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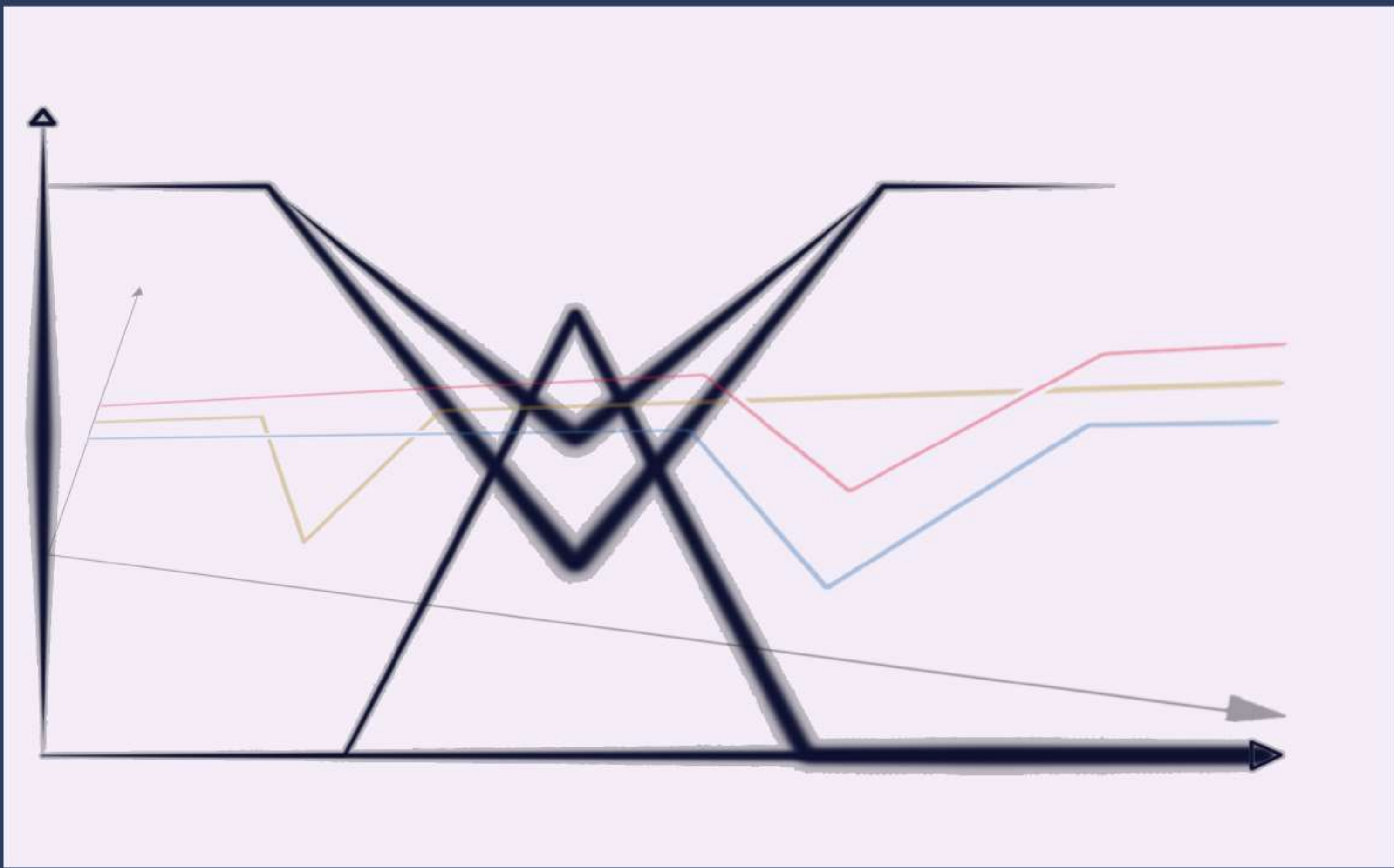

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New Characterization Theorems of the mp-Quantales

George Georgescu *

Department of Computer Science, Faculty of Mathematics and Computer Science, University of Bucharest, Bucharest, Romania; georgescu.capreni@yahoo.com

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Abstract

The mp-quantales were introduced in a previous paper as an abstraction of the lattices of ideals in mp-rings and the lattices of ideals in conormal lattices. Several properties of m-rings and conormal lattices were generalized to mp-quantales. In this paper we shall prove new characterization theorems for mp-quantales and for semiprime mp-quantales (these last structures coincide with the P F - quantales). Some proofs reflect the way in which the reticulation functor (from coherent quantales to bounded distributive lattices) allows us to export some properties from conormal lattices to mp-quantales.

Keywords: Coherent quantale. reticulation of a quantale, mp–quantales, PF-quantales, transfer properties.

1 | Introduction

The mp-quantales were introduced in [13] as an abstraction of the lattice of ideals of an mp - ring [1] and the lattice of ideals of a conormal lattice (see [7], [17], [23]). In [13] we proved some characterization theorems for mp - quantales that extend some results of [1] and [7] that describe the mp - rings, respectively the conormal lattices. The P F - quantales constitute an important class of mp - quantales (cf. [13]). They generalize the lattices of ideals in P F - rings. In fact, the P F - quantales are the semiprime P F - quantales. The paper [13] also contains several characterizations of a P F - quantale.

An important tool in proving the mentioned results was the reticulation of a coherent quantale [6] and [12] (the reticulation of a coherent quantale A is a bounded distributive lattice L(A) whose

Corresponding Author: georgescu.capreni@yahoo.com

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Stone prime spectrum SpecId, Z(L(A)) is homeomorphic with the Zariski prime spectrum SpecZ(A) of A).

The retic1 ulation construction provides a covariant functor from the category of coherent quantales to the category of bounded distributive lattices [6].

In this paper we shall obtain new characterization theorem for mp - quantales and P F - quantales. Some of these theorems contain properties expressed in terms of equations or pure and w-pure elements (see *Theorems 5 and 6*), while others (see *Theorems 7 and 8*) extend some conditions existing in some results of [27].

Now we give a short description of the content of this paper. Section 2 contains some notions and basic results in quantale theory [22] and [10]: residuation and negation operation, m - prime and minimal m - prime elements, Zariski and flat topologies on the spectra of a quantale, radical elements, etc. In Section 3 we recall from [6] and [12] the construction of the reticulation L(A) of a coherent quantale A and we present some results that describe how the reticulation functor preserves the m - prime elements, the annihilators, the pure and the w - pure elements, etc. In Section 4 we discuss the way in which the mp - quantales (defined in [13]) generalize the mp - rings [1] and the conormal lattices [7] and [23]. Some properties that characterize the mp - quantale are recalled from [13]. The main results of the paper are placed in Section 5. We prove three theorems with new algebraic and topological characterizations of the mp-quantales. Some characterization results of the P F-quantales (= the semiprime mp-quantales) are obtained as corollaries. Some of proofs reflect the way in which the reticulation functor transfer some properties of conormal lattices to mp-quantales.

2 | Preliminaries on Quantales

This section contains some basic notions and results in quantale theory [22] and [10]. Let (A, W, ∧, ·, 0, 1) be a quantale and K(A) the set of its compact elements. A is said to be integral if (A, ·, 1) is a monoid and commutative, if the multiplication is commutative. A frame is a quantale in which the multiplication coincides with the meet [17]. The quantale A is algebraic if any $a \in A$ has the form a = W X for some subset X of K(A). An algebraic quantale A is coherent if $1 \in K(A)$ and K(A) is closed under the multiplication. The set Id(R) of ideals of a (unital) commutative ring R is a coherent quantale and the set Id(L) of ideals of a bounded distributive lattice L is a coherent frame.

Throughout this paper, the quantales are assumed to be integral and commutative. We shall write ab instead of $a \cdot b$. We fix a quantale A.

On each quantale A one can consider a residuation operation (= implication) a → b = W {x | ax ≤ b} and a negation operation a ⊥ = a ⊥A, defined by a ⊥ = a → 0 = W {x ∈ A | ax = 0} (extending the terminology from ring theory [2], a ⊥ is also called the annihilator of a). Then for all a, b, c ∈ A the following residuation rule holds: a ≤ b → c if and only if ab ≤ c, so (A, ∨, ∧, ·, →, 0, 1) becomes a (commutative) residuated lattice. Particularly, we have a ≤ b ⊥ if and only if ab = 0. In this paper we shall use without mention the basic arithmetical 2 properties of a residuated lattice [11].

An element $p < 1$ of A is m-prime if for all a, b ∈ A, ab ≤ p implies a ≤ p or b ≤ p. If A is an algebraic quantale, then $p < 1$ is m-prime if and only if for all c, d ∈ K(A), cd ≤ p implies c ≤ p or d ≤ p. Let us introduce the following notations: Spec(A) is the set of m-prime elements and M ax(A) is the set of maximal elements of A. If 1 ∈ K(A) then for any $a < 1$ there exists m ∈ M ax(A) such that a ≤ m. The same hypothesis 1 ∈ K(A) implies that M ax(A) ⊆ Spec(A). We remark that the set Spec(R) of prime ideals in R is the prime spectrum of the quantale Id(R) and the set of prime ideals in L is the prime spectrum of the frame Id(L).

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Recall from [22] that the radical ϱ(a) of an element a of A is defined by ϱ(a) = V {p ∈ Spec(A) | a ≤ p}. If a = ϱ(a) then a is said to be a radical element of A. The set R(A) of the radical elements of A is a frame [22] and [23]. In [6] it is proven that Spec(A) = Spec(R(A)) and M ax(A) = M ax(R(A)).

Lemma 1. [19]. Let A be a coherent quantale and a ∈ A. Then

ϱ(a) = W {c ∈ K(A) | c k ≤ a for some integer k ≥ 1}.

For any c ∈ K(A), c ≤ ϱ(a) iff c k ≤ a for some k ≥ 1.

A is semiprime if and only if for any integer k ≥ 1, c k = 0 implies c = 0.

Let A be a quantale such that 1 ∈ K(A). For any a ∈ A, denote DA(a) = D(a) = {p ∈ Spec(A) | a 6≤ p} and VA(a) = V (a) = {p ∈ Spec(A) | a ≤ p}. Then Spec(A) is endowed with a topology whose closed sets are (V (a))a∈A. If the quantale A is algebraic then the family (D(c))c∈K(A) is a basis of open sets for this topology. The topology introduced here generalizes the Zariski topology (defined on the prime spectrum Spec(R) of a commutative ring R [2]) and the Stone topology (defined on the prime spectrum SpecId(L) of a bounded distributive lattice L [3]). Then this topology will be also called the Zariski topology of Spec(A) and the corresponding topological space will be denoted by SpecZ(A). According to [13], SpecZ(A) is a spectral space in the sense of [15]. The flat topology associated with this spectral space has as basis the family of the complements of compact open subsets of SpecZ(A)(cf.[8] and [17]). Recall from [13] that the family {V (c) | c ∈ K(A)} is a basis of open sets for the flat topology on Spec(A). We shall denote by SpecF (A) this topological space. For any p ∈ Spec(A), let us denote Λ(p) = {q ∈ Spec(A) | q ≤ p}. According to Proposition 5.6 of [13], the flat closure clF ({p}) of the set {p} is equal to Λ(p).

Let L be a bounded distributive lattice. For any x ∈ L, denote DId(x) = {P ∈ SpecId(L) | x 6∈ P} and VId(x) = {P ∈ SpecId,Z(L) | x ∈ P}. The family (DId(x))x∈L is a basis of open sets for the Stone topology on SpecId(L); this topological space will be denoted by SpecId,Z(L). We will denote by SpecId,F (L) the prime spectrum SpecId(L) endowed with the flat topology; the family (VId(x))x∈L is a basis of open sets for the flat topology.

If A is a quantale then we denote by M in(A) the set of minimal m - prime elements of A; M in(A) is called the minimal prime spectrum of A. If 1 ∈ K(A) then for any p ∈ Spec(A) there exists q ∈ M in(A) such that q ≤ p.

Proposition 1. If A is a coherent quantale and p ∈ Spec(A) then p ∈ M in(A) if and only if for all c ∈ K(A), the following equivalence holds: c ≤ p iff c → ϱ(0) 6≤ p.

Corollary 1. [18]. If A is a semiprime coherent quantale and p ∈ Spec(A) then p ∈ M in(A) if and only if for all c ∈ K(A), the following equivalence holds: c ≤ p iff c ⊥ 6≤ p.

An element e of the quantale A is a complemented element if there exists f ∈ A such that e∨f = 1 and e∧f = 0. The set B(A) of complemented elements of A is a Boolean algebra (cf. [5] and [16]). B(A) will be called the Boolean center of the quantale A.

Proposition 2. If a ∈ A then a ⊥ = V (V (a ⊥) T M in(A)).

3 | Reticulation of a Coherent Quantale

Let A be a coherent quantale and K(A) the set of its compact elements. We define the following equivalence relation on the set K(A): for all c, d ∈ K(A), c ≡ d iff ϱ(c) = ϱ(d). The quotient set L(A) = K(A)/ ≡ is a bounded distributive lattice. For any c ∈ K(A) denote by c/ ≡ its equivalence class. Consider the canonical surjection λA : K(A) ⟶ L(A) defined by λA(c) = c/ ≡, for any c ∈ K(A). The pair (L(A), λA : K(A) ⟶ L(A)) (or shortly L(A)) will be called the reticulation of A. In [6] and [12] an axiomatic definition of the reticulation of a coherent quantale was given. We remark that the reticulation L(R) of a commutative ring R (defined in [17] and [23]) is isomorphic with the reticulation L(Id(R)) of the quantale Id(R).

For any a ∈ A and I ∈ Id(L(A)) let us denote a ∗ = {λA(c) | c ∈ K(A), c ≤ a} and I∗ = W {c ∈ K(A) | λA(c) ∈ I}. The assignments a 7⟶ a ∗ and I 7⟶ I∗ define two order - preserving maps (·) ∗ : A ⟶ Id(L(A) and (·)∗ : Id(L(A)) ⟶ A. The following lemma collects the main properties of the maps (·) ∗ and (·)∗.

Lemma 2. [6]. The following assertions hold

If a ∈ A then a ∗ is an ideal of L(A) and a ≤ (a ∗)∗.

If I ∈ Id(L(A)) then (I∗) ∗ = I.

If p ∈ Spec(A) then (p ∗)∗ = p and p ∗ ∈ SpecId(L(A)).

If P ∈ SpecId((L(A)) then P∗ ∈ Spec(A).

If p ∈ K(A) then c ∗ = (λA(c)].

If c ∈ K(A) and I ∈ Id(L(A)) then c ≤ I∗ iff λA(c) ∈ I.

If a ∈ A and I ∈ Id(L(A)) then ϱ(a) = (a ∗)∗, a ∗ = (ϱ(a))∗ and ϱ(I∗) = I∗.

If c ∈ K(A) and p ∈ Spec(A) then c ≤ p iff λA(c) ∈ p ∗.

By the previous lemma one can consider the maps δA : Spec(A) ⟶ SpecId(L(A)) and A : SpecId(L(A)) ⟶ Spec(A), defined by δA(p) = p ∗ and A(I) = I∗, for all p ∈ Spec(A) and I ∈ SpecId(L(A)).

Lemma 3. [6] and [13]. The functions δA and A are homeomorphisms w.r.t. the Zariski and the flat topologies, inverse to one another.

We also observe that δA and A are order - isomorphisms. In particular, for any m - prime element p of A, we have p ∈ M in(A) if and only in p ∗ ∈ M inId(L(A)).

We denote by M inZ(A) (resp. M inF (A)) the topological space obtained by restricting the topology of SpecZ(A) (resp. SpecF (A)) to M in(A). Then M inZ(A) is homeomorphic to the space M inId,Z(L(A)) of minimal prime ideals in L(A) with the Stone topology and M inF (A) is homeomorphic to the space M inId,F (L(A)) of minimal prime ideals in L(A) with the flat topology (cf. *Lemma 3*). By [13], M inZ(A) is a zero - dimensional Hausdorff space and M inF (A) is a compact T1 space.

For a bounded distributive lattice L we shall denote by B(L) the Boolean algebra of the complemented elements of L. It is well-known that B(L) is isomorphic to the Boolean center B(Id(L)) of the frame Id(L) (see [5] and [17]). By [6], the function λA | B(A) : B(A) ⟶ B(L(A)) is a Boolean isomorphism.

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If L is a bounded distributive lattice and $I \in Id(L)$ then the annihilator of I is the following ideal of L(A): $AnnL(I) = Ann(I) = \{x \in I \,|\, x \wedge y = 0$, for all $y \in L\}$.

Let us fix a coherent quantale A.

Lemma 4. [13]. If $c \in K(A)$ and $p \in Spec(A)$ then $Ann(\lambda A(c)) \subseteq p *$ if and only if $c \to \varrho(0) \leq p$.

Proposition 3. [13]. If a is an element of a coherent quantale then $Ann(a *) = (a \to \varrho(0))*$; if A is semiprime then $Ann(a *) = (a \perp) *$.

Particularly, for any $c \in K(A)$, we have $Ann(\lambda A(c)) = (c \to \varrho(0))*$.

Proposition 4. [13]. Assume that A is a coherent quantale. If I is an ideal of L(A) then $(Ann(I))* = I* \to \varrho(0)$; if A is semiprime then $(Ann(I))* = (I*) \perp$.

An ideal I of a commutative ring R is said to be pure if for any $x \in I$ we have $I \vee Ann(x) = R$. An ideal I of a bounded distributive lattice L is said to be a σ -ideal if for any $x \in I$ we have $I \vee Ann(x) = L$. These two notions can be generalized to quantale theory: an element a of an algebraic quantale A is 5 said to be pure if for any $c \in K(A)$ we have $a \vee c \perp = 1$. We note that the σ - ideals of a bounded distributive lattice L coincide with the pure elements of the frame Id(L).

An element a of an algebraic quantale A is said to be w - pure [14] if for any $c \in K(A)$ we have $a \vee (c \to \varrho(0)) = 1$. It is easy to see that any pure element of A is w - pure.

Lemma 5. [14]. If an element a of a coherent quantale A is w - pure then $a *$ is a σ - ideal of the lattice L(A). Particulary, if a is pure then $a *$ is a σ - ideal.

Lemma 6. [14]. Let A be a coherent quantale and J a σ - ideal of L(A). Then $J*$ is a w - pure element of A.

For any $p \in Spec(A)$ let us denote $O(p) = W \{c \in K(A) \,|\, c \perp 6\leq p\}$.

Lemma 7. [14]. Let A be a coherent quantale. If $p \in Spec(A)$ and $c \in K(A)$ then $c \leq O(p)$ if and only if $c \perp 6\leq p$.

4 | From mp-Rings and Conormal Lattices to the mp–Quantales

Recall from [1] that a commutative ring R is an mp-ring if each prime ideal of R contains a unique minimal prime ideal. The following theorem of [1], that collects several characterizations of mp-rings, emphasizes some of their algebraic and topological properties.

Theorem 1. [1]. If R is a commutative ring then the following assertions are equivalent R is an mp – ring.

If P and Q are distinct minimal prime ideals of the ring R then $P + Q = R$.

R/n(R) is an mp - ring, whenever n(A) is the nil - ideal of R.

SpecF (R) is a normal space.

The inclusion M inF (R) $\subseteq$ SpecF (R) has a flat continuous retraction.

If P is a minimal prime ideal of R then VR(P) is a flat closed subset of SpecF (R).

For all a, b ∈ R, ab = 0 implies Ann(a n) + Ann(b n) = R, for some integer $n \geq 1$.

Any minimal prime ideal of A is the radical of a unique pure ideal of A.

The conormal lattices were introduced by Cornish in [7] under the name of normal lattices (a discussion of the terminology can be found in [23] and [17]). According to [23], a bounded distributive lattice L is conormal if for all a, b ∈ L such that $a \wedge b = 0$ there exist x, y ∈ L such that $a \wedge x = b \wedge y = 0$ and $x \vee y = 1$.

Theorem 2. [7]. If A is a conormal lattice then the following assertions are equivalent

A is a conormal lattice.

If P and Q are distinct minimal prime ideals of the lattice L then $P \vee Q = L$.

Any minimal prime ideal of L is a σ – ideal.

If x, y ∈ L and $x \wedge y = 0$ then $Ann(x) \vee Ann(y) = L$.

If x, y ∈ L then $Ann(x \wedge y) = Ann(x) \vee Ann(y) = L$.

Any prime ideal of L contains a unique minimal prime ideal.

For each x ∈ L, Ann(x) is a σ-ideal.

From the previous two theorems we observe that the mp - rings and the conormal lattices have similar characterizations. This remark allows us to extend these notions to quantale theory. A quantale A is said to be an mp - quantale if for any m-prime element p of A there exists a unique minimal m - prime element q such that $q \leq p$. The mp-frames are defined in a similar manner.

The following theorem establishes a strong connection between mp - quantales, mp - frames and conormal lattices.

Theorem 3. [13]. For any coherent quantale A the following assertions are equivalent

A is an mp–quantale.

R(A) is an mp–frame.

L(A) is a conormal lattice.

Proof. We know that Spec(A) = Spec(R(A)) and M in(A) = M in(R(A)), so the equivalence (1) ⇔ (2) is clear. The equivalence (1) ⇔ (3) was established in [13].

The following theorem is a generalization of some parts of *Theorems 1* and *2*.

Theorem 4. [13]. For any coherent quantale A the following are equivalent

A is an mp–quantale.

For any distinct p, q ∈ M in(A) we have $p \vee q = 1$.

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The inclusion M inF (A) ⊆ SpecF (A) has a flat continuous retraction.

SpecF (A) is a normal space.

If p ∈ Spec(A) then V (p) is a closed subset of SpecF (A).

Recall from [13] that a quantale A is a P F - quantale if for each c ∈ K(A), c ⊥ is a pure element. From [13] we know that a quantale A is a P F - quantale if and only if it is a semiprime mp-quantale.

5 | New Characterization Theorems

This section concerns the new characterization theorems for mp-quantales and P F - quantales. Our results extend some characterization theorems of mp -rings and P F - rings proven in [1], [26], [27]. Let us fix a coherent quantale A.

Recall from Section 3 that the maps δA : Spec(A) → SpecId(L(A)) and A : SpecId(L(A)) → Spec(A) are order - isomorphisms and homeomorphisms w.r.t. the Zariski and the flat topologies, inverse to one another. Then the following lemma is immediate.

Lemma 8. [13]. The functions δA | M in(A) : M in(A) → M inId(L(A)) and A | M inId(L(A)) : M inId(L(A)) → M in(A) are homeomorphisms w.r.t. the Zariski and the flat topologies, inverse to one another.

The previous lemma allows us to transfer some topological results from M inId(L(A)) to M in(A). Often we shall apply this lemma and its direct consequences without mention.

Theorem 5. The following assertions are equivalent:

A is an mp–quantale.

Any minimal m-prime element of A is w - pure.

Proof.

(1) ⇒ (2) Let p be a minimal m - prime element of A, hence p ∗ ∈ M inId(L(A)) (cf. *Lemma 8*). According to Theorem 3, L(A) is a conormal lattice, hence, by using Theorem 2, it follows that p ∗ is a σ - ideal of the lattice L(A). By *Lemmas 2* and 6, p = (p ∗)∗ is a w - pure element of A.

(2) ⇒ (1) Let P be a minimal prime ideal of L(A), so P = p ∗ , for some minimal m - prime element p of A (cf. Lemma 8). By hypothesis (2), p is a w-pure element of A. According to *Lemma 5*, P = p ∗ is a σ - ideal of L(A). Applying the implication (3) ⇒ (1) of Theorem 2, it follows that L(A) is a conormal lattice, so A is an mp - quantale (cf. *Theorem 3*).

Corollary 2. If A is semiprime then the following assertions are equivalent

A is a P F – quantale.

Any minimal m-prime element of A is pure.

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Proof.

We know from [13] that A is a P F - quantale if and only if it is a semiprime mp - quantale. In a semiprime quantale the pure and the w - pure elements coincide, so the corollary follows from *Theorem 5*.

Theorem 6. The following are equivalent

A is an mp–quantale.

For all c, d $\in$ K(A), cd $\leq \varrho(0)$ implies $(c \to \varrho(0)) \vee (d \to \varrho(0)) = 1$.

For all c, d $\in$ K(A), $cd \to \varrho(0) = (c \to \varrho(0)) \vee (d \to \varrho(0))$.

For any c $\in$ K(A), $c \to \varrho(0)$ is a w - pure element of A.

For all c, d $\in$ K(A), cd = 0 implies (c n) $\perp \vee$ (d n) $\perp$ = 1.

For any minimal m - prime element p of A there exists a unique pure element q such that $p = \varrho(q)$.

Proof.

(1) $\Rightarrow$ (2) Assume by absurdum that there exist c, d $\in$ K(A), such that $cd \leq \varrho(0)$ and $(c \to \varrho(0)) \vee (c \to \varrho(0)) < 1$, so $(c \to \varrho(0)) \vee (c \to \varrho(0)) \leq m$, for some maximal element m of A. Consider a minimal m - prime element p of A such that $p \leq m$. By *Theorem 5*, p is a w - pure element of A. From $cd \leq \varrho(0) \leq p$ we get $c \leq p$ or $d \leq p$ (because p is m - prime). Assuming that $c \leq p$ we get $p \vee (c \to \varrho(0) = 1$ (because p is a w - pure element), contradicting that $p \leq m$ and $c \to \varrho(0) \leq m$. Thus the implication (1) $\Rightarrow$ (2) is verified.

(2) $\Rightarrow$ (1) Let p, q be two distinct minimal m - prime elements of A, hence there exists d $\in$ K(A) such that $d \leq p$ and $d \not\leq q$. By Proposition 1, from $d \leq p$ it follows that $d \to \varrho(0) \not\leq p$, so there exists c $\in$ K(A) such that $c \leq d \to \varrho(0)$ and $c \not\leq p$. Then $cd \leq \varrho(0)$, hence $(c \to \varrho(0)) \vee (d \to \varrho(0)) = 1$ (by hypothesis (2). The last equality implies that there exist e, f $\in$ K(A) such that $e \leq (c \to \varrho(0))$, $f \leq (d \to \varrho(0))$ and $e \vee f = 1$. From $ce \leq \varrho(0) \leq p$ and $c \not\leq p$ we obtain $e \leq p$. Similarly, we can prove that $f \leq q$, so $p \vee q = 1$. By applying the implication (2) $\to$ (1) of *Theorem 4*, it results that A is an mp-quantale.

(2) $\Rightarrow$ (3) Firstly we shall establish the inequality $cd \to \varrho(0) \leq (c \to \varrho(0)) \vee (d \to \varrho(0))$. Let e be a compact element of A such that $e \leq cd \to \varrho(0)$, hence we get $cde \leq \varrho(0)$. In accordance with the hypothesis (2), it follows that $(ce \to \varrho(0)) \vee (d \to \varrho(0)) = 1$, so there exist x, y $\in$ K(A) such that $x \leq ce \to \varrho(0)$ and $y \leq d \to \varrho(0)$ and $x \vee y = 1$. From $x \leq ce \to \varrho(0)$ we obtain $ex \leq c \to \varrho(0)$.

Then $e = e(x \vee y) = ex \vee ey \leq ex \vee y \leq (c \to \varrho(0)) \vee (d \to \varrho(0))$, so we have proven that $cd \to \varrho(0) \leq (c \to \varrho(0)) \vee (d \to \varrho(0))$.

From $cd \leq c$ and $cd \leq d$ it results that $c \to \varrho(0) \leq cd \to \varrho(0)$ and $d \to \varrho(0) \leq cd \to \varrho(0)$, hence the converse inequality $(c \to \varrho(0)) \vee (d \to \varrho(0)) \leq cd \to \varrho(0)$ follows.

(3) $\Rightarrow$ (2) If $cd \leq \varrho(0)$ then, by using the property (3), it follows that $(c \to \varrho(0)) \vee (d \to \varrho(0)) = cd \to \varrho(0) = 1$.

(2) $\Rightarrow$ (4) Let c be a compact element of A. Assume that d is a compact element of A such that $d \leq c \to \varrho(0)$, so $cd \leq \varrho(0)$, hence $(c \to \varrho(0)) \vee (d \to \varrho(0)) = 1$ (by the condition (2)). It results that $c \to \varrho(0)$ is a

w-pure element of A.

(4) ⇒ (2) Assume that c, d ∈ K(A) and cd ≤ ϱ(0), so d ≤ c → ϱ(0). Since c → ϱ(0) is a w - pure element of A, the equality (c → ϱ(0)) ∨ (d → ϱ(0)) = 1 follows.

(1) ⇒ (5) By *Theorem 3*, the reticulation L(A) is a conormal lattice. Assume that c, d ∈ K(A) and cd = 0, hence λA(c)∧λA(d) = λA(cd) = λA(0) = 0 (by Definition 3(ii) of [6]). In accordance with Lemma 4.4 of [14], we know that the function (·) ∗ : A → Id(L(A)) preserves the joins. According to *Theorem 2*, we have Ann(λA(c)) ∨ Ann(λA(d)) = L(A), hence, by using *Proposition 3*, it follows that ((c → ϱ(0)) ∨ (c → ϱ(0)))∗ = ((c → ϱ(0))∗ ∨ (c → ϱ(0))∗ = (c → ϱ(0))⊥ ∨ (c → ϱ(0))⊥ = Ann(λA(c)) ∨ Ann(λA(d)) = L(A).

By applying *Lemma 2*, it follows that ϱ((c → ϱ(0)) ∨ (c → ϱ(0))) = (((c → ϱ(0)) ∨ (c → ϱ(0)))∗)∗ = 1, so (c → ϱ(0)) ∨ (c → ϱ(0)) = 1 (cf. Lemma 5(3) of [6]). Since 1 ∈ K(A), there exist x, y ∈ K(A) such that x ≤ c → ϱ(0), y ≤ d → ϱ(0) and x ∨ y = 1. From xc ≤ ϱ(0), yd ≤ ϱ(0) we get x nc n = 0, y nd n = 0, for some integer n ≥ 1 (cf. Lemma 1), hence x n ≤ (c n) ⊥ and y n ≤ (d n) ⊥. By Lemma 2(4) of [6], from x ∨ y = 1 it results that x n ∨ y n = 1, therefore (c n) ⊥ ∨ (d n) ⊥ = 1.

(5) ⇒ (1) According to *Theorem 3*, it suffices to show that L(A) is a conormal lattice. Let x, y ∈ L(A) such that x ∧ y = 0, so x = λA(c), y = λA(d), for some c, d ∈ K(A), hence λA(cd) = x ∧ y = 0. By Lemma 1, there exists an integer n ≥ 1 such that c nd n = 0, so (c nk) ⊥ ∨ (d nk) ⊥ = 1, for some integer k ≥ 1 (according to hypothesis (5)). Since (c nk) ⊥ ≤ c nk → ϱ(0) and (d nk) ⊥ ≤ d nk → ϱ(0), it follows that (c nk → ϱ(0)) ∨ (d nk → ϱ(0)) = 1. We know from Lemma 9(6) of [6] that λA(c nk) = λA(c) and λA(c nk) = λA(c). We recall that the map (·) ∗ preserves the joins.

In accordance with *Proposition 3*, the following equalities hold: Ann(x)∨Ann(y) = Ann(λA(c))∨Ann(λA(d)) = Ann(λA(c nk))∨Ann(λA(d nk)) = (c nk → ϱ(0))∗∨(d nk → ϱ(0))∗ = ((c nk → ϱ(0))∨((d nk → ϱ(0)))∗ = 1∗ = L(A). By applying the implication (4) ⇒ (1) of *Theorem 2*, it follows that L(A) is a conormal lattice.

(1) ⇒ (6) Let us denote by V ir(A) the set of pure elements of the quantale A. Recall that the pure elements of the frame Id(L(A)) are exactly the σ - ideals of the lattice L(A), so V ir(Id(L(A)) will be the frame of σ - ideals the lattice L(A). We recall from [12] that the map w: V ir(A) → V ir(Id(L(A)), defined by w(a) = a ∗ for any a ∈ V ir(A), is a frame isomorphism.

Let p be a minimal m - prime element of A, so p ∗ is a minimal prime ideal of the lattice L(A). Since L(A) is a conormal lattice, p ∗ is a σ-ideal of L(A) (cf. *Theorem 2*. But w is a frame isomorphism, so there exists a unique pure element q of A such that p ∗ = w(q) = q ∗. By using *Lemma 2*, it follows that p = (p ∗)∗ = (q ∗)∗ = ϱ(q).

Assume that q1, q2 are pure elements of A such that p = ϱ(q1) = ϱ(q2). By *Lemma 2*, it follows that p ∗ = (ϱ(qi))∗ = q ∗ i , for i = 1, 2. Thus w(q1) = w(q2), hence q1 = q2 (because w is a bijection).

(6) ⇒ (1) Let p, q be two distinct minimal m - prime elements of A, so there exists c ∈ K(A) such that c ≤ p and c 6≤ q. By the hypothesis (6), there exists a unique pure element r such that p = ϱ(r). From c ≤ ϱ(r) we get c n ≤ r, for some integer n ≥ 1 (cf. *Lemma 1*). But r is pure, so we get r ∨ (c n) ⊥ = 1, therefore we obtain p ∨ (c n) ⊥ = 1 (because r ≤ p). Since q is m - prime, c 6≤ q implies c n 6≤ q, hence (c n) ⊥ ≤ q. It follows that p ∨ q = 1. According to *Theorem 4*, it results that A is an mp-quantale.

The *properties (5)* and *(6)* of the previous theorem are the quantale versions of the conditions (5) and (6) of *Theorem 1*.

Corollary 3. [13]. The following are equivalent

A is a P F – quantale.

A is a semiprime mp – quantale.

For all c, d ∈ K(A), cd = 0 implies c ⊥ ∨ d ⊥ = 1.

For all c, d ∈ K(A), (cd) ⊥ = c ⊥ ∨ d ⊥.

Proof.

(1) ⇒ (2) By Lemma 8.17 of [13], A is semiprime, so for any c ∈ K(A), c → ϱ(0) = c ⊥ is a pure element, so c → ϱ(0) is w - pure. By implication (4) ⇔ (1) of *Theorem 6*, A is an mp-quantale.

(2) ⇒ (1) Let c be a compact element of A. Since A is a semiprime mp - quantale, c ⊥ = c → ϱ(0) is a w - pure element (cf. implication (1) ⇒ (4) of *Theorem 6*). But in a semiprime quantale the pure and w - pure elements concide, so c ⊥ is a pure element of A. Thus A is a P F - quantale.

(1) ⇔ (3) The condition (3) says that for any c ∈ K(A), c ⊥ is a pure element of A.

(2) ⇒ (4) Since A is semiprime we have ϱ(0) = 0, so (4) follows by using the implication (1) ⇒ (3) of *Theorem 6*.

(4) ⇒ (2) Assume that c n = 0, where c ∈ K(A) and n ≥ 1 is a natural number. By the hypothesis (3), from c n = 0 we obtain c ⊥ = (c n) ⊥ = 1, so c ≤ c ⊥⊥ = 0. It result that c = 0, so the quantale A is semiprime (by *Lemma 1*). In this case we have cd → ϱ(0) = (cd) ⊥ = c ⊥ ∨ d ⊥ = (c → ϱ(0)) ∨ (d → ϱ(0)).

By using the implication (3) → (1) of *Theorem 6*, we conclude that A is an mp - quantale.

Lemma 9. If p ∈ Spec(A) then ϱ(O(p)) ≤ p.

If p ∈ Spec(A) then p ∈ M in(A) if and only if ϱ(O(p)) = p.

Proof.

(1) If p ∈ Spec(A) then ϱ(O(p)) ≤ ϱ(p) ≤ p.

(2) Let p be an m - prime element of A. Assume that p ∈ M in(A). Firstly, from (1) we know that ϱ(O(p)) ≤ p. In order to show the converse inequality p ≤ ϱ(O(p)), suppose that c ∈ K(A) and c ≤ p. According to *Proposition 1*, c ≤ p implies c → ϱ(0) 6≤ p, so there exists d ∈ K(A) such that d ≤ c → ϱ(0) and d 6≤ p. Applying *Lemma 1*, from cd ≤ ϱ(0) we get c nd n = 0, for some integer n ≥ 1, hence d n ≤ (c n) ⊥. Since d 6≤ p and p ∈ Spec(A) we have d n 6≤ p, so (c n) ⊥ ≤ p. By using *Lemma 7*, it follows that c n ≤ O(p), therefore c ≤ ϱ(O(p)) (cf. *Lemma 1*). Conclude that p ≤ ϱ(O(p)).

Now assume that ϱ(O(p)) = p. Let us consider a minimal m - prime element q of A such that q ≤ p. We want to prove that O(p) ≤ q. For any c ∈ K(A), by using *Lemma 7*, the following implications hold: c ≤ O(p) ⇒ c ⊥ 6≤ p ⇒ c ≤ q. It follows that O(p) ≤ q, so p = ϱ(O(p)) ≤ O(q) ≤ q. Thus p = q, therefore p is a minimal m - prime element of A.

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The following result generalizes Theorem 3.2 of [27] to the framework of quantale theory.

Theorem 7. The following are equivalent

A is an mp–quantale.

If p and q are distinct minimal m - prime elements of A then we have O(p) ∨ O(q) = 1.

For any p ∈ Spec(A), ϱ(O(p)) is m – prime.

For any m ∈ M ax(A), ϱ(O(p)) is m – prime.

If p, q ∈ Spec(A) and p ≤ q then ϱ(O(p)) = ϱ(O(q)).

Proof.

(1) ⇒ (2) Assume that p, q ∈ M in(A) and p 6= q, so p ∨ q = 1 (cf. *Theorem 4*). By using *Lemma 9*, we have p = ϱ(O(p)) and q = ϱ(O(q)), hence ϱ(O(p))∨ϱ(O(q)) = 1. In accordance with Lemma 2.2 of [13], it follows that O(p) ∨ O(q) = 1.

(2) ⇒ (1) Assume that p, q ∈ M in(A) and p 6= q, so O(p) ∨ O(q) = 1. Since O(p) ≤ p and O(q) ≤ q we get p ∨ q = 1. By the implication (2) ⇒ (1) of *Theorem 4* it follows that A is an mp-quantale.

(1) ⇒ (3) Suppose that p ∈ Spec(A). In order to show that ϱ(O(p)) is m - prime, let c, d be two compact elements of A such that cd ≤ ϱ(O(p)), so there exists an integer n ≥ 1 such that c nd n ≤ O(p) (cf. *Lemma 1*). By *Lemma 7* we have (c nd n) ⊥ 6≤ p, so there exists e ∈ K(A) such that e ≤ (c nd n) ⊥ and e 6≤ p. Thus ecnedn = 0, hence (ecn → ϱ(0)) ∨ (edn → ϱ(0)) = 1 (by *Theorem 6*). Since 1 ∈ K(A) there exist two compact elements c and d such that x ≤ ecn → ϱ(0), y ≤ edn → ϱ(0) and x ∨ y = 1.

Then xecn ≤ ϱ(0) and yedn ≤ ϱ(0), so there exists an integer k ≥ 1 such that x k e k c nk = 0 and y k e kd nk = 0. By Lemma 2.1of [13], we have x k ∨y k = 1, so x k 6≤ p or y k 6≤ p. Let us assume that x k 6≤ p, hence x k e k 6≤ p (because e 6≤ p and p ∈ Spec(A) implies e k 6≤ p). From x k e k c nk = 0 we get x k e k ≤ (c nk) ⊥, hence (c nk) ⊥ 6≤ p. In virtue of *Lemma 7*, it follows that c nk ≤ O(p), so c ≤ ϱ(O(p)) (by *Lemma 1*). Similarly, y k 6≤ p implies d ≤ ϱ(O(p)). Conclude that ϱ(O(p) is m - prime.

(3) ⇒ (4) Obviously.

(4) ⇒ (1) Suppose that p ∈ Spec(A) and fix a maximal element m such that p ≤ m. Let q be a minimal m - prime element such that q ≤ p ≤ m. For any c ∈ K(A) such that c ≤ O(m) we have c ⊥ 6≤ m (by *Lemma 7*), hence c ⊥ 6≤ q. Since q is m - prime it follows that c ≤ q, so we conclude that O(m) ≤ q, so ϱ(O(m)) ≤ ϱ(q) = q. According to the hypothesis (4), ϱ(O(m)) is m - prime, therefore q = ϱ(O(m)). We have proven that there exists a unique minimal m - prime element q such that q ≤ p, so A is an mp - quantale.

(1) ⇒ (5) Assume that p, q ∈ Spec(A) and p ≤ q. Let us consider m ∈ M ax(A) and r ∈ M in(A) such that r ≤ p ≤ q ≤ m, hence, by using *Lemma 9*, we get r = ϱ(O(r)) ≤ ϱ(O(p)) ≤ ϱ(O(q)) ≤ ϱ(O(m)). According to the proof of the implication (4) ⇒ (1) we have r = ϱ(O(m)), so ϱ(O(p)) = ϱ(O(q)).

(5) ⇒ (1) Suppose that p ∈ Spec(A) and q1, q2 ∈ M in(A) such that q1 ≤ p and q2 ≤ p. Let m be a maximal element of A such that p ≤ m. Applying the hypothesis (5) and *Lemma 9* we get q1 = ϱ(O(q1)) = ϱ(O(q2)) = q2, hence there exists a unique minimal m-prime element q of A such that q ≤ p.

Now we shall present some consequences of Theorem 7 that extend Corollaries 3.3 of 3.5 of [27].

Lemma 10. V {O(m) | m ∈ M ax(A)} = 0.

Proof. Assume that c ∈ K(A) and c ≤ V {O(m) | m ∈ M ax(A)}. If c ⊥ < 1 then c ⊥ ≤ n, for some n ∈ M ax(A). By Lemma 7 we have c 6≤ O(n), contradicting the assumption that c ≤ V {O(m) | m ∈ M ax(A)}. Then c ⊥ = 1, hence c ≤ c ⊥⊥ = 0. Thus c = 0, so we conclude that V {O(m) | m ∈ M ax(A)} = 0.

Theorem 8. The following are equivalent

A is a P F – quantale.

For any p ∈ Spec(A), O(p) is m – prime.

For any m ∈ M ax(A), O(p) is m – prime.

Proof.

(1) ⇒ (2) Assume that p ∈ K(A) and c, d ∈ K(A) such that cd ≤ O(p), so (cd) ⊥ 6≤ p (cf. *Lemma 10*). By *Corollary 3* we have (cd) ⊥ = c ⊥ ∨ d ⊥, so c ⊥ 6≤ p or d ⊥ 6≤ p. By applying again *Lemma 10* it follows that c ≤ O(p) or d ≤ O(p), so O(p) is m-prime.

(2) ⇒ (3) Obviously.

(3) ⇒ (1) By the hypothesis (3), we have ϱ(O(m)) = O(m), for any m ∈ M ax(A). By using *Theorem 7* it results that A is an mp - quantale.

We shall prove that A is semiprime. Assume c ∈ K(A) and c ≤ ϱ(0), hence c n = 0 for some integer n ≥ 1 (cf. *Lemma 1*). Then for each m ∈ M ax(A) we have c n ≤ O(m), hence c ≤ O(m) (because O(m) is m - prime). Thus c ≤ V {O(m) | m ∈ M ax**(A)} = 0 (b**y *Lemma 10*), so c = 0. Thus A is a semiprime mp-quantale, so A is a P F - quantale.

By *Theorem 4*, we know that for any mp - quantale A, the inclusion M inF (A) ⊆ SpecF (A) has a flat continuous retraction. The following result establishes the form of a continuous retraction γ: SpecF (A) → M inF (A) of the inclusion M inF (A) ⊆ SpecF (A) (whenever such retraction exists).

Proposition 6. If the inclusion M inF (A) ⊆ SpecF (A) has a continuous retraction γ: SpecF (A) → M inF (A) then γ(p) = ϱ(O(p)), for all P ∈ Spec(A).

Proof. Let p be an m-prime element of A. Recall from Proposition 5.6 of [13] that the flat closure of the set {p} is given by clF ({p}) = Λ(p), where Λ(p) = {s ∈ Spec(A) | q ≤ p}. Assume that q ∈ M in(A) and q ≤ p, so q ∈ Λ(p) = clF ({p}) (cf. Proposition 5.6 of [13]). Since the map γ is a continuous retraction of the inclusion M inF (A) ⊆ SpecF (A) we have q = γ(q) ∈ clF ({γ(p)}) = Λ(γ(p)) = {γ(p)}, so q = γ(p). Thus γ(p) is the unique minimal m - prime element q such that q ≤ p. In particular, we have proven that A is an mp - quantale. According to *Lemma 9* and *Theorem 7*, from γ(p) ≤ q it follows that γ(p) = ϱ(O(γ(p))) = ϱ(O(p)).

From the previous proposition we get the uniqueness of the continuous retraction γ of the inclusion M inF (A) ⊆ SpecF (A) (whenever the retraction γ exists).

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Regular T_0-type Separations in Fuzzy Topological Spaces in the Sense of Quasi-Coincidence

Gurusamy Saravanakumar[1],*, **S. Tamilselvan**[2], A. **Vadivel**[3]

[1]Department of Mathematics, M. Kumarasamy College of Engineering, Karur, India; saravananguru2612@gmail.com
[2]Mathematics Section, Faculty of Engineering and Technology, Annamalai University, Annamalainagar, India-608002; tamil_au@yahoo.com
[3]Department of Mathematics, Government Arts College (Autonomous), Karur, India-639007; avmaths@gmail.com

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Abstract

In this paper, we introduce and study three notions of RT_0 property in fuzzy topological spaces using quasi-coincidence sense, and we relate to other such notions. Then, we show that all these notions satisfy good extension property. These concepts also satisfy hereditary, productive and projective properties. We note that all these concepts are preserved under one-one, onto, fuzzy regular open and fuzzy regular continuous mappings. Finally, we discuss initial and final fuzzy topological spaces on our concepts.

Keywords: Quasi-coincidence, regular fuzzy t_0-type separations, fuzzy topological spaces, initial and final fuzzy topology.

1 | Introduction

Chang [5] introduced the notion of fuzzy topology in 1968 by using the concept of fuzzy sets introduced by Zadeh [23] in 1965. Since then, an extensive work on fuzzy topological spaces has been carried out by many researchers like Goguen [6], Wong [20], Lowen [14], Warren [19], Hutton [8] and others. Separation axioms are important parts in fuzzy topological spaces. Many works on separation axioms have been done by researchers. Among those axioms, fuzzy T_0 type is one and it has been already introduced in the literature. There are many articles on fuzzy T_0 topological space which are created by many authors [1], [2], [7], [18], and [22]. The purpose of this paper is to further contribute to the development of fuzzy topological spaces specially on fuzzy regular T_0 topological spaces. In the present paper, fuzzy regular T_0 topological spaces are defined by using quasi-coincidence sense and relation among the given and other such notions

Corresponding Author: saravananguru2612@gmail.com
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are shown here. It is showed that the good extension property is satisfied on our notions. It is also showed that the hereditary, order preserving, productive, and projective properties hold on the new concepts.

In the last section of this paper initial and final fuzzy topological spaces are discussed on author's concept.

2 | Preliminaries

In this present paper, X and Y always denote non empty sets and $= [0,1]$, $I_1 = [0,1)$. The class of all fuzzy sets on a non empty set X is denoted by I^X and fuzzy sets on X are denoted as λ, μ, γ etc. Crisp subsets of X are denoted by capital letters U, V, W etc. throughout this paper.

Definition 1. [23]. A function λ from X into the unit interval I is called a fuzzy set in X. For every $x \in X$, $\lambda(x) \in I$ is called the grade of membership of x in λ. Some authors say that λ is a fuzzy subset of X instead of saying that λ is a fuzzy set in X. The class of all fuzzy sets from X into the closed unit interval I will be denoted by I^X.

Definition 2. [14]. A fuzzy set λ in X is called a fuzzy singleton if and only if $(x) = r$, $0 < r \leq 1$, for a certain $x \in X$ and $\lambda(y) = 0$ for all points y of X except x. The fuzzy singleton is denoted by x_r and x is its support. The class of all fuzzy singletons in X will be denoted by (X) . If $\lambda \in I^X$ and $x_r \in S(X)$, then we say that $x_r \in \lambda$ if and only if $r \leq \lambda(x)$.

Definition 3. [21]. A fuzzy set λ in X is called a fuzzy point if and only if $(x) = r$, $0 < r < 1$, for a certain $x \in X$ and $\lambda(y) = 0$ for all points y of X except x. The fuzzy point is denoted by x_r and x is its support.

Definition 4. [9]. A fuzzy singleton x_r is said to be quasi - coincidence with λ, denoted by $x_r q \lambda$ if and only if $\lambda(x) + r > 1$. If x_r is not quasi - coincidence with λ, we write $x_r \bar{q} \lambda$ and defned as $\lambda(x) + r \leq 1$.

Definition 5. [5]. Let f be a mapping from a set X into a set Y and λ be a fuzzy subset of X. Then f and λ induce a fuzzy subset μ of Y defined by

$$\mu(y) = \begin{cases} \sup \{\lambda(x)\} & \text{if } \in f^{-1}[\{y\}] \neq \phi, x \in X, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 6. [5]. Let f be a mapping from a set X into a set Y and μ be a fuzzy subset of Y. Then the inverse of μ written as f^{-1} (μ) is a fuzzy subset of X defined by $f^{-1}(\mu)(x) = \mu(f(x))$, for $x \in X$.

Definition 7. [5]. Let $I = [0,1]$, X be a non-empty set and I^X be the collection of all mappings from X into I, i.e., the class of all fuzzy sets in X. A fuzzy topology on X is defined as a family τ of members of I^X, satisfying the following conditions:

1) $1, 0 \in \tau$.
2) If $\lambda \in \tau$ for each $i \in \Lambda$, then $\bigcup_{i\in\Lambda} \lambda_i \in \tau$, where Λ is an index set.
3) If $\lambda, \mu \in \tau$ then $\lambda \cap \mu \in \tau$.

The pair (X, τ) is called a fuzzy topological space (in short fts) and members of τ are called τ - open fuzzy sets. A fuzzy set μ is called a τ -closed fuzzy set if $1 - \mu \in \tau$.

Definition 8. [4]. A fuzzy subset λ of a space X is called fuzzy regular open (resp. fuzzy regular closed) if $\lambda = intcl(A)$ (resp. $\lambda = cl(int(\lambda))$.

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The set of all fuzzy regular open sets (resp. fuzzy regular closed sets) of X is denoted by FRO(X) (resp. $FRC(X)$).

Definition 9. [15]. The function: $(X,\tau) \to (Y,\sigma)$ is called a

1) Fuzzy continuous if for every $\mu \in \sigma$, $f^{-1}(\mu) \in \tau$.
2) Fuzzy homeomorphic if f is bijective and both f and f^{-1} are fuzzy continuous.
3) Fuzzy regular continuous if for every $\mu \in \sigma$, $f^{-1}(\mu) \in FRO(X)$.

Definition 10. [13]. The function: $(X,\tau) \to (Y,\sigma)$ is called a

1) Fuzzy open if for every fuzzy open set λ in (X,τ), $f(\lambda)$ is fuzzy open set in (Y,σ).
2) Fuzzy closed if for every closed fuzzy set λ in (X,τ), $f(\lambda)$ is closed fuzzy set in (Y,σ).
3) Fuzzy regular open if for every fuzzy open set λ in (X,τ), $f(\lambda)$ is fuzzy regular open set in (X,σ).
4) Fuzzy regular closed if for every fuzzy closed set λ in (X,τ) $f(\lambda)$ is fuzzy regular closed set in (Y,σ).

Definition 11. [4]. If λ and μ are two fuzzy subsets of X and Y respectively then the Cartesian product $\lambda \times \mu$ of two fuzzy subsets λ and μ is a fuzzy subset of $X \times Y$ defined by $(\lambda \times \mu)(x,y) = min\,(\lambda(x),\mu(y))$, for each pair $(x,y) \in X \times Y$.

Definition 12. [10]. Let $\{X_i, i \in \Lambda\}$, be any class of sets and let X denotes the Cartesian product of these sets, i.e., $X = \Pi_{i\in\Lambda} X_i$. Note that X consists of all points $=< a_i, i \in \Lambda >$, where $a_i \in X_i$. Recall that, for each $j_0 \in \Lambda$, we define the projection π_{j_0} from the product set X to the coordinate space X_{j_0}, i.e. $\pi_{j_0} : X \to X_{j_0}$ by $\pi_{j_0}(< a_i, i \in \Lambda >) = a_{j_0}$. These projections are used to define the product topology.

Definition 13. [20]. Let $\{X_i, i \in \Lambda\}$ be a family of non-empty sets. Let $X = \Pi_{i\in\Lambda} X_i$ be the usual product of X_i s and let π_i be the projection from X into X_i. Further assume that each X_i is an fuzzy topological space with fuzzy topology τ_i. Now the fuzzy topology generated by $\{\pi_i^{-1}(b_i) : b_i \in \tau_i, i \in \Lambda\}$ as a sub basis, is called the product fuzzy topology on X. Clearly if w is a basis element in the product, then there exist i_1, i_2, i_3, ..., $\in \Lambda$ such that $w(x) = min\,\{b_i(x_i) : i = 1,2,3,\dots,n\}$, where $x = (x_i)_{i\in\Lambda} \in X$.

Definition 14. [16]. Let f be a real valued function on a topological space. If $\{x : f(x) > \alpha\}$ is open for every real α, then f is called lower semi continuous function.

Definition 15. [11]. Let X be a non-empty set and T be a topology on X. Let $\tau = \omega(T)$ be the set of all lower semi continuous functions from (X,τ) to I (with usual topology). Thus $\omega(T) = \{u \in I^X : u^{-1}(\alpha,1] \in T\}$ for each $\alpha \in I_1$. It can be shown that $\omega(T)$ is a fuzzy topology on X.

Let P be the property of a topological space (X,τ) and FP be its fuzzy topological analogue. Then FP is called a 'good extension' of P if and only if the statement (X,τ) has P if and only if $(X,\omega(T))$ has FP holds good for every topological space (X,τ).

Definition 16. [12]. The initial fuzzy topology on a set X for the family of fuzzy topological space $\{(X_i,\tau_i)_{i\in\Lambda}\}$ and the family of functions $\{f_i : X \to (X_i,\tau_i)\}_{i\in\Lambda}$ is the smallest fuzzy topology on X making each f_i fuzzy continuous. It is easily seen that it is generated by the family $\{f_i^{-1}(\lambda_i) : \lambda_i \in \tau_i\}_{i\in\Lambda}$.

Definition 17. [12]. The final fuzzy topology on a set X for the family of fuzzy topological spaces $\{(X_i,\tau_i)_{i\in\Lambda}\}$ and the family of functions $\{f_i : (X_i,\tau_i) \to X\}_{i\in\Lambda}$ is the finest fuzzy topology on X making each f_i fuzzy continuous.

Theorem 1. [3]. A bijective mapping from an fts (X,τ) to an fts (Y,σ) preserves the value of a fuzzy singleton (fuzzy point).

Definition 18. [17]. A fuzzy topological space (X,τ) is called

1) Fuzzy $T_0(i)$ (briefly, $T_0(i)$) if for any pair x_t, $y_{t'} \in S(X)$ with $x \neq y$, there exists $\lambda \in \tau$ such that $x_t q\lambda$, $y_{t'}\bar{q}\lambda$ or there exists $\mu \in \tau$ such that $y_{t'}q\mu$, $x_t\bar{q}\mu$.
2) Fuzzy $T_0(ii)$ (briefly, $FT_0(ii)$) if for any pair x_t, $y_{t'} \in S(X)$ with $x \neq y$, there exists $\lambda \in \tau$ such that $x_t q\lambda$, $y_{t'} \wedge \lambda = 0$ or there exists $\mu \in \tau$ such that $y_{t'}q\mu$, $x_t \wedge \mu = 0$.
3) Fuzzy $T_0(iii)$ (briefly, $FT_0(iii)$) if for any pair $x, y \in X$ with $x \neq y$, there exists $\lambda \in \tau$ such that $\lambda(x) = 1, \lambda(y) = 0$ or there exists $\mu \in \tau$ such that $\mu(y) = 1, \mu(x) = 0$.

Note: Preimage of any fuzzy singleton (fuzzy point) under bijective mapping preserves its value.

3 | Regular Fuzzy T_0-Type Separation Axiom

In this section, we introduce regular fuzzy T_0-type separation axiom and some well-known properties are discussed.

Definition 19. A fuzzy topological space (X,τ) is called

1) Regular fuzzy $T_0(i)$ (briefy, $T_0(i)$) if for any pair $x_t, y_{t'} \in S(X)$ with $x \neq y$, there exists $\lambda \in FRO(X)$ such that $x_t q\lambda$, $y_{t'}\bar{q}\lambda$ or there exists $\mu \in FRO(X)$ such that $y_{t'}q\mu$, $x_t\bar{q}\mu$.
2) Regular fuzzy $T_0(ii)$ (briefy, $RFT_0(ii)$) if for any pair x_t, $y_{t'} \in S(X)$ with $x \neq y$, there exists $\lambda \in FRO(X)$ such that $x_t q\lambda$, $y_{t'} \wedge \lambda = 0$ or there exists $\mu \in FRO(X)$ such that $y_{t'}q\mu$, $x_t \wedge \mu = 0$.
3) Regular fuzzy $T_0(iii)$ (briefy, $RFT_0(iii)$) if for any pair $x, y \in X$ with $x \neq y$, there exists $\lambda \in FRO(X)$ such that $\lambda(x) = 1, \lambda(y) = 0$ or there exists $\mu \in FRO(X)$ such that $\mu(y) = 1, \mu(x) = 0$.

Example 1. Let $X = \{a,b\}$, $\lambda_1 \in FRO(X)$, where $\lambda_1(a) = 0.9$, $\lambda_1(b) = 0$, $\lambda_2(a) = 0$, $\lambda_2(b) = 1$, $\lambda_3(a) = 0.9$, $\lambda_3(b) = 1$. Consider the fuzzy topology τ on X generated by $\{0,\lambda_1,\lambda_2,\lambda_3,1\}$. Let x_t, $y_{t'}$ be fuzzy points in X with $a \neq b$. Then $\lambda_1(a) + t > 1$ and $\lambda_1(b) + t' \leq 1$ for $0.\,2 < t \leq 1, 0 < t' \leq 1$. Therefore $x_t q\lambda_1$, $y_{t'}\bar{q}\lambda_1$. This shows that (X,τ) is $T_0(i)$. Also, as $\lambda_1(b) = 0$, $y_{t'} \wedge \lambda_1 = 0$. Thus, (X,τ) is $RFT_0(ii)$.

Theorem 2. For a fuzzy topological space (X,τ) the following implications are true: $RFT_0(ii) \Rightarrow RFT_0(i)$, $RFT_0(iii) \Rightarrow RFT_0(i)$, $RFT_0(iii) \Rightarrow RFT_0(ii)$. But, in general the converse is not true.

Proof. $RFT_0(ii) \Rightarrow RFT_0(i)$: Let (X,τ) be a fuzzy topological space and (X,τ) is $RFT_0(ii)$ we have to prove that (X,τ) is $T_0(i)$. Let x_t, $y_{t'}$ be fuzzy points in X with $x \neq y$. Since (X,τ) is $RFT_0(ii)$ fuzzy topological space, we have, there exist $\lambda \in FRO(X)$ such that $x_t q\lambda$, $y_{t'} \wedge \lambda = 0$ or there exists $\mu \in FRO(X)$ such that $y_{t'}q\mu$, $x_t \wedge \mu = 0$. To prove (X,τ) is $T_0(i)$, it is only needed to prove that $y_{t'}\bar{q}\lambda$.

Now $y_{t'} \wedge \lambda = 0 \Rightarrow \lambda(y) = 0 \Rightarrow \lambda(y) + t' \leq 1 \Rightarrow y_{t'}\bar{q}\lambda$.

It follows that there exists $\lambda \in FRO(X)$ such that $x_t q\lambda$, $y_{t'}\bar{q}\lambda$. Hence (X,τ) is $T_0(i)$.

To Show (X,τ) is $RFT_0(i) \Rightarrow (X,\tau)$ is $T_0(ii)$, we give a counter example.

Counter example (a). Let $X = \{a,b\}$, $\lambda_1 \in$ FRO (X), where $\lambda_1(a) = 0.9$, $\lambda_1(b) = 0.1$, $\lambda_2(a) = 0$, $\lambda_2(b) = 1$, $\lambda_3(a) = 0.9$, $\lambda_3(b) = 1$. Consider the fuzzy topology τ on X generated by $\{0,\lambda_1,\lambda_2,\lambda_3,1\}$. Let x_t, $y_{t'}$ be fuzzy points in X with $a \neq b$. Then $\lambda_1(a) + t > 1$ and $\lambda_1(b) + t' \leq 1$ for $0.2 < t \leq 1, 0 < t' \leq 1$. Therefore $x_t q\lambda_1$, $y_{t'}\bar{q}\lambda_1$. This shows that (X,τ) is $RFT_0(i)$. But $\lambda_1(b) \neq 0 \Rightarrow y_{t'} \wedge \lambda_1 \neq 0$. Hence (X,τ) is not $RFT_0(ii)$.

$RFT_0(iii) \Rightarrow RFT_0(i)$: Let (X,τ) be a fuzzy topological space and (X,τ) is $RFT_0(iii)$ .We have to prove that (X,τ) is $RFT_0(i)$ .Let x_t, $y_{t'}$ be fuzzy points in X with $x \neq y$. Since (X,τ) is $RFT_0(iii)$ fuzzy topological space, we have, there exists $\lambda \in FRO(X)$ such that $\lambda(x) = 1$, $\lambda(y) = 0$ or there exists $\mu \in FRO(X)$ such that $(y) = 1$, $\mu(x) = 0$. To prove (X, τ) is $T_0(i)$, it is needed to prove that $x_t q\lambda$, $y_{t'}\bar{q}\lambda$.

Now, $\lambda(x) = 1 \Rightarrow \lambda(x) + t > 1$, for any $t \in (0,1] \Rightarrow x_t q\lambda$ and $\lambda(y) = 0 \Rightarrow \lambda(y) + t' \leq 1$, for any $t' \in (0,1]$ $\Rightarrow y_{t'}\bar{q}\lambda$.

It follows that there exist $\lambda \in FRO(X)$ such that $x_t q\lambda$, $y_{t'}\bar{q}\lambda$. Hence (X, τ) is $T_0(i)$.

To show (X, τ) is $RFT_0(i) \Rightarrow (X, \tau)$ is $T_0(iii)$, we give a counter example.

Counter example (b). Consider the Counter example (a), $\lambda_1(a) \neq 1$ which implies (X, τ) is not $T_0(iii)$. $RFT_0(iii) \Rightarrow RFT_0(ii)$: Let (X, τ) be a fuzzy topological space and (X, τ) is $T_0(iii)$. We have to prove that (X, τ) is $T_0(ii)$. Let x_t, $y_{t'}$ be fuzzy points in X with $x \neq y$. Since (X, τ) is $RFT_0(iii)$ fuzzy topological space, we have, there exist $\lambda \in FRO(X)$ such that $\lambda(x) = 1$, $\lambda(y) = 0$ or there exists $\mu \in FRO(X)$ such that $(y) = 1$, $\mu(x) = 0$. To prove (X, τ) is $T_0(ii)$, it is needed to prove that $x_t q\lambda$, $y_{t'} \wedge \lambda = 0$.

Now, $\lambda(x) = 1 \Rightarrow \lambda(x) + t > 1$, for any $t \in (0,1] \Rightarrow x_t q\lambda$ and $\lambda(y) = 0 \Rightarrow y_{t'} \wedge \lambda = 0$.

It follows that there exist $\lambda \in FRO(X)$ such that $x_t q\lambda$, $y_{t'} \wedge \lambda = 0$. Hence (X, τ) is $T_0(ii)$.

To show (X, τ) is $RFT_0(ii) \Rightarrow (X, \tau)$ is $T_0(iii)$, we give a counter example.

Counter example (c). In *Example 1*, $\lambda_1(a) \neq 1$, $\lambda_1(b) = 0$, therefore (X, τ) is not $RFT_0(iii)$.

From the above definitions and examples, it is clear that the following implications is true but the converse of the implications is not true as shown by the following example.

$$\begin{tikzcd}
RFT_0(iii) \arrow[r] \arrow[d] & RFT_0(ii) \arrow[r] \arrow[d] & R\,FT_0(i) \arrow[d] \\
FT_0(iii) \arrow[r] & FT_0(ii) \arrow[r] & FT_0(i)
\end{tikzcd}$$

Example 2. Let $= \{a, b\}$, $\lambda \in \tau$, where $\lambda(a) = 1$, $\lambda(b) = 0$. Consider the fuzzy topology τ on X generated by $\{0, \lambda, 1\}$. Let x_t, $y_{t'}$ be fuzzy points in X with $a \neq b$. Then $\lambda(a) + t > 1$ and $\lambda(b) + t' \leq 1$ for , $t' \in (0,1]$. Therefore $x_t q\lambda$, $y_{t'}\bar{q}\lambda$. (i) This shows that (X, τ) is $FT_0(i)$ but it is not $RFT_0(i)$ since λ is not regular open. (ii) as $\lambda(b) = 0$, $y_{t'} \wedge \lambda = 0$ is $FT_0(ii)$ but it is not $RFT_0(ii)$. Also (iii) as $\lambda(a) = 1$, $\lambda(b) = 0$ is $FT_0(iii)$ but it is not $T_0(iii)$.

Now, we shall show that our notions satisfy the good extension property.

Theorem 3. Let (X, T) be a topological space. Consider the following statements:

1) (X, T) be a RT_0-topological space.
2) $(X, \omega(T))$ be an $RFT_0(i)$ space.
3) $(X, \omega(T))$ be an $RFT_0(ii)$ space.

Then the following implications are true. $(1) \Leftrightarrow (2)$ and $(1) \Leftrightarrow (3)$.

Proof.

(1) ⇒ (2). Let (X,T) be a topological space and (X,T) is RT_0. We have to prove that $(X,\omega(T))$ is $T_0(i)$. Let $x_t, y_{t'}$ be fuzzy points in X with $x \neq y$. Since (X,T) is RT_0 topological space, we have, there exists $U \in T$ such that $\in U$, $y \notin U$. From the definition of lower semi continuous we have $1_U \in \omega(T)$ and $1_U(x) = 1, 1_U(y) = 0$. Then $1_U(x) + t > 1 \Rightarrow x_t q 1_U$ and $1_U(y) + t' \leq 1 \Rightarrow y_{t'} \bar{q} 1_U$. It follows that there exists $1_U \in \omega(T)$ such that $x_t q 1_U$, $y_{t'} \bar{q} 1_U$. Hence $(X,\omega(T))$ is $T_0(i)$. Thus (1) ⇒ (2) holds.

(2) ⇒ (1). Let $(X,\omega(T))$ be a fuzzy topological space and $(X,\omega(T))$ is $T_0(i)$. We have to prove that (X,T) is RT_0. Let x, y be points in X with $x \neq y$. Since $(X,\omega(T))$ is $RFT_0(i)$ topological space, we have, for any fuzzy points x_t, $y_{t'}$ in X, there exists $\lambda \in FRO(X)$ such that $x_t q\lambda$, $y_{t'}\bar{q}\lambda$ or there exist $\mu \in FRO(X)$ such that $y_{t'} q\mu$, $x_t \bar{q}\mu$.

Now, $x_t q\lambda \Rightarrow \lambda(x) + t > 1 \Rightarrow \lambda(x) > 1 - t = \alpha \Rightarrow x \in \lambda^{-1}(\alpha, 1]$ and $y_{t'}\bar{q}\lambda \Rightarrow \lambda(y) + t \leq 1 \Rightarrow \lambda(y) \leq 1 - t = \alpha \Rightarrow \lambda(y) \leq \alpha \Rightarrow y \notin \lambda^{-1}(\alpha, 1]$.

Also, $\lambda^{-1}(\alpha, 1] \in T$. It follows that $\exists \lambda^{-1}(\alpha, 1] \in T$ such that $x \in \lambda^{-1}(\alpha, 1]$, $y \notin \lambda^{-1}(\alpha, 1]$. Thus (2) ⇒ (1) holds. Similarly, we can prove that (1) ⇔ (3).

Now, we shall show that the hereditary property is satisfed on our notions.

Theorem 4. Let (X,τ) be a fuzzy topological space, $B \subseteq X$, $\tau_B = \{\lambda/B : \lambda \in \tau\}$, then

1) (X,τ) is $RFT_0(i) \Rightarrow (B,\tau_B)$ is $RFT_0(i)$.
2) (X,τ) is $RFT_0(ii) \Rightarrow (B,\tau_B)$ is $RFT_0(ii)$.

Proof (1). Let (X,τ) be a fuzzy topological space and (X,τ) is $T_0(i)$. We have to prove that (B,τ_B) is $T_0(i)$. Let x_t, $y_{t'}$ be fuzzy points in B with $x \neq y$. Since $B \subseteq X$, these fuzzy points are also fuzzy points in X. Also, since (X,τ) is $RFT_0(i)$ fuzzy topological space, we have, there exists $\lambda \in FRO(X)$ such that $x_t q\lambda$, $y_{t'}\bar{q}\lambda$ or there exists $\mu \in FRO(X)$ such that $y_{t'} q\mu$, $x_t\bar{q}\mu$. For $B \subseteq X$, we have $\lambda/B \in FRO(X)$.

Now, $x_t q\lambda \Rightarrow \lambda(x) + t > 1$, $x \in X \Rightarrow \lambda/B(x) + t > 1$, $x \in B \subseteq X \Rightarrow x_t q\lambda/B$ and $y_{t'}\bar{q}\lambda \Rightarrow \lambda(y) + t' \leq 1$, $y \in X \Rightarrow \lambda/B(y) + t' \leq q$, $y \in B \subseteq X \Rightarrow y_{t'}\bar{q}\lambda/B$.

Hence, (B,τ_B) is $T_0(i)$. Proof of (2) is similar to proof of (1).

Then, we discuss the productive and projective properties on our concepts.

Theorem 5. Let (X_i,τ_i), $i \in \Lambda$ be fuzzy topological spaces and $X = \Pi_{i\in\Lambda} X_i$ and τ be the product topology on X, then

3) for all $i \in \Lambda$, (X_i,τ_i) is $RFT_0(i)$ if and only if (X,τ) is $RFT_0(i)$.
4) for all $i \in \Lambda$, (X_i,τ_i) is $RFT_0(ii)$ if and only if (X,τ) is $RFT_0(ii)$.

Proof 2. Let for all $i \in \Lambda$, (X_i,τ_i) is $RFT_0(ii)$ space. We have to prove that (X,τ) is $T_0(ii)$. Let x_t, $y_{t'}$ be fuzzy points in X with $x \neq y$. Then $(x_i)_t$, $(y_i)_{t'}$ are fuzzy points with $x_i \neq y_i$ for some $i \in \Lambda$. Since (X_i,τ_i) is $T_0(ii)$, there exists $\lambda_i \in FRO(X_i)$ such that $(x_i)_t q\lambda_i$, $(y_i)_{t'} \wedge \lambda_i = 0$ or there exists $\mu_i \in FRO(X_i)$ such that $(y_i)_{t'} q\mu_i$, $(x_i)_t \wedge \mu_i = 0$. But we have $\pi_i(x) = x_i$ and $\pi_i(y) = y_i$.

Now, $(x_i)_t q\lambda_i \Rightarrow \lambda_i(x_i) + t > 1$, $x \in X \Rightarrow \lambda_i(\pi_i(x)) + t > 1 \Rightarrow (\lambda_i \circ \pi_i)(x) + t > 1 \Rightarrow x_t q(u_i \circ \pi_i)$ and $(y_i)_{t'} \wedge \lambda_i = 0 \Rightarrow \lambda_i(y_i) = 0$, $y \in X \Rightarrow \lambda_i(\pi_i(y)) = 0$, $y \in X \Rightarrow (\lambda_i \circ \pi_i)(y) = 0 \Rightarrow y_{t'} \wedge (\lambda_i \circ \pi_i) = 0$.

It follows that there exists $(\lambda_i \circ \pi_i) \in t_i$ such that $x_t q(\lambda_i \circ \pi_i)$, $y_{t'} \wedge (\lambda_i \circ \pi_i) = 0$. Hence, (X,τ) is $T_0(ii)$.

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Conversely, let (X, τ) be a fuzzy topological space and (X, τ) is $T_0(ii)$. We have to prove that (X_i, τ_i), $i \in \Lambda$ is $T_0(ii)$. Let a_i be a fxed element in X_i. Let $A_i = \{x \in X = \Pi_{i\in\Lambda}X_i: x_j = a_j \text{ for some } i \neq j\}$. Then A_i is a subset of X, and hence (A_i, τ_{A_i}) is a subspace of (X, τ) . Since (X, τ) is $T_0(ii)$, so (A_i, τ_{A_i}) is $RFT_0(ii)$. Now we have A_i is homeomorphic image of X_i. Hence it is clear that for all $i \in \Lambda$, (X_i, τ_i) is $RFT_0(ii)$ space. Thus (2) holds. Proof of (1) is similar to proof of (2).

Now, we shall show that our notions satisfy the order preserving property.

Theorem 6. Let (X, τ) and (Y, σ) be two fuzzy topological spaces and: $X \to Y$ be a one - one, onto and regular open map then,

1) (X, τ) is $RFT_0(i) \Rightarrow (Y, \sigma)$ is $RFT_0(i)$.
2) (X, τ) is $RFT_0(ii) \Rightarrow (Y, \sigma)$ is $RFT_0(ii)$.

Proof 1. Let (X, τ) be a fuzzy topological space, and (X, τ) is $T_0(i)$. We have to prove that (Y, σ) is $T_0(i)$. Let x_t, y_t, be fuzzy points in Y with $x' \neq y'$. Since f is onto then there exist $x, y \in X$ with $f(x) = x'$, $f(y) = y'$ and $x_t, y_{t'}$ are fuzzy points in X with $x \neq y$ as f is one - one. Again since (X, τ) is $RFT_0(i)$ space, there exists $\lambda \in FRO(X)$ such that $x_t q\lambda$, $y_{t'}\bar{q}\lambda$ or there exists $\mu \in FRO(X)$ such that $y_{t'}q\mu$, $x_t\bar{q}\mu$.

Now, $x_t q\lambda \Rightarrow \lambda(x) + t > 1$ and $y_{t'}\bar{q}\lambda \Rightarrow \lambda(y) + t' \leq 1$.

Now, $(\lambda)(x') = \{ sup\, \lambda(x): f(x) = x'\} \Rightarrow f(\lambda)(x') = \lambda(x)$, for some x and $(\lambda)(y') = \{ sup\, \lambda(y) : f(y) = y'\} \Rightarrow f(\lambda)(y') = \lambda(y)$, for some y.

Also, since f is regular open map then $f(\lambda) \in FRO(Y)$ as $\lambda \in \tau$.

Again, $\lambda(x) + t > 1 \Rightarrow f(\lambda)(x') + t > 1 \Rightarrow x'_t qf(\lambda)$ and, $\lambda(y) + t' \leq 1 \Rightarrow f(\lambda)(y') + t' \leq 1 \Rightarrow y'_{t'}\bar{q}f(\lambda)$.

It follows that there exists $f(\lambda) \in FRO(Y)$ such that $x'_t qf(\lambda)$, $y'_{t'}\bar{q}f(\lambda)'$. Hence, it is clear that (Y, σ) is $RFT_0(i)$ space. Proof of (2) is similar to proof of (1).

Theorem 7. Let (X, τ) and (Y, σ) be two fuzzy topological spaces and: $X \to Y$ be a one - one, onto and regular continuous map then,

1) (Y, σ) is $RFT_0(i) \Rightarrow (X, \tau)$ is $RFT_0(i)$.
2) (Y, σ) is $RFT_0(ii) \Rightarrow (X, \tau)$ is $RFT_0(ii)$.

Proof 2. Let (Y, σ) be a fuzzy topological space and (Y, σ) is $T_0(ii)$. We have to prove that (X, τ) is $T_0(ii)$. Let x_t, $y_{t'}$ be fuzzy points in X with $x \neq y$. Then $(f(x))_t$, $(f(y))_{t'}$ are fuzzy points in Y with $f(x) \neq f(y)$ as f is one - one. Again, since (Y, σ) is $RFT_0(ii)$ space, there exists $\lambda \in FRO(Y)$ such that $(f(x))_t q\lambda$, $(f(y))_{t'} \wedge \lambda = 0$ or there exist $\mu \in FRO(Y)$ such that $(f(y))_{t'}q\mu$, $(f(x))_t \wedge \mu = 0$.

Now, $(f(x))_t q\lambda \Rightarrow \lambda(f(x)) + t > 1 \Rightarrow f^{-1}(\lambda(x)) + t > 1 \quad \Rightarrow (f^{-1}(\lambda))(x) + t > 1 \Rightarrow x_t qf^{-1}(\lambda)$ and, $(f(y))_{t'} \wedge \lambda = 0 \Rightarrow \lambda(f(y)) = 0 \Rightarrow f^{-1}(\lambda(y)) = 0 \Rightarrow (f^{-1}(\lambda))(y) = 0 \Rightarrow y_{t'} \wedge f^{-1}(\lambda) = 0$.

Now, since f is regular continuous map and $\lambda \in \sigma$ then $f^{-1}(\lambda) \in FRO(X)$. It follows that there exist $f^{-1}(\lambda) \in FRO(X)$ such that $x_t qf^{-1}(\lambda)$, $y_{t'} \wedge f^{-1}(\lambda) = 0$. Hence it is clear that (X, τ) is $RFT_0(ii)$ space. Proof of (1) is similar to proof of (2).

Theorem 8. If $\{(X_i, \tau_i)\}_{i\in\Lambda}$ is a family of $RFT_0(ii)$ fts and $\{f_i : X \to (X_i, \tau_i)\}_{i\in\Lambda}$, a family of one - one and fuzzy regular continuous functions, then the initial fuzzy topology on X for the family $\{f_i\}_{i\in\Lambda}$ is $RFT_0(ii)$.

Proof. Let τ be the initial fuzzy topology on X for the family $\{f_i\}_{i\in\Lambda}$. Let x_t, $y_{t'}$ be fuzzy points in X with $x \neq y$. Then $f_i(x)$, $f_i(y) \in X_i$ and $f_i(x) \neq f_i(y)$ as f_i is one - one. Since (X_i, τ_i) is $T_0(ii)$, then for every two distinct fuzzy points $(f_i(x))_t$, $(f_i(y))_{t'}$ in X_i, there exists regular fuzzy sets λ_i or $\mu_i \in FRO(X_i)$ such that $(f_i(x))_t q\lambda_i$, $(f_i(y))_{t'} \wedge \lambda_i = 0$ or $(f_i(y))_{t'} q\mu_i$, $(f_i(x))_t \wedge \mu_i = 0$.

Now, $(f_i(x))_t q\lambda_i$ and $(f_i(y))_{t'} \wedge \lambda_i = 0$. That is $\lambda_i(f_i(x)) + t > 1$ and $\lambda_i(f_i(y)) = 0$. That is $f_i^{-1}(\lambda_i)(x) + t > 1$ and $f_i^{-1}(\lambda_i)(y) = 0$.

This is true for every $i \in \Lambda$. So, $\inf f_i^{-1}(\lambda_i)(x) + t > 1$ and $\inf f_i^{-1}(\lambda_i)(y) = 0$. Let $= inf\ f_i^{-1}(\lambda_i)$. Then $\lambda \in FRO(X)$ as f_i is fuzzy regular continuous. So, $\lambda(x) + t > 1$ and $\lambda(y) = 0$. Hence, $x_t q\lambda$ and $y_{t'} \wedge \lambda = 0$. Therefore, (X, τ) is must $T_0(ii)$. Thus, the proof is complete.

Theorem 9. If $\{(X_i, \tau_i)\}_{i\in\Lambda}$ is a family of $RFT_0(ii)$ fts and $\{f_i(X_i, \tau_i) \to X\}_{i\in\Lambda}$, a family of fuzzy regular open and bijective function, then the fnal fuzzy topology on X for the family $\{f_i\}_{i\in\Lambda}$ is $RFT_0(ii)$.

Proof. Let τ be the fnal fuzzy topology on X for the family $\{f_i\}_{i\in\Lambda}$. Let x_t, $y_{t'}$ be fuzzy points in X with $x \neq y$. Then $f_i^{-1}(x)$, $f_i^{-1}(y) \in X_i$ and $f_i^{-1}(x) \neq f_i^{-1}(y)$ as f_i is bijective. Since (X_i, τ_i) is $T_0(ii)$, then for every two distinct fuzzy points $(f_i^{-1}(x))_t$, $(f_i^{-1}(y))_{t'}$ in X_i, there exists regular fuzzy sets λ_i or $\mu_i \in FRO(X_i)$ such that $(f_i^{-1}(x))_t q\lambda_i$, $(f_i^{-1}(y))_{t'} \wedge \lambda_i = 0$ or $(f_i^{-1}(y))_{t'} q\mu_i$, $(f_i^{-1}(x))_t \wedge \mu_i = 0$.

Now, $(f_i^{-1}(x))_t q\lambda_i$ and $(f_i^{-1}(y))_{t'} \wedge \lambda_i = 0$. That is $\lambda_i(f_i^{-1}(x)) + t > 1$ and $\lambda_i\left(f_i^{-1}(y)\right) = 0$. That is $f_i(\lambda_i)(x) + t > 1$ and $f_i(\lambda_i)(y) = 0$.

This is true for every $i \in \Lambda$. So, $inf\ f_i(\lambda_i)(x) + t > 1$ and $\inf f_i(\lambda_i)(y) = 0$. Let $= inf\ f_i(\lambda_i)$. Then $\lambda \in FRO$ (X) as f_i is fuzzy regular open. So, $\lambda(x) + t > 1$ and $\lambda(y) = 0$. Hence, $x_t q\lambda$ and $y_{t'} \wedge \lambda = 0$. Therefore, (X, τ) is must $T_0(ii)$. Thus, the proof is complete.

4 | Conclusion

In this paper, we introduce and study notion of RT_0 separation axiom in fts in quasi-coincidence sense. We have shown that all of our concepts are good extension of their counterparts and are stronger than other such notion. Further, we have shown that hereditary, productive, projective and other preserving properties hold on our concepts. Finally, initial and final fuzzy topologies are studied on one of our notions. We hope that the results of this paper will aid researchers in developing a general structure for fuzzy mathematics expansion.

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New View of Fuzzy Aggregations. Part I: General Information Structure for Decision-Making Models

Gia Sirbiladze*

Department of Computer Sciences, Javakhishvili Tbilisi State University, Tbilisi; gia.sirbiladze@tsu.ge

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Abstract

The Ordered Weighted Averaging (OWA) operator was introduced by Yager [57] to provide a method for aggregating inputs that lie between the max and min operators. In this article two variants of probabilistic extensions the OWA operator-POWA and FPOWA (introduced by Merigo [26] and [27]) are considered as a basis of our generalizations in the environment of fuzzy uncertainty (parts II and III of this work), where different monotone measures (fuzzy measure) are used as uncertainty measures instead of the probability measure. For the identification of "classic" OWA and new operators (presented in parts II and III) of aggregations, the Information Structure is introduced where the incomplete available information in the general decision-making system is presented as a condensation of uncertainty measure, imprecision variable and objective function of weights.

Keywords: Mean aggregation operators, Fuzzy aggregations, Fuzzy measure, Fuzzy numbers, Fuzzy decision making.

1 | Introduction

It is well recognized that intelligent decision support systems and technologies have been playing an important role in improving almost every aspect of human society. Intensive study over the past several years has resulted in significant progress in both the theory and applications of optimization and decision sciences.

Optimization and decision-making problems are traditionally handled by either the deterministic or the probabilistic approach. When working with complex systems in parallel with classical approaches of their modelling, the most important matter is to assume fuzziness ([3], [6], [13], [15]-[32], [35]-[43], [49]-[62] and others). All this is connected to the complexity of study of complex and vague processes and events in nature and society, which are caused by lack or

Corresponding Author: gia.sirbiladze@tsu.ge

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shortage of objective information and when expert data are essential for construction of credible decisions. With the growth of complexity of information our ability to make credible decisions from possible alternatives with complex states of nature reduces to some level, below which some dual characteristics such as precision and certainty become mutually conflicting ([3], [11], [20]-[22], [36]-[38], [41], [49], [51], [54], [55] and others). When working on real, complex decision systems using an exact or some stochastic quantitative analysis is often less convenient, concluding that the use of fuzzy methods is necessary, because systems approach for development of information structure of investigated decision system [20], [36], [37] with combined fuzzy-stochastic uncertainty enables us to construct convenient intelligent decision support instruments. Obviously, the source for obtaining combined objective + fuzzy + stochastic samplings is the populations of fuzzy-characteristics of expert's knowledge ([22], [36], [38], [42], [51] and others). Our research is concerned with quantitative-information analysis of the complex uncertainty and its use for modelling of more precise decisions with minimal decision risks from the point of view of systems research. The main objects of our attention are 1) the analysis of Information Structures of expert's knowledge, its uncertainty measure and imprecision variable and 2) the construction of instruments of aggregation operators, which condense both characteristics of incomplete information - an uncertainty measure and an imprecision variable in the scalar ranking values of possible alternatives in the decision-making system. The first problem is considered in this paper. The second problem will be presented in the Parts II and III of this work.

Making decisions under uncertainty is a pervasive task faced by many Decision-Making Persons (DMP), experts, investigators or others. The main difficulty is that a selection must be made between alternatives in which the choice of alternative doesn't necessarily lead to well determined payoffs (experts' valuations, utilities and so on) to be received as a result of selecting an alternative. In this case DMP is faced with the problem of comparing multifaceted objects whose complexity often exceeds his/her ability to compare of uncertain alternatives. One approach to addressing this problem is to use valuation functions (or aggregation operators). These valuation functions convert the multifaceted uncertain outcome associated with an alternative into a single (scalar) value. This value provides a characterization of the DMP or expert perception of the worth the possible uncertain alternative being evaluated. The problems of Decision Making Under Uncertainty (DMUU) [51] were discussed and investigated by many well-known authors ([1]-[6], [9], [10], [13], [15]-[18], [23]-[60], [62] and others). In this work our focus is directed on the construction of new generalizations of the aggregation Ordered Weighted Averaging (OWA) operator in the fuzzy-probabilistic uncertainty environment.

In Section 2 some preliminary concepts are presented on the OWA operator; on the arithmetic of the triangular fuzzy numbers; on some extensions of the OWA operator – POWA and FPOWA operators in the probabilistic uncertainty (developed by Merigo [26] and [27]) and their information measures (see Section 3). In Section 4 a new conceptual Information Structure (IS) of a General Decision-Making System (GDMS) with fuzzy-probabilistic uncertainty is defined. This IS classifying some aggregation operators and new generalizations of the OWA operator defined in the parts II and III of this work.

2 | On the OWA Operator and Its Some Fuzzy-Probabilistic Generalizations

In this type of problem, the DMP has a collection $D=\{d_1,d_2,...,d_n\}$ of possible uncertain alternatives from which he must select one or some ranking of decisions by some expert's preference relation values. Associated with this problem is a variable of characteristics, activities, symptoms and so on, which acts on the decision procedure. This variable is normally called the state of nature, which affects the payoff, utilities, valuations and others to the DMP's preferences or subjective activities. This variable is assumed to take its values (states of nature) from some set $S=\{s_1,s_2,...,s_m\}$. As a result, the DMP knows that if he selects d_i and the state of nature assumes the value s_j then his payoff (valuation, utility and so on) is a_{ij}. The objective of the decision is to select the "best" alternative, get the biggest payoff (valuation, utility and so on). But in DMUU [51] the selection procedure becomes more difficult. In this case each alternative can be seen as corresponding to a row vector of possible payoffs. To make a choice the DMP must compare these vectors, a problem which generally doesn't lead to a compelling solution. Assume d_i and d_k are two alternatives such that for all $j, j=1,2,...,m$ $a_{ij} \geq a_{kj}$ (*Table 1*). In this case there is no reason to select d_k. In this situation we shall say d_i dominates $d_k (d_i \succeq d_k)$. Furthermore, if there exists one alternative that dominates all the alternatives then it will be optimal solution and as a result, we call this the Pareto optimal. Faced with the general difficulty of comparing vector payoffs we must provide some means of comparing these vectors. Our focus in this work is on the construction of valuation function (aggregation operator) F that can take a collection of m values and convert it into a single value,

$$F : R^m \Rightarrow R^1.$$

Once we apply this function to each of the alternatives, we select the alternative with the *largest scalar value*. The construction of F involves considerations of two aspects. The first being the satisfaction of some rational, objective properties naturally required of any function used to convert (aggregate) a vector of payoffs (valuations, utilities and so on) into an equivalent scalar value. The second aspect being the inclusion of characteristics particular to the DMP's subjective properties or preferences, dependences with respect to risks and other main external factors.

Table 1. Decision matrix.

D \ S	s_1	s_2	…	s_K	…	s_m
d_1	a_{11}	a_{12}	…	a_{1k}	…	a_{1m}
d_2	a_{21}	a_{22}	…	a_{2k}	…	a_{2m}
…	…	…	…	…	…	…
d_3	a_{i1}	a_{i2}	…	a_{ik}	…	a_{im}
…	…	…	…	…	…	
d_n	a_{n1}	a_{n2}	…	a_{nk}	…	a_{nm}

First, we shall consider the objective properties required of the valuation function (aggregation operator) F [51].

1) The first property is the satisfaction of Pareto optimality. To insure this, we require that if for j=1,2,...,m, then

$$F(a_{i1}, a_{i2}, ..., a_{im}) \geq F(a_{k1}, a_{k2}, ..., a_{km}). \quad (1)$$

An aggregation operator satisfying this condition is said to be monotonic.

2) A second condition is that the value of an alternative should be bounded by its best payoffs (valuations, utilities) and worst possible one. $\forall i = 1,2,...,n$.

$$\min\{a_{ij}\} \leq F(a_{i1}, a_{i2}, ..., a_{im}) \leq \max\{a_{ij}\} \ \underset{j=1,m}{\underline{\min}}\{a_{ij}\} \leq F(a_{i1}, a_{i2}, ..., a_{im}) \leq \underset{j=1,m}{\underline{\max}}\{a_{ij}\} \quad (2)$$

This condition is said to be *bounded.*

3) Remark: if $a_{ij} \equiv a_i$ for all j, then from *Eq. (2)*

$$\underline{\min} \ \underset{j=1,m}{\underline{\min}}\{a_{ij}\} = \underset{j=1,m}{\underline{\max}}\{a_{ij}\} \text{ and } F(a_{i1}, a_{i2}, ..., a_m) = a_i.$$

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This condition is said to be idempotent.

4) The final objective condition is that the indexing of the states of nature shouldn't affect the answer:

$$F(a_{i1},a_{i2},...,a_{im}) = F(Permutation(a_{i1},a_{i2},...,a_{im})), \tag{3}$$

where $permutation(.)$ is some permutation of the set $\{a_{i1},a_{i2},...,a_{im}\}$. An aggregation function satisfying this is said to be symmetric (or commutative).

Finally, we have required that our aggregation function satisfy four conditions: monotonicity, boundedness, idempotency and symmetricity. Such functions are called mean or averaging operators [51].

In determining which of the many possible aggregation operators to select as our valuation function, we need some guidance from the DMP. The choice of a valuation function, from among the aggregation operators is essentially a "subjective" act reflecting the preferences of the DMP for one vector of payoffs over another. What is needed are tools and procedure to enable a DMP to reflect their subjective preferences into valuations. There are important problems in expert knowledge engineering for which we often use such intelligent technologies as neural networks, machine learning, fuzzy logic control systems, knowledge representations and others.

These problems may be solved by introducing information measures of aggregation operators ([1], [2], [4], [12], [13], [15], [16], [26]-[33], [35], [38], [40]-[42], [45]-[60], [62] and others). In this paper we will present new extensions of information measures of operators constructed bellow.

As an example, we present some mean aggregation operators. Assume we have an m-tuple of values $\{a_1,a_2,...,a_m\}$.

Then $F(a_1,a_2,...,a_m) = \min_{i=1m}\{a_i\}$ is one mean aggregation operator. The use of the operator Min corresponds to a pessimistic attitude, one in which the DMP assumes the worst thing will happen. Another example of a mean aggregation operator is $F(a_1,a_2,...,a_m) = Max\{a_i\}$.

Here we have very optimistic valuations. Another example is the simple average: $Mean(a_1,a_2,...,a_m) = \frac{1}{m}\sum_{i=1}^{m} a_i$.

In [58] Yager introduced a class of mean operators called OWA operator.

Definition 1. [57]. An OWA operator of dimension m is mapping $OWA: R^m \Rightarrow R^1$ that has an associated weighting vector W of dimension m with $w_j \in [0;1]$ and $\sum_{j=1}^{m} w_j = 1,$ such that

$$OWA(a_1,...,a_m) = \sum_{j=1}^{m} w_j b_j, \tag{4}$$

where b_j is the jth largest of the $\{a_i\}$, $i=1,2,...,m$.

Note that different properties could be studied such as the distinction between descending and ascending orders, different measures for characterizing the weighting vector and different families of the OWA operator ([1], [4], [26]-[33], [45], [47]-[52], [56], [57], [59], [60], [62] and others).

The OWA operator and its modifications are among the most known mean aggregation operators to the construction of DMUU valuation functions. These aggregations are generalizations of known instrument as Choquet Integral ([5], [7], [23], [38], [41], [51], [53], [54], [57] and others), Sugeno integral ([14], [17], [24], [25], [36], [42], [44] and others) or induced mean functions ([2], [12], [60], [62] and others).

The Fuzzy Numbers (FN) have been studied by many authors ([11] and [19] and others). It can represent in a more complete way as an imprecision variable of the incomplete information because it can consider the maximum and minimum and the possibility that the interval values may occur.

Definition 2. [19]. $\tilde{a}(t): R^1 \Rightarrow [0;1]$ is called the *FN* which can be considered as a generalization of the interval number:

$$\tilde{a}(t)=\begin{cases} 1 & \text{if } t\in\left[a_2',a_2''\right] \\ \dfrac{t-a_1}{a_2'-a_1} & \text{if } t\in\left[a_1,a_2'\right] \\ \dfrac{a_3-t}{a_3-a_2''} & \text{if } t\in\left[a_2'',a_3\right] \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

where $a_1 \le a'_2 \le a''_2 \le a_3 \in R^1$.

In the following, we are going to review the triangular FN (TFN) [20] arithmetic operation as follows (in *Eq. (5)* $a'_2 = a''_2$). Let $\tilde{a}$ and $\tilde{b}$ be two TFNs, where $\tilde{a}=(a_1,a_2,a_3)$ and $\tilde{b}=(b_1,b_2,b_3)$. Then

$$\begin{aligned}
&1:\ \tilde{a}+\tilde{b}=(a_1+b_1,\ a_2+b_2,\ a_3+b_3).\\
&2:\ \tilde{a}-\tilde{b}=(a_1-b_3,\ a_2-b_2,\ a_3-b_1).\ \tilde{a}-\tilde{b}=\left(a_1-b_3,\ a_2-b_2,\ a_3-b_1\right).\\
&3:\ \tilde{a}\times k=(ka_1,\ ka_2,\ ka_3),\ k>0.\\
&4:\ \tilde{a}^k=(a_1^k,a_2^k,a_3^k),\ k>0,\ a_i>0.\\
&5:\ \tilde{a}.\tilde{b}=(a_1b_1,\ a_2b_2,a_3b_3),\ a_i>0,\ b_i>0.\\
&6:\ \tilde{b}^{-1}=\{\frac{1}{b_3},\frac{1}{b_2},\frac{1}{b_1}\},\ b_i>0.\\
&7:\ \tilde{a}>\tilde{b} \text{ if } a_2>b_2 \text{ and if } a_2=b_2 \text{ than } \tilde{a}>\tilde{b} \text{ if } \frac{a_1+a_3}{2}>\frac{b_1+b_3}{2} \text{ otherwise } \tilde{a}=\tilde{b}.\\
&\tilde{a}>\tilde{b} \text{ if } a_2>b_2 \text{ and if } a_2=b_2 \text{ then } \tilde{a}>\tilde{b} \text{ if } \frac{a_1+a_3}{2}>\frac{b_1+b_3}{2} \text{ otherwise } \tilde{a}=\tilde{b}.
\end{aligned} \tag{6}$$

The set of all TFNs is denoted by ψ and positive TFNs ($a_i > 0$) by ψ^+.

Note that other operations and ranking methods could be studied ([19] and others).

Now we consider some extensions of the OWA operator, mainly developed by [26], [27], and [29], because our future investigations concern with extensions of Merigo's aggregation operators constructed on the basis of the OWA operator.

Definition 3. [29]. Let ψ be the set of TFNs. A fuzzy OWA operator - FOWA of dimension m is a mapping $FOWA: \psi^M \Rightarrow \psi$ that has an associated weighting vector w of dimension m with $w_j \in [0,1]$, $\sum_{j=1}^{m} w_j = 1$ and

$$FOWA(\tilde{a}_1, \tilde{a}_2, ... \tilde{a}_m) = \sum_{j=1}^{m} w_j \tilde{b}_j. \quad (7)$$

where $\tilde{b}_j$ is the jth largest of the $\{\tilde{a}_i\}_{i=1}^{m}$, and $a_i \in \psi$, $i=1,2,...,m$.

The FOWA operator is an extension of the OWA operator that uses imprecision information in the arguments represented in the form of TFNs. The reason for using this aggregation operator is that sometimes the available information presented by the DMP and formalized in payoffs (valuations, utilities and others) can't be assessed with exact numbers and it is necessary to use other techniques such as TFNs. So, in this aggregation incomplete information is presented by imprecision variable of expert's reflections and formalized in TFNs. Sometimes the available information presented by the DMP (or expert) also has an uncertain character, which is presented by the probability distribution on the states of nature consequents on the payoffs of the DMP.

The fuzzy-probability aggregations based on the OWA operator was constructed by Merigo and others. One of the variants we present here:

$$POWA(a_1, a_2, ..., a_m) = \sum_{j=1}^{m} p_j \hat{b}_j. \quad (8)$$

where b_j is the jth largest of the $\{a_i\}$, $i=1,2,...,m;$; each argument a_i has an associated probability p_i with $\sum_{i=1}^{m} p_i = 1$, $0 \le p_i \le 1$, $\hat{P}_j = \beta w_j + (1-\beta) p_j$ with $\beta \in [0,1]$ and p_j is the probability p_i ordered according to b_j, that is according to the jth largest of the a_i.

Note that if $\beta = 0$, we get the usual probabilistic mean aggregation (mathematical expectation - E_p with respect to probability distribution $\{p_i\}_{i=1}^{m}$), and if $\beta = 1$, we get the OWA operator. Equivalent representation of *Eq. (8)* may be defined as:

$$\begin{aligned} POWA(a_1, a_2, ... a_m) &= \beta \sum_{j=1}^{n} w_j b_j + \\ &+ (1-\beta) \sum_{i=1}^{m} p_i a_i = \beta . OWA(a_1, a_2, ..., a_m) + \\ &+ (1-\beta) . E_P(a_1, a_2, ..., a_m). \end{aligned} \quad (9)$$

We often use probabilistic information in the decision-making systems and consequently in their aggregation operators. Many fuzzy-probabilistic aggregations have been researched in OWA and other operators ([5], [17], [18], [26]-[32], [35]-[42], [49]-[53], [59], [60], [62] and others). In the following we present one of them defined in [27]:

Definition 4. [27]. Let ψ be the set of TFNs. A fuzzy-probabilistic OWA operator - FPOWA of dimension m is a mapping $FPOWA: \psi^m \Rightarrow \psi$ that associated a weighting vector w of dimension m such that $w_j \in [0,1]$, $\sum_{j=1}^{m} w_j = 1$, according to the following formula:

$$FPOWA(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m) = \sum_{j=1}^{n} \hat{p}_j \tilde{b}_j. \quad (10)$$

where $\tilde{b}_j$ is the jth largest of the $\{a_i\}_{i=1}^{m}$ are TFNs and each one has an associated probability $p_i \equiv P(\tilde{a} = \tilde{a}_i)$, with $\sum_{j=1}^{m} p_j = 1$, $0 \le p_j \le 1$, $\hat{p}_j = \beta w_j + (1-\beta) p_j'$, $\beta \in [0,1]$ and p_j' is the probability ordered according to $\tilde{b}_j (p'_j = P(\tilde{a} = \tilde{b}_j))$ that is according to the jth largest of the $\{\tilde{a}_i\}_{i=1}^{m}$.

Analogously to *Eq. (9)* we present the equivalent form of the FPOWA operator as a weighted sum of the OWA operator and the mathematical expectation - E_p:

$$\begin{aligned}&\mathrm{FPOWA}\left(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m\right) = \beta\sum_{j=1}^{n} w_j \tilde{b}_j + \\ &+(1-\beta)\sum_{i=1}^{m} p_i \tilde{a}_i = \beta \cdot \mathrm{OWA}\left(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m\right) + \\ &+(1-\beta)\cdot E_p\left(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m\right).\end{aligned} \tag{11}$$

In [27] the Semi-boundary condition of the *aggregation operator (11)* was proved. Semi-boundary condition of some operator F if defined as:

$$\begin{aligned}&\beta \times \min_i\{\tilde{a}_i\} + (1-\beta)\cdot E_p\left(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m\right) \le \\ &\le F\left(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m\right) \le \beta \times \max_i\{\tilde{a}_i\} + \\ &+(1-\beta)\cdot E_p\left(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_m\right).\end{aligned} \tag{12}$$

So, the FPOWA operator is monotonic, bounded, idempotent, symmetric and semi-bounded.

3 | on the Information Measures of the POWA and FPOWA Operators

As preliminary concepts of our investigation we present four probabilistic information measures of the POWA and FPOWA operators defined in [27] following similar methodology developed for the OWA operator ([1], [2], [3], [6], [47], [48], [50], [52] and others):

- The Orness parameter classifies the POWA and FPOWA operators in regard to their location between and and *or*:

$$\begin{aligned}&\alpha(\hat{p}_1, \hat{p}_2, ..., \hat{p}_m) = \\ &= \beta\sum_{j=1}^{m} w_j\left(\frac{m-j}{m-1}\right) + \\ &+(1-\beta)\sum_{j=1}^{m} p_j\left(\frac{m-j}{m-1}\right).\end{aligned} \tag{13}$$

- The Entropy (dispersion) measures the amount of information being used in the aggregation:

$$\begin{aligned}&H(\hat{p}_1, \hat{p}_2, ..., \hat{p}_m) = \\ &= -\left\{\beta\sum_{j=1}^{m} w_j \ln w_j + (1-\beta)\sum_{i=1}^{m} p_i \ln p_i\right\}.\end{aligned} \tag{14}$$

- The divergence of weighted vector w measures the divergence of the weights against the degree of Orness:

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$$\begin{aligned}&\mathrm{Div}(\hat{p}_1,\hat{p}_2,\ldots,\hat{p}_m)=\\&=\beta\left\{\sum_{j=1}^{m}w_j\left(\frac{m-j}{m-1}-\alpha(W)\right)^2\right\}+\\&+(1-\beta)\left\{\sum_{j=1}^{m}p'_j\left(\frac{m-j}{m-1}-\alpha(P)\right)^2\right\}\end{aligned} \tag{15}$$

where $a(W)$ is an *Orness* measure of the OWA or FOWA operators $(\beta=1)$:

$$\alpha(W)=\sum_{j=1}^{m}w_j\left(\frac{m-j}{m-1}\right), \tag{16}$$

and $a(p)$ is an *Orness* measure of the fuzzy-probabilistic aggregation $(\beta=0)$:

$$\alpha(P)=\sum_{j=1}^{m}p'_j\cdot\left(\frac{m-j}{m-1}\right). \tag{17}$$

– The balance parameter measures the balance of the weights against the Orness or the andness

$$\begin{aligned}&\mathrm{Bal}(\hat{p}_1,\hat{p}_2,\ldots,\hat{p}_m)=\\&=\beta\left\{\sum_{j=1}^{m}w_j\left(\frac{m+1-2j}{m-1}\right)\right\}+\\&+(1-\beta)\left\{\sum_{j=1}^{m}p'_j\left(\frac{m+1-2j}{m-1}\right)\right\}.\end{aligned} \tag{18}$$

4 | General Decision-Making System (GDMS) and Its Information Structure (IS)

In the parts II and III of this work we will focus on the construction of new generalizations of the POWA and FPOWA fuzzy-probabilistic aggregation operators induced by the ME (Choquet Integral [5], [7], [23], [38], [41], [51], [53], [54], [57] and others), or the FEV (Sugeno integral [14], [17], [24], [25], [36], [42], [44] and others) with respect to different monotone measures (fuzzy measure [8], [14], [21], [22], [36]-[38], [43], [44], [53]-[55], [61] and others). When trying to functionally describe insufficient expert data, in many real situations the property of additivity remains unrevealed for a measurable representation of a set and this creates an additional restriction. Hence, to study such data, it is better to use monotone measures (estimators) instead of additive ones. So, we will construct new generalizations of the POWA and FPOWA operators with respect to different monotone measures (instead of the probability measure) and different mean operators.

We introduce the definition of a monotone measure (fuzzy measure) [44] adapted to the case of a finite referential.

Definition 5. Let $S=\{s_1,s_2,\ldots,s_m\}$ be a finite set and g be a set function $g:2^s \Rightarrow [0,1]$. We say g is a monotone measure on S if it satisfies

(i) $g(\varnothing)=0;\ g(S)=1;$
(ii) $\forall A,B\subseteq S,\ if\ A\subseteq B,\ then\ g(A)\le g(B).$

A monotone measure is a normalized and monotone set function. It can be considered as an extension of the probability concept, where additivity is replaced by the weaker condition of monotonicity. Non-additive but monotone measures were first used in the fuzzy analysis in the 1980s [44] and are well investigated ([8], [14], [21], [22], [36]-[38], [43], [44], [53]-[55], [61] and others). Therefore, in order to classify OWA-type aggregation operators with probabilistic (POWA, FPOWA operators and others) or fuzzy uncertainty (defined in parts II and III) it is necessary to define an information structure of these operators. The different cases of incompleteness (uncertainty measure + imprecision variable) and objectivity (objective weighted function) will be considered in our new aggregation operators. Therefore, from the point of view of systems approach it is necessary to describe and formally present the scheme of GDMS in uncertain – objective environment. GDMS gives us the possibility to identify the different cases of levels of incompleteness and objectivity of available information which in whole defines the aggregation procedure.

Now we define the general decision-making system and its information structure which will be considered in the aggregation problems of parts II and III.

Definition 6. The GDMS that will combine decision-making technologies and methods of construction of decision functions (aggregation operators) may be presented by the following 8-tuple

$$\langle \mathrm{D, S, a, g, W, I, F, Im} \rangle, \tag{19}$$

where $D=\{d_1, d_2, ..., d_n\}$ is a set of all possible alternatives (decisions, diagnosis and so on) that are made by a Decision-Making Person (DMP).

$S=\{s_1, s_2, ..., s_m\}$ is a set of systems states of nature (actions, activities, factors, symptoms and so on) that are act on the possible alternatives in the decision procedure?

a - is an imprecision on precision variable of payoffs (utilities, valuations, some degrees of satisfaction to a fuzzy set, prices and so on), which will by defined by DMP's subjective properties of preferences, dependences with respect to risks and other external factors. As a result, variable a constructs some decision matrix (binary relation) on $D \times S$.

g is an uncertainty measure on $2^s\,(g: 2^s \Rightarrow [0,1])$. In our case it may be some monotone measure. W is an objective weighted function (or vector) on the states of nature - S

I is the Information Structure on the data of states of nature. Cases of different levels of information incompleteness (uncertainty measure + imprecision variable) and objectivity (objective weighted function) on the states of nature will be considered as:

I = Information Structure (on S): =imprecision (on S) + uncertainty (on S) + objectivity (on S), where:

- Imprecision on S may be presented by some inexact (stochastic, fuzzy, fuzzy-stochastic or other) variable.
- Uncertainty on S may be presented by the levels of belief, credibility, probability, possibility and other monotone measures on 2^s . These levels identify the possibility of occurrence of some groups (events, focal elements and others) on the states of nature.
- Objectivity on S is defined by the objective importance of states of nature in the procedure of decision making. As usual the objective function is presented by a weighted function (vector) $W: S \Rightarrow R_0^+$.

Now we may classify cases of the Information Structure – I:

I1: The case:

- Imprecision is presented by some exact variable $a: S \Rightarrow R^1$.
- The measure of uncertainty does not exist.

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- Objectivity is presented by the weights $W=\{w_1,w_2,...,w_m\}$.

Examples: OWA and MEAN operators belong to I1.

I2: The case:

- Imprecision is presented by some fuzzy variable: $\tilde{a}\in\psi;\tilde{a}:S\Rightarrow[0,1]$.
- The measure of uncertainty does not exist.
- Objectivity is presented by the weights $W=\{w_1,w_2,...,w_m\}$.

Examples: FOWA operator belongs to I2.

I3: The case:

- Imprecision is presented by some stochastic variable: $a:S\Rightarrow R^1$.
- The measure of uncertainty is presented by concerning probability distribution on S ($P:2^S\Rightarrow[0,1]$) $p_i=P\{s_i\}, i=1,2,...,m.$
- Objectivity is presented by the weights $W=\{w_1,w_2,...,w_m\}$.

Example: POWA operator belongs to I3.

I4: The case:

- Imprecision is presented by the some fuzzy-stochastic variable:

$\tilde{a}\in\psi;\tilde{a}:S\Rightarrow[0,1]$.

- Uncertainty measure is presented by the concerning probability distribution on S ($P:2^s\Rightarrow[0,1]$) $p_i=P\{s_i\}, i=1,2,...,m.$
- Objectivity is presented by the weights $W=\{w_1,w_2,...,w_m\}$.

Example: FPOWA operator belongs to I4.

I5: The case:

- Imprecision is presented by some exact variable: $a:S\Rightarrow R^1$.
- The measure of uncertainty defined by some monotone measure (possibility measure [11], [14], [21], [22], λ-additive measure ([44] and so on) $g:2^s\Rightarrow[0,1]$.
- Objectivity is presented by the weights $W=\{w_1,w_2,...,w_m\}$.

Examples: SEV (Yager [51]) operator belongs to I5; SEV-POWA, AsPOWA, SA-POWA, SA-AsPOWA (will be defined in the part II of this work) operators belong to I5.

I6: The case:

- Imprecision is presented by some fuzzy variable: $\tilde{a}\in\psi;\tilde{a}:S\Rightarrow[0,1]$.
- The measure of uncertainty is presented by some monotone measure $g:2^s\Rightarrow[0,1]$.
- Objectivity is presented by the weights $W=\{w_1,w_2,...,w_m\}$.

Examples. SEV-FOWA, AsFPOWA, and SA-AsFPOWA operators (will be defined in the part III of this work) belong to I6.

Note that some other cases may be considered in the Information Structure – I (for an example, the cases when the weights in structure are not present and others).

7) F – is an aggregation (in our case OWA-type) operator for ranking of possible alternatives by its outcome values calculated by the F. Following the Information Structure I on the states of nature for all possible alternatives $d \in D, F(d)$ is a ranking value. In general, $F(d)$ is defined as converted (or condensed) information of imprecision values plus uncertainty measure and objective weights.

$$F(d) = aggregation(a(d), g, w).$$

We say – that alternative d_j is more prefered (dominated) than

$$d_k, d_j \succ d_k, \text{ if } F(d_j) \rangle F(d_k),$$

and d_j is equivalent to d_k, $d_j \equiv d_k$, if $F(d_j) = F(d_k)$. So, the aggregation operator F induces some preference binary relation $\succeq$ on the all-possible alternatives - D.

8) Im is a set of information measures of an aggregation operator F:

$$Im = \{Orness, \ Dispersion, Divergence, Balance\}. \quad (20)$$

In order to classify OWA-type aggregation operators $\{F\}$ it is necessary to investigate information measures (*Eq.(20)*). This analysis also gives us some information on the inherent subjectivity of the choice of the decision aggregation operator by DMP [6].

5 | Conclusion

This paper has a conceptual and introductory character. The main preliminary concepts were presented. Definitions of the OWA operator and the POWA and FPOWA operators as some fuzzy-probabilistic extensions of the OWA operator were introduced. Their information measures as - *Orness, Enropy, Divergence* and *Balance* were considered. From the point of view of systems approach the scheme of GDMS in uncertain – objective environment and its Information Structure was described and formally presented. New GDMS gives us the possibility to identify the different cases of levels of incompleteness and objectivity of available information which in whole defines the aggregation procedure. The main results on the constructions of new generalizations of the POWA and FPOWA operators will be presented in Parts II and III of this work.

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Well Drilling Fuzzy Risk Assessment Using Fuzzy FMEA and Fuzzy TOPSIS

Mazdak Khodadadi-Karimvand [1,*], Hadi Shirouyehzad[2]

[1] Department of Industrial Engineering, Najafabad Branch, Islamic Azad University, Iran; mazdak.kh@gmail.com
[2] Department of Industrial Engineering, Najafabad Branch, Islamic Azad University, Iran; hadi.shirouyehzad@gmail.com

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Abstract

One of the most important issues that organizations have to deal with is the timely identification and detection of risk factors aimed at preventing incidents. Managers' and engineers' tendency towards minimizing risk factors in a service, process or design system has obliged them to analyze the reliability of such systems in order to minimize the risks and identify the probable errors. Concerning what was just mentioned, a more accurate Failure Mode and Effects Analysis (FMEA) is adopted based on fuzzy logic and fuzzy numbers. Fuzzy TOPSIS is also used to identify, rank, and prioritize error and risk factors. This paper uses FMEA as a risk identification tool. Then, Fuzzy Risk Priority Number (FRPN) is calculated and triangular fuzzy numbers are prioritized through Fuzzy TOPSIS. In order to have a better understanding toward the mentioned concepts, a case study is presented.

Keywords: Failure mode and effects analysis, Fuzzy risk priority number, Fuzzy technique for order of preference by similarity to ideal solution; Risk priority number

1 | Introduction

Failure Mode and Effects Analysis (FMEA), is a proper approach to assessing system, design, process or services. It can uncover paths, including problems, errors and risks, ending to a failure. FMEA is a preventive action with a teamwork approach. It was first developed as a design methodology in the aerospace industry to meet security and reliability requirements and then was broadly adopted in the industry field to assure the security and reliability of products [21]. It is an effective tool for predicting errors and finding the minimum cost of error-avoidance solution. FMEA is a structured technique for initializing design step or reviewing and developing product and service design in the organization. It is used to link the key parameters of an organization, related documents, design and implementation and so on [7]. Generally, FMEA avoids the

Corresponding Author: mazdak.kh@gmail.com
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occurrence of errors, promotes the creation and development of a great product or service and records related parameters and indices of designs and developments, process or services [1]. The output of FMEA intends to answers the following questions: what kind of errors, problems or risks are there? Which one of the identified errors, problems or risks has the highest importance (risk)? What are the remedies for reducing the occurrence probability of such cases? FMEA systematically takes the control in order to provide a correct answer to these questions. In addition to the identification of errors, problems or latent risks of a process, it prioritizes them relying on the knowledge and proficiency of a work team. Risk analysis is a part of FMEA technique applied to prevent the occurrence of a problem.

Today, organizations have discovered by experience that the concept of zero risk is no longer available and the occurrence of a problem is always probable. By improving control systems, therefore, they try to reduce the occurrence probability of problems and accidents in the work place and entrust remainder possible risks, known as residual risks in the insurance literature, to insurance companies [17]. Organizations implement FMEA for different reasons. Dale and Shaw [6] conducted a study on Ford Company. According to their results, companies implement FMEA in order to satisfy customers' needs, improve the quality and reliability of products and improve the process and safety of production [23]. FMEA is a proper methodology for engineers by which they create a structured approach in the following mental thought [21]:

- What may be done by mistake?
- What may serve as the cause of a mistake?
- What are the consequences of mistakes?

Ireson et al. believe that FMEA is an effective preventive methodology which can be easily connected to many engineering and reliability methodologies. FMEA creates an effective risk management environment through influencing the probable deficiencies of a product/service and providing planned reactions to such deficiencies [12]. According to Chrysler [6] FMEA may be described as a group of regular activities identifying and assessing the probable deficiencies of a product/service. Furthermore, it identifies those activities which can reduce or eliminate failure opportunities within a given period of time. In addition, it helps users to identify the main aspect of a design or process to be particularly controlled for production purposes and to realize those areas showing an advanced control or performance. Reviewing related literatures many studies have been carried out to strengthen FMEA using artificial intelligence, AI, modeling techniques [5]. Russomanno et al. [19] suggested in their works the application of AI systems in FMEA. Bowles and Pelaez used fuzzy logic to improve deficiency risk assessment and FMEA prioritization capability [2].

2 | Risk Priority Number

This method uses 0-10 scoring system. Every number stands for a specific level of severity, probability or detection of a problem. RPN is, indeed, the product of severity, probability and detection and depends on the following three factors [10].

$$\text{RPN} = \text{Severity} \times \text{Occurrence} \times \text{Detection}.$$

Severity (assessment and measurement of failure result) stands for the severity of a potential failure effects. It is actually a kind of assessing and measuring the consequences of a failure. The extent of severity indicates the extent of the effect of a potential failure or incident. Severity is a numerical number where the more important the effect, the higher the severity. Severity number varies from 1 to 10.

Probability (the probability of the occurrence of a failure or incidence or in other words counting the number of failures) stands for rank (value) which is used to estimate the occurrence probability of a failure or incident or, in other words, to count the number of failures. Mathematics, process capability index, reliability and probability rules can be adopted to determine the probability of each process. Probability is assessed using numerical values ranging from 1 to 10.

Detection stands for detecting an incident prior to realizing the consequences of its occurrence. RPN is assessed by a number ranging from 1 to 1000. It is used to classify required corrective actions for reducing or eliminating potential failure or incident modes. Failure/incident modes with higher RPN numbers should be assessed at the first priority. However, paying attention to the severity number of each class is of high importance. If the severity number of a class is 9 or 10, the cause should be urgently assessed regardless of the total RPN number of that class [1].

2 | Literature Review

The application of fuzzy set theory has been broadly studied due to the ambiguity of risk analysis in different engineering fields. Lee [14] adopted fuzzy set theory for comprehensive risk assessment of software development. Sadiq and Husain [20] employed a fuzzy-based method for comprehensive environmental risk assessment of drilling time loss during drilling operation. Wang and Elhag [24] used fuzzy group decision making method for bridging risk analysis purposes. In an article, Pillay and Wang [18] used fuzzy logic and FMEA grey relational analysis in navigation industry to overcome the traditional weaknesses of FMEA in risk assessment. Xu et al. [26] suggested a fuzzy FMEA estimation for engine systems in their works. Guimara and Lapa [9] adopted an absolute fuzzy logic system in the inlet water system of a reserved steam boiler of a nuclear power plant in order to improve risk ranking. Sankar and Prabhu [21] criticized RPNs due to combining P, S and D. Wang et al. [25] suggested Fuzzy Weighted Geometric Mean (FWGM), for calculating Fuzzy Risk Priority Number (FRPNs) and centroid defuzzification method for finding the centroid of fuzzy number. In an article, Bowles and Pelaez [13] used fuzzy cognitive maps to demonstrate the relationships between the causes of errors effects. They argued that fuzzy cognitive maps are an appropriate diagnosis tool in FMEA because they can demonstrate the proportions and relationships between causes and effects. This study uses a combined triangular-trapezoidal membership function. In another article, Bowles and Pelaez [2] used “if-then” logic to develop FMEA in fuzzy environment where all possible modes between severity, probability and detection parameters are studied using “if-then” logic. For instance, if severity and probability are high and detection is low, then the priority of risk will be high. This model uses a triangular-trapezoidal membership function. The study of Chang et al. [4] is another work in the field of FMEA where they introduced a relatively easy defuzzifier model to obtain the accurate value of linguistic variables. They first allocate a linguistic variable to severity, probability and detection parameters and then allocate a fuzzy number to each linguistic variable using a triangular membership function. Afterward, they defuzzify them using their defuzzifier model and calculate a relative relationship degree for each cause of the three parameters. The stronger the relationship, the weaker the effect of cause. Therefore, any increase in the relationship degree indicates improved risk priority.

4 | Methodology

In this study, selection of fuzzy membership function is done first, then fuzzy risk priority numbers calculations are done by multiplying the membership functions of Severity, Probability and Detection, finally fuzzy risk priority numbers as fuzzy numbers are prioritized using Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS).

4.1 | Explanation of a Number of Models

Fuzzy quantities are ranked based on one or more features of fuzzy numbers including the center of gravity, the area below membership function or intersection points of sets. In one ranking model, a particular property of fuzzy number is selected and variables are ranked in terms of this property. Therefore, the first rational conclusion is that we should not expect that different ranking methods assign the same ranks to the same samples of fuzzy numbers.

Methods for ranking fuzzy number are divided into two groups:

- Some methods convert a fuzzy number to a non-fuzzy number using a mapping function, F. In other words, if $\widetilde{A}$ is a fuzzy number, then F ($\widetilde{A}$) =a will be a non-fuzzy number. Then, they rank fuzzy numbers by ranking corresponding non-fuzzy numbers derived from this function. The center of gravity, the maximum membership function and left and right scores are among the techniques of this group.
- Some methods conduct a pairwise comparison on fuzzy numbers using fuzzy relations and states results with linguistic words. For example, results will be similar to this sentence: "fuzzy number $\widetilde{A}$ is better then fuzzy number $\widetilde{B}$ to some extent".

However, each method has its own advantages and disadvantages. Regarding group 1, it is argued that the conversion of a fuzzy number to a non-fuzzy number may result in the loss of a large number of data deliberately kept during calculation process. On the other hand, such methods rank considered fuzzy numbers in a stable manner. In other words, if $A^{\sim}$ is larger than $B^{\sim}$ and $B^{\sim}$ is larger than $C^{\sim}$, then $A^{\sim}$ will be always larger than $C^{\sim}$. Furthermore, there will always be a fuzzy number in ranked numbers which are introduced as the best, the second best and the third best and so on. Maintaining the linguistic words during comparison process, group 2 methods survive fuzzy information of a problem. Nevertheless, it may be impossible to determine the total rank of a fuzzy number among other fuzzy numbers using pairwise comparisons. This means that if $A^{\sim}$ is better than $B^{\sim}$ and $B^{\sim}$ is better than $C^{\sim}$, then $A^{\sim}$ might not be better than $C^{\sim}$.

The inherent complexity of techniques for ranking fuzzy numbers is not limited to this. In simple problems, the majority of techniques perform a stable ranking. Nevertheless, in more complex problems, different ranking techniques lead to different results. This means that if for some values of x, the membership functions of fuzzy numbers overlap with each other (intersect) or even if there is a slight difference between the support sets of fuzzy numbers, then different methods will most likely assign different ranks to fuzzy numbers [8].

In an article, Lavasani et al. [16] assessed offshore wells risks. The majority of offshore wells data are unknown and ambiguous data and discovering their mechanisms is a difficult and complex problem. They stated every basic risk item using a trapezoidal fuzzy number which was a combination of probability and severity.

Tay and Lim relied in their article on fuzzy inference techniques as a way for overcoming the weaknesses of classic FMEA systems. Compared with classic FMEA, fuzzy methods assess the risks of failure modes and ranks them based on expert knowledge. This article introduces a general method for modeling RPN function. Fuzzy FMEA assumes three inlet factors of RPN function, i.e., severity, probability and detection as the input factors of fuzzy RPN function. In this way, a Fuzzy Inference System (FIS) is generated along with a set of fuzzy production rules, FPRs, in order to infer input factors [15].

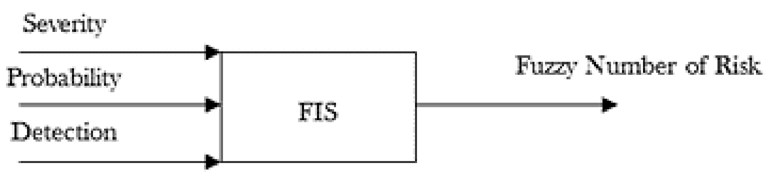

Fig. 1. FRPN model.

Ilangkumaran and Thamizhselvan identified and ranked risks in petrochemical industry. They used hazard and operability study method (HAZOP) and FMEA in order to identify and prioritize probable latent defects of a system [11].

HAZOP is an old methodology. It systematically and effectively identifies all important latent defect modes endangering human, environment, facilities and process. It was used to score FRPN which is used in FMEA. This proposed technique is used to find a better rank for defect modes. The number of risk priorities and the fuzzy adjusted geometric mean of risk are used to improve risk assessment efficiency. This makes the effective assessment of malfunctioning systems easier. The higher the fuzzy centroid value, the higher the overall risk and the higher the risk priority. All failure modes can be prioritized in terms of the fuzzified centroid values of their FRPNs [11].

In their article, Shirouyehzad et al. [22] defuzzified triangular fuzzy numbers of FRPN and then ranked them. This paper used left and right scores technique to defuzzify numbers.

4.2 | Fuzzy Risk Priority Number

After obtaining the rates of severity, probability and detection from *Tables 1*, *2* and *Table 3*, this method obtains FRPN by selecting a fuzzy membership function for each rate and forming a membership function by multiplying fuzzy membership functions.

4.2.1 | Selection of fuzzy membership function

Five linguistic variables i.e., very low (VL), low (L), moderate (M), high (H) and very high (VH) were assigned to all influential factors of risk bearing degree i.e., severity, probability and detection. These variables are assigned to the ranks as per the following *Table 1*.

Table 1. Fuzzy numbers of linguistic variables corresponding to ranks 1 to 10.

Fuzzy Number	Verbal Variable	Rank
(0.9,1,1)	VH	9,10
(0.7,0.85,1)	H	7,8,9,10
(0.4,0.6,0.8)	M	4,5,6,7,8
(0.2,0.35,0.5)	L	2,3,4,5
(0,0.15,0.3)	VL	1,2,3

{VL, L, M, H, VH} = T(x) = set of linguistic variables values,

[0, 1] = U= variation amplitude of the reference set.

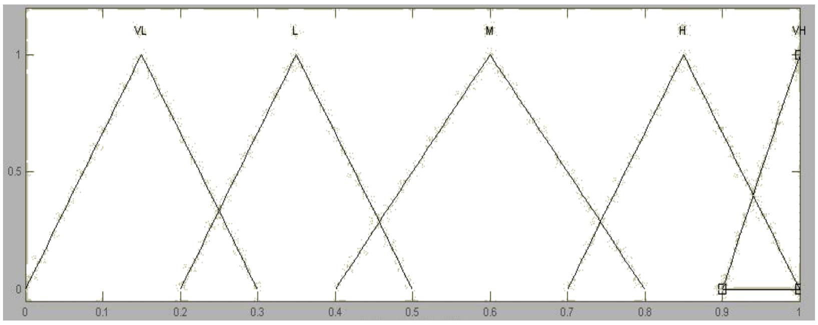

Fig. 2. Membership function of linguistic variables.

4.2.2 | Forming a membership function by multiplying the membership functions of severity, probability and detection

FRPN is calculated from the following relation by multiplying the membership functions of severity, probability and detection. If M is a linguistic variable, its triangular fuzzy number may be defined as follows:

$$M= (l, m, u).$$

Where u, l and m are the upper limit, the lower limit and the mean of u, respectively where the membership degree of l is 1.

Algebraic operations rules are applied on triangular numbers as follows to calculate RPN:

$$RPN = S \times P \times D,$$

$$FRPN = (l1, m1, u1) \times (l2, m2, u2) \times (l3, m3, u3) = (l1l2l3, m1m2m3, u1u2u3). \quad (1)$$

4.3 | Prioritization of Fuzzy Numbers Using TOPSIS

Classis TOPSIS uses accurate and precise values to determine the weight of criteria and rank options. Fuzzy TOPSIS assesses the elements of decision-making matrix or the weight of criteria, or both, using linguistic variables offered by fuzzy numbers.

This article uses Cheng and Hwang [3] technique in the case of triangular fuzzy numbers. Then, decision matrix is formed as follows:

$$\tilde{D}=\begin{bmatrix}\tilde{x}_{11} & \cdots & \tilde{x}_{1n}\\ \vdots & \ddots & \vdots \\ \tilde{x}_{m1} & \cdots & \tilde{x}_{mn}\end{bmatrix}, \quad \tilde{x}_{ij}=\left(a_{ij},b_{ij},c_{ij}\right), \quad i = (1, 2\ldots m), \quad j = (1, 2\ldots n).$$

Criteria weight matrix is defined as follows:

$$\widetilde{W}=\left[\widetilde{w}_1,\widetilde{w}_2,\widetilde{w}_3,\ldots,\widetilde{w}_n\right].$$

$$\widetilde{w}_j=\left(w_{j1},w_{j2},w_{j3}\right).$$

Then, fuzzy decision matrix is de-scaled:

$$\tilde{x}_{ij}=\left(\frac{a_{ij}}{c_j^*},\frac{b_{ij}}{c_j^*},\frac{c_{ij}}{c_j^*}\right). \quad (2)$$

$$c_j^*=\max c_{ij}, \quad (3)$$

$$\widetilde{R}=[\tilde{r}_{ij}]_{m\times n} \quad i=1, 2\ldots m;\ j=1, 2,\ldots,n.$$

The fuzzy decision matrix is weighted.

$$\tilde{v}_{ij}=r_{ij}.\widetilde{w}_j, \quad (4)$$

$$\widetilde{V}=[\tilde{v}_{ij}]_{m\times n} \quad i=1, 2\ldots m;\ j=1, 2\ldots n.$$

Then, fuzzy ideal and non-ideal solutions are found:

$$A^*=\{\tilde{v}_1^*,\tilde{v}_2^*,\tilde{v}_3^*,\ldots,\tilde{v}_n^*\}. \quad (5)$$

$$A^-=\{\tilde{v}_1^-,\tilde{v}_2^-,\tilde{v}_3^-,\ldots,\tilde{v}_n^-\}. \quad (6)$$

$$\tilde{v}_j^*=\max\{\tilde{v}_{ij}\} \quad i=1, 2\ldots m;\ j=1, 2\ldots n. \quad (7)$$

$$\tilde{v}_j^-=\min\{\tilde{v}_{ij}\} \quad i=1, 2\ldots m;\ j=1, 2\ldots n. \quad (8)$$

The closeness to ideal and non-ideal solutions is calculated:

$$S_i^*=\sum_{j=1}^{n} d(\tilde{v}_{ij},\tilde{v}_1^*)\ ,\ i=1,2,\ldots,m. \quad (9)$$

$$S_i^-=\sum_{j=1}^{n} d(\tilde{v}_{ij},\tilde{v}_1^-)\ ,\ i=1,2,\ldots,m. \quad (10)$$

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$$d_v=\sqrt{\frac{1}{3}[(a_1\text{-}a_2)^2+(b_1\text{-}b_2)^2+(c_1\text{-}c_2)^2}. \quad (12)$$

Finally, similarity index is calculated and alternatives are ranked:

$$CC_i=\frac{S_i^-}{S_i^*+S_i^-}\ i=1,2,\ldots,m. \quad (12)$$

5 | Case Study

The following case study was conducted on gas and oil wells drilling operations using FRPN techniques. According to the following table, there are 8 potential failure modes each has different effects, causes and detection probabilities determined by FMEA team. Severity, probability and detection numbers are defined using related tables and by the aid of FMEA team. The RPN of all 8 potential failure modes is determined. The last column of *Table 2* shows the control actions required for each mode.

After determining corresponding linguistic variables for the values, the fuzzy numbers of severity, probability and detection is defined using membership function and in accordance with *Table 1*. Then, FRPN is calculated as per *Table 4*. Finally, fuzzy TOPSIS is used to prioritize them.

Table 2. Formation of fuzzy membership function for severity, probability, and detection for all 8 potential failure modes.

#	Severity	Probability	Detection
1	(0.4,0.6,0.8)	(0.9,1,1)	(0,0.15,0.3)
2	(0.7,0.85,1)	(0,0.15,0.3)	(0.2,0.35,0.5)
3	(0.4,0.6,0.8)	(0.2,0.35,0.5)	(0.4,0.6,0.8)
4	(0.7,0.85,1)	(0.4,0.6,0.8)	(0.2,0.35,0.5)
5	(0.4,0.6,0.8)	(0.4,0.6,0.8)	(0,0.15,0.3)
6	(0.4,0.6,0.8)	(0.4,0.6,0.8)	(0,0.15,0.3)
7	(0.7,0.85,1)	(0.7,0.85,1)	(0,0.15,0.3)
8	(0.7,0.85,1)	(0.7,0.85,1)	(0,0.15,0.3)

After determining the linguistic variables of severity, probability and detection for all 8 potential failure modes, fuzzy values are substituted as per *Table 3*. Then, FRPNs are calculated using *Eq. (1)* and in accordance with *Table 4*.

Now, all 8 potential failure modes are prioritized using fuzzy TOPSIS. Criteria are considered positive values and each member is divided into the maximum number of each column as per *Eq. (2)* in order to normalize and de-scale them. Then, they are multiplied by the weight matrix as per *Eq. (4)*. Since there is the same number of criteria, the balanced matrix will be similar to the previous matrix. *Table 5* shows obtained values.

$$\tilde{R}=\tilde{V}.$$

Table 3. FMEA table.

#	Process Function	Potential Failure Mode	Potential Effect(s) of Failure	Severity	Potential Cause(s)/ Mechanism(s) of Failure	Probability	Current Process Controls	Detection	RPN
1	Check top drive & run in hole	Collapse/part of 9-5/8" casing	Technical non-productive time (NPT) for contractor.	5	Poor cementing / poor design of casing.	9	Pressure test of annulus both positive& negative run in hole with bit.	1	45
2	Drilling 6-1/8" hole section	Low progress	More time than program.	8	Deviated hole& abnormal parameters.	1	Monitoring surface parameters& optimization.	3	24
3	Side track the well	Low progress in side tracking	More time than program.	6	Utilizing heavy duty material in 4-3/4" positive displacement motors hardness of formation& low motor efficiency.	3	Calculate time more than the previous drilling operation.	4	72
4	5" liner lap leakage	Leakage in liner lap	More time than program.	7	Poor cementing of liner.	5	Abnormal surface pressure.	2	70
5	Fish in hole	Fish in hole while drilling cement plugs	More cost for contractor due to remedial actions.	5	Hard cement & harsh parameters.	4	Drill out cement to evaluate quality.	1	20
6	Opening window	More time in opening window & rupture of tri mill	More time than program.	5	Low quality of tri mill strength of casing due to high thickness.	4	Casing coupling log (CCL) & segmented bond tool (SBT) log.	1	20
7	Flowing the well	Well does not flow normally	Excess operations & cost.	7	Formation damage& skin.	7	Poor wellhead pressure after perforation.	1	49
8	Running 7" liner	Liner stuck	More time than program.	8	Poor hole condition& dog leg severity.	7	Monitoring of condition trips determining tight holes.	1	56

Table 4. Formation of membership function by multiplying the membership functions of severity, probability and detection.

#	FRPN
1	(0,0.09,0.24)
2	(0,0.0446,0.15)
3	(0.032,0.126,0.32)
4	(0.056,0.1785,0.4)
5	(0,0.054,0.192)
6	(0,0.054,0.192)
7	(0,0.108,0.3)
8	(0,0.108,0.3)

Table 5. Formation of membership function by multiplying the membership functions of severity, probability and detection.

#	FRPN ($\tilde{V}$)
1	(0,0.5042,0.6)
2	(0,0.2499,0.375)
3	(0.5714,0.7059,0.8)
4	(1,1,1)
5	(0,0.3025,0.48)
6	(0,0.3025,0.48)
7	(0,0.6067,0.75)
8	(0,0.6067,0.75)
$\tilde{v}^*$	(1,1,1)
$\tilde{v}^-$	(0,0,0)

At this point, the closeness to ideal and non-ideal solutions and similarity index are calculated in accordance with relations 11, 10, 9, 12.

Table 6. Calculating closeness to ideal and non-ideal solutions and similarity index.

#	CC_i	S_i^-	S_i^*
1	0.34	0.4525	0.8918
2	0.21	0.2602	0.9656
3	0.49	0.6988	0.7154
4	1	1	0
5	0.26	0.3276	0.9448
6	0.26	0.3276	0.9448
7	0.4	0.5570	0.8305
8	0.4	0.5570	0.8305

The priority of activities is obtained as follows:

Activity 4, activity 3, activities 7 and 8, activity 1, activities 5 and 6, activity 2.

The comparison of the obtained results with those of non-fuzzy RPN technique demonstrates ranking difference.

Result Of non-fuzzy RPN Ranking were:

Activity 3, activity 4, activity 8, activity 7, activity 1, activities 2, activities 5 and 6.

6 | Conclusion

Given the importance of recognizing risk factors and their multiplicity, it is important to prioritize them. This article first identified hazards using FMEA tool and conducting teamwork for each activity. Second, it examined the relative priority of these factors with the help of the RPN, the FRPN, and TOPSIS. Extensive research has been done for improving the FMEA methodology using such techniques as fuzzy logic. This article tried to maintain the fuzzy information of problem by maintaining fuzzy logic values equivalent to linguistic terms in comparison process. In simple problems, most methods perform ranking, but in more complex problems, they lead to different results. In other words, for some X values, if fuzzy number membership functions overlap (intersect) with each other or even if fuzzy number support sets slightly differ with each other, various methods will most likely assign different rankings to fuzzy numbers.

The fuzzy number ranking results presented in this article can be challenged through comparing them with the results of the Fuzzy Techniques for Order of Preference by Similarity to Ideal Solution (FTOPSIS) method. The same method can also be used for calculating the fuzzy risk level in qualitative and
semi-quantitative risk assessment techniques with two and more than two dimensions.

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Application of Fuzzy Algebraic Model to Statistical Analysis of Neuro-Psychopathology Data

Semiu Ayinla Alayande[1,*], Ezekiel Akande[2], Amanze Egere[3]

[1] Department of Mathematical Sciences, College of Natural Sciences, Redeemer's University Ede, Osun state Nigeria.; alayandesa@run.edu.ng;

[2] Department of Mathematical Sciences, College of Natural Sciences, Redeemer's University Ede, Osun state Nigeria.; karile20baba@gmail.com;

[3] Department of Mathematical Sciences, College of Natural Sciences, Redeemer's University Ede, Osun state Nigeria.; egerea@run.edu.ng.

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Abstract

The purpose of this paper is to describe and present applications of fuzzy logic in analysis of certain neuro-psychopathological symptoms. These symptoms have been linked to conditions relating to occupational hazards. Our method of data analysis which is based on Hamacher operation on picture fuzzy sets is then applied to analyse such occupational hazards. Our result proves to be effective and applicable in medical decision processes especially in situations where such neuro-psychopathological symptoms are detectable by first-aid diagnostic machines.

Keywords: Fuzzy logic, Hamacher sum, Hamacher product, Psychopathology.

1 | Introduction

Psychopathology is a subject that deals with problems related to mental health: how to understand them, how to classify them, and how to fix them. Thus, the topic of psychopathology extends from research to treatment and covers every step in between [4]. Proper understanding of causes of mental disorders therefore allows for effective treatments. The major problem aspect of work is psychopathology (emotional or behaviour disorders) which results from inhuman work conditions such as profit maximization and exploitation, poor leadership particularly task-oriented leadership, poor followership, selfishness and laziness. Occupation, when not managed very well, affects, ab initio, individual's personal, family, physical, social, and psychological health. In fact, work can cause permanent injury to man's physical and psychological health and can also reduce one's life span [3].

Corresponding Author: alayandesa@run.edu.ng
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Occupational stress occurs when some elements of work generate negative impacts on an employee's physical and mental well-being. World Health Organisation [9] estimates that about 154 million people suffer from depression, 25 million people suffer from schizophrenia, 91 million people are affected by alcohol use disorder and 15 million people suffer from drug use disorder. Hence, psychopathology is a global problem.

The precipitating and maintenance factors of some psychopathological symptoms are: hypertension, diabetics, drug related symptoms, depressions, schizophrenia, anxiety and other neurotic symptoms, sexual and reproductive dysfunctions, poor intra and interpersonal relationships, marital stress and family crises have been linked to conditions relating to occupational hazards and demands including occupational stress and betrayal of trust by the employer or employee ([1] and [2]). Due to the nature of the problem stated above, this paper proposes a model that can be used to address such problems in any organization to prevent poison to the workers.

Psychiatry is the most multifarious domain in medical sciences. Psychiatry diseases are not directly measured due to unclear symptomatic presentations. Results of investigation and treatment are physically correlated with the course of morbidity and such correlations can head to biased decision making. It is often noticed that different diseases present with analogous types of symptoms and vice versa. Also behind deposit of symptoms, there is possibility of multiple diseases evident in case of psychopathology with mania.

Researchers attempted to use hard computing techniques, such as heuristic and probabilistic algorithm in diagnosing psychiatry diseases. But they found that such algorithm are rigid to actually evaluate morbidity [6]. On the other hand, the merit of fuzzy logics lies on competence to handle non-discrete quantitative inputs and analyze like human beings.

General causes of occupational psychopathology can be divided into two major groups namely work related factors and individual factors. The work-related factors are: work overload, time pressures, bad relations with supervisor, change of work, role ambiguity, frustration, conflict at work and, job design and harassment. The individual factors can be enumerated as follows; financial worries, marital problems, pregnancy, problems with children clash of spouse, personal traits and excessive consumptions of alcohol.

The data for this work is obtained from two different hospitals, using different case files that span over five years. Medical results, together with the associated symptoms of patients whose cases are of interest to this study were collected for analysis.

2. Methodological Approach

This study applies fuzzy algebraic model to analyze the data. We assume that all a levels are the same, that is, the two major criteria are of the same significant level. A fuzzy set is a pair (X,μ) where $X \neq \phi$ and μ a membership function. The reference set X is called universe of discourse, and for each $x \in X$ the value μ_x is called the grade of membership of $x \in (X,\mu)$. The function $\mu = \mu_A$ is called the membership function of the fuzzy set $A = (X,\mu)$. The fuzzy set is said to be convex if and only if $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq min\{\mu_{\bar{A}}(x_1), \mu_{\bar{A}}(x_2)\}$, $x_1, x_2 \in X, \lambda \in [0,1]$, where μ_A and $\mu_{\bar{A}}$ are membership functions of A and its complement $\bar{A}$ respectively. The typical pairs of non-parameterized t-norms(τN) and t-conorms$(\tau(N)$ are given respectively by the Hamacher product

$$t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \frac{\mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)}{\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x) . \mu_{\tilde{B}}(x)}. \quad (1)$$

The Hamacher sum is given by

$$s(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x))=\frac{\mu_{\tilde{A}}(x)+\mu_{\tilde{B}}(x)-2\mu_{\tilde{A}}(x).\mu_{\tilde{B}}(x)}{1-\mu_{\tilde{A}}(x).\mu_{\tilde{B}}(x)}. \quad (2)$$

We recall some basic definitions and algebra of Intuitionistic Fuzzy Sets (IFS) and Picture Fuzzy Sets (PFS):

– Intuitionistic fuzzy sets

Let $X \neq \emptyset$ be the universe of discourse. The Intuitionistic IFS over X is defined by:

$$A = \{[\hat{\mu}_A(x), \hat{\mu}_B(x)] | x \in X\}, \quad (3)$$

where $\hat{\mu}_A: X \to [0,1]$ *and* $\hat{\mu}_B: X \to [0,1]$ are respectively membership and non-membership function of IFS A. Here $0 \leq \hat{\mu}_A + \hat{\mu}_B \leq 1$ *for all* $x \in X$ and $\pi := 1 - (\hat{\mu}_A(x) + \hat{\mu}_B(x))$ is called the degree of indeterminacy of $x \in X$ in IFS A. The pair $(\hat{\mu}_A, \hat{\mu}_B)$ is called Intuitionistic Fuzzy Value or Intuitionistic Fuzzy Number (IFN) by Xu [10].

– Picture fuzzy sets

Let *X* be a universe of discourse objects. A picture fuzzy set over *X*, denoted by $\hat{P}$ is defined as follows:

$$\hat{P} = \{[\hat{\mu}_{A\hat{P}}(x), \hat{\theta}_{\hat{P}}(x), \hat{\mu}_{B\hat{P}}(x)] | x \in X\}, \quad (4)$$

where $\hat{\mu}_{A\hat{P}}: X \to [0,1]$, $\hat{\theta}_{\hat{P}}: X \to [0,1]$ *and* $\hat{\mu}_B: X \to [0,1]$ are called positive neutral and negative degree of membership of picture fuzzy set $\hat{P}$, respectively. Here, $0 \leq \hat{\mu}_{A\hat{P}}(x) + \hat{\theta}_{\hat{P}}(x) + \hat{\mu}_{B\hat{P}}(x) \leq 1$ *for all* $x \in X$. Besides, $\theta_{\hat{P}}(x)$ denotes degree of refusal of $x \in X$ and is defined by $\pi := 1 - \left(\hat{\mu}_{A\hat{P}}(x) + \hat{\theta}_{\hat{P}}(x) + \hat{\mu}_{B\hat{P}}(x)\right)$. The pair $(\hat{\mu}_{A\hat{P}}, \hat{\theta}_{\hat{P}}, \hat{\mu}_{B\hat{P}})$ is called Picture Fuzzy Value (PFV) or Picture Fuzzy Element (PFE).

2.1. Hamacher Operation on Picture Fuzzy Set

2.1.1. Hamacher operations

The TN and TCN are useful notions in fuzzy set theory, which are used to define general union and intersection of fuzzy set. The definition and conditions of TN and TCN are proposed by Roychowdhury and Wang [11]. The generalized union and generalized intersection of intuitionistic fuzzy sets based on TN and TCN were provided by Deschrijver and Kerre [5]. In 1978, Hamatcher introduced HOs known as Hamacher sum $(\oplus)$ and Hamacher product $(\otimes)$, which are examples of TN and TCN respectively and are given by

$$T_H(u_A, u_B) = u_A \oplus u_B = \frac{u_A + u_B - u_A u_B - (1-\varphi) u_A u_B}{1 - (1-\varphi) u_A u_B}. \quad (5)$$

$$T_H^* = (u_A, u_B) = u_A \otimes u_B) = \frac{u_A u_B}{\varphi + (1-\varphi)(u_A + u_B - u_A u_B}. \quad (6)$$

For $\varphi = 1$, the Hamacher TN and TCN reduce to the forms:

$$T_H = (u_A, u_B) = u_A \oplus u_B = u_A + u_B - u_A u_B. \quad (7)$$

$$T_H^* = (u_A, u_B) = u_A \otimes u_B = u_A u_B. \quad (8)$$

T_H and T_H^* represent algebraic TN and TCN respectively. When $\xi = 2$, the Hamacher TN and TCN will conclude to the form

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$$T_H = (u_A, u_B) = u_A \oplus u_B = \frac{u_A + u_B}{1 + u_A u_B}. \quad (9)$$

$$T_H^* = (u_A, u_B) = u_A \otimes u_B = \frac{u_A u_B}{1 + (1 - u_A)(1 - u_B)}. \quad (10)$$

We make use of the Hamacher operations on PFNs provided by Wei [7] and Wu & Wei [8]. Let *X* and *Y* be two PFSs, then Hamacher Sums and Products of the two PFSs *X* and *Y* are denoted by $(\hat{p}_1 \oplus \hat{p}_2)$ and $(\hat{p}_1 \otimes \hat{p}_2)$, respectively, and defined by

$$\hat{p}_1 \oplus \hat{p}_2 = \left(\begin{array}{c} \dfrac{\hat{\mu}_{A1} + \hat{\mu}_{A2} - \hat{\mu}_{A1}\hat{\mu}_{A2} - (1-\varphi)\hat{\mu}_{A1}\hat{\mu}_{A2}}{1 - (1-\varphi)\hat{\mu}_{A1}\hat{\mu}_{A2}}, \dfrac{\hat{\theta}_1\hat{\theta}_2}{\varphi + (1-\varphi)(\hat{\theta}_1 + \hat{\theta}_2 \widehat{-} \hat{\theta}_1\hat{\theta}_2)}, \\ \dfrac{\hat{\mu}_{B1}\hat{\mu}_{B2}}{\varphi + (1-\varphi)(\hat{\mu}_{B1} + \hat{\mu}_{B2} - \hat{\mu}_{B1}\hat{\mu}_{B2}} \end{array} \right) \quad (11)$$

And

$$\hat{p}_1 \otimes \hat{p}_2 = \left(\begin{array}{c} \dfrac{\hat{\mu}_{A1}\hat{\mu}_{A2}}{\varphi + (1-\varphi)(\hat{\mu}_{A1} + \hat{\mu}_{A2} - \hat{\mu}_{A1}\hat{\mu}_{A2}}, \dfrac{\hat{\theta}_1 + \hat{\theta}_2 - \hat{\theta}_1\hat{\theta}_2 - (1-\varphi)\hat{\theta}_1\hat{\theta}_2}{1 - (1-\varphi)\hat{\theta}_1\hat{\theta}_2}, \\ \dfrac{\hat{\mu}_{B1} + \hat{\mu}_{B2} - \hat{\mu}_{B1}\hat{\mu}_{B2} - (1-\varphi)\hat{\mu}_{B1}\hat{\mu}_{B2}}{1 - (1-\varphi)\hat{\mu}_{B1}\hat{\mu}_{B2}} \end{array} \right) \quad (12)$$

2.2. Model for Multiple Component Analysis (MCA)

In this section, Multiple Component Analysis (MCA) method using picture fuzzy information model is proposed based on the operators where weights are real numbers and values of attributes are PFNs. To illustrate effectiveness of the proposed MCA method, an application of neuro-psychopathology under picture fuzzy information is given. Let $Y = (y_1, y_2, \dots, y_r)$ be the discrete set of alternatives and $X = (x_1, x_2, \dots, x_s)$ be the set of attributes.

Let $\Phi = (\phi_1, \phi_2, \dots, \phi_s)$be the weight vector of the attribute such that $\phi_b > 0, (b = 1,2, \dots, s)$ and $\sum_{b=1}^{s} \phi_b = 1$, and let

$R = (\hat{\mu}_{ab}, \hat{\varphi}_{ab}, \hat{v}_{ab})$, be an r by s picture fuzzy decision matrix. Here, $\hat{\mu}_{ab}$ is the degree of the positive membership for which alternative Y_a satisfies the attribute X_b given by decision. $\hat{\varphi}_{ab}$ denotes the degree of neutral membership such that alternative Y_a does not satisfy the attribute X_b. $\hat{v}_{ab}$ provides the degree that the alternative Y_a does not satisfy the attribute X_b given by the decision.

Let $\hat{P} = (\hat{\mu}_{ab}, \hat{\varphi}_{ab}, \hat{v}_{ab})$ be a PFI. Then the score function is given by $\hat{S}_p$ is defined as

$$\hat{S}_p = \hat{\mu}_{ab} - \hat{v}_{ab}, \text{ where } \hat{S}_p \in [-1, 1]. \quad (13)$$

2.2.1. Steps in using MCA

Step 1. Construction of matrix R by decision under PF-Information

$$\begin{pmatrix} \hat{\alpha}_{11} & \hat{\alpha}_{12} \cdots & \hat{\alpha}_{1r} \\ \hat{\alpha}_{21} & \hat{\alpha}_{22} \cdots & \hat{\alpha}_{2r} \\ \vdots & \vdots \quad \ddots & \vdots \\ & \hat{\alpha}_{a1}\hat{\alpha}_{a2}\hat{\alpha}_{ar} & \end{pmatrix}. \quad (14)$$

Step 2. Finding the values of $\hat{\alpha}_a$ $(a = 1,2, \dots, r)$ based on decision matrix R. These values are found by using PFH.

= PFH $(\hat{\alpha}_{a1}, \hat{\alpha}_{a2}, \dots, \hat{\alpha}_{ab})$=$\bigoplus_{b=1}^{s} (\phi_b \hat{\alpha}_{ab})$ =

$$\left(\begin{array}{c}\dfrac{\prod_{b=1}^{s}(1+(\xi-1)\hat{\mu}_b)^{\phi_b}-\prod_{b=1}^{s}(1-\hat{\mu}_b)^{\phi_b}}{\prod_{b=1}^{s}(1+(\xi-1)\hat{\mu}_b)^{\phi_b}+(\xi-1)\prod_{b=1}^{s}(1-\hat{\mu}_b)^{\phi_b}} \\ ,\dfrac{\xi\prod_{b=1}^{s}(\hat{\varphi}_b)^{\phi_b}}{\prod_{b=1}^{s}(1+(\xi-1)\hat{\varphi}_b)^{\phi_b}+\prod_{b=1}^{s}(\hat{\varphi}_b)^{\phi_b}}, \\ \dfrac{\xi\prod_{b=1}^{s}(\hat{\upsilon}_b)^{\phi_b}}{\prod_{b=1}^{s}(1+(\xi-1)\hat{\upsilon}_b)^{\phi_b}+\prod_{b=1}^{s}(\hat{\upsilon}_b)^{\phi_b}}\end{array}\right)\hat{\alpha}_a$$

Step 3. Calculate the score $\hat{S}(\hat{\alpha}_a), (1,2, \dots, r)$.

Step 4. Rank them $\phi_a > 0, (a = 1,2, \dots, r)$

Step 5. Arrange them in ascending order.

Step 6. Stop.

2.3. Numerical Example and Data Collection

Seven major causes of occupational psychopathology have been considered in this study according to DSM-IV-TR guidelines. These symptoms denote independent factor. Data of total number of 400 adult cases have been collected from two different hospitals. A control group (none) has been considered to validate the model. The grades of these cases are unknown. Appropriate ethical measures have been taken to preserve data privacies. Our study excludes Patients whose ages are below the working-class age (35 -60).

Table 1. Occupational Psychopathology (OCP) and the corresponding abbreviations.

Serial No	Symptoms	Abbreviation
1	Tension and anxiety	TAT
2.	Anger	ANG
3.	Aggression	AGG
4.	High blood pressure	HBP
5.	Inability to relax	ITR
6.	Excessive alcohol/tobacco use	EAL
7.	Forgetfulness & increased absenteeism	FIA

Step 1. Decision matrix R is constructed by a professional or expert under PF information as follows.

Step 2. Let $\xi = 4$. By using the PFH operator of the overall performances values $\hat{\alpha}_a$ of probable occupational stress, φ_a $(a = 1,2,3,4,5,6,7)$ are obtained as follows:

$\hat{\mu}_{ab}$=(0.45,0.9,0.2,0.35), $\hat{\varphi}_{ab}$ =(0.24,0.03,0.25,0.64) and $\hat{\upsilon}_{ab}$ = (0.10,0.04,0.29,0.05) with assign weight

ϕ_b =(0.15,0.20,0.25,0.18)

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$$\hat{\alpha}_1=\left(\begin{array}{c}\dfrac{\left[(1+3\times0.45)^{0.15}(1+3\times0.9)^{0.2}(1+3\times0.2)^{0.25}(1+3\times0.35)^{0.18}\right]-\left[(1-0.45)^{0.15}(1-0.9)^{0.2}(1-0.2)^{0.25}(1-0.35)^{0.18}\right]}{\left[(1+3\times0.45)^{0.15}(1+3\times0.9)^{0.2}(1+3\times0.2)^{0.25}(1+3\times0.35)^{0.18}\right]+2\left[(1-0.45)^{0.15}(1-0.9)^{0.2}(1-0.2)^{0.25}(1-0.35)^{0.18}\right]},\\[2ex] \dfrac{4\left[(0.24)^{0.15}(0.03)^{0.2}(0.25)^{0.25}(0.64)^{0.18}\right]}{\left[(1+3\times(1-0.24))^{0.15}(1+3\times(1-0.03))^{0.2}(1+3\times(1-0.25))^{0.25}(1+3\times(1-0.64))^{0.18}\right]-2\left[(0.24)^{0.15}(0.03)^{0.2}(0.25)^{0.25}(0.64)^{0.18}\right]},\\[2ex] \dfrac{3\times\left[(0.25)^{0.15}\times(0.07)^{0.2}\times(0.49)^{0.25}\times(0.01)^{0.18}\right]}{\left[(1+3\times(1-0.25))^{0.15}(1+3\times(1-0.07))^{0.2}(1+3\times(1-0.49))^{0.25}(1+3\times(1-0.01))^{0.18}\right]-2\left[(0.25)^{0.15}(0.07)^{0.2}(0.49)^{0.25}(0.01)^{0.18}\right]}\end{array}\right)$$

=(0.5153,0.1583,0.1679). By similar way $\hat{\alpha}_2, \hat{\alpha}_3$ $\hat{\alpha}_4$, $\hat{\alpha}_5$, $\hat{\alpha}_6$ *and* $\hat{\alpha}_7$ are obtained as follows: $\hat{\alpha}_2$= (0.4108, 0.5328, 0.0247), $\hat{\alpha}_3$ = (0.5922, , 0.1175, 0.2894), $\hat{\alpha}_4$ = (0.3511, 0.2458, 0.1681), $\hat{\alpha}_5$= (0.4882,0.1249,0.2344), $\hat{\alpha}_6$ =(0.3542, 0.2349, 0.1067) and $\hat{\alpha}_7$ = (0.2486, 0.1354, 0.0158).

Step 3. By using the equation above 13, the score values $\hat{S}(\hat{\alpha}_a), (a = 1,2,3,4,5,7)$ of the overall PFIs $\hat{\alpha}_a$, $(a = 1,2,3,4,5,7)$ are obtained as follows

$\hat{S}(\hat{\alpha}_1)$ = 0.5153 – 0.1679 = 0.3474. Similarly, $\hat{S}(\hat{\alpha}_2)$ = 0.3861, $\hat{S}(\hat{\alpha}_3)$ = 0.3028, $\hat{S}(\hat{\alpha}_4)$ = 0.1830, $\hat{S}(\hat{\alpha}_5)$ = 0.2538, $\hat{S}(\hat{\alpha}_6)$ = 0.2457 *and* $\hat{S}(\hat{\alpha}_7)$ = 0.2300.

Step 4. The ranking order of φ_a $(a = 1,2,3,4,5,6,7)$ in accordance with the value of the score functions $\hat{S}(\hat{\alpha}_a), (a = 1,2,3,4,5,7)$ of the overall PFIs is as follows: $\varphi_2 > \varphi_1 > \varphi_3 > \varphi_5 > \varphi_6 > \varphi_7 > \varphi_4$.

Step 5. the first diagnosis will be focused on φ_2.

Step 6. Stop.

In comparisons with other existing methods (manual methods), the ranking order of alternatives is slightly different but the optimum alternative is more desirable and focuses on this method. Thus, the proposed method is stable and can be applied by any professional that has the machine that gives first aid diagnosis.

3. Conclusion

Our system has better advantages over the traditional approach of analyzing psychopathological conditions. The result is efficient in detecting the areas where the diagnosis should be focused. It thus proves to be a better tool that will aid physicians in medical decision process.

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Fuzzy Hypersoft Sets and Its Weightage Operator for Decision Making

Somen Debnath *

Department of Mathematics, Umakanta Academy, Agartala-799001, Tripura, India; somen008@rediffmail.com.

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Abstract

Hypersoft set is an extension of the soft set where there is more than one set of attributes occur and it is very much helpful in multi-criteria group decision making problem. In a hypersoft set, the function F is a multi-argument function. In this paper, we have used the notion of Fuzzy Hypersoft Set (FHSS), which is a combination of fuzzy set and hypersoft set. In earlier research works the concept of Fuzzy Soft Set (FSS) was introduced and it was applied successfully in various fields. The FHSS theory gives more flexibility as compared to FSS to tackle the parameterized problems of uncertainty. To overcome the issue where FSS failed to explain uncertainty and incompleteness there is a dire need for another environment which is known as FHSS. It works well when there is more complexity involved in the parametric data i.e the data that involves vague concepts. This work includes some basic set-theoretic operations on FHSSs and for the reliability and the authenticity of these operations, we have shown its application with the help of a suitable example. This example shows that how FHSS theory plays its role to solve real decision-making problems.

Keywords: Fuzzy set, Soft set, Hypersoft set, Decision making.

1 | Introduction

Uncertainty is a part and parcel of our daily life activities and it exists in various forms. The classical set-theoretic approach could not find any way to deal with incomplete information i.e the information which is blurred. For the scientific computation of vague data, we need a powerful tool that gives us a precise idea about objects so that we get insight into the objects and classify them into different groups. Finally, fuzzy set theory was introduced by Zadeh [24] in 1965 for dealing with uncertain, incomplete, indeterministic information in a systematic way. After the introduction of fuzzy set theory, it has been used successfully in various fields such as engineering, social science, computer science, control theory, game theory, pattern recognition, logic, etc. In the fuzzy set, every object has some membership value and it is called the degree of membership and each membership value belongs to the unit closed interval [0, 1]. So, a fuzzy set is an extension of a crisp set where there are only two choices (0 or 1) to denote the membership of an object i.e

Corresponding Author: somen008@rediffmail.com
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we choose 1 for belongingness and 0 for non-belongingness of an object. Compared to classical set, by using fuzzy set theory we are enabled to extend the range of domain under the fuzzy environment where each object is a fuzzy word or fuzzy sentence or fuzzy axiom, etc. It gives a general formula to model vague or uncertain or indeterministic or incomplete concepts lucidly. But we know that every theory has its limitations so does fuzzy theory. Using fuzzy theory, we only determine the degree of membership of an object but, there is no scope of non-membership degree. We have experienced the co-existence of two opposite concepts like agreement-disagreement, truth-falsity, success-failure, yes-no, attraction-repulsion in a real-life scenario to make a balance. Like this, there is a demand for the coexistence of membership value and non-membership value. To realize the importance of non-membership value along with membership value another set-theoretical notion known as Intuitionistic Fuzzy Set (IFS) was introduced by Atanassov [5] in 1986. In IFS, every object has two values i.e membership value and non-membership value, and their sum range from 0 to 1. For the need of the hour, fuzzy set theory has been applied successfully to develop new theories, propositions, axioms, etc. Some of them are given in [6], [10], [11], [12], [25], [26].

Due to the more complex like uncertainty in data the fuzzy set and its variants are not sufficient for mathematical modeling. It is due to the parameters involved in an attribute. To handle parametric data comprehensively we need another tool to solve the issue. This creates the invention of the Soft Set (SS). Soft set theory was introduced by Molodtsov [16] in 1999. It gives more rigidity to model the vague concept in a parametric way. It is the more general framework as compared to the fuzzy set and its variants. The soft set has been progressed more rapidly and applied in different fields with great success. Some of the novel's work on soft set theory given in [3], [4], [7], [8], [14], [15]. In 2001, a new concept known as Fuzzy Soft Set (FSS) was introduced by Maji et al. [13]. FSS is a combination of Fuzzy Set (FS) and SS. Some applications and extensions of FSS are given in [9], [17], [21], [22].

Recently, Smarandache [20] generalize the concept of the soft set to the hypersoft set, where the function F is transformed into a multi-attribute function. The main motivation of using Hypersoft Set (HSS) is that when the attributes are more than one and further bisected, the SS environment cannot be applied to handle such types of cases. So, there is a worth need to define a new approach to solve these. Afterward, Saeed et al. [19] studied the fundamentals of hypersoft set theory, Abbas et al. [1] defined the basic operations on hypersoft sets, Yolcu and Ozturk [23] introduced fuzzy hypersoft set and its application in decision-making, Ajay and Charisma [2] defined neutrosophic hypersoft topological spaces, aggregate operators of neutrosophic hypersoft set studied in [18], Extension of TOPSIS method under intuitionistic fuzzy hypersoft set environment is discussed in [27], some fundamental operations on interval-valued neutrosophic hypersoft set are discussed in [28].

In this work, we have used the notion of the Fuzzy Hypersoft Set (FHSS), which is an amalgamation of the FS and HSS. Afterward, we define different set-theoretic operations on them, and then there is an attempt to use this concept effectively in multi-criteria decision-making problems using weightage aggregate operator.

2 | Preliminaries

This section includes some basic definitions with examples that are relevant for subsequent discussions.

Definition 1. [24] and [26]. Let X be a non-empty set. Then a fuzzy set A, defined on X, is a set having the form $A=\{(x,\mu_A(x)):x\in X\}$, where the function $\mu_A:X\to[0,1]$ is called the membership function and $\mu_A(x)$ is called the degree of membership of each element $x\in X$.

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Definition 2. [16]. Let U be an initial universe and E be a set of parameters. Let $P(U)$ denotes the power set of U, and $A \subseteq E$. Then the pair (F, A) is called a soft set over U, where F is a mapping given by $F: A \rightarrow P(U)$.

Definition 3. [13]. Let U be an initial universe and E be a set of parameters. Let I^U be the set of all fuzzy subsets of U, and $A \subseteq E$. Then the pair (F, A) is called a fuzzy soft set over U, where F is a mapping given by $F: A \rightarrow I^U$.

Definition 4. [20]. Let ξ be the set of the universe and $P(\xi)$ denotes the power set of ξ. Consider $l^1, l^2, \ldots\ldots, l^n$, for $n \geq 1$, be n well-defined attributes, whose corresponding values are respectively the set $L^1, L^2, \ldots\ldots, L^n$ with $L^i \cap L^j = \varnothing$, for $i \neq j$ and $i, j \in \{1, 2, \ldots\ldots, n\}$, then the pair $(F, L^1 \times L^2 \times \ldots\ldots \times L^n)$ is said to be hypersoft set over ξ, where $F: L^1 \times L^2 \times \ldots\ldots \times L^n \rightarrow P(\xi)$.

Example 1. Let $U = \{c_1, c_2, c_3, c_4\}$ be the set of the universe of cars under consideration and $A = \{c_1, c_3\} \subset U$. We consider the attributes to be $x_1 =$ size, $x_2 =$ color, $x_3 =$ cost price (in a dollar), $x_4 =$ mileage,and $x_5 =$ model, and their respective values are given by Size= $X_1 =$ {small, medium, big}; color $= X_2 =$ {white, black, red}; cost price (in dollar)= $X_3 =$ {1000,1050,1080}; model= $X_4 =$ {honda amaze, tata tigor, ford figo}.

Let the function be: $F: X^1 \times X^2 \times X^3 \times X^4 \rightarrow P(U)$. In respect of A, one has assumed that $F =$ ({small, white, 1000, Honda amaze}, {small, white, 1000, tata tigor}, {small, white, 1000, ford figo}, {small, white, 1050, Honda amaze}, {small, white, 1050, tata tigor}, {small, white, 1050, ford figo}, {small, white, 1080, Honda amaze}, {small, white, 1080, tata tigor}, {small, white, 1080, ford figo}, {small, blue, 1000, Honda amaze}, {small, blue, 1000, tata tigor}, {small, blue, 1000, ford figo}, {small, blue, 1050, Honda amaze}, {small, blue, 1050, tata tigor}, {small, blue, 1050, ford figo}, {small, blue, 1080, Honda amaze}, {small, blue, 1080, tata tigor}, {small, blue, 1080, ford figo}, {small, red, 1000, Honda amaze}, {small, red, 1000, tata tigor}, {small, red, 1000, ford figo}, {small, red, 1050, Honda amaze}, {small, red, 1050, tata tigor}, {small, red, 1050, ford figo}, {small, red, 1080, Honda amaze}, {small, red, 1080, tata tigor}, {small, red, 1080, ford figo}, {medium, white, 1000, Honda amaze}, {medium, white, 1000, tata tigor}, {medium, white, 1000, ford figo}, {medium, white, 1050, Honda amaze}, {medium, white, 1050, tata tigor}, {medium, white, 1050, ford figo}, {medium, white, 1080, Honda amaze}, {medium, white, 1080, tata tigor}, {medium, white, 1080, ford figo}, {medium, blue, 1000, Honda amaze}, {mediuml, blue, 1000, tata tigor}, {medium, blue, 1000, ford figo}, {medium, blue, 1050, Honda amaze}, {medium, blue, 1050, tata tigor}, {medium, blue, 1050, ford figo}, {medium, blue, 1080, Honda amaze}, {medium, blue, 1080, tata tigor}, {medium, blue, 1080, ford figo}, {medium, red, 1000, Honda amaze}, {medium, red, 1000, tata tigor}, {medium, red, 1000, ford figo}, {medium, red, 1050, Honda amaze}, {medium, red, 1050, tata tigor}, {medium, red, 1050, ford figo}, {medium, red, 1080, Honda amaze}, {medium, red, 1080, tata tigor}, {medium, red, 1080, ford figo}, {big,white, 1000, Honda amaze}, {big, white, 1000, tata tigor}, {big, white, 1000, ford figo}, {big, white, 1050, Honda amaze}, {big, white, 1050, tata tigor}, {big,white, 1050, ford figo}, {big,white, 1080, Honda amaze}, {big, white, 1080, tata tigor}, {big, white, 1080, ford figo}, {big, blue, 1000, Honda amaze}, {big, blue, 1000, tata tigor}, {big, blue, 1000, ford figo}, {big, blue, 1050, Honda amaze}, {big, blue, 1050, tata tigor}, {big, blue, 1050, ford figo}, {big, blue, 1080, Honda amaze}, {big, blue, 1080, tata tigor}, {big, blue, 1080, ford figo}, {big, red, 1000, Honda amaze}, {big, red, 1000, tata tigor}, {big, red, 1000, ford figo}, {big, red, 1050, Honda amaze}, {big, red, 1050, tata tigor}, {big, red, 1050, ford figo}, {big, red, 1080, Honda amaze}, {big, red, 1080, tata tigor}, {big, red, 1080, ford figo})= $\{c_1, c_3\}$.

Thus, there are 81 possible hypersoft sets to describe $\{c_1, c_3\}$.

Definition 5. [23]. Let ς be the universe of discourse and $P(\varsigma)$ be the power set of ς. Suppose $l^1, l^2, \ldots\ldots, l^n$, for $n \geq 1$, be n well-defined attributes, whose corresponding values are respectively the set $L^1, L^2, \ldots\ldots, L^n$ with $L^i \cap L^j = \varnothing$, for $i \neq j$ and $i, j \in \{1, 2, \ldots\ldots, n\}$, and $L^1 \times L^2 \times \ldots\ldots \times L^n = S$, then the pair (F, S) is said to be the FHSS over ς, where $F : L^1 \times L^2 \times \ldots\ldots \times L^n \rightarrow P(\varsigma)$ and $\Gamma_S = F\left(L^1 \times L^2 \times \ldots\ldots \times L^n\right) = \left\{\left\langle x, \mu\left(F(S)\right)\right\rangle : x \in \varsigma\right\}$, where μ is the membership function which determines the value of the degree of belongingness and $\mu : \varsigma \rightarrow [0,1]$.

Example 2. Let us consider an example where we have proposed a data set that is suitable for selecting a plot of land by the decision-makers. Suppose ς be the set of decision-makers to decide the best plot. We consider $\varsigma = \left\{d^1, d^2, d^3, d^4, d^5\right\}$ and $A = \left\{d^1, d^3, d^5\right\} \subset \varsigma$.

Now we consider the sets of attributes as P^1 = plot size (in sq. ft), P^2 = plot location, P^3 = cost of the plot (in dollars), and P^4 = landmark surrounding of a plot. Their corresponding values are given as

$P^1 = \{2000, 1745, 1100, 900, 1245\}$, P^2 = {Agartala, Lucknow, Amritsar, Greater Noida, Hooghly},

$P^3 = \{4135, 3812, 3907, 2547\}$ and P^4 = {shopping mall, Railway Station, Airport, Multi specialist Hospital, Highway}.

Therefore, $F : P^1 \times P^2 \times P^3 \times P^4 \rightarrow P(\varsigma)$.

We consider the following tables to assign their membership values:

Table 1. Decision making fuzzy values for the size of the plot.

(Plot size in sq. ft)	d^1 d^5	d^2	d^3	d^4	
2000	0.6	0.5	0.7	0.4	0.8
1745	0.6	0.9	0.1	0.5	0.5
1100	0.6	0.7	0.4	0.8	0.5
900	0.3	0.2	0.5	0.6	0.2
1245	0.5	0.7	0.8	0.6	0.8

Table 2. Decision making fuzzy values for the location of the plot.

(Plot location)	d^1	d^2	d^3	d^4	d^5
Agartala	0.8	0.9	0.7	0.8	0.9
Lucknow	0.7	0.8	0.6	0.7	0.6
Amritsar	0.6	0.8	0.5	0.6	0.8
Greater Noida	0.4	0.3	0.2	0.3	0.2
Hooghly	0.8	0.6	0.5	0.7	0.9

Table 3. Decision making fuzzy values for the cost of the plot.

(Plot cost in sq. ft)	d^1	d^2	d^3	d^4	d^5
4135	0.5	0.4	0.8	0.5	0.7
3812	0.8	0.6	0.8	0.6	0.8
3907	0.8	0.4	0.8	0.6	0.9
2547	0.6	0.6	0.5	0.7	0.6

Table 4. Decision making fuzzy values for the landmark surrounding of a plot.

(Landmark surrounding of a plot)	d^1	d^2	d^3	d^4	d^5
Shopping Mall	0.7	0.6	0.8	0.5	0.8
Railway Station	0.6	0.4	0.5	0.8	0.7
Airport	0.5	0.7	0.6	0.8	0.7
Multispecialist Hospital	0.3	0.7	0.5	0.4	0.7
Highway	0.4	0.7	0.6	0.6	0.8

Then for the set $A=\{d^1,d^3,d^5\}$, we define the fuzzy hypersoft in the following way:

$$F=\{1100,\text{ Agartala, }3812,\text{ shopping mall}\}=\left\{\begin{array}{l}\langle d^1,(1100,0.6),(Agartala,0.8),(3812,0.8),(Shopping\ Mall,0.7)\rangle,\\ \langle d^3,(1100,0.4),(Agartala,0.7),(3812,0.8),(Shopping\ Mall,0.8)\rangle,\\ \langle d^5,(1100,0.5),(Agartala,0.9),(3812,0.8),(Shopping\ Mall,0.8)\rangle\end{array}\right\}$$

Similarly, we can construct $5\times5\times4\times5=500$ such fuzzy hypersoft sets for the set A as per the attribute values are concerned. To find these 500 sets manually is a very tedious job but with the blessings of Data Science the computing and the storage process are very easy to practice and we use it for various practical purposes. So, with an aid of hypersoft set, there is a lot of choices made by the decision-makers among which we fix with one choice that is suitable for use in all perspective. Such type of facility is not available if we use the fuzzy soft set. So, the concept of a fuzzy hypersoft set gives us a scope to enhance our critical thinking systematically. Also, by using it we redefine or extend the earlier concepts by introducing various properties on fuzzy hypersoft sets and all these properties are more generalized as compared to the other existing set theories.

It is to be noted that the set of all fuzzy hypersoft sets over ς is denoted by $FHSS(\varsigma)$.

3 | Different Types of Fuzzy Hypersoft Sets and Their Properties

Definition 6. Let $\Gamma_S\in FHSS(\varsigma)$, where $L^1\times L^2\times.......\times L^n=S$. If $\forall x\in\varsigma$, $S=\varnothing$ then Γ_S is called an FHS-null set and it is denoted by $\Gamma_\varnothing$.

Definition 7. Let $\Gamma_S\in FHSS(\varsigma)$, where $L^1\times L^2\times.......\times L^n=S$. If S is a crisp set and $\forall x\in\varsigma$, $\Gamma_S=\varsigma$ then Γ_S is called the FHS-universal set and it is denoted by Γ_U.

Definition 8. Let $\Gamma_S,\Gamma_T\in FHSS(\varsigma)$. Then Γ_S is said to be an FHS-subset of Γ_T i.e $\Gamma_S\subseteq\Gamma_T$ iff $S\subseteq T$, and $\mu(F(S))\le\mu(F(T))$.

Proposition 1. For $\Gamma_S,\Gamma_T,\Gamma_R\in FHSS(\varsigma)$, we have the following results:

1) $\Gamma_S\subseteq\Gamma_T$ and $\Gamma_T\subseteq\Gamma_S\Rightarrow\Gamma_S=\Gamma_T$.
2) $\Gamma_S\subseteq\Gamma_T$ and $\Gamma_T\subseteq\Gamma_R\Rightarrow\Gamma_S\subseteq\Gamma_R$.
3) $\Gamma_\varnothing\subseteq\Gamma_S$ and $\Gamma_S\subseteq\Gamma_U$

Definition 9. Let $\Gamma_S\in FHSS(\varsigma)$. Then the complement of Γ_S is denoted by $(\Gamma_S)^c$ and it is defined as $(\Gamma_S)^c=\{\langle x,1-\mu(F(S))\rangle : x\in\varsigma\}$.

Definition 10. Let $\Gamma_S, \Gamma_T \in FHSS(\varsigma)$. Then their union is denoted by $\Gamma_S \cup \Gamma_T$ and is defined by

$$\Gamma_S \cup \Gamma_T = \begin{Bmatrix} \langle x, \max(\mu(F(S)), \mu(F(T)))\rangle : \forall x \in S \wedge T \\ \langle x, \mu(F(S))\rangle, \text{ if } x \in S \\ \langle x, \mu(F(T))\rangle, \text{if } x \in T \end{Bmatrix}$$

Definition 11. Let $\Gamma_S, \Gamma_T \in FHSS(\varsigma)$. Then their union is denoted by $\Gamma_S \cap \Gamma_T$ and is defined by

$$\Gamma_S \cap \Gamma_T = \begin{Bmatrix} \langle x, \min(\mu(F(S)), \mu(F(T)))\rangle : \forall x \in S \wedge T \\ \langle x, \mu(F(S))\rangle \text{ if } x \in S \\ \langle x, \mu(F(T))\rangle \text{ if } x \in T \end{Bmatrix}$$

.

Proposition 2. We have the following propositions that are based on FHS-complementary set:

1) $\left(\Gamma^c{}_S\right)^c = \Gamma_S$.
2) $(\Gamma_\varnothing)^c = \varsigma$ (in case of crisp set).
3) (iii) $(\Gamma_S \cup \Gamma_T)^c = \Gamma^c{}_S \cap \Gamma^c{}_T$ and $(\Gamma_S \cap \Gamma_T)^c = \Gamma^c{}_S \cup \Gamma^c{}_T$ (De Morgan Laws).
4) (iv) $\Gamma_S \cup (\Gamma_S \cap \Gamma_T) = \Gamma_S$ and $\Gamma_S \cap (\Gamma_S \cup \Gamma_T) = \Gamma_S$ (Absorption Laws).

Proposition 3. For $\Gamma_S, \Gamma_T, \Gamma_R \in FHSS(\varsigma)$, we have the following results:

1) $\Gamma_S \cup \Gamma_T = \Gamma_T \cup \Gamma_S$ and $\Gamma_S \cap \Gamma_T = \Gamma_T \cap \Gamma_S$ (Idempotent Laws).
2) $\Gamma_S \cup (\Gamma_T \cup \Gamma_R) = (\Gamma_S \cup \Gamma_T) \cup \Gamma_R$ and $\Gamma_S \cap (\Gamma_T \cap \Gamma_R) = (\Gamma_S \cap \Gamma_T) \cap \Gamma_R$ (Associative Laws).
3) $\Gamma_S \cup (\Gamma_T \cap \Gamma_R) = (\Gamma_S \cup \Gamma_T) \cap (\Gamma_S \cup \Gamma_R)$ and $\Gamma_S \cap (\Gamma_T \cup \Gamma_R) = (\Gamma_S \cap \Gamma_T) \cup (\Gamma_S \cap \Gamma_R)$ (Distributive Laws).

4 | Weightage of Fuzzy Hypersoft Set in Decision Making

In example 2, we have determined only one fuzzy hypersoft set though there are 500 different choices available. As we know that computer science is an integral part of Mathematics, then by using suitable software or programming we can easily enumerate all these 500 different sets within few minutes as it is quite difficult to do manually. That's why here we avoid such lengthy calculations. But we need an operator by which we can select the best alternative for the customer among these 500 cases. Without any proper decision-making of all the items we do not say, this or that is the best choice or there may be multiple choices for the customer. For that purpose, we need to find the weightage of each fuzzy hypersoft set by using the following operator.

$$\Omega = \frac{1}{|FHSS(\varsigma)|} \sum_{i=1}^{n} \max\left(\mu\left(d^i\right)\right) |\Xi_{FV}|.$$

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Where $|FHSS(\varsigma)|$=number of fuzzy hypersoft sets over ς, $max\left(\mu\left(d^i\right)\right)$=maximum fuzzy membership value concerning for to the decision-makers, and $|\Xi_{FV}|$=number of fuzzy attribute values corresponding to the set of attributes.

We have to determine Ω for every each possible fuzzy hypersoft set of the given problem. Afterward, we choose the best choice for the client which has the maximum weightage. In case of a tie, there may be multiple choices for a client. In such a case, he or she may choose any one of the suitable choices.

5 | Application

There is a huge scope of the use of hypersoft set in different areas such as forecasting, business management, traffic control, similarity measures, neural networking, data science, sociology, etc.

6 | Conclusions

Here we have used the novel concept known as a hypersoft set which was introduced by F. Smarandache in 2018. By combining the fuzzy set and hypersoft set a new theory called fuzzy hypersoft set is introduced in [23]. The Fuzzy hypersoft set is a more generalized form of fuzzy set, soft set, fuzzy soft set, etc. In this article, an attempt has been made to study fuzzy hypersoft set and their kinds and discuss some set-theoretic operations and propositions on them. With the help of a valid and concrete example, it also shown that how this concept is more effective in multi-criteria decision-making problems compared to other existing theories. In future, it has a wide range of application in the fields of topology, game theory, computer science, neural network, decision-making etc.

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Conflict of interest

Author declared no conflict of interest regarding the publication of the paper.

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On the Prediction of COVID-19 Time Series: An Intuitionistic Fuzzy Logic Approach

Imo Eyo [1],*, Jeremiah Eyoh[2], Uduak Umoh[3]

[1] Department of Computer Science, University of Uyo, Uyo, Akwa Ibom State, Nigeria; imoheyoh@uniuyo.edu.ng;
[2] School of Electrical, Electronics and Systems Engineering, AVRRC, Loughborough University, Loughborough, UK; j.e.eyoh@lboro.ac.uk
[3] Department of Computer Science, University of Uyo, Uyo, Akwa Ibom State, Nigeria; j.e.eyoh@lboro.ac.uk.

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Abstract

This paper presents a time series analysis of a novel coronavirus, COVID-19, discovered in China in December 2019 using intuitionistic fuzzy logic system with neural network learning capability. Fuzzy logic systems are known to be universal approximation tools that can estimate a nonlinear function as closely as possible to the actual values. The main idea in this study is to use intuitionistic fuzzy logic system that enables hesitation and has membership and non-membership functions that are optimized to predict COVID-19 outbreak cases. Intuitionistic fuzzy logic systems are known to provide good results with improved prediction accuracy and are excellent tools for uncertainty modelling. The hesitation-enabled fuzzy logic system is evaluated using COVID-19 pandemic cases for Nigeria, being part of the COVID-19 data for African countries obtained from Kaggle data repository. The hesitation-enabled fuzzy logic model is compared with the classical fuzzy logic system and artificial neural network and shown to offer improved performance in terms of root mean squared error, mean absolute error and mean absolute percentage error. Intuitionistic fuzzy logic system however incurs a setback in terms of the high computing time compared to the classical fuzzy logic system.

Keywords: Pandemic; Coronavirus; Hesitation index; Gradient descent backpropagation algorithm.

1 | Introduction

The emergence of COVID-19 (coronavirus disease 2019) in December, 2019 has shaken and brought the whole world for some weeks/months of lockdown due to extreme loss of lives [1]. COVID-19 is a severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2) [2]. SARS-CoV-2 is highly contagious and has presented a major global health threat [3]. Reports from the World Health Organization (WHO) indicate that the world records 10,357,662 confirmed cases of COVID-19 with 508,055 deaths until July 1st, 2020, 6:08 pm CEST [1]. It is therefore incumbent

Corresponding Author: imoheyoh@uniuyo.edu.ng
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on the Governments, private organizations and individuals to take necessary steps to combat this global pandemic.

COVID-19 is known to emanate from Wuhan, China with rapid spread to surrounding countries such as Korea, Thailand and Japan, and from there to Europe, America, and later to Africa [4]. The most affected countries in Africa are South Africa, Egypt, Nigeria and Ghana respectively. Here, the focus is on Nigeria, the most populous country in Africa with population of over 200 million people, which contributes to about 2.64% of the world population. The Nigeria Center for Disease Control (NCDC) recorded the first case of coronavirus in Nigeria on February 28th, 2020 and the first death on March 23rd, 2020. Currently, Nigeria is experiencing a steady but exponential growth in the confirmed cases of COVID-19 across the country. As reported on July 1st, 2020, 6:08 pm CEST, the number of confirmed cases of COVID-19 in Nigeria have risen to 25,694 with 590 deaths, making Nigeria the third most affected country in Africa.

To curb the propagation of COVID-19, cities in Nigeria and other African countries have been locked down for weeks until recently, with a gradual easing of the lockdown. In Nigeria, the NCDC provided strict preventive measures, such as washing of hands thoroughly and frequently with soap under running water, quarantining symptomatic persons and isolating infected persons, promoting social distancing and wearing of facemask especially in public places for self-protection. Other ways to curtail the spread of COVID-19 included restrictions on public gathering, travelling (banned interstate travelling) except for essential workers, closing of schools, and offices. Exclusions were however granted to grocery stores, pharmacies, public markets, and other stores selling food and essential products. There was a complete lockdown in major cities like Abuja - the Federal Capital Territory, Lagos and Ogun states and later in all the states of the federation. Despite these preventive efforts, the COVID-19 cases in Nigeria are gradually increasing and steps must be taken to accurately predict the COVID-19 pandemic. In this study, the use of intuitionistic fuzzy set to predict COVID-19 pandemic cases in Nigeria is proposed. The objective of this study is to ascertain the performance of hesitation-enabled intuitionistic fuzzy set on the prediction of COVD-19 pandemic and to compare its performance with the traditional fuzzy set and artificial neural network. To the best knowledge of the authors, this is the first study that predict COVID-19 pandemic cases using intuitionistic fuzzy logic system that utilizes intuitionistic fuzzy sets with optimized parameters.

The rest of the paper is organized as follows: Section 2 has the literature review while Section 3 discusses the methodology adopted to solve the COVID-19 prediction problem. Performance evaluation is presented in Section 4 while conclusion is drawn in Section 5.

2 | Literature Review

Many studies have been conducted for the prediction of COVID-19 pandemic all over the world. For instance, Bastos and Cajueiro [5] forecasted the early evolution of COVID-19 in Brazil using two modified versions of the Susceptible-Infected Recovered (SIR) epidemic model. The data for the forecast was collected from February 25th, 2020 to March 30th, 2020 and the results from their short-term forecast were in tandem with the collected data. In the same vein, Pandey et al. [3] proposed the use of Susceptible, Exposed, Infectious, recovered (SEIR) and regression models to predict the COVID-19 confirmed cases in India. The two models were found to effectively analyze and predict COVID-19 disease in India.
However, according to [6], COVID-19 has some characteristic features that are quite distinct from other existing infectious diseases. These features make it difficult to apply SIR and SEIR models directly to COVID-19 data. Therefore, Zhao and Chen [6] proposed the Susceptible, Un-quarantined infected, quarantined infected, Confirmed infected (SUQC) model. The authors noted that the SUQC is able to characterize the dynamics of COVID-19 and provided accurate prediction on the test data better than other epidemic models. Patra et al. [2] presented long short-term memory (LSTM) networks for the prediction of COVID-19 data in India, USA, Argentina and Brazil. The authors adopted 90% of

the data for the countries under study as training data while 10% was used as test data. The results of LSTM were compared with convolutionary neural network and nonlinear autoregressive time series and found to outperform both in terms of the nine-error metrics adopted for the study. Roosa et al. [7] proposed a COVID-19 epidemic forecast in China that operates in real time from February 5th to February 24th, 2020 using the sub-epidemic model. Their proposed sub-epidemic model was compared with generalized logistic growth model and Richards**'s** model and found to provide a good forecast in terms of the mean squared error. Anastassopoulou et al. [8] adopted the Susceptible-Infectious-Recovered-Dead (SIRD) epidemic model in the prediction of COVID-19 outbreak in Hubei, China. The data was collected from a publicly available database from January 11th to February 10th, 2020 and analysis of results show that the evolution of the COVID-19 pandemic was within the bounds of the forecast.

As a global pandemic, prediction of COVID-19 outbreak has been conducted for several other countries including Canada [9], Saudi Arabia [10], Italy, Spain and France [11], Brazil [5] and [12], Hungary [13], Italy [14] and [15] Malaysia [16], Japan [17], Iran [18] and more. Petropoulos and Makridakis [19] presented a statistical forecast of COVID-19 confirmed cases using robust time series. The COVID-19 data collected consisted of cumulative daily figures aggregated globally and captured three cases namely: confirmed cases, deaths and recoveries. The data was obtained from John Hopkins University of daily cumulative cases from January 22, 2020 to March 11, 2020. Simple time series from the family of exponential smoothing was adopted and shown to produce good forecast. According to [8], the official data provided for COVID-19 is highly uncertain and according to [20], fuzzy logic is a concept that connotes uncertainty and can adequately model the same. This calls for the utilization of fuzzy logic tools that can adequately cope with uncertainty in the COVID-19 data. To achieve this, many researchers have adopted and integrated fuzzy logic in the prediction models. For instance, Patra et al. [2] has proposed the use of multiple ensemble neural network models with fuzzy response aggregation for the prediction of the COVID-19 time series in Mexico. The main essence of the integration of fuzzy response aggregation was to manage the uncertainty occasioned by the individual networks, thus leading to lower uncertainty. The proposed approach was shown to provide good estimation when compared with the actual values and other prediction models. Al-Qaness et al. [21] proposed the use of adaptive neuro-fuzzy inference system (ANFIS) optimized with flower pollination algorithm (FPA) and salp swarm algorithm (SSA) to estimate and forecast the confirmed cases of COVID-19 in China. According to the authors, the performance of FPASSA-ANFIS in terms of the predicted values of the confirmed COVID-19 is very high and outperforms other models in terms of RMSE, MAE, MAPE, root mean squared relative error (RMSRE), coefficient of determination (R2) and computing time. Other studies such as Dhiman and Sharma [22] proposed a fuzzy logic inference for identification and prevention of COVID-19. Fong et al. [23] proposed the use of hybridized deep learning and fuzzy rule induction for the analysis of COVID-19 outbreak. Fatima et al. [24] presented Internet of Things (IoT) which enabled smart monitoring of COVID-19 with associated fuzzy inference system. Verma et al. [25] applied arima and fuzzy time series models while Van Tinh [26] utilized fuzzy time series model in combination with particle swarm optimization for COVID-19 prediction.

All the previous works make use of classical type-1 fuzzy logic systems (FLSs) for the prediction of COVID-19 outbreak with the aim of modelling uncertainty in the data. The classical FLS can only handle uncertainty by defining membership functions with the assumption that every non-membership function is complementary to the membership function. This assumption may not always be correct, as there may be some hesitations surrounding membership and non-membership functions of an element to a set. Kumar [27] put it clearly that the hesitation occurring in the membership degrees cannot be integrated in a fuzzy set theory.

To this end, the use of intuitionistic fuzzy set (IFS) introduced in 1999 by Atanassov [28] for the prediction of COVID-19 pandemic cases in Nigeria is proposed in this study. An IFS is a fuzzy set format that is defined using both Membership Functions (MFs) and Non-Membership Functions (NMFs), which are independent from each other, with extra parameter known as the hesitation degree (index).

Literature is replete with studies involving IFS such as prediction and time series forecasting [20,] [29], [30], [31], [32], [33], [34], and [35], multi-criteria decision making [36], control [37], temporal fault trees analysis [38], system failure probability analysis [39], data envelopment analysis [40], estimating correlation coefficient between IFSs with hesitation index [41] and more. The motivation behind this study is that by using IFLS to analyze COVID-19 pandemic data, more information will be captured and uncertainty efficiently handled. Moreover, the IFSs enable hesitation which is preponderant in humanlanguage representation, thus providing more adequate and concordant solutions to the real-world (COVID-19) problem than its classical counterpart in terms of providing better advantages in handling vagueness and uncertainty. For instance, Khatibi and Montazer [42] adopted the classical FS and IFS in medical diagnosis for the detection of intestinal bacteria that causes typhoid fever and dysentery by using different similarity measures of FS and IFS. According to the authors, although both Fs and IFS are strong tools for uncertainty modeling, analyses in Khatibi and Montazer [42] show that IFS provided more accurate results than the classical FS. According to [39], the IFS, defined with separate membership and non-membership degrees has much wider range of applicability than traditional fuzzy set theory. In otherwords, as Rahman et al. [43] state, IFS stands as an important tool in managing with imprecision.

Themaincontribution of this paper, therefore is the adoption of parameter optimized intuitionistic fuzzy logicsystem (IFLS) which captures some level of hesitation in the MFs and NMFs. The inclusion of the intuitionistic fuzzy index in the COVID-19 pandemic prediction provides flexibility and tends to agree with human reasoning and information representation better. The integration of hesitation index component in the modelling of uncertainty in COVID-19 data is an interesting direction followed in this analysis. To aid comparison, the traditional type-1 fuzzy logic system is also constructed and evaluated using the COVID-19 pandemic cases.

3 | Methodology

In this section, the traditional type-1 FS is briefly discussed. The IFS, IFLS and parameter update for IFLS MF and NMF are derived. The datasets used for evaluating the proposed model are also described.

3.1 | Fuzzy Sets (Classical Fuzzy Set and Intuitionistic Fuzzy Set)

The classical FS introduced by Zadeth [44] is an extended version of the traditional binary set. Unlike binary set with 0 or 1 membership value, FS membership falls in a closed interval [0, 1].

Definition 1. A classical FS is characterised by only the MF, $\mu_A(x)$ which specifies the degree of belonging of an element to a set, i.e., $A=\{(x,\mu_A(x))|\forall x\in X\}$.

Any system that adopts one or more type-1 FS is known as type-1 FLS. This assumption may not be applied to every situation as they may be some hesitation from the expert in determining the degree of membership of an element to a set. This extra parameter may not simply be classified as MF or NMF. This calls for another kind of FS known as the IFS which provides some flexibility in terms of the hesitation degree. Thus, the IFS is an extended version of the traditional type-1 FS.

Definition 2. [28]. An IFS is defined by both MF, $\mu_{A^*}(x)\in[0,1]$ and NMF, $\nu_{A^*}(x)\in[0,1]$ such that $0\le\mu_{A^*}(x)+\nu_{A^*}(x)\le 1$.

An IFS has an additional parameter called the hesitation index, $\pi(x)$ such that $\pi_{A^*}(x)=1-(\mu_{A^*}(x)+\nu_{A^*}(x))$. Obviously, when $(\mu_{A^*}(x)+\nu_{A^*}(x))=1$, a traditional type-1 FS is obtained. Radhika and Parvathi [45], Hájek and Olej [46] and Mahapatra and Roy [47] have formulated ways of defining the MF and NMF of an IFS. In this work, the MF and NMF (see *Fig. 1*) are defined following the approach in [45] using Gaussian

$$\mu_{ik}(x_i)=\exp\left(-\frac{(x_i-c_{ik})^2}{2\sigma^2}\right)-\pi, \tag{1}$$

$$\nu_{ik}(x_i)=1-\exp\left(-\frac{(x_i-c_{ik})^2}{2\sigma^2}\right), \tag{2}$$

function as follows:

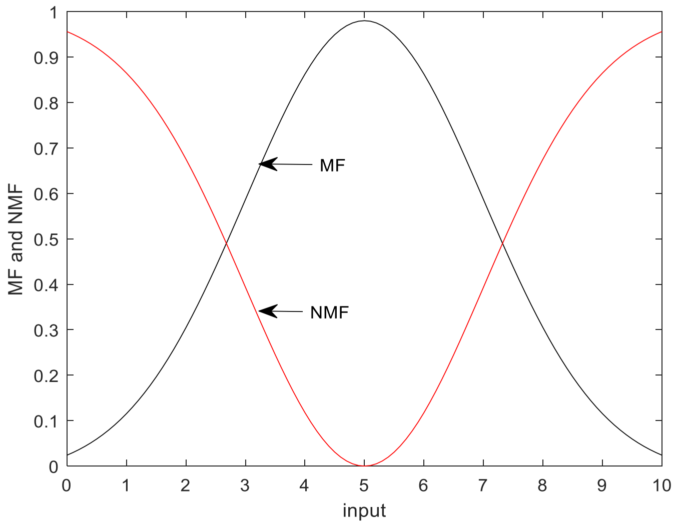

Fig. 1. Intuitionistic fuzzy set [48].

where $\mu(x)$ is the MF and $\nu(x)$ is the NMF, x is the input, σ and c are the standard deviation and center of the IFS respectively while $\pi\in[0,1]$ is the hesitation index, otherwise known as intuitionistic fuzzy index. For all the experiments, the hesitation index was chosen as 0.1. A system that uses IFS in either its antecedent and/or consequent part(s) is known as IFLS.

3.2 | Intuitionistic Fuzzy Logic System

An IFLS (see *Fig. 2*) possesses the same functionalities as the traditional FLS namely: the fuzzifier, rule based, inference engine and defuzzifier. The only exception is that the different parts are intuitionistic based (with hesitation indices).

3.2.1 | Fuzzification

Similar to the classical FS, fuzzification involves converting crisp inputs into MFs and NMFs which are fed into the intuitionistic inference engine and translated into intuitionistic fuzzy output set. Here, singleton fuzzification is assumed. That is, $\mu_{A^*}(x)=\begin{cases}1/1 & if\, x=x' \\ 1/0 & if\, x\neq x'\end{cases}$.

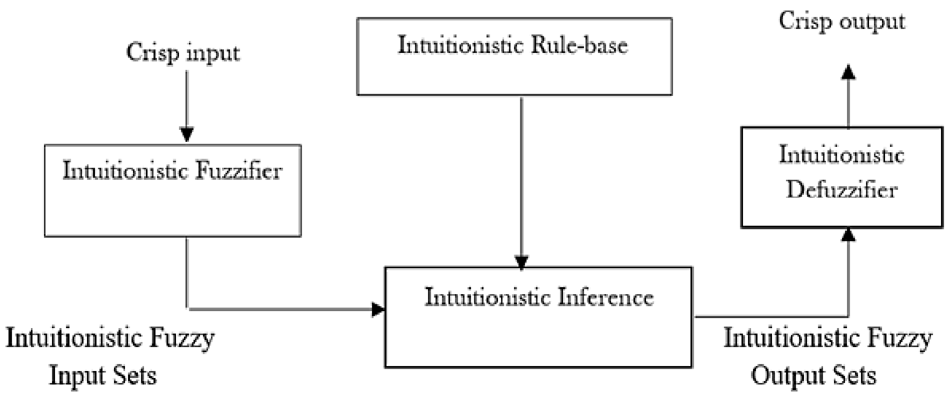

Fig. 2. Intuitionistic fuzzy logic system [48].

3.2.2 | Rules

The generic rule structure of IFLSs is as below

$$R_k : \text{if } x_i \text{ is } A^*_{ik} \text{ and ... and } x_n \text{ is } A^*_{nk} \text{ then } y_k = \sum_{i=1}^{n} w_{ik} x_i + b_k . \quad (3)$$

Which can be reformulated for MF and NMF as follows:

$$R^\mu_k : \text{if } x_i \text{ is } A^{*\mu}_{ik} \text{ and ... and } x_n \text{ is } A^{*\mu}_{nk} \text{ then } y^\mu_k = \sum_{i=1}^{n} w^\mu_{ik} x_i + b^\mu_k , \quad (4)$$

$$R^\nu_k : \text{if } x_i \text{ is } A^{*\nu}_{ik} \text{ and ... and } x_n \text{ is } A^{*\nu}_{nk} \text{ then } y^\nu_k = \sum_{i=1}^{n} w^\nu_{ik} x_i + b^\nu_k . \quad (5)$$

Where x's represent the inputs, y_k's are rule's outputs, A^*'s are IFSs, w represents the weight and b, the bias. Once the intuitionistic fuzzy rules are established, the IFLS can be seen as a mapping from inputs to outputs with the mapping quantitatively represented as $y = f(x)$.

3.2.3 | Inference

This study adopts a Takagi-Sugeno-Kang (TSK) inference. Here the IF-THEN rules in the rule base are combined into a mapping from an input linguistic vector to an output variable, y. For TSK inference, the output is a linear combination of the inputs.

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3.2.4 | Defuzzification

In order to obtain a crisp value for the output of a FLS, the defuzzification procedure is often employed. This work adopts the defuzzification method proposed in [49] where the outputs of each subsystems (MF and NMF) are computed and then combined to produce the final output. Hájek and Olej [49] defined the final output of a TSK-type IFLS as follows:

$$y = (1-\beta)\sum_{k=1}^{M} \tilde{f}^{\mu} y_k^{\mu} + \beta \sum_{k=1}^{M} \tilde{f}^{\nu} y_k^{\nu}. \tag{6}$$

Where:

$$\tilde{f}^{\mu} = \frac{f_k^{\mu}}{\sum_{k=1}^{M} f_k^{\mu}}, \tag{7}$$

and

$$\tilde{f}^{\nu} = \frac{f_k^{\nu}}{\sum_{k=1}^{M} f_k^{\nu}}. \tag{8}$$

And $\tilde{f}^{\mu}$ and $\tilde{f}^{\nu}$ are normalized firing magnitude for MFs and NMFs respectively while β is the user defined parameter which controls how much MF and NMF support the final output. The MF alone contributes to the final output if β is 0 and NMF alone contributes to the final output if β is 1. However, when $0 \le \beta \le 1$, the output is formed by both MFs and NMFs.

3.3 | Parameter Update

The problem under investigation is an optimization problem and requires adjustment of the parameters of the MF and NMF of the IFLS. The popular Gradient Descent (GD) back propagation algorithm is used to optimize these parameters. The cost function for a single output is defined as

$$E = \frac{1}{2}\left(y^a - y\right)^2. \tag{9}$$

Where y^a is the actual output and y is the predicted output. The parameters of IFLS to be updated include the center, c, standard deviation, σ, weight, w, bias, b and β.

For GD optimization, any generic parameter, θ, can be updated as follows

$$\theta_{ik}(t+1) = \theta_{ik}(t) - \gamma \frac{\delta E}{\delta \theta_{ik}}. \tag{10}$$

Where γ is the learning rate that controls the learning process and must be chosen carefully to avoid instability or slow learning. The parameters of the consequent parts include the weights (w) and biases (b) and updated as follows:

$$w_{ik}(t+1)=w_{ik}(t)-\gamma\frac{\delta E}{\delta w_{ik}}, \tag{11}$$

and

$$b_{ik}(t+1)=b_{ik}(t)-\gamma\frac{\delta E}{\delta b_{ik}}. \tag{12}$$

Respectively, the derivative with respect to the weight is computed as in *Eq. (13)* and *Eq. (14)*.

$$\frac{\delta E}{\delta w_{ik}}=\frac{\delta E}{\delta y}\frac{\delta y}{\delta y_k}\frac{\delta y_k}{\delta w_{ik}}=\sum_{k=1}^{M}\frac{\delta E}{\delta y}\left[\frac{\delta y}{\delta y_k^{\mu}}\frac{\delta y_k^{\mu}}{\delta w_{ik}^{\mu}}+\frac{\delta y}{\delta y_k^{\nu}}\frac{\delta y_k^{\nu}}{\delta w_{ik}^{\nu}}\right]. \tag{13}$$

$$\left(y(t)-y^{a}(t)\right)\left[(1-\beta)\left(\frac{f_k^{\mu}}{\sum_{k=1}^{M}f_k}\right)+\beta\left(\frac{f_k^{\nu}}{\sum_{k=1}^{M}f_k}\right)\right]x_i. \tag{14}$$

While the derivative with respect to the bias is as in *Eq. (15)* and *Eq. (16)* respectively.

$$\frac{\delta E}{\delta b_{ik}}=\frac{\delta E}{\delta y}\frac{\delta y}{\delta y_k}\frac{\delta y_k}{\delta b_{ik}}=\sum_{k=1}^{M}\frac{\delta E}{\delta y}\left[\frac{\delta y}{\delta y_k^{\mu}}\frac{\delta y_k^{\mu}}{\delta b_{ik}^{\mu}}+\frac{\delta y}{\delta y_k^{\nu}}\frac{\delta y_k^{\nu}}{\delta b_{ik}^{\nu}}\right], \tag{15}$$

$$\left(y(t)-y^{a}(t)\right)\left[(1-\beta)\left(\frac{f_k^{\mu}}{\sum_{k=1}^{M}f_k}\right)+\beta\left(\frac{f_k^{\nu}}{\sum_{k=1}^{M}f_k}\right)\right]x_i. \tag{16}$$

The Gaussian function is adopted to construct the MF.

$$\mu_{ik}(x_i)=\exp\left(-\frac{(x_i-c_{ik})^2}{2\sigma^2}\right). \tag{17}$$

The Gaussian function in *Eq. (17)* is modified as in *Eq. (18)* and *Eq. (19)* to reflect membership and non-membership functions of IFS respectively.

$$\mu_{ik}(x_i)=\exp\left(-\frac{(x_i-c_{ik})^2}{2\sigma^2}\right)-\pi, \tag{18}$$

$$\nu_{ik}(x_i)=1-\exp\left(-\frac{(x_i-c_{ik})^2}{2\sigma^2}\right). \tag{19}$$

Where π is the intuitionistic fuzzy index defining the hesitation of the expert in specifying MFs and NMFs. The antecedent parameters are the center (c) and standard deviation (σ) which are updated as in *Eq. (20)* and *Eq. (21)* respectively.

$$c_{ik}(t+1)=c_{ik}(t)-\gamma\frac{\delta E}{\delta c_{ik}}, \tag{20}$$

and

$$\sigma_{ik}(t+1)=\sigma_{ik}(t)-\gamma\frac{\delta E}{\delta \sigma_{ik}}. \tag{21}$$

Where the derivative $\frac{\delta E}{\delta c_{ik}}$ in *Eq. (20)* is calculated as follows:

$$\frac{\delta E}{\delta c_{ik}}=\sum_{k=1}^{M}\frac{\delta E}{\delta y}\left[\frac{\delta y}{\delta f_k^{\mu}}\frac{\delta f_k^{\mu}}{\delta \mu_{ik}}\frac{\delta \mu_{ik}}{\delta c_{ik}}+\frac{\delta y}{\delta f_k^{\nu}}\frac{\delta f_k^{\nu}}{\delta \nu_{ik}}\frac{\delta \nu_{ik}}{\delta c_{ik}}\right], \tag{22}$$

and the derivative in *Eq.(21)* is computed as follows:

$$\frac{\delta E}{\delta \sigma_{ik}}=\sum_{k=1}^{M}\frac{\delta E}{\delta y}\left[\frac{\delta y}{\delta f_k^{\mu}}\frac{\delta f_k^{\mu}}{\delta \mu_{ik}}\frac{\delta \mu_{ik}}{\delta \sigma_{ik}}+\frac{\delta y}{\delta f_k^{\nu}}\frac{\delta f_k^{\nu}}{\delta \nu_{ik}}\frac{\delta \nu_{ik}}{\delta \sigma_{ik}}\right]. \tag{23}$$

The parameters of the classical type-1 FS are also updated the same way using the generic GD backpropagation algorithm in *Eq. (10)*. However, for classical FS, only the MF parameters are optimized. Shown in Algorithm 1 is the complete procedure for GD learning of the parameters of IFLS. The same procedure applies to classical type-1 FLS. The IFLS-GD was implemented in MATLAB® 2020.

Algorithm 1: IFLS-GD Learning Procedure

INPUT: training set, centre (*c*), standard deviation (*σ*), weight (*w*), bias (*b*), hesitation index (*π*), user

defined parameter (*β*), learning rate (*γ*)
(1) Set initial training epoch to 1
(2) Set training data to 1
(3) Propagate the training data through the IFLS model.
(4) Using Eq. (11) and (12), tune the consequent parameters of IFLS.
(5) Calculate the output of IFLS using Eq. (6)
(6) Calculate the difference between the actual output and predicted output of IFLS with root mean

squared error (RMSE) as the cost function.
(7) Backpropagate the error and tune the antecedent parameters using Eq. (20) and (21).
(8) Increment the training data by 1. If training data ≤ total number of training samples, go to step 3

else increment training epoch by 1
(9) If maximum epoch is reached END; else,
(10) Go to step 3.
OUTPUT: Prediction error

3.4 | Dataset Description

The Nigeria COVID-19 pandemic cases used in this study are extracted from Kaggle, a publicly available data repository [50] which houses COVID-19 data for all African countries. The dataset was captured from February 15th (as other African countries had confirmed cases from this day, however, the first case in Nigeria was reported on February 28th, 2020) to June 24th, 2020. The dataset contains 5 cases of COVID-19 outbreak in Nigeria namely: daily cases, daily deaths, active cases, total cases and total deaths. In this study, prediction is done for each COVID-19 case in Nigeria using present and past values to predict a one-step future value. According to [51], prediction can be qualitative, explanatory or time series in nature. In this study, each of the COVID-19 case dataset is modelled as time series which involves sequential collections of data over time [20]. The task here is a short-term forecast where a day-ahead prediction is carried out. The time series is represented as:

$$Y(t+1)=f\left[x(t),x(t-1),...,x(t-s+1)\right], \tag{24}$$

Where f is a function representing the model of prediction and s is the input size. For four inputs adopted in this study, the current input and three previous inputs of the time series are utilized giving the input generating vector as $\left[x(t);x(t-1);x(t-2);x(t-3)\right]$ while $Y(t+1)$ represents the output. Whilst the current value of the time series helps to keep an up-to-date measurement of COVID-19 case, the previous values keep track of the trend. Before the analysis, the collected COVID-19 cases data are normalized to a small range between 0 and 1 using the min-max normalization as follows:

$$x_{new}=\frac{x_i-\min(X)}{\max(X)-\min(X)}. \tag{25}$$

Where x is the data instant of input variable, X, $min(X)$ and $max(X)$ represent the minimum and maximum values of variable, X. To obtain the actual predicted (non-normalized) values, the normalized predicted outputs are converted back to the original scale using *Eq. (26)*.

$$Xnew=IFLS_{predictedOutput}*(\max(trainingData)-\min(trainingData))+\min(trainingData). \tag{26}$$

Shown in *Fig. 3* is the structure of IFLS with two inputs and three MFs and NMFs.

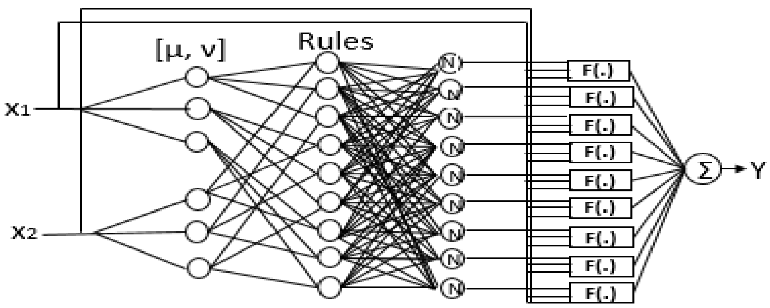

Fig. 3. Architecture of IFLS [48].

The time series are split into 70% training and 30% testing instances respectively. For an objective evaluation of the cases, the experiments are conducted 10 times and the average results are computed.

The epoch was kept at 100 and the learning rate chosen as 0.1. The normalized training data are then propagated into the IFLS as shown in *Fig. 3*. As shown in *Fig. 3*, the inputs are first passed forward into the fuzzifier to obtain the MF (μ) and NMF (ν) of IFLS, the rules are generated, and depending on the firing strength, the outputs are obtained. *Table 1* is a snapshot of the different COVID-19 cases from the first day (28th February) of confirmed case in Nigeria up to March 31st, 2020. *Fig. 4* shows the trend of the COVID-19 outbreak in Nigeria for the period of February 15th, 2020 to June 24th, 2020.

Table 1. Snapshot of COVID-19 cases in Nigeria from 28th February to 31st March, 2020.

Date	Daily cases	Daily Deaths	Active Cases	Total Cases	Total Deaths
Feb-28	1	0	1	1	0
Feb-29	0	0	1	1	0
Mar-01	0	0	1	1	0
Mar-02	0	0	1	1	0
Mar-03	0	0	1	1	0
Mar-04	0	0	1	1	0
Mar-05	0	0	1	1	0
Mar-06	0	0	1	1	0
Mar-07	0	0	1	1	0
Mar-08	0	0	1	1	0
Mar-09	1	0	2	2	0
Mar-10	0	0	2	2	0
Mar-11	0	0	2	2	0
Mar-12	0	0	2	2	0
Mar-13	0	0	2	2	0
Mar-14	0	0	2	2	0
Mar-15	0	0	1	2	0
Mar-16	0	0	1	2	0
Mar-17	1	0	2	3	0
Mar-18	5	0	7	8	0
Mar-19	4	0	11	12	0
Mar-20	0	0	11	12	0
Mar-21	10	0	21	22	0
Mar-22	8	0	28	30	0
Mar-23	10	1	37	40	1
Mar-24	4	0	41	44	1
Mar-25	7	0	48	51	1
Mar-26	14	0	61	65	1
Mar-27	5	0	66	70	1
Mar-28	27	0	93	97	1
Mar-29	14	0	107	111	1
Mar-30	20	1	121	131	2
Mar-31	4	0	125	135	2

Source: https://www.kaggle.com

As shown in the figure, COVID-19 total and active cases in Nigeria started to escalate from April 18th, 2020.

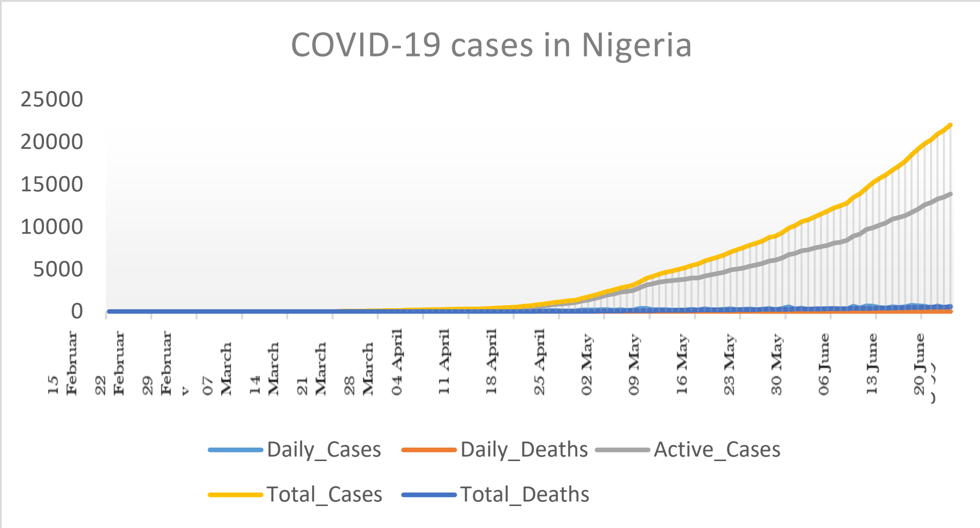

Fig. 4. Chart showing the trend of COVID-19 cases in Nigeria from June 15th to 24th, 2020.

4 | Performance Evaluation

The metrics employed to evaluate the performance of the models are the root mean squared error (RMSE), Mean Absolute Error (MAE) and Mean Absolute Percentage Error (MAPE).

$$\mathrm{RMSE} = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(y^{a} - y\right)^{2}}. \tag{27}$$

$$\mathrm{MAE} = \frac{1}{T}\sum_{t=1}^{T}\left|y^{a} - y\right|. \tag{28}$$

$$\mathrm{MAPE} = \frac{1}{T}\sum_{t=1}^{T}\frac{\left|y^{a} - y\right|}{y^{a}} * 100. \tag{29}$$

Where y^a is the real output and y is the predicted output of the different prediction models.

Shown in *Fig. 5* to *Fig. 9* are the prediction performances of IFLS and FLS. As shown in most of the figures, the predicted outputs of IFLS tend to follow the actual outputs as closely as possible compared to the classical FLS. In particular, *Fig. 7* shows the classical FLS performing poorly in the prediction of the active COVID-19 pandemic cases. This is an indication that the classical FLS may not be a very robust model that can provide more accurate estimates in the face of uncertainty in most cases. However, a closer look at *Fig. 5* shows that the traditional FLS aligns closely with the actual values more than the IFLS. This is also revealed in *Table 2* with FLS yielding lower absolute average prediction error than IFLS. Shown in *Fig. 10* is a single instance of the adaptation of the user defined parameter, β, of IFLS.

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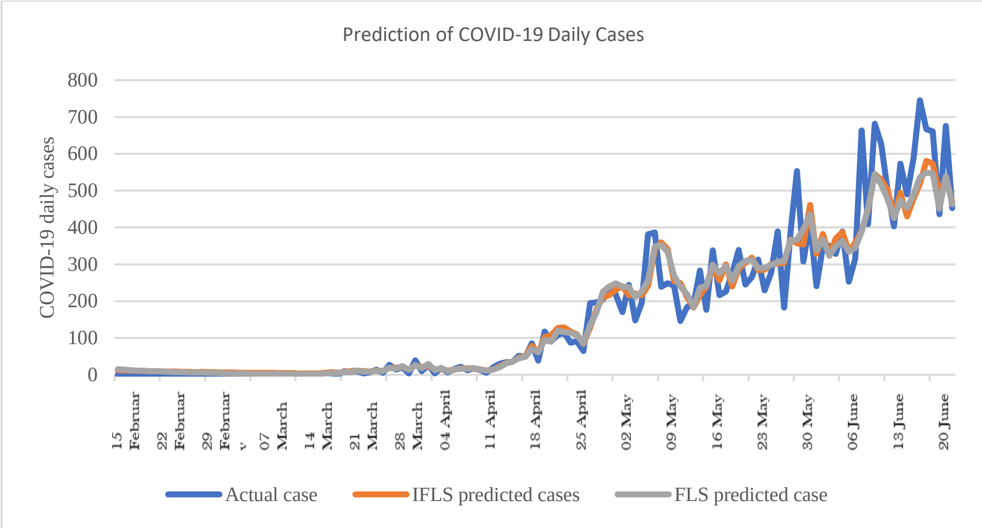

Fig. 5. Comparison of actual and predicted daily cases of COVID-19 in Nigeria using IFLS and FLS.

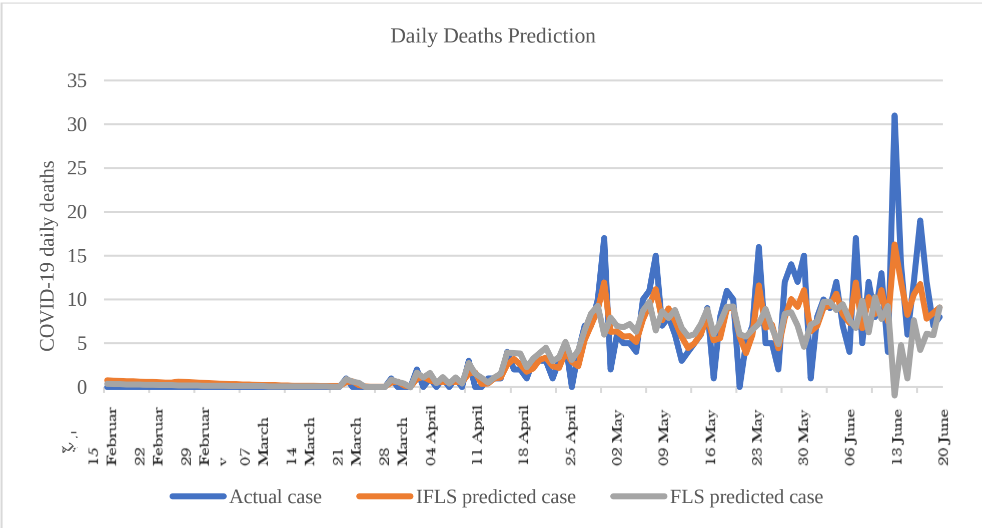

Fig. 6. Actual and predicted daily deaths from COVID-19.

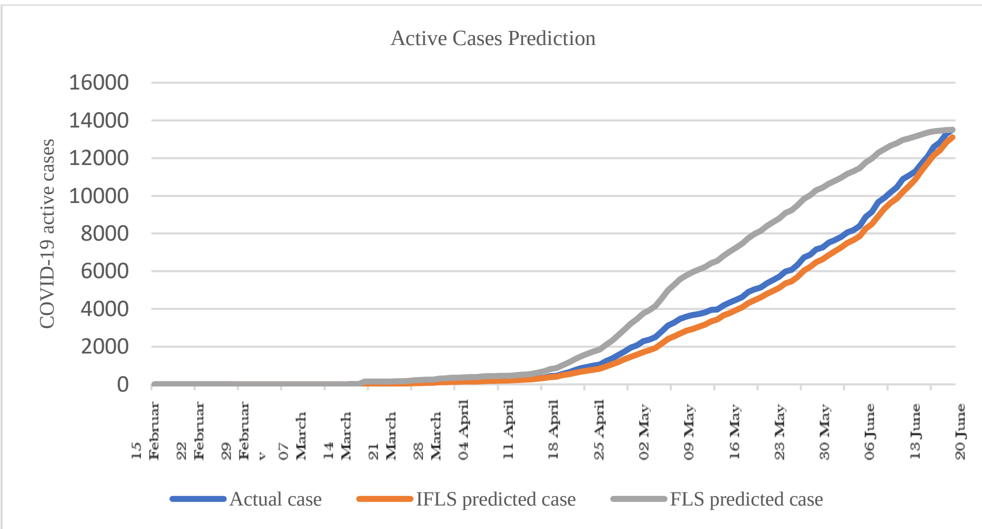

Fig. 7. Actual and predicted active cases of COVID-19.

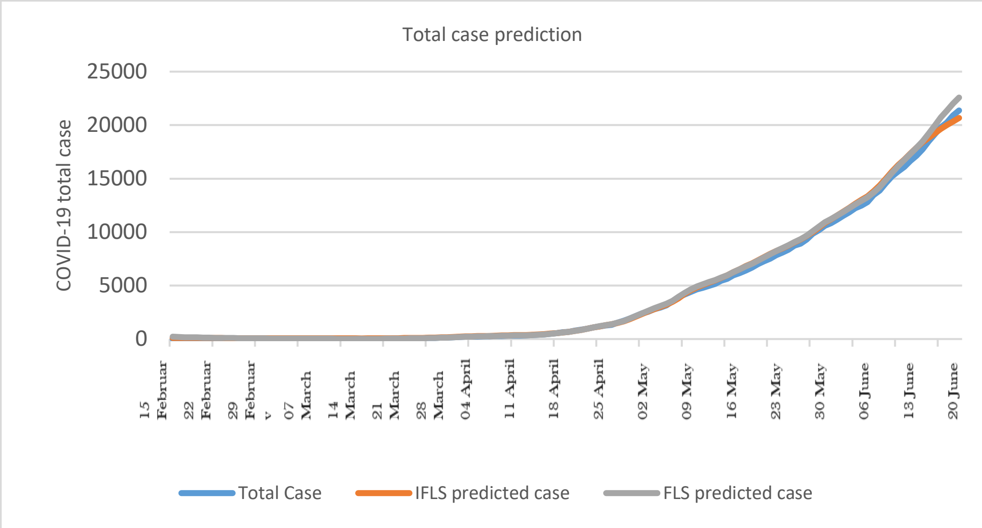

Fig. 8. Graph showing actual and predicted outputs of COVID-19 total cases.

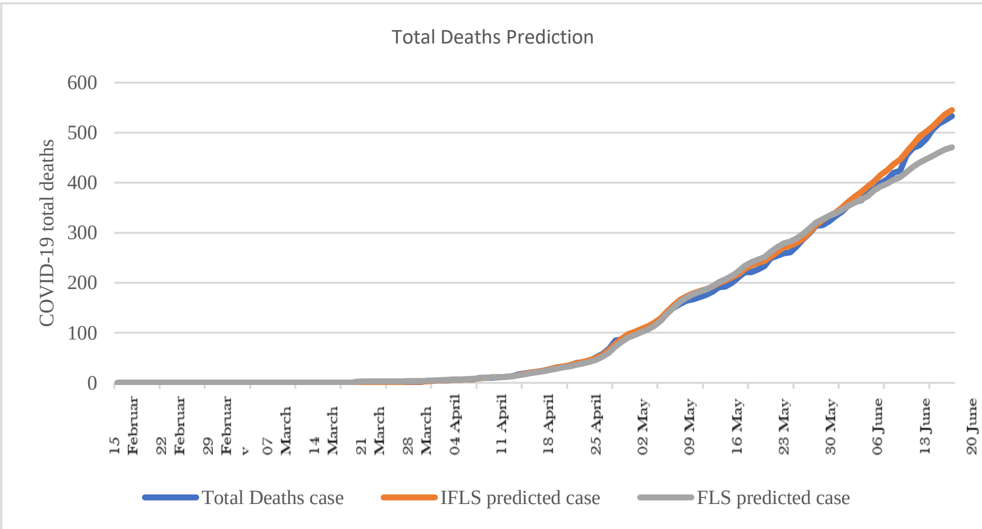

Fig. 9. Graph showing actual and predicted values of total deaths from COVID-19.

Tables 2-8 show the comparison of the actual and predicted numbers of the different cases of COVID-19 pandemic in Nigeria using classical FLS and IFLS with their corresponding absolute prediction errors. Interestingly, IFLS performs better overall as shown in the actual and predicted number of cases and the lower average absolute prediction errors (see *Tables 3-6*).

Table 2. Comparison of actual and predicted COVID-19 daily cases using IFLS and traditional type-1 FLS.

Day	Actual case	IFLS predicted case	FLS predicted case	IFLS predicted error	FLS predicted error
14-Jun	403	431.7257	425.6794	28.7257	22.6794
15-Jun	573	493.8253	473.382	79.1747	99.618
16-Jun	490	429.6738	453.1943	60.3262	36.8057
17-Jun	587	477.6494	489.0567	109.3506	97.9433
18-Jun	745	518.0091	536.6998	226.9909	208.3002
19-Jun	667	580.5806	548.7646	86.4194	118.2354
20-Jun	661	572.9636	546.8389	88.0364	114.1611
21-Jun	436	488.4975	450.5051	52.4975	14.5051
22-Jun	675	537.1518	538.3174	137.8482	136.6826
23-Jun	452	468.4112	462.9581	16.4112	10.9581
	Average error			88.57808	85.98889

Table 3. Comparison of actual and predicted COVID-19 daily deaths using IFLS and traditional Type-1 FLS.

Day	Actual case	IFLS predicted case	FLS predicted case	IFLS predicted error	FLS predicted error
14-Jun	13	11.0428	7.78	1.9572	5.22
15-Jun	4	7.4865	9.2303	3.4865	5.2303
16-Jun	31	16.2782	-0.9636	14.7218	31.9636
17-Jun	14	11.9934	4.7803	2.0066	9.2197
18-Jun	6	8.2303	0.9676	2.2303	5.0324
19-Jun	12	10.5317	7.637	1.4683	4.363
20-Jun	19	11.7382	4.243	7.2618	14.757
21-Jun	12	7.814	6.0818	4.186	5.9182
22-Jun	7	8.4769	5.9029	1.4769	1.0971
23-Jun	8	9.1006	9.0768	1.1006	1.0768
	Average error			3.9896	8.38781

Table 4. Comparison of actual and predicted COVID-19 active cases using IFLS and traditional Type-1 FLS.

Day	Actual case	IFLS predicted case	FLS predicted case	IFLS predicted error	FLS predicted error
14-Jun	10445	9861.75	12792.46	583.25	2347.462
15-Jun	10885	10212.75	12955.82	672.25	2070.819
16-Jun	11070	10512.45	13049.24	557.55	1979.235
17-Jun	11299	10872.9	13145.2	426.1	1846.199
18-Jun	11698	11329.2	13256.55	368.8	1558.55
19-Jun	12079	11738.25	13346.67	340.75	1267.674
20-Jun	12584	12143.25	13425.99	440.75	841.9872
21-Jun	12847	12410.55	13461.73	436.45	614.7342
22-Jun	13285	12825	13492.13	460	207.131
23-Jun	13500	13101.75	13500	398.25	0
	Average error			468.415	1273.379

Table 5. Comparison of Actual and Predicted COVID-19 Total Cases using IFLS and Traditional Type-1 FLS.

Day	Actual case	IFLS predicted case	FLS predicted case	IFLS predicted error	FLS predicted error
14-Jun	16085	16785	16761	700	676
15-Jun	16658	17339	17319	681	661
16-Jun	17148	17894	17872	746	724
17-Jun	17735	18395	18491	660	756
18-Jun	18480	18807	19189	327	709
19-Jun	19147	19231	19943	84	796
20-Jun	19808	19641	20715	167	907
21-Jun	20244	20009	21362	235	1118
22-Jun	20919	20349	22009	570	1090
23-Jun	21371	20671	22575	700	1204
	Average error			487	864.1

Table 6. Comparison of actual and predicted COVID-19 total deaths using IFLS and traditional Type-1 FLS.

Day	Actual case	IFLS predicted case	FLS predicted case	IFLS predicted error	FLS predicted error
14-Jun	420	437.2488	405.9877	17.2488	14.0123
15-Jun	424	446.2144	411.2714	22.2144	12.7286
16-Jun	455	461.3492	421.4614	6.3492	33.5386
17-Jun	469	476.0808	431.2186	7.0808	37.7814
18-Jun	475	492.2251	440.7627	17.2251	34.2373
19-Jun	487	501.7825	446.6526	14.7825	40.3474
20-Jun	506	512.3679	453.2103	6.3679	52.7897
21-Jun	518	525.1286	460.3934	7.1286	57.6066
22-Jun	525	536.7947	466.5433	11.7947	58.4567
23-Jun	533	545.0093	470.7669	12.0093	62.2331
	Average error			12.22013	40.37317

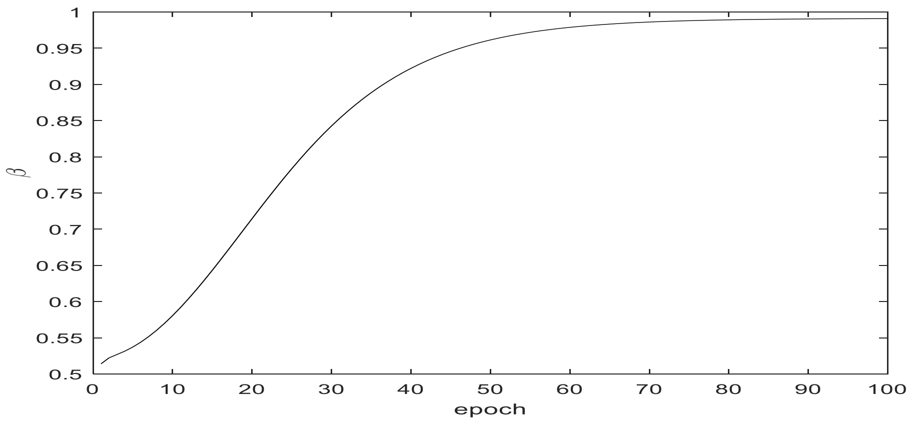

Fig. 10. A scenario showing the adaptation of the user defined parameter, β, of IFLS.

For further comparison, an experiment is conducted to compare the performances of the FLS approaches with ANN, where ANN forms an integral part of these FLSs. The GD-backpropagation is used to learn the parameters of the ANN. However, the number of hidden neurons for the ANN is set to 5 as it provided the smallest errors. Every other computational set-up is the same as those for the FLSs. Shown in *Table 7* are the errors for the different models and for the different cases of COVID-19 in Nigeria. As shown in the table, IFLS with MFs and NMFs together with the hesitation indices exhibits more acceptable performance in terms of RMSE, MAE and MAPE with reduced average absolute errors compared to traditional FLS with only MFs. The IFLS also outperforms the standalone

ANN. The integration of ANN in the FLSs (IFLS and FLS), however, provided a synergistic capability for effective handling of uncertainty than the standalone ANN. In the overall, the FLSs provided better performances than the ANN. The plot of the variations in the RMSE of the different models for different COVID-19 cases are shown in *Fig. 11*. The lower the RMSE, the better the performance.

Table 7. Performance of FLS, IFLS and ANN on cases of COVID-19 based on different performance metrics.

COVID-19 cases	Metrics	FLS	ANN	IFLS
Daily cases	RMSE	106.4437	233.5901	104.6956
	MAE	87.3931	193.9882	87.4217
	MAPE (%)	17.038	32.5901	16.1937
Daily deaths	RMSE	7.4546	7.8208	6.7965
	MAE	4.9938	5.338	4.4439
	MAPE (%)	41.4715	45.6242	43.1968
Active cases	RMSE	1718.404	2870.6	1530.1769
	MAE	1481.424	2614.1	1286.7551
	MAPE (%)	13.4793	21.051	11.3811
Total cases	RMSE	1598.213	2063.1	1505.5709
	MAE	1313.287	1811.5	1254.4038
	MAPE (%)	8.2875	9.8775	7.3991
Total deaths	RMSE	67.192	178.8472	57.4778
	MAE	60.2266	173.0478	45.3706
	MAPE (%)	13.5011	35.4769	10.4515

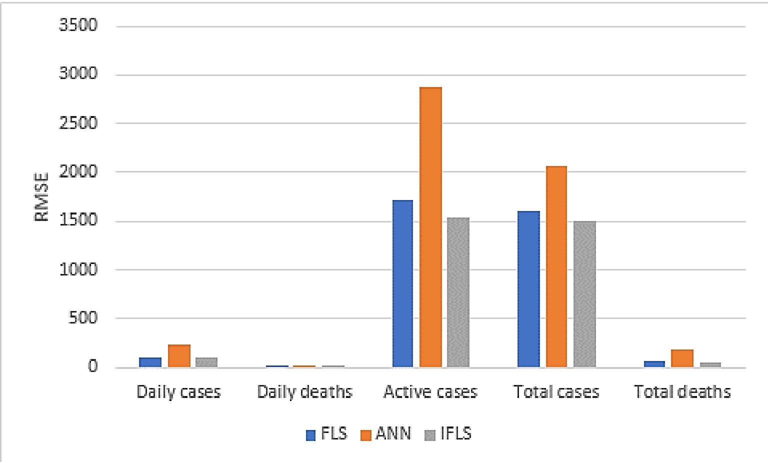

Fig. 11. RMSE for each model and COVID-19 case.

Analysis is also conducted to compare the average running times of the various models in the prediction of COVID-19 cases as depicted in *Table 8*.

Table 8. Comparison of running time of FLS, IFLS and ANN.

Model	Average running time (sec)
FLS	4.38
IFLS	10.33
ANN	13.28

As shown in *Table 8*, classical fuzzy logic system exhibits the lowest computational time compared to IFLS and ANN. This implies that if running time is of essence, then traditional FLS may be a good choice in these problem cases.

5 | Conclusion

In this study, IFLS was applied to analyze the prediction capability using COVID-19 data in Nigeria, the second most affected country with COVID-19 in Africa. To aid comparison, classical type-1 FLS and traditional neural networks were also employed. As shown in the tables, IFLS with MFs and NMFs outperforms the two competing models (FLS and ANN) in four of the COVID-19 cases based on the error metrics with decreasing errors. The presence of NMFs and hesitation indices provides more design degrees of freedom and flexibility for IFLS to handle uncertainty and vagueness well. Moreover, IFLS is an adaptive system, allowing the system to cope with the changing nature of COVI-19 pandemic. Optimizing the parameters of the IFLS helps to enhance prediction and generalization capability of the model. IFLS can therefore stand as a robust model for the prediction of COVID-19 pandemic cases. IFLS however incurs more computational cost than the classical FLS and may not be applicable in situation where running time is paramount. Overall, the FLS models outperform the single neural network model both in terms of accuracy and running time. However, IFLS has MF and NMFs that are precise and may not handle uncertainty well in many situations. Hence, in the future, we intend to use higher order fuzzy logic systems such as classical type-2 FLS with fuzzy MFs and type-2 intuitionistic FLS with fuzzy MFs and NMFs for the analysis of the COVID-19 pandemic cases. These higher order FLSs are expected to efficiently handle uncertainties and minimize their effects on the predicted COVID-19 pandemic cases. A study will also be conducted to include other African countries mostly affected by the COVID-19 pandemic.

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