onma	Val	(2,4.5,6.5,8)	(2,5,5.7,7)
		Act	(7,9)	(0,3,9)
		Dom	(2,4,7,9)	(2,5,5.5,6)
4	Ujo	Val	(8,10)	(0,3,10)
		Act	(7,8)	(0,2,8)
		Dom	(8,9)	(0,2,9)
5	Ntukwasi Obi	Val	(8,10)	(0,3,10)
		Act	(7,8)	(0,3,8)
		Dom	(0,3,10)	(0,6,9,10)

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5| Result and Discussion

5.1| The Fuzzy Set Part

The IT2FL model used to analyze the Igbo emotion words are simulated using Matlab, Microsoft excel and Netbeans. The data as collected are input into a spreadsheet, preprocessed as discussed in the paper. Then, the model is run on the data yielding the MFs and the FOUs of each of the first 5 emotion words. Parts of the results of simulation for dimensions Valence, Activation and Dominance are shown in *Figs. (4)-(16)*.

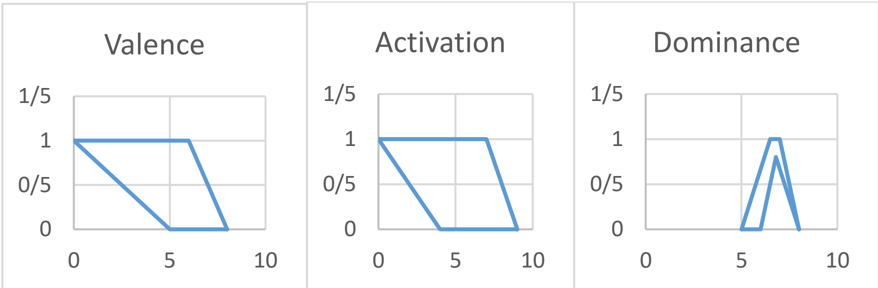

Fig. 4. The FOU of Iwe (Anger) for dimensions valence, activation and dominance.

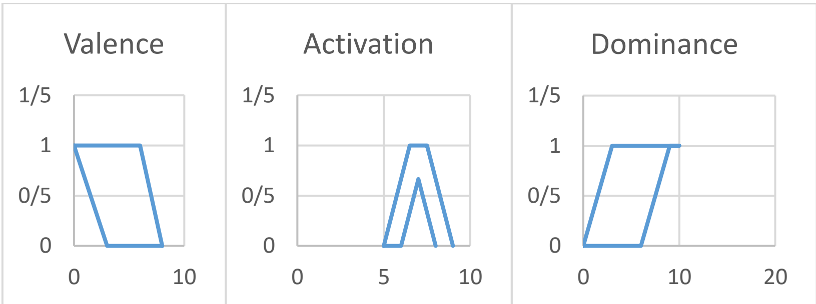

Fig. 5. The FOU of Obi Uto (Happiness) for dimensions valence, activation and dominance.

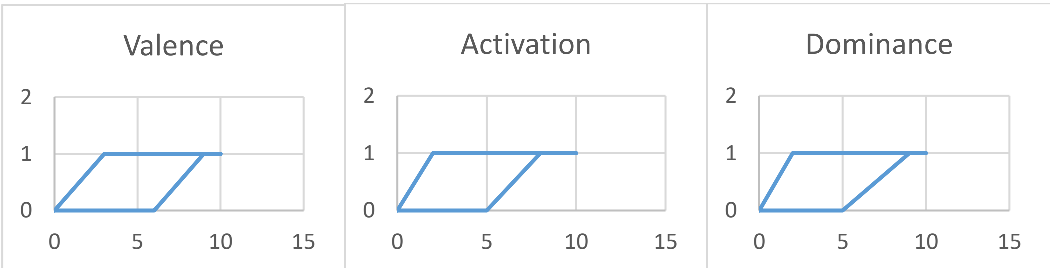

Fig. 6. The FOU of Ihere (Shame) for dimensions valence, activation and dominance.

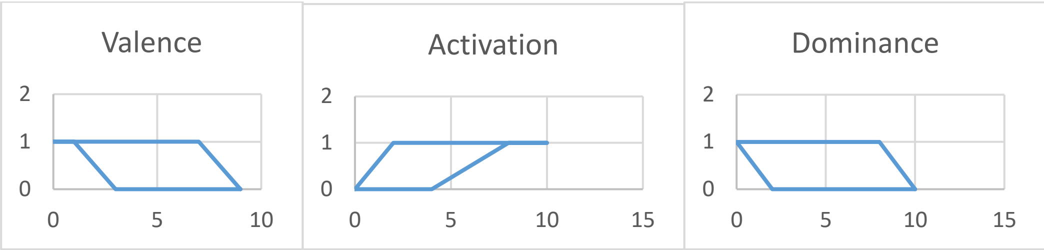

Fig. 7. The FOU of Akwa Uta (Remorse) for dimensions valence, activation and dominance.

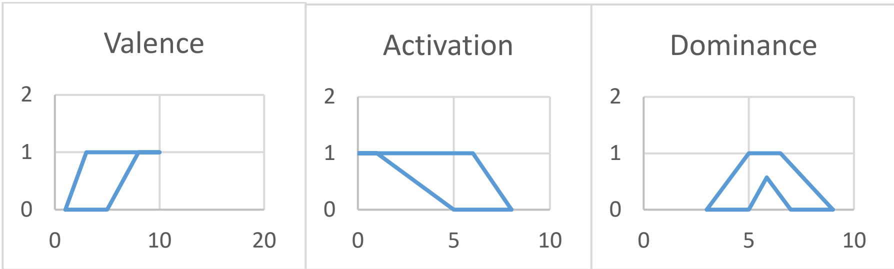

Fig. 8. The FOU of Ara (Mad) for dimensions valence, activation and dominance.

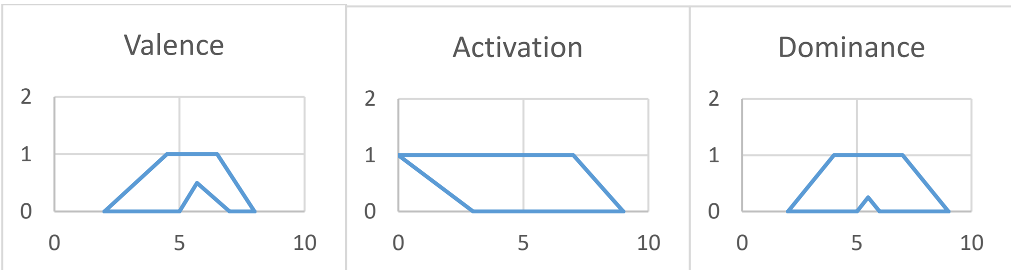

Fig. 9. The FOU of Onuma (Wrath) for dimensions valence, activation and dominance.

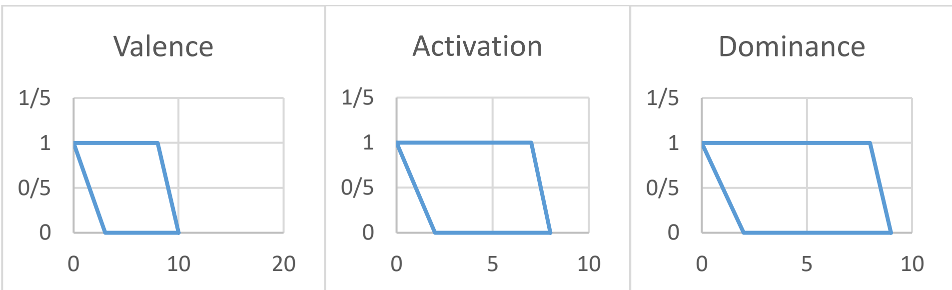

Fig. 10. The FOU of Ujo (Fear) for dimensions valence, activation and dominance.

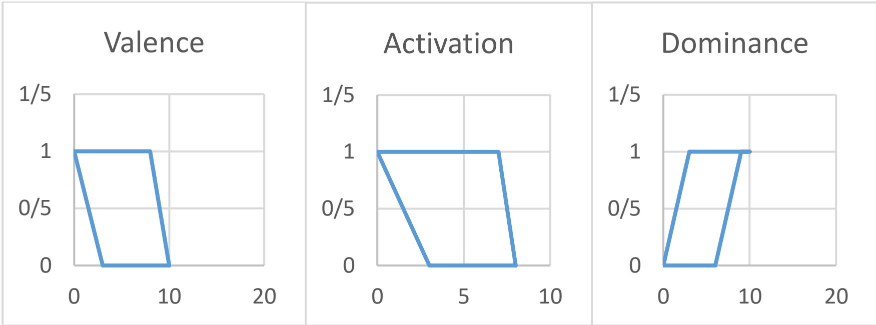

Fig. 11. The FOU of Ntukwasi (Trust) for dimensions valence, activation and dominance.

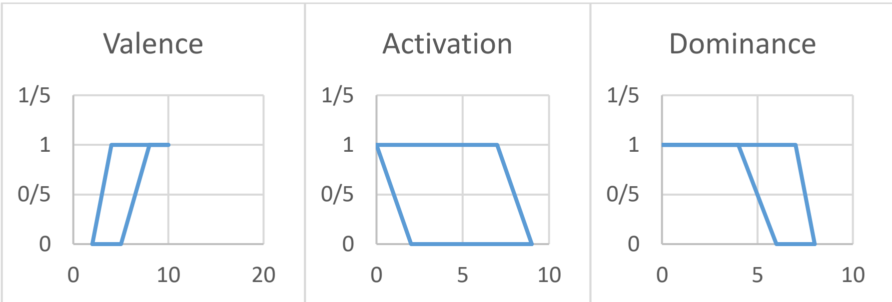

Fig. 12. The FOU of Ojo (Wicked) for dimensions valence, activation and dominance.

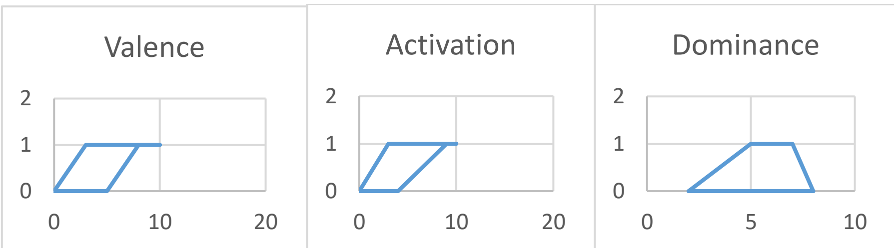

Fig. 13. The FOU of Onu (Joy) for dimensions valence, activation and dominance.

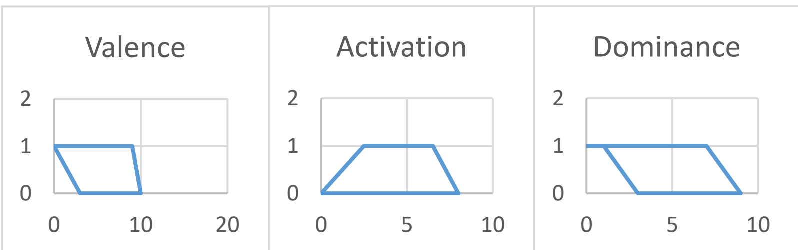

Fig. 14. The FOU of Anuri (Glad) for dimensions valence, activation and dominance.

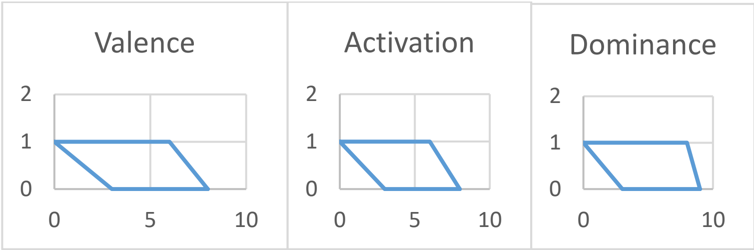

Fig. 15. The FOU of Egwu (Dread) for dimensions valence, activation and dominance.

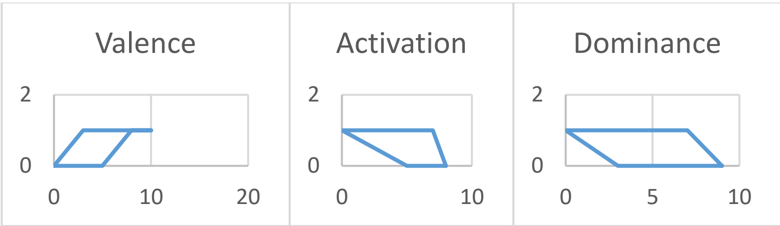

Fig. 16. The FOU of Ihunaya (Love) for dimensions valence, activation and dominance.

5.2| Discussion

Figs. (4) – (16) show parts of the MFs that were calculated from the survey data. These Figures indicate 3 general tendencies of graph: the MF could be either steeper and more peripheral if the emotion words are in accordance with their real meaning, or less steep and more central if the emotion words have ambiguous and undetermined meaning, or shifted to the opposite side of at least one of the scales if the emotion words do not conform with the real meaning. The valence dimension for the word "Iwe (Anger)" is not so broad, indicating a value a bit low. The activation dimension indicates a high value while the dominance dimension is narrow and has a small footprint of uncertainty. This means the word "Iwe (Anger)" carries a meaning that is well determined by the valence and dominance dimensions but is not well determined by the activation dimension. The word "Obi Uto (Happiness)" has a high value in its valence dimension, low value in its activation dimension and high value in its dominance dimension. This implies that the meaning of the word "Obi Uto (Happiness)" is well determined by its activation dimension. For the word "Ihere (Shame)", the valence dimension has a high value. The activation and dominance dimensions have low values. This indicates that the meaning of the word "Ihere (Shame)" is well determined by its activation and dominance dimensions and not the valence dimension. The word "AkwaUta (Remorse)" has a high value in its valence dimension, a low value in its activation dimension and a high value in its dominance dimension. This indicates that the meaning of the word "AkwaUta (Remorse)" is well determined by all the three dimensions. For the word "Ara (Mad)", the valence dimension has a low value while the activation dimension has a high value. Its dominance dimension though not so broad has a small footprint of uncertainty. This shows that the meaning of the word "Ara (Mad)" is well determined by all the three characteristics.

5| Conclusion

This paper involves the implementation of the IT2FL model for analyzing thirty (30) Igbo emotion words. Interval survey is conducted using Igbo native speakers to collect human intuition about fuzzy predicate which is emotion. Using an interval approach, user data are associated with each emotion word with intervals on the three scales of emotional characteristics-valence, activation, and dominance which are collected and used to estimate IT2F MFs for each scale. Results indicate that the study is able to demonstrate that the use of the proposed system will be of immense benefit to every aspect of Natural

Language Processing (NLP) and affective computing and that the IT2FL model for words is more suitable for any purpose in which emotion words may be computed. This is because of the interval approach method used to analyze the words to yield IT2FSs which captures most uncertainty that are contained in an emotion word. Also, the study will help the users in selecting specific Igbo emotion words for easy communication and understanding.

In the future, more emotion words can be added to the system and IT2FL tool can be employed in the translation of Igbo emotion words in English language.

References

[1] Lakoff, G., & Johnson, M. (1980). The metaphorical structure of the human conceptual system. *Cognitive science*, *4*(2), 195-208.

[2] Russell, J. A. (1980). A circumplex model of affect. *Journal of personality and social psychology*, *39*(6), 1161-1178.

[3] Russell, J. A., & Mehrabian, A. (1977). Evidence for a three-factor theory of emotions. *Journal of research in personality*, *11*(3), 273-294.

[4] Barrett, L. F. (2004). Feelings or words? Understanding the content in self-report ratings of experienced emotion. *Journal of personality and social psychology*, *87*(2), 266–281

[5] [5] Klir G. and T. Folger (1988). *Fuzzy sets, uncertainty, and information*. Prentice-Hall, Englewood cliffs, NJ.

[6] Mendel, J. M. (2003, May). Fuzzy sets for words: a new beginning. *The 12th IEEE international conference on fuzzy systems, FUZZ'03*. (Vol. 1, pp. 37-42). IEEE.

[7] Kazemzadeh, A. (2010, July). Using interval type-2 fuzzy logic to translate emotion words from Spanish to english. *International conference on fuzzy systems* (pp. 1-8). IEEE. DOI:10.1109/FUZZY.2010.5584884

[8] Cakmak, O., Kazemzadeh, A., Can, D., Yildirim, S., & Narayanan, S. (2012). Root-word analysis of Turkish emotional language. *Corpora for research on emotion sentiment & social signals*. Retrieved from https://www.researchgate.net/profile/Serdar_Yildirim/publication/264855680_Root-Word_Analysis_of_Turkish_Emotional_Language/links/549951d80cf2d6581ab02839/Root-Word-Analysis-of-Turkish-Emotional-Language.pdf

[9] Kazemzadeh A. (2010). An interval type-2 fuzzy logic system to trans- late emotion words from Spanish to English. *Proceedings of FUZZ-IEEE at the world conference on computational intelligence (WCCI)*. Barcelona, Spain: IEEE.

[10] Oflazer, K. (1994). Two-level description of Turkish morphology. *Literary and linguistic computing*, *9*(2), 137-148.

[11] Kazemzadeh, A., Lee, S., Georgiou, P. G., & Narayanan, S. S. (2011, October). Emotion twenty questions: Toward a crowd-sourced theory of emotions. *International conference on affective computing and intelligent interaction* (pp. 1-10). Springer, Berlin, Heidelberg.

[12] Cakmak, O., Kazemzadeh, A., Yildirim, S., & Narayanan, S. (2012, December). Using interval type-2 fuzzy logic to analyze Turkish emotion words. *Proceedings of the 2012 Asia pacific signal and information processing association annual summit and conference* (pp. 1-4). IEEE.

[13] Jang, H., & Shin, H. (2010, August). Language-specific sentiment analysis in morphologically rich languages. In *Coling 2010: Posters* (pp. 498-506). Coling 2010 Organizing Committee.

[14] Kazemzadeh, A., Lee, S., & Narayanan, S. (2013). Fuzzy logic models for the meaning of emotion words. *IEEE Computational intelligence magazine*, *8*(2), 34-49. DOI: 10.1109/MCI.2013.2247824

[15] Qamar, S., & Ahmad, P. (2015). Emotion detection from text using fuzzy logic. *International journal of computer applications*, *121*(3). Retrieved from https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.695.7970&rep=rep1&type=pdf

[16] Howells, K., & Ertugan, A. (2017). Applying fuzzy logic for sentiment analysis of social media network data in marketing. *Procedia computer science*, *120*, 664-670.

[17] Brainerd, C. J. (2018). The emotional-ambiguity hypothesis: A large-scale test. *Psychological science*, *29*(10), 1706-1715. DOI: 10.1177/0956797618780353

[18] Furht, B. (Ed.). (2012). Multimedia tools and applications (Vol. 359). Springer Science & Business Media.

[19] Onwuejeogwu, M. A. (1972). The Ikenga: The cult of individual achievements and advancements. *Afr. notes*, *6*(2), 92.

[20] Uchendu V. (1965). *The Igbo of South-East Nigeria*. Holt Rinehart & Winston, New York.

[21] Zadeh, L. A. (1965). Fuzzy sets. *Information and control*, *8*(3), 338-353.

[22] Mendel, J. M., & Liu, F. (2007). Super-exponential convergence of the Karnik–Mendel algorithms for computing the centroid of an interval type-2 fuzzy set. *IEEE transactions on fuzzy systems*, *15*(2), 309-320.

[23] Mendel, J. M. (2007). Computing with words and its relationships with fuzzistics. *Information sciences*, *177*(4), 988-1006.

[24] Liu, F., & Mendel, J. M. (2008). Encoding words into interval type-2 fuzzy sets using an interval approach. *IEEE transactions on fuzzy systems*, *16*(6), 1503-1521.

[25] Kazemzadeh, A., Lee, S., & Narayanan, S. (2013). Fuzzy logic models for the meaning of emotion words. *IEEE computational intelligence magazine*, *8*(2), 34-49. DOI: 10.1109/MCI.2013.2247824

[26] Liu, F., & Mendel, J. M. (2007, July). An interval approach to fuzzistics for interval type-2 fuzzy sets. *2007 IEEE international fuzzy systems conference* (pp. 1-6). IEEE.

[27] Liu, F., & Mendel, J. M. (2008). Encoding words into interval type-2 fuzzy sets using an interval approach. *IEEE transactions on fuzzy systems*, *16*(6), 1503-1521.

[28] Guzmán, J. C., Miramontes, I., Melin, P., & Prado-Arechiga, G. (2019). Optimal genetic design of type-1 and interval type-2 fuzzy systems for blood pressure level classification. *Axioms*, *8*(1), 1-35. DOI: 10.3390/axioms8010008

[29] Dirik, M., Castillo, O., & Kocamaz, A. F. (2019). Visual-servoing based global path planning using interval type-2 fuzzy logic control. *Axioms*, *8*(2), 1-16. DOI: 10.3390/axioms8020058

[30] Damirchi-Darasi, S. R., Zarandi, M. F., Turksen, I. B., & Izadi, M. (2019). Type-2 fuzzy rule-based expert system for diagnosis of spinal cord disorders. *Scientia Iranica. Transaction E, industrial engineering*, *26*(1), 455-471. DOI: 10.24200/SCI.2018.20228

[31] Rahimi D., Mohammad H., Burhan T. (2018). Type-2 fuzzy rule-based expert system for diagnosis of spinal cord disorders. *Scientia Iranica*, *26*(1). 455-471. DOI: 10.24200/SCI.2018.20228

[32] Wu, D. (2012). On the fundamental differences between interval type-2 and type-1 fuzzy logic controllers. *IEEE transactions on fuzzy systems*, *20*(5), 832-848.

[33] Almahasneh, R., Tüű-Szabó, B., Kóczy, L. T., & Földesi, P. (2020). Optimization of the time-dependent traveling salesman problem using interval-valued intuitionistic fuzzy sets. *Axioms*, *9*(2), 53.

[34] Ruba A., Boldizsár T, László T. & Péter f. (2020). Optimization of the time-dependent traveling problem using interval-valued intuitionistic fuzzy sets. *Axioms* *9*(53), 1-14. DOI: 10.3390/axioms9020053

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Multi-Item Inventory Model Include Lead Time with Demand Dependent Production Cost and Set-Up-Cost in Fuzzy Environment

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Abstract

In this paper, we have developed the multi-item inventory model in the fuzzy environment. Here we considered the demand rate is constant and production cost is dependent on the demand rate. Set-up- cost is dependent on average inventory level as well as demand. Lead time crashing cost is considered the continuous function of leading time. Limitation is considered on storage of space. Due to uncertainty all cost parameters of the proposed model are taken as generalized trapezoidal fuzzy numbers. Therefore this model is very real. The formulated multi objective inventory problem has been solved by various techniques like as Geometric Programming (GP) approach, Fuzzy Programming Technique with Hyperbolic Membership Function (FPTHMF), Fuzzy Nonlinear Programming (FNLP) technique and Fuzzy Additive Goal Programming (FAGP) technique. An example is given to illustrate the model. Sensitivity analysis and graphical representation have been shown to test the parameters of the model.

Keywords: Inventory, Multi-Item, Leading time; Generalized trapezoidal fuzzy number, Fuzzy techniques; GP technique.

1 | Introduction

Inventory models deal with decisions that minimize the total average cost or maximize the total average profit. In that way to construct a real life mathematical inventory model on based on various assumptions and notations and approximations. Multi-item is also an important factor in the inventory control system. The basic well known Economic Order Quantity (EOQ) model was first introduced by Harris in 1913; Abou-el-ata and Kotb studied a multi-item EOQ inventory model with varying holding costs under two restrictions with a geometric programming approach [1]. Chen [7] presented an optimal determination of quality level, selling quantity and purchasing price for intermediate firms. Liang and Zhou [11] discussted two warehouse inventory model for deteriorating items and stock dependent demand under conditionally permissible delay in payment.

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Das et al. [12] developed a multi-item inventory model with quantity dependent inventory costs and demand-dependent unit cost under imprecise objectives and restrictions with a geometric programming approach. Das and Islam [13] considered a multi-objective two echelon supply chain inventory model with customer demand dependent purchase cost and production rate dependent production cost. Shaikh et al. [30] discussed an inventory model for deteriorating items with preservation facility of ramp type demand and trade credit.

The concept of fuzzy set theory was first introduced by Zadeh [27]. Afterward, Zimmermann [28] applied the fuzzy set theory concept with some useful membership functions to solve the linear programming problem with some objective functions. Bit [2] applied fuzzy programming with hyperbolic membership functions for multi objective capacitated transportation problems. Bortolan and Degani [4] discussed a review of some methods for ranking fuzzy subsets. Maiti [19] developed a fuzzy inventory model with two warehouses under possibility measure in fuzzy goals. Mandal et al. [22] presented a multi-objective fuzzy inventory model with three constraints with a geometric programming approach. Shaikh et al. [26] developed a fuzzy inventory model for a deteriorating Item with variable demand, permissible delay in payments and partial backlogging with Shortage Following Inventory (SFI) policy. Garai et al. [29] discussed multi-objective inventory model with both stock-dependent demand rate and holding cost rate under fuzzy random environment.

In the global market system lead time is an important matter. Ben-Daya and Rauf [3] considered an inventory model involving lead-time as a decision variable. Chuang et al. [8] presented a note on periodic review inventory model with controllable setup cost and lead time. Hariga and Ben-Daya [14] discussed some stochastic inventory models with deterministic variable lead time. Ouyang et al. [20] studied mixture inventory models with backorders and lost sales for variable lead time. Ouyang and Wu [21] established a min-max distribution free procedure for mixed inventory models with variable lead time. Sarkar et al. [24] developed an integrated inventory model with variable lead time and defective units and delay in payments. Sarkar et al. [25] studied quality improvement and backorder price discount under controllable lead time in an inventory model.

Geometric Programming (GP) is a powerful optimization technique developed to solve a class of non-linear optimization programming problems especially found in engineering design and manufacturing. Multi objective geometric programming techniques are also interesting in the EOQ model. GP was introduced by Duffin et al. in 1966 [10] and published a famous book in 1967 [9]. Beightler et al. [5] applied GP. Biswal [6] considered fuzzy programming techniques to solve multi-objective geometric programming problems. Islam [16] discussed multi-objective geometric-programming problem and its application. Mandal et al. [22] developed a multi-objective fuzzy inventory model with three constraints with a geometric programming approach. Mandal et al. [23] discussed an inventory model of deteriorating items with a constraint with a geometric Programming approach. Islam [17] studied a multi-objective marketing planning inventory model with a geometric programming approach. Kotb et al. [18] presented a multi-item EOQ model with both demand dependent on unit cost and varying lead time via geometric programming.

In this paper, we have developed an inventory model of multi-item with space constraint in a fuzzy environment. Here we considered the constant demand rate and production cost is dependent on the demand rate. Set-up- cost is dependent on average inventory level as well as demand. Lead time crashing cost is considered the continuous function of leading time. Due to uncertainty all cost parameters are taken as generalized trapezoidal fuzzy numbers. The proposal has been solved by various techniques like GP approach, FPTHMF, FNLP, and FAGP. Numerical example is given to illustrate the model. Finally sensitivity analysis and graphical representation have been shown to test the parameters of the model.

2 | Mathematical Model

2.1 | Notations

h_i: Holding cost per unit per unit time for ith item.

T_i: The length of cycle time for i^{th}item, $T_i > 0$.

D_i : Demand rate per unit time for the i^{th} item.

L_i : Rate of leading time for the i^{th} item.

SS: Safety stock.

k: Safety factor.

$I_i(t)$: Inventory level of the i^{th} item at time t.

C_p^i : Unit production cost of i^{th} item.

${S_c}^i(Q_i, D_i)$: Set up cost for i^{th} item.

$R^i(L_i)$: Lead time crashing cost for the i^{th} item.

Q_i: The order quantity for the duration of a cycle of length T_i for i^{th} item.

$TAC_i(D_i, Q_i, L_i)$: Total average profit per unit for the i^{th} item.

w_i: Storage space per unit time for the i^{th} item.

W: Total area of space.

$\widetilde{w}_i$: Fuzzy storage space per unit time for the i^{th} item.

$\widetilde{h}_i$: Fuzzy holding cost per unit per unit time for the i^{th} item.

$\widetilde{TAC}_i(D_i, Q_i, L_i)$: Fuzzy total average cost per unit for the i^{th} item.

$\widehat{w}_i$: Defuzzyfication of the fuzzy number $\widetilde{w}_i$.

$\widehat{h}_i$: Defuzzyfication of the fuzzy number $\widetilde{h}_i$.

$\widehat{TAC}_i(D_i, Q_i, L_i)$: Defuzzyfication of the fuzzy number $\widetilde{TAC}_i(D_i, Q_i, L_i)$.

2.2 | Assumptions

- *Multi-item is considered.*
- *The replenishment occurs instantaneously at infinite rate.*
- *The lead time is considered.*
- *Shortages are not allowed.*
- *Production cost is inversely related to the demand. Here considered* $C_p^i(D_i) = \alpha_i {D_i}^{-\beta_i}$, *where* $\alpha_i > 0$ *and* $\beta_i > 1$ *are constant real numbers.*

- *The set up cost is dependent on the demand as well as average inventory level. Here considered* $S_c^{\ i}(Q_i, D_i) = \gamma_i\left(\frac{Q_i}{2}\right)^{\delta_i} D_i^{\sigma_i}$ *where* $0 < \gamma_i$, $0 < \delta_i \ll 1$ *and* $0 < \sigma_i \ll 1$ *are constant real numbers.*
- *Lead time crashing cost is dependent on the lead time by a function of the form* $R^i(L_i) = \rho_i L_i^{-\tau_i}$, *where* $\rho_i > 0$ *and* $0 < \tau_i \le 0.5$ *are constant real numbers.*
  Li.
- *Deterioration is not allowed.*

2.3| Formulation of the Model

The inventory level for i^th^ item is illustrated in *Fig. 1*. During the period $[0, T_i]$ the inventory level reduces due to demand rate. In this time period, the governing differential equation is

$$\frac{dI_i(t)}{dt} = -D_i, 0 \le t \le T_i. \tag{1}$$

With boundary condition, $I_i(0) = Q_i$, $I_i(T_i) = 0$.

Solving *Eq. (1)* we have,

$$I_i(t) = Q_i - D_i t,\ \ 0 \le t \le T_i\ \ . \tag{2}$$

$$T_i = \frac{Q_i}{D_i}. \tag{3}$$

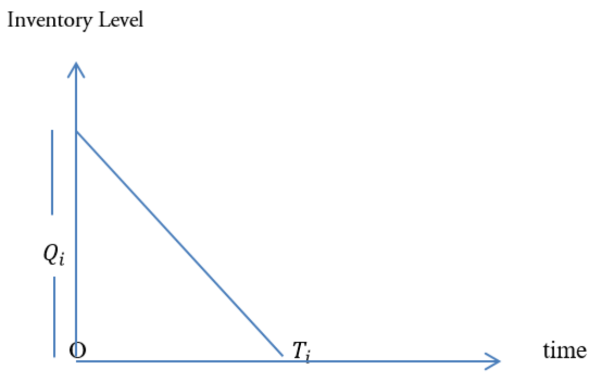

Fig. 1. Inventory level for the i^th^ item.

Now calculating various averages cost for i^th^ item,

Average production cost $(PC_i) = \frac{Q_i C_p^i(D_i)}{T_i} = \alpha_i D_i^{(1-\beta_i)}$;

Average holding cost$(HC_i) = \frac{1}{T_i}\int_0^{T_i} h_i I_i(t)dt + h_i k\omega\sqrt{L_i}\ = h_i\left(\frac{Q_i}{2} + k\omega\sqrt{L_i}\right)$;

Average set-up-cost $(SC_i)\ =\ \frac{1}{T_i}\left[\gamma_i\left(\frac{Q_i}{2}\right)^{\delta_i} D_i^{\sigma_i}\right] = \frac{\gamma_i Q_i^{\delta_i - 1} D_i^{\sigma_i + 1}}{2^{\delta_i}}$;

Average lead time crashing cost $(CC_i) = \frac{\rho_i L_i^{-\tau_i}}{T_i} = \frac{D_i \rho_i L_i^{-\tau_i}}{Q_i}$.

Total average cost for i^th^ item is

$$\mathrm{TAC_i(D_i, Q_i, L_i) = (PC_i + HC_i + SC_i + CC_i) = \alpha_i D_i^{(1-\beta_i)} + h_i \left(\frac{Q_i}{2} + k\omega\sqrt{L_i}\right) + \frac{\gamma_i Q_i^{\delta_i - 1} D_i^{\sigma_i + 1}}{2^{\delta_i}} + \frac{D_i \rho_i L_i^{-\tau_i}}{Q_i}}. \quad (4)$$

A Multi-Objective Inventory Model (MOIM) can be written as:

$$\mathrm{Min}\ \{\mathrm{TAC_1, TAC_2, TAC_3, \dots\dots\dots\dots\dots, TAC_n}\},$$

$$\mathrm{TAC_i(D_i, Q_i, L_i) = \alpha_i D_i^{(1-\beta_i)} + h_i \left(\frac{Q_i}{2} + k\omega\sqrt{L_i}\right) + \frac{\gamma_i Q_i^{\delta_i - 1} D_i^{\sigma_i + 1}}{2^{\delta_i}} + \frac{D_i \rho_i L_i^{-\tau_i}}{Q_i}}, \quad (5)$$

Subject to

$$\sum_{i=1}^{n} \mathrm{w_i Q_i \le W, D_i > 0, Q_i > 0, L_i > 0}, \text{ for } i = 1,2, \dots\dots\dots..n.$$

2.4 | Fuzzy Model

Due to uncertainty, we consider all the parameters $(\alpha_i, \beta_i, h_i, \rho_i, \gamma_i, \delta_i, \sigma_i, \tau_i)$of the model and storage space w_i as Generalized Trapezoidal Fuzzy Number (GTrFN)$(\widetilde{\alpha_i}, \widetilde{\beta_i}, \widetilde{h_i}, \widetilde{\rho_i}, \widetilde{\gamma_i}, \widetilde{\delta_i}, \widetilde{\sigma_i}, \widetilde{w_i}, \widetilde{\tau_i})$. Here

$$\widetilde{\alpha_i} = \left(\alpha_i^1, \alpha_i^2, \alpha_i^3, \alpha_i^4; \varphi_{\alpha_i}\right), 0 < \varphi_{\alpha_i} \le 1; \widetilde{h_i} = \left(h_i^1, h_i^2, h_i^3, h_i^4; \varphi_{h_i}\right), 0 < \varphi_{h_i} \le 1;$$

$$\widetilde{\beta_i} = \left(\beta_i^1, \beta_i^2, \beta_i^3, \beta_i^4; \varphi_{\beta_i}\right), 0 < \varphi_{\beta_i} \le 1; \widetilde{\rho_i} = \left(\rho_i^1, \rho_i^1, \rho_i^1, \rho_i^1; \varphi_{\rho_i}\right), 0 < \varphi_{\rho_i} \le 1;$$

$$\widetilde{\gamma_i} = \left(\gamma_i^1, \gamma_i^2, \gamma_i^3, \gamma_i^4; \varphi_{\gamma_i}\right), 0 < \varphi_{\gamma_i} \le 1; \widetilde{w_i} = \left(w_i^1, w_i^2, w_i^3, w_i^4; \varphi_{w_i}\right), 0 < \varphi_{w_i} \le 1;$$

$$\widetilde{\delta_i} = \left(\delta_i^1, \delta_i^2, \delta_i^3, \delta_i^4; \varphi_{\delta_i}\right), 0 < \varphi_{\delta_i} \le 1; \widetilde{\sigma_i} = \left(\sigma_i^1, \sigma_i^2, \sigma_i^3, \sigma_i^4; \varphi_{\sigma_i}\right), 0 < \varphi_{\sigma_i} \le 1;$$

$$\widetilde{\tau_i} = \left(\tau_i^1, \tau_i^2, \tau_i^3, \tau_i^4; \varphi_{\tau_i}\right), 0 < \varphi_{\tau_i} \le 1; (\ i = 1,2, \dots\dots\dots, n).$$

Then the above inventory *Model* (5) becomes the fuzzy inventory model as

$$\mathrm{Min}\quad \left\{\widetilde{\mathrm{TAC}}_1, \widetilde{\mathrm{TAC}}_2, \widetilde{\mathrm{TAC}}_3, \dots\dots\dots\dots\dots\dots\dots., \widetilde{\mathrm{TAC}}_n\right\},$$

Subject to

$$\sum_{i=1}^{n} \widetilde{\mathrm{w_i}} \mathrm{Q_i} \le \mathrm{W}, \text{ for } i = 1,2, \dots\dots\dots..n. \quad (6)$$

Where

$$\mathrm{TAC_i}(\widetilde{\mathrm{D_i}, \mathrm{Q_i}, \mathrm{L_i}}) = \widetilde{\alpha_i} \mathrm{D_i}^{(1-\widetilde{\beta_i})} + \widetilde{h_i}\left(\frac{\mathrm{Q_i}}{2} + \mathrm{k}\omega\sqrt{\mathrm{L_i}}\right) + \frac{\widetilde{\gamma_i} \mathrm{Q_i}^{\widetilde{\delta_i}-1} \mathrm{D_i}^{\widetilde{\sigma_i}+1}}{2^{\widetilde{\delta_i}}} + \frac{\mathrm{D_i} \widetilde{\rho_i} \mathrm{L_i}^{-\widetilde{\tau_i}}}{\mathrm{Q_i}}.$$

$\lambda -$Integer method is used to defuzzify the fuzzy number. In this method the defuzzify value of the fuzzy number $\tilde{A} = (a, b, c, d; \varphi)$ *is* $\varphi\left(\frac{a+b+c+d}{4}\right)$. So using the defuzzified values $(\widehat{\alpha_i}, \widehat{\beta_i}, \widehat{h_i}, \widehat{\rho_i}, \widehat{\gamma_i}, \widehat{\delta_i}, \widehat{\sigma_i}, \widehat{w_i}, \widehat{\tau_i})$ of the GTrFN $(\widetilde{\alpha_i}, \widetilde{\beta_i}, \widetilde{h_i}, \widetilde{\rho_i}, \widetilde{\gamma_i}, \widetilde{\delta_i}, \widetilde{\sigma_i}, \widetilde{w_i}, \widetilde{\tau_i})$, the above fuzzy inventory *Model* (6) reduces to

$$\text{Min}\left\{\widehat{TAC_1}, \widehat{TAC_2}, \widehat{TAC_3}, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .., \widehat{TAC_n}\right\},$$

Subject to

$$\sum_{i=1}^{n} \widehat{w_i} Q_i \leq W,$$

Where

$$TAC_i(\widehat{D_i, Q_i}, L_i) = \widehat{\alpha_i} D_i^{(1-\widehat{\beta_i})} + \widehat{h_i}\left(\frac{Q_i}{2} + k\omega\sqrt{L_i}\right) + \frac{\widehat{\gamma_i} Q_i^{\widehat{\delta_i}-1} D_i^{\widehat{\sigma_i}+1}}{2^{\widehat{\delta_i}}} + \frac{D_i \widehat{\rho_i} L_i^{-\widehat{\tau_i}}}{Q_i},$$

$$D_i > 0, Q_i > 0, L_i > 0, \text{ for } i = 1,2, \ldots \ldots \ldots .. n. \tag{7}$$

3| Fuzzy Programming Techniques to Solve MOIM

Solve the MOIM as a single objective NLP using only one objective at a time and ignoring the others. So we get the ideal solutions. Using the ideal solutions the pay-off matrix as follows:

$$\begin{array}{c} \\ (D_1^1, Q_1^1, L_1^1) \\ (D_2^2, Q_2^2, L_2^2) \\ \\ \\ (D_n^n, Q_n^n, L_n^n) \end{array} \begin{pmatrix} TAC_1(D_1, Q_1, L_1) & TAC_2(D_2, Q_2, L_2) \ldots & \ldots TAC_n(D_n, Q_n, L_n) \\ TAC_1^*(D_1^1, Q_1^1, L_1^1) & TAC_2(D_1^1, Q_1^1, L_1^1) \ldots & \ldots TAC_n(D_1^1, Q_1^1, L_1^1) \\ TAC_1(D_2^2, Q_2^2, L_2^2) & TAC_2^*(D_2^2, Q_2^2, L_2^2) \ldots & TAC_n(D_2^2, Q_2^2, L_2^2) \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ TAC_1(D_n^n, Q_n^n, L_n^n) & TAC_2(D_n^n, Q_n^n, L_n^n) \ldots & TAC_n^*(D_n^n, Q_n^n, L_n^n) \end{pmatrix}, \tag{8}$$

Let $U^k = \max\{TAC_k(D_i^i, Q_i^i, L_i^i), i = 1,2, \ldots, n\}$ for $k = 1,2, \ldots, n$ and

$L^k = TAC_k^*(D_k^k, Q_k^k, L_k^k)$ for $k = 1,2, \ldots, n$.

Hence U^k, L^k are identified, $L^k \leq TAP_k(D_i^i, Q_i^i, L_i^i) \leq U^k$, for $i = 1,2, \ldots, n$; $k = 1,2, \ldots, n$.

3.1| Fuzzy Programming Technique Using Hyperbolic Membership Function (FPTHMF)

Now fuzzy non-linear hyperbolic membership functions $\mu_{TAC_k}^H(TAC_k(D_k, Q_k, L_k))$ for the k^th^ objective functions $TAC_k(D_k, Q_k, L_k)$ respectively for $k = 1,2, \ldots, n$ are defined as follows:

$$\mu_{TAC_k}^H(TAC_k(D_k, Q_k, L_k)) = \frac{1}{2}\tanh\left(\left(\frac{U^k + L^k}{2} - TAC_k(D_k, Q_k, L_k)\right)\sigma_k\right) + \frac{1}{2}.$$

Where α_k is a parameter, $\sigma_k = \frac{3}{(U^k - L^k)/2} = \frac{6}{U^k - L^k}$.

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In this technique the problem is defined as follows:

Max λ,

Subject to

$$\frac{1}{2}\tanh\left(\left(\frac{U^k+L^k}{2}-TAC_k(D_k,Q_k,L_k)\right)\sigma_k\right)+\frac{1}{2}\geq\lambda,$$

$$\sum_{i=1}^{n}\widehat{w_i}Q_i\leq W,\ \lambda\geq 0,\ \ D_k>,Q_k>0,L_k>0,\text{ for } k=1,2,\dots\dots\dots..n.$$

After simplification the above problem can be written as

Max y ,

Subject to

$$y+\sigma_k TAC_k(D_k,Q_k,L_k)\leq\frac{U^k+L^k}{2}\sigma_k,$$

$$\sum_{i=1}^{n}\widehat{w_i}Q_i\leq W,\ y\geq 0,\ D_k>,Q_k>0,L_k>0\text{ for } k=1,2,\dots\dots\dots..n.$$

Now the above problem can be freely solved by suitable mathematical programming algorithm and then we shall get the appropiet solution of the MOIM.

3.2 | Fuzzy Non-Linear Programming (FNLP) Technique based on Max-Min

In this technique fuzzy membership function $\mu_{TAC_k}\big(TAC_k(Q_k,D_k)\big)$ for the kth objective function $TAC_k(D_k,Q_k,L_k)$ respectively for $k=1,2,\dots.,n$ are defined as follows:

$$\mu_{TAC_k}\big(TAC_k(D_k,Q_k,L_k)\big)=\begin{cases}1 & \text{for } TAC_k(D_k,Q_k,L_k)<L^k\\ \dfrac{U^k-TAC_k(D_k,Q_k,L_k)}{U^k-L^k} & \text{for } L^k\leq TAC_k(D_k,Q_k,L_k)\leq U^k\\ 0 & \text{for } TAC_k(D_k,Q_k,L_k)>U^k\end{cases}$$

for $k=1,2,\dots.,n$.

In this technique the problem is defined as follows:

Max α',

Subject to

$$TAC_k(D_k,Q_k,L_k)+\alpha'\big(U^k-L^k\big)\leq U^k\quad,\qquad \text{for } k=1,2,\dots.,n,$$

$$\sum_{i=1}^{n}\widehat{w_i}Q_i\leq W,0\leq\alpha'\leq 1,\ D_k>,Q_k>0,L_k>0.$$

Now the above problem can be freely solved by suitable mathematical programming algorithm and then we shall get the required solution of the MOIM.

3.3| Fuzzy Additive Goal Programming (FAGP) Technique Based on Additive Operator

Using the above membership function, fuzzy non-linear programming problem is formulated as

$$\text{Max} \sum_{k=1}^{n} \frac{U^k - TAC_k(D_k, Q_k, L_k)}{U^k - L^k},$$

Subject to

$$U^k - TAC_k(D_k, Q_k, L_k) \leq U^k - L^k,$$

$$\sum_{i=1}^{n} \widehat{w_1} Q_i \leq W, \ D_k >, Q_k > 0, L_k > 0 \text{ for } k = 1,2, \ldots, n.$$

Now the above problem can be solved by suitable mathematical programming algorithm and then we shall get the solution of the MOIM.

4| Geometric Programming Technique

Let us consider a Multi Objective Geometric Programming (MOGP) problem is as follows

$$\text{Minimize } g_{s0}(t) = \sum_{k=1}^{T_{s0}} c_{s0k} \prod_{j=1}^{m} t_j^{\alpha_{s0kj}}, s = 1,2,3, \ldots \ldots \ldots, n,$$

Subject to

$$g_r(t) = \sum_{k=1}^{l_r} c_{rk} \prod_{j=1}^{m} t_j^{\alpha_{rkj}} \leq 1, r = 1,2,3, \ldots \ldots \ldots, p,$$

$$t_j > 0, j = 1,2, \ldots, m.$$

Where $c_{rk}, c_{s0k}(> 0)$, α_{rkj} and α_{s0kj} $(j = 1,2, \ldots, m; r = 0,1,2, \ldots, p; \ k = 1,2, \ldots . l_r \ ; \ s = 1,2,3, \ldots \ldots \ldots, n)$are all real numbers. T_{s0} is the number of terms in the s^{th} objective function and l_r is the number of terms in the r^{th} constraint.

Now introducing the weights w_i $(i = 1,2,3, \ldots \ldots \ldots, n)$, the above MOGP converted into the single objective geometric programming problem as following

Primal Problem.

$$\text{Minimize } g(t) = \sum_{s=1}^{n} w_s \sum_{k=1}^{T_{s0}} c_{s0k} \prod_{j=1}^{m} t_j^{\alpha_{s0kj}}, s = 1,2,3, \ldots \ldots \ldots, n,$$

$$\text{i.e.} = \sum_{s=1}^{n} \sum_{k=1}^{T_{s0}} w_s c_{s0k} \prod_{j=1}^{m} t_j^{\alpha_{s0kj}},$$

Subject to

$$g_r(t) = \sum_{k=1}^{l_r} c_{rk} \prod_{j=1}^{m} t_j^{\alpha_{rkj}} \leq 1, r = 1,2,3, \ldots \ldots \ldots, p, \tag{9}$$

$$t_j > 0, j = 1,2, \ldots, m,$$

$$\sum_{i=1}^{n} w_i = 1, w_i > 0, i = 1,2,3, \ldots \ldots \ldots, n.$$

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Let T be the total numbers of terms (including constraints), number of variables is m. Then the degree of the difficulty (DD) is $T-(m+1)$.

Dual Program.

The dual programming of *Eq. (9)* is given as follows:

$$\text{Maximize } v(\theta)=$$
$$\prod_{s=1}^{n}\prod_{k=1}^{T_{so}}\left(\frac{w_s c_{s0k}}{\theta_{0sk}}\right)^{\theta_{0sk}}\prod_{r=1}^{p}\prod_{k=1}^{l_r}\left(\frac{c_{rk}}{\theta_{rk}}\right)^{\theta_{rk}}\left(\sum_{k=1}^{l_r}\theta_{rk}\right)^{\sum_{k=1}^{l_r}\theta_{rk}},$$

Subject to

$$\sum_{s=1}^{n}\sum_{k=1}^{T_{s0}}\theta_{0sk}=1,\ \text{(Normality condition)}$$

$$\sum_{r=1}^{p}\sum_{k=1}^{l_r}\alpha_{rkj}\theta_{rk}+\sum_{s=1}^{n}\sum_{k=1}^{T_{s0}}\alpha_{s0kj}\theta_{0sk}=0,\ (j=1,2,\ldots,m)$$
$$\text{(Orthogonality conditions)}$$

$$\theta_{0sk},\theta_{rk}>0,\ (r=0,1,2,\ldots,p;\ k=1,2,\ldots.l_r;\ s=1,2,3,\ldots\ldots\ldots,n).$$
$$\text{(Positivity conditions)}$$

Now here three cases may arises

Case I. $T_0\ =\ m+1$, (i.e. DD=0). So DP presents a system of linear equations for the dual variables. So we have a unique solution vector of dual variables.

Case II. $T_0\ >\ m+1$, So a system of linear equations is presented for the dual variables, where the number of linear equations is less than the number of dual variables. So it is concluded that dual variables vector have many solutions.

Case III. $T_0\ <\ m+1$, so a system of linear equations is presented for the dual variables, where the number of linear equations is greater than the number of dual variables. It is seen that generally no solution vector exists for the dual variables here.

4.1| Solution Procedure of My Proposed Problem

Primal Problem.

$$\begin{aligned}\text{Minimize TAC}(D,Q,L)&=\sum_{i=1}^{n}w_i'\left(\hat{\alpha}_i D_i^{(1-\hat{\beta}_i)}+\hat{h}_i\left(\frac{Q_i}{2}+k\omega\sqrt{L_i}\right)+\frac{\hat{\gamma}_i {Q_i}^{\hat{\delta}_i-1}D_i^{\hat{\sigma}_i+1}}{2^{\hat{\delta}_i}}\right.\\&\left.+D_i\hat{\rho}_i{L_i}^{-\hat{\tau}_i}{Q_i}^{-1}\right),\end{aligned}\tag{10}$$

Subject to

$$\sum_{i=1}^{n}\frac{\widehat{w}_i}{W}Q_i\le 1,$$

$$\sum_{i=1}^{n}w_i'=1,w_i'>0,i=1,2,3,\ldots\ldots\ldots,n.$$

Dual Program.

The dual programming of *Eq. (10)* is given as follows:

$$\text{Maximize } v(\theta) = \prod_{i=1}^{n}\left(\frac{w_i'\widehat{\alpha}_1}{\theta_{i1}}\right)^{\theta_{i1}}\left(\frac{w_i'\widehat{h}_1}{2\theta_{i2}}\right)^{\theta_{i2}}\left(\frac{w_i'k\omega}{\theta_{i3}}\right)^{\theta_{i3}}\left(\frac{w_i'\widehat{\gamma}_1}{2^{\widehat{\delta}_1}\theta_{i4}}\right)^{\theta_{i4}}\left(\frac{w_i'\widehat{\rho}_1}{\theta_{i5}}\right)^{\theta_{i5}}\left(\frac{\widehat{w}_1}{W\theta_{i1}'}\right)^{\theta_{i1}'}\left(\sum_{i=1}^{n}\theta_{i1}'\right)^{\sum_{i=1}^{n}\theta_{i1}'},$$

Subject to

$$\begin{aligned}
&\theta_{i1}+\theta_{i2}+\theta_{i3}+\theta_{i4}+\theta_{i5}=1,\\
&\left(1-\widehat{\beta}_1\right)\theta_{i1}+(\widehat{\sigma}_1+1)\theta_{i4}+\theta_{i5}=0,\\
&\theta_{i2}+\left(\widehat{\delta}_1-1\right)\theta_{i4}-\theta_{i5}+\theta_{i1}'=0,\\
&\frac{\theta_{i3}}{2}-\widehat{\tau}_1\theta_{i5}=0\,,\\
&\sum_{i=1}^{n}w_i'=1, w_i'>0,\\
&\theta_{i1},\theta_{i2},\theta_{i3},\theta_{i4},\theta_{i5},\theta_{i1}'\geq 0 \text{ for } i=1,2,3,\dots\dots\dots..,n.
\end{aligned} \tag{11}$$

Solving the above linear equations we have

$$\theta_{i1}=\frac{\widehat{\sigma}_\imath+1}{\widehat{\beta}_\imath-1}y_i+\frac{1}{\widehat{\beta}_\imath-1}x_i,\ \theta_{i2}=1-\left\{(1+2\widehat{\tau}_\imath)+\frac{1}{\widehat{\beta}_\imath-1}x_i\right\}-\frac{\widehat{\sigma}_\imath+\widehat{\beta}_\imath}{\widehat{\beta}_\imath-1}y_i,\ \theta_{i3}=2\widehat{\tau}_\imath x_i,\ \theta_{i4}=y_i,\ \theta_{i5}=x_i,$$

$$\theta_{i1}'=1-\left(2\widehat{\tau}_\imath+\frac{1}{\widehat{\beta}_\imath-1}\right)x_i-\left(\frac{\widehat{\sigma}_\imath+\widehat{\beta}_\imath}{\widehat{\beta}_\imath-1}+\widehat{\delta}_\imath-1\right)y_i\,.$$

Putting the above values in *Eq. (11)* we have

$$\begin{aligned}
&\text{Maximize } v(x,y)\\
&=\prod_{i=1}^{n}\left(\frac{w_i'\left(\widehat{\beta}_1-1\right)\widehat{\alpha}_1}{(\widehat{\sigma}_1+1)y_i+x_i}\right)^{\frac{\widehat{\sigma}_1+1}{\widehat{\beta}_1-1}y_i+\frac{1}{\widehat{\beta}_1-1}x_i}\left(\frac{w_i'\widehat{h}_1}{2\left\{1-\left\{(1+2\widehat{\tau}_1)+\frac{1}{\widehat{\beta}_1-1}x_i\right\}-\frac{\widehat{\sigma}_1+\widehat{\beta}_1}{\widehat{\beta}_1-1}y_i\right\}}\right)^{1-\left\{(1+2\widehat{\tau}_1)+\frac{1}{\widehat{\beta}_1-1}x_i\right\}-\frac{\widehat{\sigma}_1+\widehat{\beta}_1}{\widehat{\beta}_1-1}y_i}\\
&\quad\left(\frac{w_i'k\omega}{2\widehat{\tau}_1x_i}\right)^{2\widehat{\tau}_1x_i}\left(\frac{w_i'\widehat{\gamma}_1}{2^{\widehat{\delta}_1}y_i}\right)^{y_i}\left(\frac{w_i'\widehat{\rho}_1}{x_i}\right)^{x_i}\left(\frac{\widehat{w}_1}{Wz_i}\right)^{z_i}\left(\textstyle\sum_{i=1}^{n}z_i\right)^{\sum_{i=1}^{n}z_i},\\
&\sum_{i=1}^{n}w_i'=1,\\
&\text{Where } z_i=1-\left(2\widehat{\tau}_1+\frac{1}{\widehat{\beta}_1-1}\right)x_i-\left(\frac{\widehat{\sigma}_1+\widehat{\beta}_1}{\widehat{\beta}_1-1}+\widehat{\delta}_1-1\right)y_i,\\
&\qquad x_i,y_i>0, w_i'>0 \text{ and } x=(x_1,x_2,x_3,\dots..,x_n),\ y=(y_1,y_2,y_3,\dots\dots\dots,y_n).
\end{aligned}$$

Now using the primal-dual relation we have

$$TAC^*(D,Q,L) = n\left(v^*(x,y)\right)^{1/n};$$

$$w_i'\widehat{\alpha}_1 D_i^{*(1-\widehat{\beta}_1)} = \theta_{i1}^*\left(v^*(x,y)\right)^{1/n};$$

$$\frac{w_i'\widehat{h}_1 Q_i^*}{2} = \theta_{i2}^*\left(v^*(x,y)\right)^{1/n};$$

wi′hikωLi∗=θi3∗v∗x,y1/n, for i=1,2,3,……….,n.

5| Numerical Example

Here we consider an inventory system which consists of two items with following parameter values in proper units. Total storage area $W = 500\ Sq.ft.$ and $k = 3, \omega = 5, w_1' = 0.5, w_1' = 0.5$.

Table 1. Input imprecise data for shape parameters.

Parameters	Items	
	I	II
$\widetilde{\alpha}_1$	(200,205,210,215; 0.9)	(215,220,225,230; 0.8)
$\widetilde{\beta}_1$	(4,5,6,7; 0.8)	(5,6,7,8; 0.8)
$\widetilde{h}_1$	(2,4,5,6; 0.9)	(2,2.5,3,3.5; 0.8)
$\widetilde{\rho}_1$	(2,2.3,2.4,2.5; 0.9)	(3,3.1,3.2,3.3; 0.9)
$\widetilde{\gamma}_1$	(90,95,100,105; 0.7)	(92,95,98,102; 0.8)
$\widetilde{\delta}_1$	(0.02,0.03,0.04,0.05; 0.8)	(0.04,0.05,0.06,0.07; 0.8)
$\widetilde{\sigma}_1$	(0.2,0.3,0.4,0.5; 0.8)	(0.2,0.3,0.4,0.5; 0.9)
$\widetilde{w}_1$	(1.5,1.6,1.7,1.8; 0.7)	(1.7,1.8,1.9,2.0; 0.9)
$\widetilde{\tau}_1$	$\left(\frac{1}{6},\frac{1}{5},\frac{1}{4},\frac{1}{3}; 0.9\right)$	$\left(\frac{1}{7},\frac{1}{6},\frac{1}{5},\frac{1}{4}; 0.9\right)$

Approximate value of the above parameter is

Table 2. Defuzzification of the fuzzy numbers.

Defuzzification of the Fuzzy Numbers	Items	
	I	II
$\widehat{\alpha}_1$	186.75	178
$\widehat{\beta}_1$	4.4	5.2
$\widehat{h}_1$	3.825	2.2
$\widehat{\rho}_1$	2.07	2.835
$\widehat{\gamma}_1$	68.25	77.4
$\widehat{\delta}_1$	0.028	0.044
$\widehat{\sigma}_1$	0.28	0.315
$\widehat{w}_1$	1.155	2.115
$\widehat{\tau}_1$	0.21375	0.17089

Table 3. Optimal solutions of MOIM using different methods.

Methods	D_1^*	Q_1^*	L_1^*	TAC_1^*	D_2^*	Q_2^*	L_2^*	TAC_2^*
FPTHMF	2.50	11.51	0.34×10^{-3}	53.95	2.28	15.54	0.30×10^{-3}	41.03
FNLP	2.50	11.51	0.34×10^{-3}	53.95	2.30	15.57	0.33×10^{-3}	41.03
FAGP	2.49	11.35	0.35×10^{-3}	53.96	2.30	15.63	0.37×10^{-3}	41.03
GP	2.58	10.07	0.27×10^{-3}	54.58	2.20	17.58	0.42×10^{-3}	41.52

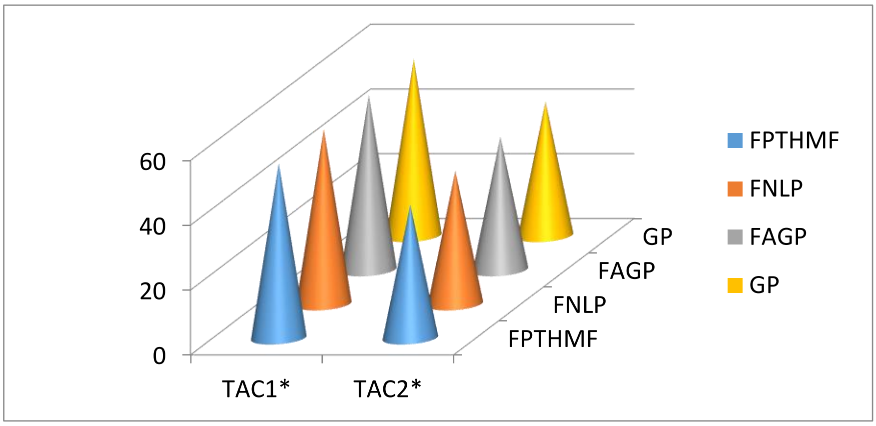

Fig. 2. Minimizing cost of both items using different methods.

From the above figure shows that GP, FPTHMF, FNLP, and FAGP methods almost provide the same results.

6| Sensitivity Analysis

In the sensitivity analysis the optimal solutions have been found buy using FNLP method.

Table 4. Optimal solution of MOIM for different values of α_1, α_2.

Method	α_1, α_2	D_1^*	Q_1^*	L_1^*	TAC_1^*	D_2^*	Q_2^*	L_2^*	TAC_2^*
	-20%	2.36	11.12	0.33×10^{-3}	52.14	2.19	15.12	0.29×10^{-3}	39.82
	-10%	2.43	11.33	0.34×10^{-3}	53.09	2.24	15.36	0.33×10^{-3}	40.45
FNLP	10%	2.56	11.68	0.35×10^{-3}	54.75	2.34	15.78	0.37×10^{-3}	41.55
	20%	2.61	10.84	0.35×10^{-3}	55.49	2.38	17.97	0.42×10^{-3}	42.03

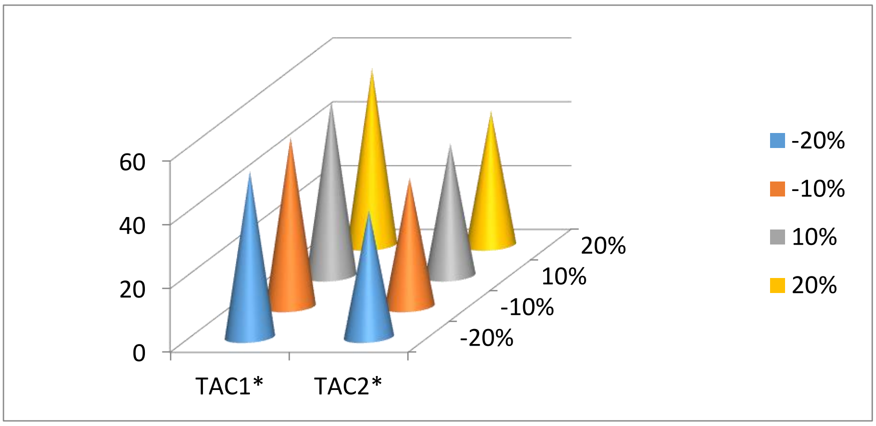

Fig. 3. Minimizing cost of both items for different values of α_1, α_2.

From the *Fig. 3* suggests that the minimum cost of both items is increased when values of α_1, α_2 are increased.

Table 5. Optimal solution of MOIM for different values of β_1, β_2.

Method	β_1, β_2	D_1^*	Q_1^*	L_1^*	TAC_1^*	D_2^*	Q_2^*	L_2^*	TAC_2^*
	-20%	2.93	12.73	0.37×10^{-3}	62.82	2.67	17.19	0.32×10^{-3}	47.24
	-10%	2.69	12.05	0.35×10^{-3}	57.77	2.46	16.30	0.31×10^{-3}	43.71
FNLP	10%	2.35	11.07	0.33×10^{-3}	50.98	2.16	15.00	0.29×10^{-3}	38.92
	20%	2.22	10.70	0.32×10^{-3}	48.59	2.06	14.52	0.28×10^{-3}	37.23

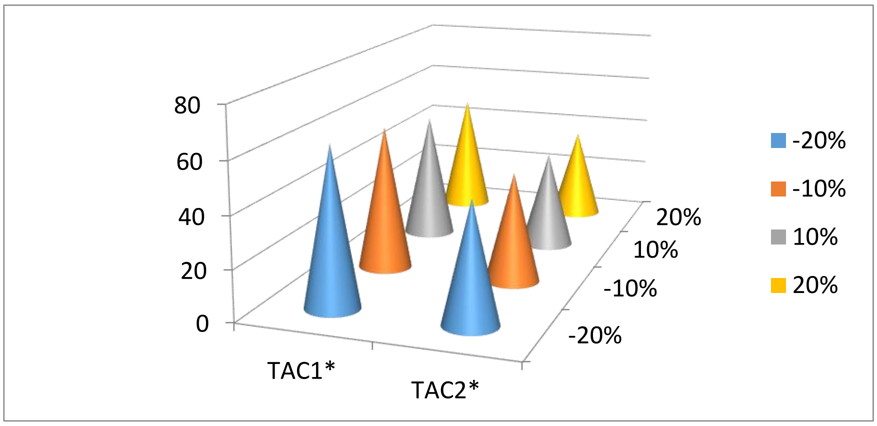

Fig. 4. Minimizing cost of 1st and 2nd items for different values of β_1, β_2.

From the *Fig. 4* suggests that the optimal cost of both items is decreased when values of β_1, β_2 are increased.

Table 6. Optimal solution of MOIM for different values of γ_1, γ_2.

Method	γ_1, γ_2	D_1^*	Q_1^*	L_1^*	TAC_1^*	D_2^*	Q_2^*	L_2^*	TAC_2^*
FNLP	-20%	2.57	10.58	0.40×10^{-3}	49.70	2.34	14.26	0.35×10^{-3}	37.60
	10%-	2.53	11.06	0.37×10^{-3}	51.89	2.32	14.94	0.32×10^{-3}	39.36
	10%	2.47	11.94	0.32×10^{-3}	55.92	2.27	16.19	0.28×10^{-3}	42.60
	20%	2.45	12.35	0.30×10^{-3}	57.78	2.25	16.77	0.26×10^{-3}	44.11

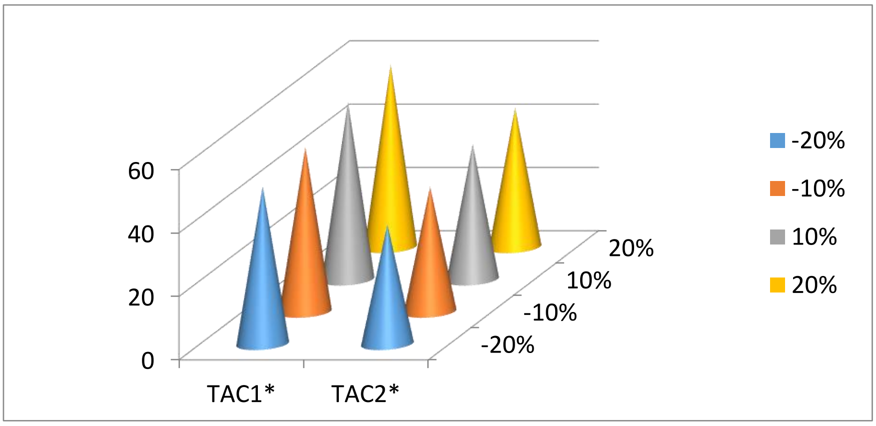

Fig. 5. Minimizing cost of both items for different values of γ_1, γ_2.

From the above *Fig. 5* suggests that the optimal cost of both items is increased when values of γ_1, γ_2 are increased.

Table 7. Optimal solutions of MOIM for different values of σ_1, σ_2.

Method	σ_1, σ_2	D_1^*	Q_1^*	L_1^*	TAC_1^*	D_2^*	Q_2^*	L_2^*	TAC_2^*
	-20%	2.54	11.36	0.35×10^{-3}	52.92	2.33	15.35	0.31×10^{-3}	40.18
	-10%	2.52	11.44	0.35×10^{-3}	53.44	2.31	15.46	0.31×10^{-3}	40.60
FNLP	10%	2.48	11.59	0.33×10^{-3}	54.47	2.28	15.70	0.29×10^{-3}	41.45
	20%	2.46	11.67	0.33×10^{-3}	54.99	2.26	15.81	0.29×10^{-3}	41.87

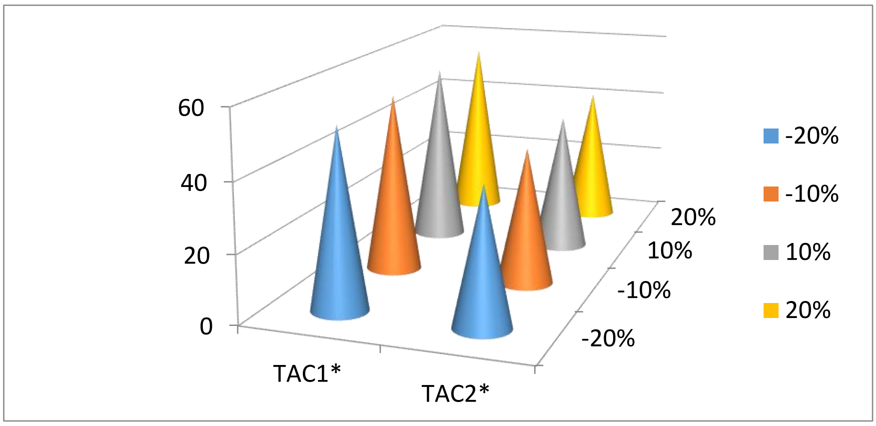

Fig. 6. Minimizing cost of 1st and 2nd items for different values of σ_1, σ_2.

From the **above *Fig. 6*** suggests that the minimum cost of both items is increased when values of σ_1, σ_2 are increased.

Table 8. Optimal solutions of MOIM for different values of ρ_1, ρ_2.

Method	ρ_1, ρ_2	D_1^*	Q_1^*	L_1^*	TAC_1^*	D_2^*	Q_2^*	L_2^*	TAC_2^*
	-20%	2.50	11.42	0.25×10^{-3}	53.44	2.29	15.47	0.23×10^{-3}	40.68
	-10%	2.50	11.47	0.30×10^{-3}	53.70	2.29	15.53	0.26×10^{-3}	40.86
FNLP	10%	2.50	11.55	0.39×10^{-3}	54.20	2.29	15.64	0.34×10^{-3}	41.19
	20%	2.50	11.60	0.43×10^{-3}	54.45	2.29	15.69	0.39×10^{-3}	41.35

Table 9. Optimal solutions of MOIM for different values of τ_1, τ_2.

Method	τ_1, τ_2	D_1^*	Q_1^*	L_1^*	TAC_1^*	D_2^*	Q_2^*	L_2^*	TAC_2^*
	-20%	2.50	11.40	0.15×10^{-3}	53.15	2.29	15.47	0.14×10^{-3}	40.58
	-10%	2.50	11.46	0.23×10^{-3}	53.54	2.29	15.52	0.21×10^{-3}	40.80
FNLP	10%	2.50	11.57	0.48×10^{-3}	54.39	2.29	15.64	0.42×10^{-3}	41.26
	20%	2.50	11.62	0.67×10^{-3}	54.83	2.29	15.69	0.57×10^{-3}	41.50

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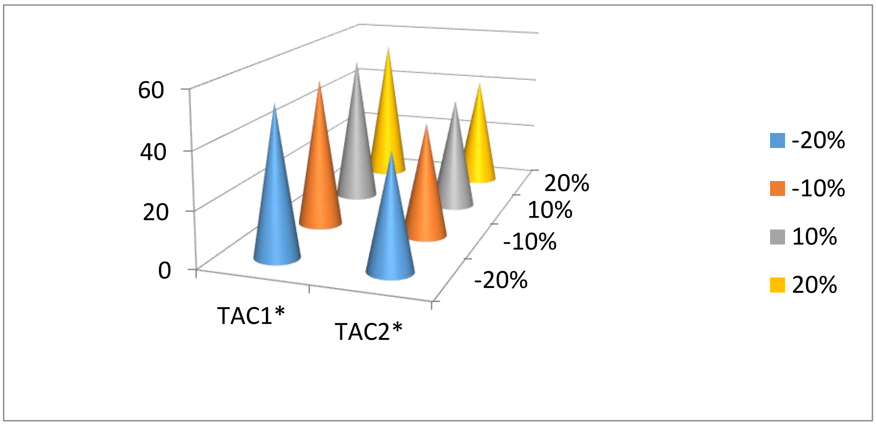

Fig. 7. **Minimizing cost of both items for different values of ρ_1, ρ_2.**

From the above *Fig. 7* suggests that the minimum cost of both items is increased when values of ρ_1, ρ_2 are increased.

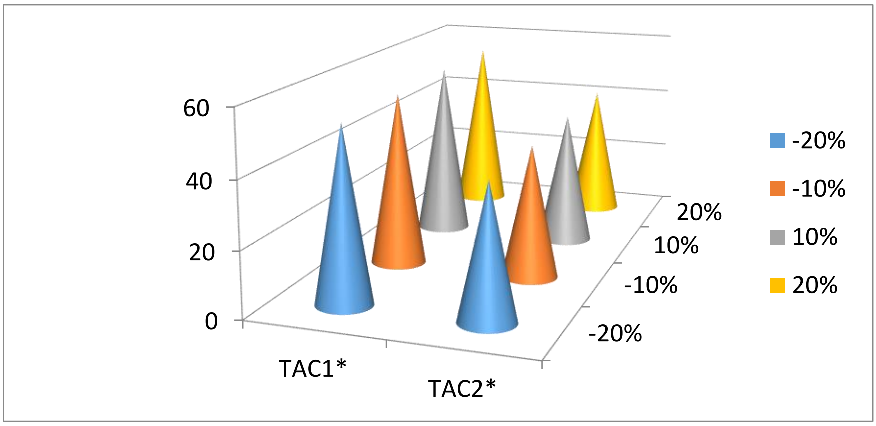

Fig. 8. Minimizing cost of 1st and 2nd items for different values of τ_1, τ_2.

From the above *Fig. 8* suggests that the minimum cost of both items is increased when values of τ_1, τ_2 are increased.

7| Conclusion

In this article, we have developed an inventory model of multi-item with limitations on storage space in a fuzzy environment. Here we considered the constant demand rate and production cost is dependent on the demand rate. Set-up- cost is dependent on average inventory level as well as demand. Lead time crashing cost is considered the continuous function of leading time. Due to uncertainty all cost parameters are taken as a generalized trapezoidal fuzzy number. The formulated problem has been solved by various techniques like GP approach, FPTHMF, FNLP, and FAGP. Numerical example is given under considering two items to illustrate the model. A numerical problem is solved by using LINGO13 software.

This paper will be extended by using linear, quadratic demand, ramp type demand, power demand, and stochastic demand etc., introduce shortages, generalize the model under two-level credit period strategy etc. Inflation plays a crucial position in Inventory Management (IM) but here it is not considered. So inflation can be used in this model for practical. Also other types of fuzzy numbers like triangular fuzzy numbers; PfFN, pFN, etc. may be used for all cost parameters of the model.

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References

[1] Abou-El-Ata, M. O., & Kotb, K. A. M. (1997). Multi-item EOQ inventory model with varying holding cost under two restrictions: a geometric programming approach. *Production planning and control*, *8*(6), 608-611.

[2] Bit, A. K. (2004). Fuzzy programming with hyperbolic membership functions for multiobjective capacitated transportation problem. *Opsearch*, *41*(2), 106-120.

[3] Ben-Daya, M. A., & Raouf, A. (1994). Inventory models involving lead time as a decision variable. *Journal of the operational research society*, *45*(5), 579-582.

[4] Bortolan, G., & Degani, R. (1985). A review of some methods for ranking fuzzy subsets. *Fuzzy sets and systems*, *15*(1), 1-19.

[5] Beightler, C., & Phillips, D. T. (1976). *Applied geometric programming*. John Wiley & Sons.

[6] Biswal, M. P. (1992). Fuzzy programming technique to solve multi-objective geometric programming problems. *Fuzzy sets and systems*, *51*(1), 67-71.

[7] Chen, C. K. (2000). Optimal determination of quality level, selling quantity and purchasing price for intermediate firms. *Production planning and control*, *11*(7), 706-712.

[8] Chuang, B. R., Ouyang, L. Y., & Chuang, K. W. (2004). A note on periodic review inventory model with controllable setup cost and lead time. *Computers and operations research*, *31*(4), 549-561.

[9] Duffin, R. J. (1967). *Geometric programming-theory and application* (No. 04; QA264, D8.). New York Wiley196278 p.

[10] Duffin, R. J., Peterson, E. L. & Zener, C. (1966). *Geometric programming theory and applications.* Wiley, New York.

[11] Liang, Y., & Zhou, F. (2011). A two-warehouse inventory model for deteriorating items under conditionally permissible delay in payment. *Applied mathematical modelling*, *35*(5), 2221-2231.

[12] Das, K., Roy, T. K., & Maiti, M. (2000). Multi-item inventory model with quantity-dependent inventory costs and demand-dependent unit cost under imprecise objective and restrictions: a geometric programming approach. *Production planning and control*, *11*(8), 781-788.

[13] Das, S. K., & Islam, S. (2019). Multi-objective two echelon supply chain inventory model with lot size and customer demand dependent purchase cost and production rate dependent production cost. *Pakistan journal of statistics and operation research*, *15*(4), 831-847.

[14] Hariga, M., & Ben-Daya, M. (1999). Some stochastic inventory models with deterministic variable lead time. *European journal of operational research*, *113*(1), 42-51.

[15] Harri, F. (1913). How many parts to make at once factory. *Mag. Mannage*, (10), 135-136.

[16] Islam, S. (2016). Multi-objective geometric programming problem and its applications. *Yugoslav journal of operations research*, *20*(2). http://www.yujor.fon.bg.ac.rs/index.php/yujor/article/view/353

[17] Islam, S. (2008). Multi-objective marketing planning inventory model: A geometric programming approach. *Applied mathematics and computation*, *205*(1), 238-246.

[18] Kotb, K. A., & Fergany, H. A. (2011). Multi-item EOQ model with both demand-dependent unit cost and varying leading time via geometric programming. *Applied mathematics*, *2*(5), 551-555.

[19] Maiti, M. K. (2008). Fuzzy inventory model with two warehouses under possibility measure on fuzzy goal. *European journal of operational research*, *188*(3), 746-774.

[20] Ouyang, L. Y., Yeh, N. C., & Wu, K. S. (1996). Mixture inventory model with backorders and lost sales for variable lead time. *Journal of the operational research society*, *47*(6), 829-832.

[21] Ouyang L., & Wu K. (1998). A min-max distribution free procedure for mixed inventory model with variable lead time. *Int J Pro Econ*, 56(1), 511-516.

[22] Mandal, N. K., Roy, T. K., & Maiti, M. (2005). Multi-objective fuzzy inventory model with three constraints: a geometric programming approach. *Fuzzy sets and systems*, *150*(1), 87-106.

[23] Mandal, N. K., Roy, T. K., & Maiti, M. (2006). Inventory model of deteriorated items with a constraint: A geometric programming approach. *European journal of operational research*, *173*(1), 199-210.

[24] Sarkar, B., Gupta, H., Chaudhuri, K., & Goyal, S. K. (2014). An integrated inventory model with variable lead time, defective units and delay in payments. *Applied mathematics and computation*, *237*, 650-658.

[25] Sarkar, B., Mandal, B., & Sarkar, S. (2015). Quality improvement and backorder price discount under controllable lead time in an inventory model. *Journal of manufacturing systems*, *35*, 26-36.

[26] Shaikh, A. A., Bhunia, A. K., Cárdenas-Barrón, L. E., Sahoo, L., & Tiwari, S. (2018). A fuzzy inventory model for a deteriorating item with variable demand, permissible delay in payments and partial backlogging with shortage follows inventory (SFI) policy. *International journal of fuzzy systems*, *20*(5), 1606-1623.

[27] Zadeth, L. A. (1965). Fuzzy sets. *Information and control*, *8*, 338-353.

[28] Zimmermann, H. J. (1985). Applications of fuzzy set theory to mathematical programming. *Information sciences*, *36*(1-2), 29-58.

[29] Garai, T., Chakraborty, D., & Roy, T. K. (2019). Multi-objective inventory model with both stock-dependent demand rate and holding cost rate under fuzzy random environment. *Annals of data science*, *6*(1), 61-81.

[30] Shaikh, A. A., Panda, G. C., Khan, M. A. A., Mashud, A. H. M., & Biswas, A. (2020). An inventory model for deteriorating items with preservation facility of ramp type demand and trade credit. *International journal of mathematics in operational research*, *17*(4), 514-551.

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Paper Type: Research Paper

Some Similarity Measures of Spherical Fuzzy Sets Based on the Euclidean Distance and Their Application in Medical Diagnosis

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Abstract

Similarity measure is an important tool in multiple criteria decision-making problems, which can be used to measure the difference between the alternatives. In this paper, some new similarity measures of Spherical Fuzzy Sets (SFS) are defined based on the Euclidean distance measure and the proposed similarity measures satisfy the axiom of the similarity measure. Furthermore, we apply the proposed similarity measures to medical diagnosis decision making problem; the numerical example is used to illustrate the feasibility and effectiveness of the proposed similarity measures of SFS, which are then compared to other existing similarity measures.

Keywords: Spehrical fuzzy sets, Euclidean distance; Proposed similarity measures medical diagnosis.

1 | Introduction

The concept of Fuzzy Set (FS) $A = \{< x_i, \mu_{A_{x_i}} > | x_i \in X\}$ in $X = \{x_1, x_2, \dots, x_n\}$ was proposed by Zadeh [1], where the membership degree $\mu_{A_{x_i}}$ is a single value between zero and one. The FS has been widely applied in many fields, such as medical diagnosis, image processing, supply decision-making [2]-[4], and so on. In some uncertain decision-making problems, the degree of membership is not exactly as a numerical value but as an interval. Hence, Zadeh [5] proposed the Interval-Valued Fuzzy Sets (IVFS). However, the FS and the IVFS only have the membership degree, and they cannot describe the non-membership degree of the element belonging to the set.

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Then, Atanassov [6] proposed the Intuitionistic Fuzzy Set (IFS) $E = \{< x_i, \mu_E(x_i), \vartheta_E(x_i) > |x_i \in X\}$, where $\mu_E(x_i)$ $(0 \le \mu_E(x_i) \le 1)$ and $\vartheta_E(x_i)$ $(0 \le \vartheta_E(x_i) \le 1)$ represent the membership and the non-membership degree, respectively, and the indeterminacy- membership degree $\pi_E(x_i) = 1 - \mu_E(x_i) - \vartheta_E(x_i)$. The IFS is more effective to deal with the vague information more than the FS and IVFS.

Yang and Chiclana [7] proposed a spherical representation, which allowed us to define a distance function between intuitionistic fuzzy sets. In the spherical representation, hesitancy can be calculated based on the given membership and non-membership values since they only consider the surface of the sphere. Besides, they measure the spherical arc distance between two IFSs. Furthermore, Gong et al. [8] introduced an approach generalizing Yang and Chiclana's work.

The Spherical Fuzzy Sets (SFSs) are based on the fact that the hesitancy of a decision maker can be defined independently from membership and non-membership degrees, satisfying the following condition:

$$0 \le \mu_{\tilde{A}}^2(u) + \vartheta_{\tilde{A}}^2(u) + \pi_{\tilde{A}}^2(u) \le 1. \qquad \forall\ u \in U. \tag{1}$$

On the surface of the sphere, *Eq. (1)* becomes

$$\mu_{\tilde{A}}^2(u) + \vartheta_{\tilde{A}}^2(u) + \pi_{\tilde{A}}^2(u) = 1. \qquad \forall\ u \in U. \tag{2}$$

On the other hand, similarity measure is an important tool in multiple-criteria decision making problems, which can be used to measure the difference between the alternatives. Many studies about the similarity measures have been obtained. For example, Beg and Ashraf [9] proposed a similarity measure of fuzzy sets based on the concept of $\epsilon -$ fuzzy transitivity and discussed the degree of transitivity of different similarity measures. Song et al. [4] considered the similarity measure and proposed corresponding distance measure between intuitionistic fuzzy belief functions. In addition, cosine similarity measure is also an important similarity measure, and it can be defined as the inner product of two vectors divided by the product of their lengths. There are some scholars who studied the cosine similarity measures [10]-[15]**.** Various forms of Spherical fuzzy sets which are applied in Multi–attribute decision making problems are developed in [16]-[18].

In this paper, we propose a new method to construct the similarity measure of SFSs. They play an important role in practical application, especially in pattern recognition, medical diagnosis, and so on. Furthermore, the proposed similarity measure can be applied more widely in the field of decision–making problems.

2| Preliminaries

Definition 1. [19]. A SFS $\tilde{A}_s$ of the universe of discourse U is given by,

$\tilde{A}_s = \{\langle \mu_{\tilde{A}_s}(u), \vartheta_{\tilde{A}_s}(u), \pi_{\tilde{A}_s}(u) | u \in U \rangle\}$, *where* $\mu_{\tilde{A}_s}: U \to [0,1]$, $\vartheta_{\tilde{A}_s}: U \to [0,1]$, $\pi_{\tilde{A}_s}: U \to [0,1]$ *and* $0 \le \mu_{\tilde{A}_s}^2(u) + \vartheta_{\tilde{A}_s}^2(u) + \pi_{\tilde{A}_s}^2(u) \le 1.\quad \forall\ u \in U.$

For each u, the numbers $\mu_{\tilde{A}_s}(u), \vartheta_{\tilde{A}_s}(u)$ *and* $\pi_{\tilde{A}_s}(u)$ are the degree of membership, non-membership and hesitancy of u to $\tilde{A}_s$, respectively.

Definition 2. [9]. Basic operators of spherical fuzzy sets:

Union. $\tilde{A}_s \cup \tilde{B}_s = \{max\{\mu_{\tilde{A}_s}, \mu_{\tilde{B}_s}\}, min\{\vartheta_{\tilde{A}_s}, \vartheta_{\tilde{B}_s}\}, min\{\pi_{\tilde{A}_s}, \pi_{\tilde{B}_s}\}\}$.

Intersection. $\tilde{A}_s \cap \tilde{B}_s = \left\{min\left\{\mu_{\tilde{A}_s}, \mu_{\tilde{B}_s}\right\}, max\left\{\vartheta_{\tilde{A}_s}, \vartheta_{\tilde{B}_s}\right\}, max\left\{\pi_{\tilde{A}_s}, \pi_{\tilde{B}_s}\right\}\right\}$.

Addition. $\tilde{A}_s \bigoplus \tilde{B}_s = \left\{\left(\mu_{\tilde{A}_s}{}^2 + \mu_{\tilde{B}_s}{}^2 - \mu_{\tilde{A}_s}{}^2\mu_{\tilde{B}_s}{}^2\right)^{1/2}, \vartheta_{\tilde{A}_s}\vartheta_{\tilde{B}_s}, \ \pi_{\tilde{A}_s}\pi_{\tilde{B}_s}\right\}$.

Multiplication. $\tilde{A}_s \bigotimes \tilde{B}_s = \left\{\mu_{\tilde{A}_s}\mu_{\tilde{B}_s}, \ \left(\vartheta_{\tilde{A}_s}{}^2 + \vartheta_{\tilde{B}_s}{}^2 - \vartheta_{\tilde{A}_s}{}^2\vartheta_{\tilde{B}_s}{}^2\right)^{1/2}, \pi_{\tilde{A}_s}\pi_{\tilde{B}_s}\right\}$.

Multiplication by a scalar, $\boldsymbol{\lambda > 0}$. $\lambda. \tilde{A}_s = \left\{\left(1 - \left(1 - \mu_{\tilde{A}_s}{}^2\right)^{\lambda}\right)^{1/2}, \ \vartheta_{\tilde{A}_s}{}^{\lambda}, \ \pi_{\tilde{A}_s}{}^{\lambda}\right\}$.

Power of $\boldsymbol{\tilde{A}_s}$, $\boldsymbol{\lambda > 0}$. $\tilde{A}_s{}^{\lambda} = \left\{\mu_{\tilde{A}_s}{}^{\lambda}, \left(1 - \left(1 - \vartheta_{\tilde{A}_s}{}^2\right)^{\lambda}\right)^{1/2}, \ \pi_{\tilde{A}_s}{}^{\lambda}\right\}$.

3| Several New Similarity Measures

The similarity measure is a most widely used tool to evaluate the relationship between two sets. The following axiom about the similarity measure of IVSFSs should be satisfied:

Lemma 1. Let $X = \{x_1, x_2, \dots, x_n\}$ be the universal set [12] if the similarity measure S (A, B) between SFSs A and B satisfies the following properties:

$0 \le S(A, B) \le 1$;

$S(A, B) = 1$ if and only if $A = B$;

$S(A, B) = S(B, A)$.

Then, the similarity measure $S(A, B)$ is a genuine similarity measure.

3.1| The New Similarity Measures between SFSs

Definition 3. Let $X = \{x_1, x_2, \dots, x_n\}$ be the universal set for any two SFSs $A = \{< x_i, \mu_{A_{x_i}}, \vartheta_{A_{x_i}}, \pi_{A_{x_i}} > | x_i \in X\}$ and $B = \{< x_i, \mu_{B_{x_i}}, \vartheta_{B_{x_i}}, \pi_{B_{x_i}} > | x_i \in X\}$; then the Euclidean distance between SFSs A and B is defined as follows:

$$D_{SFSs}(A, B) = \sqrt{\frac{\sum_{i=1}^{n}\left[\left(\mu_A{}^2(x_i) - \mu_B{}^2(x_i)\right)^2 + \left(\vartheta_A{}^2(x_i) - \vartheta_B{}^2(x_i)\right)^2 + \left(\pi_A{}^2(x_i) - \pi_B{}^2(x_i)\right)^2\right]}{3n}}. \tag{3}$$

Now, we construct new similarity measures of SFSs based on the Euclidean distance measures.

Definition 4. Let $X = \{x_1, x_2, \dots, x_n\}$ be the universal set for any two SFSs $A = \{< x_i, \mu_{A_{x_i}}, \vartheta_{A_{x_i}}, \pi_{A_{x_i}} > | x_i \in X\}$ and $B = \{< x_i, \mu_{B_{x_i}}, \vartheta_{B_{x_i}}, \pi_{B_{x_i}} > | x_i \in X\}$; the similarity measure of SFSs between A and B is defined as follows:

$$S_{1SFSs}(A, B) = \frac{\sum_{i=1}^{n}\left(\min\left(\mu_A{}^2(x_i), \mu_B{}^2(x_i)\right) + \min\left(\vartheta_A{}^2(x_i), \vartheta_B{}^2(x_i)\right) + \min\left(\pi_A{}^2(x_i), \pi_B{}^2(x_i)\right)\right)}{\sum_{i=1}^{n}\left(\max\left(\mu_A{}^2(x_i), \mu_B{}^2(x_i)\right) + \max\left(\vartheta_A{}^2(x_i), \vartheta_B{}^2(x_i)\right) + \max\left(\pi_A{}^2(x_i), \pi_B{}^2(x_i)\right)\right)}. \tag{4}$$

The similarity measure S_{1SFSs} satisfies the properties in *Lemma 1*.

Next, we propose a new method to construct a new similarity measure of SFSs, and the Euclidean distance, it can be defined as follows:

Definition 5. Let $X = \{x_1, x_2, \dots, x_n\}$ be the universal set for any two SFSs $A = \{< x_i, \mu_{A_{x_i}}, \vartheta_{A_{x_i}}, \pi_{A_{x_i}} > | x_i \in X\}$ and $B = \{< x_i, \mu_{B_{x_i}}, \vartheta_{B_{x_i}}, \pi_{B_{x_i}} > | x_i \in X\}$; a new similarity measure $S^*_{1SFSs}(A, B)$ is defined as follows:

$$S^*_{1SFSs}(A,B) = \frac{1}{2}\left(S_{1SFSs}(A,B) + 1 - D_{SFSs}(A,B)\right). \tag{5}$$

The proposed similarity measure of SFSs satisfies the *Theorem 1*.

Theorem 1. The similarity measure $S^*_{1SFSs}(A, B)$ between $A = \{< x_i, \mu_{A_{x_i}}, \vartheta_{A_{x_i}}, \pi_{A_{x_i}} > | x_i \in X\}$ and $B = \{< x_i, \mu_{B_{x_i}}, \vartheta_{B_{x_i}}, \pi_{B_{x_i}} > | x_i \in X\}$ satisfies the following properties:

$0 \le S^*_{1SFSs}(A, B) \le 1$

$S^*_{1SFSs}(A, B) = 1$ if and only if $A = B$

$S^*_{1SFSs}(A, B) = S^*_{1SFSs}(B, A)$.

Proof. Because $D_{SFSs}(A, B)$ is an Euclidean distance measure, obviously, $0 \le D_{SFSs}(A, B) \le 1$. Furthermore, according to lemma 1, we know that $0 \le S_{1SFSs}(A, B) \le 1$. Then, $0 \le \frac{1}{2}\left(S_{1SFSs}(A, B) + 1 - D_{SFSs}(A, B)\right) \le 1$, *i.e.,* $0 \le S^*_{1SFSs}(A, B) \le 1$.

If $S^*_{1SFSs}(A, B) = 1$, we have $S_{1SFSs}(A, B) + 1 - D_{SFSs}(A, B) = 2$, that is $S_{1SFSs}(A, B) = 1 + D_{SFSs}(A, B)$. Because $D_{SFSs}(A, B)$ is the Euclidean distance measure $0 \le D_{SFSs}(A, B) \le 1$. Furthermore, $0 \le S_{1SFSs}(A, B) \le 1$, then $S_{1SFSs}(A, B) = 1$ and $D_{SFSs}(A, B) = 0$ should be established at the same time. If the Euclidean distance measure $D_{SFSs}(A, B) = 0, A = B$ is obvious. According to lemma 1, when $S_{1SFSs}(A, B) = 1, A = B$; so if $S^*_{1SFSs}(A, B) = 1, A = B$ is obtained.

On the other hand, when $A = B$, according to *Eqs. (3)* and *(4)* $D_{SFSs}(A, B) = 0$ and $S_{1SFSs}(A, B) = 1$ are obtained respectively. Furthermore, we can get $S^*_{1SFSs}(A, B) = 1$. $S^*_{1SFSs}(A, B) = S^*_{1SFSs}(B, A)$ is straightforward.

From *Theorem 1*, we know that the proposed new similarity measure $S^*_{1SFSs}(A, B)$ is a genuine similarity measure. On the other hand, cosine similarity measure is also an important similarity measure. The cosine similarity measure between SFSs is as follows:

Definition 6. Let $X = \{x_1, x_2, \dots, x_n\}$ be the universal set for any two SFSs $A = \{< x_i, \mu_{A_{x_i}}, \vartheta_{A_{x_i}}, \pi_{A_{x_i}} > | x_i \in X\}$ and $B = \{< x_i, \mu_{B_{x_i}}, \vartheta_{B_{x_i}}, \pi_{B_{x_i}} > | x_i \in X\}$; the cosine similarity measure of SFSs between A and B is defined as follows:

$$S_{2SFSs}(A,B) = \frac{1}{n}\sum_{i=1}^{n} \frac{\left(\left(\mu_A{}^2(x_i)\mu_B{}^2(x_i)\right) + \left(\vartheta_A{}^2(x_i)\,\vartheta_B{}^2(x_i)\right) + \left(\pi_A{}^2(x_i)\,\pi_B{}^2(x_i)\right)\right)}{\sqrt{(\mu_A{}^2(x_i))^2 + (\vartheta_A{}^2(x_i))^2 + (\pi_A{}^2(x_i))^2}\sqrt{(\mu_B{}^2(x_i))^2 + (\vartheta_B{}^2(x_i))^2 + (\pi_B{}^2(x_i))^2}}. \tag{6}$$

Now, we are going to propose another similarity measure of SFSs based on the cosine similarity measure and the Euclidean distance D_{SFSs}. It considers the similarity measure not only from the point of view of algebra but also from the point of view of geometry, which can be defined as:

Definition 7. Let $X = \{x_1, x_2, \dots, x_n\}$ be the universal set for any two SFSs

$A = \{< x_i, \mu_{A_{x_i}}, \vartheta_{A_{x_i}}, \pi_{A_{x_i}} > | x_i \in X\}$ and $B = \{< x_i, \mu_{B_{x_i}}, \vartheta_{B_{x_i}}, \pi_{B_{x_i}} > | x_i \in X\}$; a new similarity measure $S^*_{2SFSs}(A, B)$ is defined as follows:

$$S^*_{2SFSs}(A, B) = \frac{1}{2}\left(S_{2SFSs}(A, B) + 1 - D_{SFSs}(A, B)\right). \tag{7}$$

Theorem 2. The similarity measure $S^*_{2SFSs}(A, B)$ between $A = \{< x_i, \mu_{A_{x_i}}, \vartheta_{A_{x_i}}, \pi_{A_{x_i}} > | x_i \in X\}$ and

$B = \{< x_i, \mu_{B_{x_i}}, \vartheta_{B_{x_i}}, \pi_{B_{x_i}} > | x_i \in X\}$ satisfies the following properties:

$0 \le S^*_{2SFSs}(A, B) \le 1$;

$S^*_{2SFSs}(A, B) = 1$ if and only if $A = B$;

$S^*_{2SFSs}(A, B) = S^*_{2SFSs}(B, A)$.

Proof. Because $D_{SFSs}(A, B)$ is an Euclidean distance measure, obviously, $0 \le D_{SFSs}(A, B) \le 1$. Furthermore, according to lemma 1, we know that $0 \le S_{SFSs}(A, B) \le 1$. Then, $0 \le \frac{1}{2}\left(S_{2SFSs}(A, B) + 1 - D_{SFSs}(A, B)\right) \le 1$, i.e., $0 \le S^*_{1SFSs}(A, B) \le 1$.

If $S^*_{2SFSs}(A, B) = 1$, we have $S_{2SFSs}(A, B) + 1 - D_{SFSs}(A, B) = 2$, that is $S_{2SFSs}(A, B) = 1 + D_{SFSs}(A, B)$. Because $D_{SFSs}(A, B)$ is the Euclidean distance measure $0 \le D_{SFSs}(A, B) \le 1$. Furthermore, $0 \le S_{2SFSs}(A, B) \le 1$, then $S_{2SFSs}(A, B) = 1$ and $D_{SFSs}(A, B) = 0$ should be established at the same time. When $S_{2SFSs}(A, B) = 1$, we have $\mu_A(x_i) = k\mu_B(x_i)$, $\vartheta_A(x_i) = k\vartheta_B(x_i), and\ \pi_A(x_i) = k\pi_B(x_i)$ (k is a constant). When the Euclidean distance measure $D_{SFSs}(A, B) = 0, A = B$. Then $A = B$ is obtained.

On the other hand, when $A = B$, according to *Eqs. (3)* and *(6)* if $A = B$, $D_{SFSs}(A, B) = 0$ and $S_{2SFSs}(A, B) = 1$ are obtained respectively. Furthermore, we can get $S^*_{2SFSs}(A, B) = 1$. $S^*_{2SFSs}(A, B) = S^*_{2SFSs}(B, A)$ is straightforward.

Thus $S^*_{2SFSs}(A, B)$ satisfies all the properties of the *Theorem 2*.

In the next section, we will apply the proposed new similarity measures to medical diagnosis decision problem; numerical examples are also given to illustrate the application and effectiveness of the proposed new similarity measures.

4| Applications of the Proposed Similarity Measures

4.1| The Proposed Similarity Measures between SFSs for Medical Diagnosis

We first give a numerical example medical diagnosis to illustrate the feasibility of the proposed new similarity measur $S^*_{1SFSs}(A, B)$e and $S^*_{2SFSs}(A, B)$ between SFSs.

Example 1. Consider a medical diagnosis decision problem; Suppose a set of diagnosis $Q = \{Q_1$(viral fever), Q_2(malaria), Q_3(typhoid), Q_4(Gastritis), Q_5(stenocardia)$\}$ and a set of symptoms $S = \{S_1$(fever), S_2(headache), S_3(stomach), S_4(cough), S_5(chestpain)$\}$. Assume a patient P_1 has all the symptoms in the process of diagnosis, the SFS evaluate information about P_1 is

$P_1(Patient) = \{< S_1, 0.8, 0.2, 0.1 >, < S_2, 0.6, 0.3, 0.1 >, < S_3, 0.2, 0.1, 0.8 >, < S_4, 0.6, 0.5, 0.1 >, < S_5, 0.1, 0.4, 0.6 >\}$.

The diagnosis information $Q_i(i = 1,2, \dots ,5)$ with respect to symptoms $S_i(i = 1,2, \dots ,5)$ also can be represented by the SFSs, which is shown in *Table 1*.

Table 1. Diagnosis information.

	S_1	S_2	S_3	S_4	S_5
Q_1	[0.4, 0.6,0.0]	[0.3, 0.2, 0.5]	[0.1, 0.3, 0.7]	[0.4, 0.3, 0.3]	[0.1, 0.2, 0.7]
Q_2	[0.7, 0.3, 0.0]	[0.2, 0.2, 0.6]	[0.0, 0.1, 0.9]	[0.7, 0.3, 0.0]	[0.1, 0.1, 0.8]
Q_3	[0.3, 0.4, 0.3]	[0.6, 0.3, 0.1]	[0.2, 0.1, 0.7]	[0.2, 0.2, 0.6]	[0.1, 0.0, 0.9]
Q_4	[0.1, 0.2, 0.7]	[0.2, 0.2, 0.4]	[0.8, 0.2, 0.0]	[0.2, 0.1, 0.7]	[0.2, 0.1, 0.7]
Q_5	[0.1, 0.1, 0.8]	[0.0, 0.2,0.8]	[0.2, 0.0, 0.8]	[0.3, 0.1, 0.8]	[0.8, 0.1, 0.1]

By applying *Eqs. (5)* and *(7)* we can obtain the similarity measure values $S^*{}_{1SFSs}(P_1, Q_i)$ and $S^*{}_{2SFSs}(P_1, Q_i)$; the results are shown in *Table 2*.

Table 2. Similarity measures.

	Q_1	Q_2	Q_3	Q_4	Q_5
$S^*{}_{1SFSs}(P_1, Q_i)$	0.5980	0.6801	0.5729	0.3919	0.3820
$S^*{}_{2SFSs}(P_1, Q_i)$	0.4277	0.4581	0.4024	0.3514	0.3155

From the above two similarity measures $S^*{}_{1SFSs}$ and $S^*{}_{2SFSs}$, we can conclude that the diagnoses of the patient P_1 are all malaria (Q_2). The proposed two similarity measures are feasible and effective.

4.2| Comparative Analysis of Existing Similarity Measures

To illustrative the effectiveness of the proposed similarity measures for medical diagnosis, we change the existing similarity measures for SFS and thus will apply the existing similarity measures for comparative analyses.

At first, we introduce the existing similarity measures between SFSs as follows:

Let $A = \{< x_i, \mu_{A_{x_i}}, \vartheta_{A_{x_i}}, \pi_{A_{x_i}} > |x_i \in X\}$ $an\, B = \{< x_i, \mu_{B_{x_i}}, \vartheta_{B_{x_i}}, \pi_{B_{x_i}} > |x_i \in X\}$d be two SFSs in $X = \{x_1, x_2, \dots , x_n\}$, the existing measures between A and B are defined as follows:

Broumi et al. [20] proposed the similarity measure SM_{SFS}:

$$\mathrm{SM_{SFS}(A, B) = 1 - D_{SFS}(A, B)}. \quad (8)$$

Sahin and Küçük [21] proposed the similarity measure SD_{SFS}:

$$\mathrm{SD_{SFS}} = \frac{1}{1+\mathrm{D_{SFS}(A,B)}}. \quad (9)$$

Ye [22] proposed the improved cosine similarity measure SC_{1SFS} and SC_{2SFS}:

$$SC_{1SFS}(A,B) = \frac{1}{n}\sum_{i=1}^{n} \cos\left[\frac{\pi.\max\left(\left|\mu_A{}^2(x_i) - \mu_B{}^2(x_i)\right|, \left|\vartheta_A{}^2(x_i) - \vartheta_B{}^2(x_i)\right|, \left|\pi_A{}^2(x_i) - \pi_B{}^2(x_i)\right|\right)}{2}\right]. \quad (10)$$

$$SC_{2SFS}(A,B) = \frac{1}{n}\sum_{i=1}^{n} \cos\left[\frac{\pi.\left(\left|\mu_A{}^2(x_i) - \mu_B{}^2(x_i)\right| + \left|\vartheta_A{}^2(x_i) - \vartheta_B{}^2(x_i)\right| + \left|\pi_A{}^2(x_i) - \pi_B{}^2(x_i)\right|\right)}{6}\right].$$

Yong-Wei et al. [23] proposed the similarity measure $SY_{SFS}(A,B)$:

$$SY_{SFS}(A,B) = \frac{SC_{SFS}(A,B)}{SC_{SFS}(A,B) + D_{SFS}(A,B)}. \quad (12)$$

Example 2. We apply *Eqs. (4)*, *(6)* and *(8) – (12)* to calculate *Example 1* again; the similarity measure values between P_1 and Q_i ($i = 1,2,..,5$) are shown on *Table 3*.

As we can see from *Table 3*, the patient P_1 is still assigned to malaria (Q_2), and the results are same as the proposed similarity measures in this paper, which means the proposed similarity measures are feasible and effective.

Table 3. Similarity values between patient and symptoms.

	Q_1	Q_2	Q_3	Q_4	Q_5
SM_{SFS}	0.8003	0.8314	0.7449	0.6388	0.6007
SD_{SFS}	0.8335	0.8557	0.7967	0.7346	0.7146
SC_{1SFS}	0.8555	0.9325	0.6469	0.7324	0.6391
SC_{2SFS}	0.9648	0.9759	0.7531	0.885	0.8585
SY_{SFS}	0.8107	0.8468	0.7171	0.6697	0.6154
S_{1SFS}	0.3958	0.5289	0.4010	0.1451	0.1633
S_{2SFS}	0.0551	0.0849	0.0600	0.0191	0.0304

The proposed similarity measures in the paper have some advantages in solving multiple criteria decision making problems. They are constructed based on the existing similarity measures and Euclidean distance, which not only satisfy the axiom of the similarity measure but also consider the similarity measure from the points of view of algebra and geometry. Furthermore, they can be applied more widely in the field of decision making problems.

5| Conclusion

The similarity measure is widely used in multiple criteria decision making problems. This paper proposed a new method to construct the similarity measures combining the existing cosine similarity measure and the Euclidean distance measure. And, the similarity measures are proposed not only from the points of view of algebra and geometry but also satisfy the axiom of the similarity measure. Furthermore, we apply the proposed similarities measures to the medical diagnosis decision problems, and the numerical example is used to illustrate the feasibility and effectiveness of the proposed similarity measure, which are then compared to other existing similarity measures.

References

[1] Zadeh, L. A. (1996). Fuzzy sets. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh* (pp. 394-432). https://doi.org/10.1142/9789814261302_0021

[2] De, S. K., Biswas, R., & Roy, A. R. (1998). Multicriteria decision making using intuitionistic fuzzy set theory. *Journal of fuzzy mathematics*, *6*, 837-842.

[3] Phuong, N. H., Thang, V. V., & Hirota, K. (2000). Case based reasoning for medical diagnosis using fuzzy set theory. *International journal of biomedical soft computing and human sciences: the official journal of the biomedical fuzzy systems association*, *5*(2), 1-7.

[4] Song, Y., Wang, X., & Zhang, H. (2015). A distance measure between intuitionistic fuzzy belief functions. *Knowledge-based systems*, *86*, 288-298.

[5] Zadeh, L. A. (1975). The concept of a linguistic variable and its application to approximate reasoning—I. *Information sciences*, *8*(3), 199-249.

[6] Atanassov, K. (2016). Intuitionistic fuzzy sets. *International journal bioautomation*, *20*, 1-6.

[7] Yang, Y., & Chiclana, F. (2009). Intuitionistic fuzzy sets: Spherical representation and distances. *International journal of intelligent systems*, *24*(4), 399-420.

[8] Gong, Z., Xu, X., Yang, Y., Zhou, Y., & Zhang, H. (2016). The spherical distance for intuitionistic fuzzy sets and its application in decision analysis. *Technological and economic development of economy*, *22*(3), 393-415.

[9] Beg, I., & Ashraf, S. (2009). Similarity measures for fuzzy sets. *Appl. Comput. Math*, *8*(2), 192-202.

[10] Jun, Y. E. (2016). Multicriteria decision-making method based on cosine similarity measures between intervalvalued fuzzy sets with risk preference. *Economic computation & economic cybernetics studies & research*, *50*(4), 205-215.

[11] Ye, J. (2011). Cosine similarity measures for intuitionistic fuzzy sets and their applications. *Mathematical and computer modelling*, *53*(1-2), 91-97.

[12] Ye, J. (2014). Vector Similarty meaures of simplified neutrosophic sets and their application in multicriteria decision making, *International journal of fuzzy systems*, 16(2), 204-211.

[13] Ye, J. (2015). Improved cosine similarity measures of simplified neutrosophic sets for medical diagnoses. *Artificial intelligence in medicine*, *63*(3), 171-179.

[14] Liu, D., Chen, X., & Peng, D. (2017). Interval-valued intuitionistic fuzzy ordered weighted cosine similarity measure and its application in investment decision-making. *Complexity*. https://doi.org/10.1155/2017/1891923

[15] Liu, D., Chen, X., & Peng, D. (2018). The intuitionistic fuzzy linguistic cosine similarity measure and its application in pattern recognition. *Complexity*, *2018*. *Complexity*. https://doi.org/10.1155/2018/9073597

[16] Ashraf, S., & Abdullah, S. (2019). Spherical aggregation operators and their application in multiattribute group decision-making. *International journal of intelligent systems*, *34*(3), 493-523.

[17] Ashraf, S., Abdullah, S., Mahmood, T., Ghani, F., & Mahmood, T. (2019). Spherical fuzzy sets and their applications in multi-attribute decision making problems. *Journal of intelligent & fuzzy systems*, *36*(3), 2829-2844. 1–16. DOI: 10.3233/JIFS-172009

[18] Ashraf, S., Abdullah, S., & Mahmood, T. (2018). GRA method based on spherical linguistic fuzzy Choquet integral environment and its application in multi-attribute decision-making problems. *Mathematical sciences*, *12*(4), 263-275.

[19] Kutlu Gündoğdu, F., & Kahraman, C. (2019). Spherical fuzzy sets and spherical fuzzy TOPSIS method. *Journal of intelligent & fuzzy systems*, *36*(1), 337-352.

[20] Broumi, S., & Smarandache, F. (2013). Several similarity measures of neutrosophic sets. In *Neutrosophic sets and systems* (pp. 54-62). University of New Mexico

[21] Şahin, R., & Küçük, A. (2014). On similarity and entropy of neutrosophic soft sets. *Journal of intelligent & fuzzy systems*, *27*(5), 2417-2430.

[22] Ye, J. (2015). Improved cosine similarity measures of simplified neutrosophic sets for medical diagnoses. *Artificial intelligence in medicine*, *63*(3), 171-179.

[23] Yong-Wei, Y. A. N. G., Zhang, R. L., & Jing, G. U. O. (2017). A multrattribute decision-making approach based on hesitantneutrosophic sets. *Fuzzy systems and mathematics*, *31*(2), 114-123.

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