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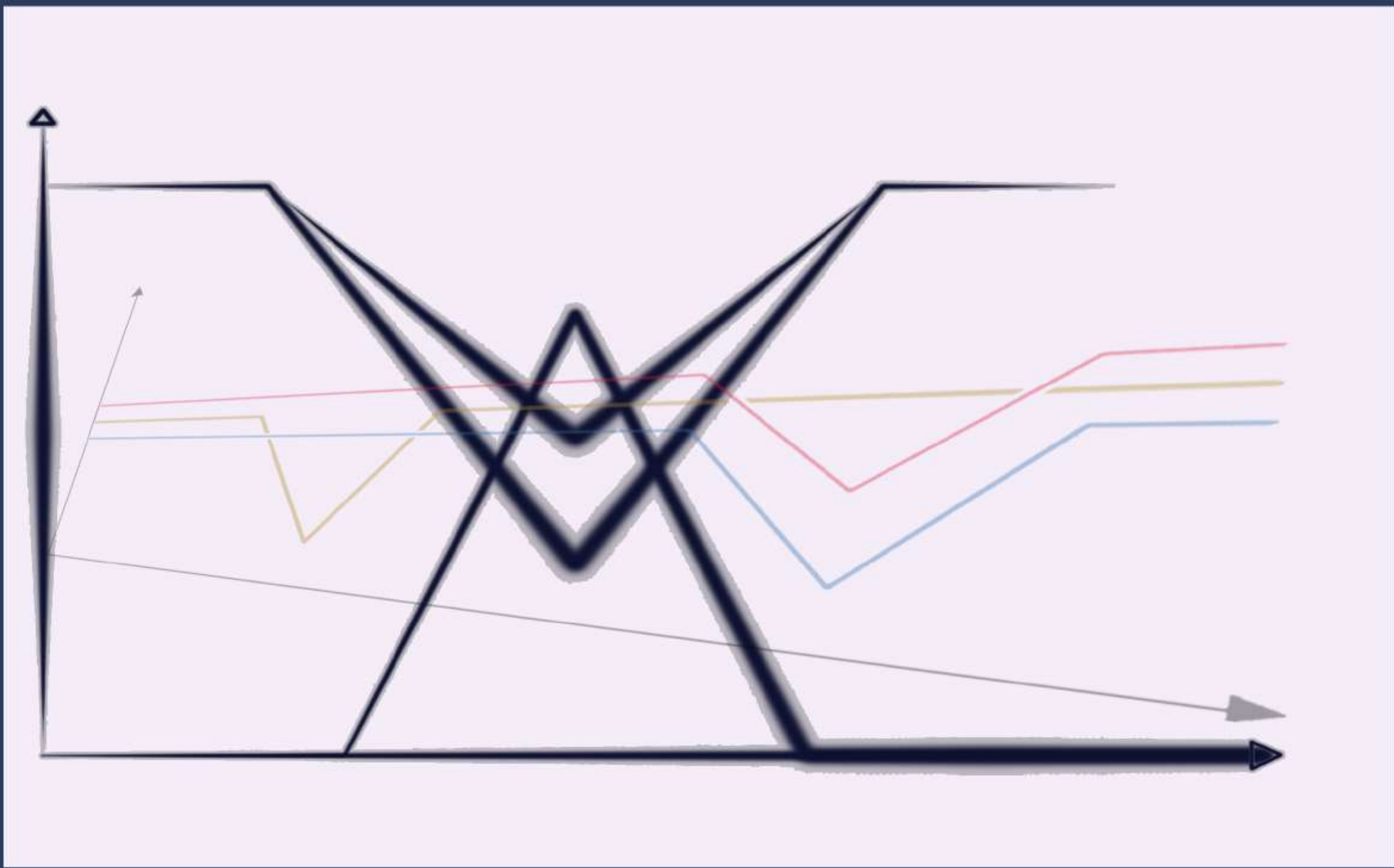

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On Algebraic Aspects of η-Fuzzy Subgroups

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Abstract

In this paper, we define the term " η-fuzzy subgroup" and show that every fuzzy subgroup is a η-fuzzy subgroup. We define some of the algebraic properties of the concept of η-fuzzy cosets. Furthermore, we initiate the study of the η-fuzzy normal subgroup and the quotient group with respect to the η-fuzzy normal subgroup and demonstrate some of their various group theoretical properties.

Keywords: Fuzzy subgroup, η-fuzzy subgroup, η-fuzzy coset, η-fuzzy normal subgroup.

1 | Introduction

Fuzzy sets were first studied by Zadeh in [12], and since there has been an incredible attention in this particular branch of mathematics because of its many applications in fields like engineering and computer science as well as the analysis of social and economic behaviour. Rosenfeld [8] introduced the concept of fuzzy groups on fuzzy sets in and developed number of basic results for fuzzy groups. In fact, the fuzzy subgroups admit many algebraic properties of the groups. For more details, we refer to [9], [10]. Anthony in [2] redefined the concept of fuzzy subgroup. Later, Das [4] modified Zadeh and Rosenfeld's work by defining the level subgroups of a given group. The concept of fuzzy homomorphism between two groups was defined by Chakraborty and Khare [3], they also examined how it affected fuzzy subgroups. Additionally, Ajmal [1] presented the concept of the typical kernel of a group homomorphism in fuzzy subgroups. The most recent research on the use of fuzzy sets in various algebraic structures may be found in [11], [13]-[16]. Gupta and Qi [5] developed the notion of T operators on fuzzy sets. The theory of fuzzy operators plays a key role in various disciplines, specifically in the field of engineering and artificial intelligence. This significant application of fuzzy operators motivates us to familiarize the concept of a fuzzy set based on these operators.

In this paper, a fuzzy set is defined in relation to a N_T-operator. With the help of fuzzy subset, we propose a new version of fuzzy subgroup called it η-fuzzy subgroup and analyse its supplementary

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theory, derive analogues for several fundamental group theoretic results. Using the classical homomorphism, we demonstrate that the homomorphic image (pre-image) of η-fuzzy subgroup is η-fuzzy subgroup. Furthermore, we introduce the concept of η–fuzzy cosets and fuzzy normal subgroup. We also define isomorphism between the quotient group with respect to the normal subgroup $G_\rho\eta$. Since η –fuzzy subgroup is more abstract structure then the fuzzy subgroup and the results in this version are more general than the existing results in the literature. Throughout in this paper, we will refer to $FS(G)$ and $FNS(G)$ as the fuzzy subgroup and fuzzy normal subgroup of a group G, respectively.

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2 | Preliminaries

We review some of these core concepts which are relevant to the rest of our discussion.

Definition 1 ([6]). Let E be a nonempty set. A mapping $\rho: E \longrightarrow [0,1]$ is called a fuzzy subset of E.

Definition 2 ([6]). Let ρ and σ be fuzzy sets of a set E. Their intersection $\rho \cap \sigma$ and union $\rho \cup \sigma$ are fuzzy sets of E defined by

I. $(\rho \cap \sigma)(a_1) = min\{\rho(a_1), \sigma(a_1)\}$ $for\ all$ $a_1 \in E$.
II. $(\rho \cup \sigma)(a_1) = max\{\rho(a_1), \sigma(a_1)\}$ $for\ all$ $a_1 \in E$.

Definition 3 ([6]). Let ρ be a fuzzy set of a set E. For $\gamma \in [0,1]$, the set $\rho_\gamma = \{a_1 : a_1 \in E, \rho(a_1) \geq \gamma\}$ is called level subset of ρ.

Definition 4 ([6]). Let ρ be a fuzzy subset of a group G. Then ρ is called a $FS(G)$ if

I. $\rho(a_1 a_2) \geq min\{\rho(a_1), \rho(a_2)\}$ $for\ all$ $a_1, a_2 \in G$.
II. $\rho(a_1^{-1}) \geq \rho(a_1)$ $for\ all$ $a_1 \in G$.

Lemma 1 ([6]). Let $\rho : G \to [0,1]$ be a $FS(G)$, for all $a_1 \in G$, we have

I. $\rho(e) \geq \rho(a_1)$ $for\ all$ $a_1 \in G$.
II. $\rho(a_1^{-1}) = \rho(a_1)$.

Theorem 1 ([4]). Let ρ be a fuzzy subset of group G then ρ is $FS(G)$ if and only if the level subset ρ_γ, for $\gamma \in [0,1]$, $\rho(e) \geq \gamma$, is subgroup of G, where e is an identity of G.

Definition 5 ([7]). A $FS(G)$ ρ is called a $FNS(G)$ if $\rho(a_1 a_2) = \rho(a_2 a_1)$ $for\ all$ $a_1, a_2 \in G$.

Definition 6 ([7]). Let ρ be a $FS(G)$. For any $a_1 \in G$, define a map $\rho_{a_1} : G \to [0,1]$ as follows:

I. $\rho_{a_1}(g) = \rho(g a_1^{-1})$ $for\ all$ $g \in G$.
II. ρ_{a_1} is called fuzzy coset of G determined by a_1 and ρ.

Definition 7 ([5]). A map T: $[0,1] \times [0,1] \longrightarrow [0,1]$ define by $(a_1, a_2) \longmapsto \min\{a_1, a_2\}$ is T–norm iff $for\ all$ $a_1, a_2, a_3, a_4 \in [0,1]$

I. $T(a_1, a_2) = T(a_2, a_1)$,
II. $T(a_1, T(a_2, a_3)) = T(T(a_1, a_2), a_3)$,
III. $T(a_1, 1) = T(1, a_1) = 1$,
IV. If $a_1 \leq a_3$ and $a_2 \leq a_4$ then $T(a_1, a_2) \leq T(a_3, a_4)$.

3 | η-Fuzzy Subsets and Their Properties

Definition 8. Let N_T be an operator defined as $N_T : [0,1] \times [0,1] \longrightarrow [0,1]$ by

$$N_T(a_1, a_2) = \min\{1 - a_1, 1 - a_2\} \quad \text{for all } a_1,\ a_2 \in [0,1\ ].$$

Infect, N_T admits the properties below, *for all* $a_1,\ a_2, a_3, a_4 \in [0,1\ ]$:

I. $N_T(a_1, a_2) = N_T(a_2, a_1)$,
II. $N_T(a_1, 1) = N_T(1, a_1) = 0$,
III. If $a_1 \le a_3$ and $a_2 \le a_4$, then $N_T(a_1, a_2) \ge N_T(a_3, a_4)$.

The operator N_L is non-associative.

Definition 9. Let $\rho: X \to [0,1]$ be fuzzy subset of X and $\eta \in [0,1]$, the fuzzy subset ρ^η of X (w. r. t fuzzy set $\rho: X \to [0,1]$) denotes the η-fuzzy subset of X and is defined as follows:

$$\rho^\eta(a_1) = \min\left\{1 - \rho(a_1), 1 - \eta\right\} \text{ for all } a_1 \in X.$$

Example 1. Let A = {set of young people} define ρ fuzzy set on A as follows:

$$\rho_A(a_1) = \begin{cases} 1, & \text{if } \ a_1 < 25, \\ \dfrac{40 - a_1}{15}, & \text{if } \ 25 \le a_1 \le 40, \\ 0, & \text{if } \ a_1 > 40. \end{cases}$$

Take $\eta = 0.6$, now for $a_1 = 20$, we have $\rho^\eta{}_A(a_1) = 0$. For $a_2 = 30$, we have $\rho^\eta{}_A(a_2) = 0.333$ and for $a_3 = 45$, we have $\rho^\eta{}_A(a_3) = 0.4$.

Remark 1: It is important to note that one can obtain the negation of classical fuzzy subset $\rho(a_1)$ by choosing the value of $\eta = 0$ in above definition whereas the case become crisp for the choice of $\eta = 1$. These algebraic facts lead to note that the case illustrates the η-fuzzy version with respect to any fuzzy subset for the value of η, when $\eta \in (0,1)$.

Definition 10. Let $\xi: G \to G'$ where G, G' are groups and ρ and σ be η-fuzzy subsets of G and G' respectively. Then $\xi(\rho^\eta)$ and $\xi^{-1}(\sigma^\eta)$ are the image of η-fuzzy subset ρ^η and the inverse image of η-fuzzy subset σ^η respectively, defined as:

I. $\xi(\rho^\eta)(a_2) = \begin{cases} sup\, \rho^\eta(a_1) : a_1 \in \xi^{-1}(a_2), & if\ \xi^{-1}(a_2) \neq \varnothing, \\ 0\ , & if\ \xi^{-1}(a_2) = \varnothing. \end{cases}$
II. $\xi^{-1}(\sigma^\eta)(a_1) = \ \sigma^\eta(\xi(a_1)) \ \ for\ all\ a_1 \in G.$

Example 2. Let $\xi: V_4 \ \to \mathbb{R}$ where $V_4 = \{1,\ a_1,\ a_2,\ a_1a_2\}$ defined as follows:

$\xi(1) = 1,\ \xi(a_1) = 2,\ \xi(a_2) = -2$ and $\xi(a_1a_2) = 4$. Define fuzzy set ρ on V_4 given by $\rho(1) = 1,\ \rho(a_1) = 0.8$, $\rho(a_2) = 0.4$ and $\rho(a_1a_2) = 0.5$ define fuzzy set σ on $\mathbb{R}$ as follows:

$$\sigma(a_1) = \frac{1}{|a_1|}.$$

Take $\eta = 0.3$, so $\xi\left(\rho^\eta(a_2)\right) = \{0,\ 0.2,\ 0.7,\ 0.5\ \}$ and $\xi^{-1}\left(\sigma^\eta(a_1)\right) = \{0,\ 0.5,\ 0.7\}$

Theorem 2.

I. Let ρ and σ be any two fuzzy subsets of a set E then $(\rho\ \cap \sigma)^\eta = \rho^\eta \cap \sigma^\eta$.
II. Let ρ and σ be two fuzzy subsets of a set P and Q respectively and $\xi : P \longrightarrow Q$ be a mapping, then

- $\xi(\rho^\eta) = \left(\xi(\rho)\right)^\eta$.
- $\xi^{-1}(\rho^\eta) = \left(\xi^{-1}(\rho)\right)^\eta$.

Proof: By *Definition 9*, we have

$(\rho \cap \sigma)^\eta(a_1) = \min\left\{1 - (\rho \cap \sigma)(a_1), 1 - \eta\right\}$, where $a_1 \in E$ and $\eta \in [0,1]$.

$= \min\left\{1 - \min\left\{\rho(a_1),\ \sigma(a_1)\right\}, 1 - \eta\right\}$

$= \min\left\{\min\left\{1 - \rho(a_1),\ 1 - \eta\right\},\ \min\{1 - \sigma(a_1), 1 - \eta\}\right\}$

$= \min\{\rho^\eta(a_1), \sigma^\eta(a_1)\} = (\rho^\eta \cap \sigma^\eta)(a_1)$ *for all* $a_1 \in E$.

Consequently, $(\rho^\eta \cap \sigma^\eta)(a_1) = \rho^\eta \cap \sigma^\eta$.

I. $\xi(\rho^\eta)(a_2) = \sup\{\rho^\eta(a_1) : \xi(a_1) = a_2\}$

$= \sup\left\{\min\left\{1 - \rho(a_1), 1 - \eta\right\}\right\}$

$= \min\left\{\sup\left\{1 - \rho(a_1), 1 - \eta\right\}\right\}$

$= \min\{1 - \xi(\rho)(a_2), 1 - \eta\}$

$= \left(\xi(\rho)\right)^\eta$ *for all* $a_2 \in Q$.

Hence, $\xi(\rho^\eta) = \left(\xi(\rho)\right)^\eta$.

II. From *Definition 9*, we have

$$\xi^{-1}(\rho^\eta)(a_1) = (\rho^\eta)\xi(a_1) = \min\left\{1 - \rho\left(\xi(a_1)\right), 1 - \eta\right\} = \min\left\{1 - \xi^{-1}\rho(a_1), 1 - \eta\right\}$$

$$= \left(\xi^{-1}(\rho)\right)^\eta(a_1) \ \textit{for all}\ a_1 \in P.$$

Hence, $\xi^{-1}(\rho^\eta) = \left(\xi^{-1}(\rho)\right)^\eta$.

4 | η-Fuzzy Subgroups

This section deals with the concept of η-$FS(G)$ and η-$FNS(G)$. We prove that every $FS(G)$ $\left(FNS(G)\right)$ is also η-$FS(G)$ $(FNS(G))$ but converse need not to be true. The concept of η-fuzzy coset is defined and discussed deeply. Moreover, applying the idea of $\eta - FNS(G)$, we introduced the quotient group with respect to $FNS(G)$. This leads us to develop a natural homomorphism with respect to $\eta - FNS(G)$ from a group G to its quotient group. Additionally, we discover the homomorphic image and pre-image of $\eta - FS(G)$ $(\eta - FNS(G)$. We conclude this section by establishing an isomorphism between the quotient group $\frac{G}{\rho^\eta}$ and $\frac{G}{G_{\rho^\eta}}$.

Definition 11. Let G be a group and $\rho: G \longrightarrow [0,1]$ be a fuzzy subset G. Let $\eta \in [0,1]$, then ρ is called η-$FS(G)$ if ρ^η is $FS(G)$. In other words, ρ is η-$FS(G)$ if ρ^η admits the following properties, *for all* $a_1,\ a_2 \in G$:

I. $\rho^{\eta}(a_1a_2) \geq \min \{ \rho^{\eta}(a_1), \rho^{\eta}(a_2)\}$.
II. $\rho^{\eta}\left(a_1{}^{-1}\right) = \rho^{\eta}(a_1)$.

Example 3. Let ρ be a fuzzy subset of the group $G = V_4 = \{1, a_1, a_2, a_1a_2\}$ defined as $\rho(1) = 0.7$ and $\rho(a_1) = \rho(a_2) = \rho(a_1a_2) = 0.9$.

Define η-fuzzy subset ρ^{η} of G for $\eta = 0.8$ as follows:

$\rho^{\eta}(1) = 0.2$ and $\rho^{\eta}(a_1) = \rho^{\eta}(a_2) = \rho^{\eta}(a_1a_2) = 0.1$.

Clearly, ρ^{η} is η-$FS(G)$.

Remark 2: Note that, ρ is η-$FS(G)$ for any choice of η in each of the following case

I. $\rho(a_1a_2) \geq \eta > \min\left\{\rho(a_1), \rho(a_2)\right\}$.
II. $\eta \geq \rho(a_1a_2) > \min\left\{\rho(a_1), \rho(a_2)\right\}$.
III. $\rho(a_1a_2) > \min\left\{\rho(a_1), \rho(a_2)\right\} > \eta$.
IV. For $\eta = 0$, we get the complement of classical fuzzy subgroup.

Proposition 1. Let ρ be η-$FS(G)$. Then the following statements hold:

- $\rho^{\eta}(a_1) \leq \rho^{\eta}(e)$ *for all* $a_1 \in G$ *and* e *is identity element of* G.
- $\rho^{\eta}(a_1a_2{}^{-1}) = \rho^{\eta}(e)$ *gives* $\rho^{\eta}(a_1) = \rho^{\eta}(a_2)$ *for all* $a_1, a_2 \in G$.

Proof:

I. Since $\rho^{\eta}\left(a_1a_1{}^{-1}\right) = \rho^{\eta}(e)$ and also $\rho^{\eta}\left(a_1a_1{}^{-1}\right) = min\left\{\rho^{\eta}(a_1), \rho^{\eta}\left(a_1{}^{-1}\right)\right\}$

$= min\left\{\rho^{\eta}(a_1), \rho^{\eta}(a_1)\right\} = \rho^{\eta}(a_1)$.

This implies that $\rho^{\eta}(a_1) \leq \rho^{\eta}(e)$, *for all* $a_1 \in G$.

II. Since we have $\rho^{\eta}(a_1) = \rho^{\eta}(a_1a_2{}^{-1}a_2) \geq \min\left\{\rho^{\eta}\left(a_1a_2{}^{-1}\right), \rho^{\eta}(a_2)\right\}$.

Then by our assumption we have $\rho^{\eta}(a_1) \geq \min\left\{\rho^{\eta}(e), \rho^{\eta}(a_2)\right\}$ which implies that

$\rho^{\eta}(a_1) \geq \rho^{\eta}(a_2)$.

Similarly, $\rho^{\eta}(a_2) = \rho^{\eta}(a_2a_1{}^{-1}a_1) \geq \min\left\{\rho^{\eta}\left(a_2a_1{}^{-1}\right), \rho^{\eta}(a_1)\right\}$ then by our assumption, we have $\rho^{\eta}(a_2) \geq \min\left\{\rho^{\eta}(e), \rho^{\eta}(a_1)\right\} = \rho^{\eta}(a_1)$, which implies that $\rho^{\eta}(a_2) \geq \rho^{\eta}(a_1)$.

Hence, $\rho^{\eta}(a_1) = \rho^{\eta}(a_2)$.

The next result leads to note that every $FS(G)$ is η-$FS(G)$.

Proposition 2. Every $FS(G)$ is also $\eta - FS(G)$.

Proof: Let ρ be a $FS(G)$. Consider, $\rho^{\eta}(a_1a_2) = \min\left\{1 - \rho(a_1a_2), 1 - \eta\right\}$, where $\eta \in [0,1]$ and $a_1, a_2 \in G$.

$\geq \min\left\{1 - \min\left\{\rho(a_1), \rho(a_2)\right\}, 1 - \eta\right\}$

$= \min\left\{\min\{1 - \rho(a_1),\ 1 - \eta\},\ \min\{1 - \rho(a_2), 1 - \eta\}\right\} = \min\left\{\rho^{\eta}(a_1), \rho^{\eta}(a_2)\right\}.$

Thus, we have $\rho^{\eta}(a_1 a_2) \geq \min\left\{\rho^{\eta}(a_1), \rho^{\eta}(a_2)\right\}$.

Moreover, $\rho^{\eta}\left(a_1^{-1}\right) = \min\left\{1 - \rho\left(a_1^{-1}\right), 1 - \eta\right\} = \min\left\{1 - \rho(a_1), 1 - \eta\right\} = \rho^{\eta}(a_1)$.

This implies that ρ is η-$FS(G)$.

Remark 3: The converse of the aforementioned proposition must not be true.

Example 4. Let $G = S_3 = \{(1), (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$,

$\rho((1)) = 0.4$, $\rho\big((1\ 2)\big) = \rho\big((1\ 3)\big) = \rho\big((2\ 3)\big) = 0.5$,

And

$$\rho\big((1\ 2\ 3)\big) = \rho((1\ 3\ 2)) = 0.6.$$

Consider the η-fuzzy set for $\eta = 0.55$ as follows:

$\rho^{\eta}\big((1)\big) = 0.45, \rho^{\eta}\big((1\ 2)\big) = \rho^{\eta}\big((1\ 3)\big) = \rho^{\eta}\big((2\ 3)\big) = 0.45,$

And

$$\rho^{\eta}\big((1\ 2\ 3)\big) = \rho^{\eta}\big((1\ 3\ 2)\big) = 0.4.$$

Clearly, the fuzzy subset ρ is $\eta - FS(G)$.

Moreover, ρ is not $FS(G)$ because all possible level subset $\rho_{0.4} = \{(1), (1\ 2), (1\ 3), (2\ 3)\}$, $\rho_{0.5} = \{(1\ 2), (1\ 3), (2\ 3)\}$ and $\rho_{0.6} = \{(1\ 2\ 3), (1\ 3\ 2)\}$.

$\rho_{0.4}, \rho_{0.5}$ and $\rho_{0.6}$ are not subgroups of $S_3 = G$.

Proposition 3. Let G be a group and ρ be its fuzzy subset such that $\rho(a_1) = \rho\left(a_1^{-1}\right)$ *for all* $a_1 \in G$. Let $\eta \geq m$, where $\eta \in [0,1]$ and $m = \sup\left\{\rho(a_1), a_1 \in G\right\}$, then ρ is also η-$FS(G)$.

Proof: Since, we have $\eta \geq m$. So, $\eta \geq \sup\left\{\rho(a_1), a_1 \in G\right\}$, which implies $\eta \geq \rho(a_1)$ *for all* $a_1 \in G$.

So, we have $\rho^{\eta}(a_1) = \min\left\{1 - \rho(a_1), 1 - \eta\right\} = 1 - \eta$ *for all* $a_1 \in G$.

This implies that $\rho^{\eta}(a_1 a_2) \geq \min\left\{\rho^{\eta}(a_1), \rho^{\eta}(a_2)\right\}$ *for all* $a_1, a_2 \in G$.

Also, $\rho^{\eta}\left(a_1^{-1}\right) = \rho^{\eta}(a_1)$.

Hence, ρ is η-$FS(G)$.

Proposition 4. Let ρ and σ be any two η-$FS(G)$. Then $\rho \cap \sigma$ is also η-$FS(G)$.

Proof: Let ρ and σ be two η-$FS(G)$ of a group G and let $a_1, a_2 \in G$.

Since, $(\rho \cap \sigma)^{\eta}(a_1) = (\rho^{\eta} \cap \sigma^{\eta})(a_1)$ hold, then we have

$$(\rho \cap \sigma)^{\eta}(a_1a_2) = (\rho^{\eta} \cap \sigma^{\eta})(a_1a_2).$$

So, $(\rho^{\eta} \cap \sigma^{\eta})(a_1a_2) = \min\{\rho^{\eta}(a_1a_2), \sigma^{\eta}(a_1a_2)\}$

$$\geq \min\{\min\{\rho^{\eta}(a_1), \rho^{\eta}(a_2)\}, \min\{\sigma^{\eta}(a_1), \sigma^{\eta}(a_2)\}\}$$

$$= \min\{\min\{\rho^{\eta}(a_1), \sigma^{\eta}(a_1)\}, \min\{\rho^{\eta}(a_2), \sigma^{\eta}(a_2)\}\}$$

$$= \min\{(\rho^{\eta} \cap \sigma^{\eta})(a), (\rho^{\eta} \cap \sigma^{\eta})(a_2)\}.$$

This implies that $(\rho^{\eta} \cap \sigma^{\eta})(ab) \geq \min\{(\rho^{\eta} \cap \sigma^{\eta})(a), (\rho^{\eta} \cap \sigma^{\eta})(a_2)\}$.

Moreover, $(\rho \cap \sigma)^{\eta}(a_1{}^{-1}) = (\rho^{\eta} \cap \sigma^{\eta})(a_1{}^{-1}) = \min\{\rho^{\eta}(a_1),$

$$\rho^{\eta}(a_1{}^{-1})\} = \min\{\rho^{\eta}(a_1), \rho^{\eta}(a_1)\}.$$

We have $(\rho \cap \sigma)^{\eta}(a_1{}^{-1}) = (\rho \cap \sigma)^{\eta}(a_1)$.

Consequently, $\rho \cap \sigma$ is η-$FS(G)$.

Corollary 1. The intersection of any finite number of η-$FS(G)$ is also η-$FS(G)$.

Proposition 5. Let ρ and σ be any two η-$FS(G)$. Then $\rho \cup \sigma$ need not to be η-$FS(G)$.

Example 5. Let $G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. Take two subgroups of G that are

$H_1 = \{\pm 1, \pm i\}$ and $H_2 = \{\pm 1, \pm j\}$.

Let ρ and σ be two fuzzy subsets of G as:

$$\rho(a_1) = \begin{cases} 0.2, & \text{if } a_1 \in H_1, \\ 0.9, & \text{otherwise.} \end{cases}$$

And

$$\sigma(a_1) = \begin{cases} 0.3, & \text{if } a_1 \in H_2, \\ 1, & \text{otherwise.} \end{cases}$$

Since, $\rho^{\eta}(a) = \min\{1 - \rho(a), 1 - \eta\}$.

Define η-fuzzy subsets ρ^{η} and σ^{η} for $\eta = 0$, as follows:

$$\rho^{0}(a_1) = \begin{cases} 0.8, & \text{if } a_1 \in H_1, \\ 0.1, & \text{otherwise.} \end{cases}$$

And

$$\sigma^{0}(a_1) = \begin{cases} 0.7, & \text{if } a_1 \in H_2, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that ρ^0 and σ^0 are 0-$FS(G)$.

Now, we define $\rho^0 \cup \sigma^0$ as:

$(\rho^0 \cup \sigma^0)(a_1) = \max\{\rho^0(a_1), \sigma^0(a_1)\}$,

So, we have

$$(\rho^0 \cup \sigma^0)(a_1) = \begin{cases} 0.8, & if\ a_1 \in H_1, \\ 0.7, & if\ a_1 \in H_2 \backslash H_1, \\ 0.1, & otherwise. \end{cases}$$

Let $a_1 = i$ and $a_2 = j$,

Observe that $(\rho^0 \cup \sigma^0)(i) = 0.8$ and $(\rho^0 \cup \sigma^0)(j) = 0.7$.

So, $\min\{(\rho^0 \cup \sigma^0)(i), (\rho^0 \cup \sigma^0)(j)\} = 0.7$, but $(\rho^0 \cup \sigma^0)(ij) = (\rho^0 \cup \sigma^0)(k) = 0.1$.

This implies that

$$(\rho^0 \cup \sigma^0)(ij) \leq \min\{(\rho^0 \cup \sigma^0)(i), (\rho^0 \cup \sigma^0)(j)\}.$$

Hence, $\rho^0 \cup \sigma^0$ is not 0-$FS(G)$.

Example 6. Let $G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. Take a subgroup of G that is $H_1 = \{\pm 1, \pm i\}$.

Let ρ and σ be two fuzzy subsets of G as:

$$\rho(a_1) = \begin{cases} 0.3, & \text{if } a_1 \in H_1, \\ 0.9, & \text{otherwise.} \end{cases}$$

And

$$\sigma(a_1) = \begin{cases} 0.2, & \text{if } a_1 \in H_1, \\ 1, & \text{otherwise.} \end{cases}$$

Then we have ρ^0 and σ^0 as follows:

$$\rho^0(a_1) = \begin{cases} 0.7, & \text{if } a_1 \in H_1, \\ 0.1, & \text{otherwise.} \end{cases}$$

And

$$\sigma^0(a_1) = \begin{cases} 0.8, & \text{if } a_1 \in H_1, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$(\rho^0 \cup \sigma^0)(a_1) = \begin{cases} 0.8, & if\ a_1 \in H_1, \\ 0.1, & otherwise. \end{cases}$$

It can be easily seen that $\rho^0 \cup \sigma^0$ is $0 - FS(G)$.

Definition 12. Let ρ be η-$FS(G)$, for any $a_1 \in G$ define η-fuzzy left coset $a_1\rho^\eta$ of ρ in G as follows:

$$a_1\rho^\eta(x) = \min\{1 - \rho(a_1^{-1}x), 1 - \eta\}\ for\ all\ a_1, x \in G.$$

Similarly, we define η-fuzzy right coset $\rho^\eta a_1$ of ρ in G as follows:

$$\rho^\eta a_1(x) = \min\{1 - \rho(xa_1^{-1}), 1 - \eta\}\ for\ all\ a_1, x \in G.$$

Example 7. Let ρ be a fuzzy subset of the group $G = Z_4 = \{0, 1, 2, 3\}$ defined as

$\rho(0) = 0.2$, $\rho(2) = 0.4$ and $\rho(1) = \rho(3) = 0.4$.

Define η-fuzzy subset ρ^η of G for $\eta = 0.5$ as follows:

$\rho^\eta(0) = 0.5$, $\rho^\eta(2) = 0.5$ and $\rho^\eta(1) = \rho^\eta(3) = 0.4$.

Clearly, ρ^{η} is η-$FS(G)$. Consider η-fuzzy left coset of ρ by the element $2 \in Z_4$ as follows:

$$2 + \rho^{\eta}(a_1) = \min\{1 - \rho(2 + a_1), 1 - \eta\} = \begin{cases} 0.4, & if\ a_1 \in \{0,2\}, \\ 0.5, & otherwise. \end{cases}$$

Similarly, define η-fuzzy right coset of G.

Proposition 6. Let ρ be η- $FS(G)$. Then ρ be $\eta - FNS(G)$ if and only if $a_1\rho^{\eta}(x) = \rho^{\eta}a_1(x)\ for\ all\ a_1 \in G$.

Note: $a_1\rho^{\eta}(x) = \rho^{\eta}\left(a_1{}^{-1}x\right)$ and $\rho^{\eta}a_1(x) = \rho^{\eta}\left(xa_1{}^{-1}\right)\ \ for\ all\ x \in G$.

The following result leads to note that every $FNS(G)$ is η- $FNS(G)$.

Example 8. In view of *Example 7* η- $FS(G)$ is a η-$FNS(G)$, because its all η-fuzzy left cosets and η-fuzzy right cosets are equal. For instance, consider

$$2 + \rho^{\eta}(a_1) = \rho^{\eta}(a_1) + 2 = \begin{cases} 0.4, & \text{if } a_1 \in \{0,2\}, \\ 0.5, & \text{otherwise.} \end{cases}$$

Proposition 7. Every $FNS(G)$ is also η-$FNS(G)$.

Proof: Suppose that ρ is $FNS(G)$ which implies that $a_1\rho = \rho a_1$.

Then for any $x \in G$ we have $\rho\left(a_1{}^{-1}x\right) = \rho\left(xa_1{}^{-1}\right)$. So, we have

$$\min\{1 - \rho\left(a_1{}^{-1}x\right), 1 - \eta\} = \min\{1 - \rho\left(xa_1{}^{-1}\right), 1 - \eta\}.$$

This implies that $a_1\rho^{\eta}(x) = \rho^{\eta}a_1(x)\ for\ all\ x \in G$. Consequently, ρ is η-$FNS(G)$.

Note that the converse of above result need not to be true.

Example 9. Let $G = D_3 = < a, b :\ a^3 = b^2 = e,\ ba = a^2b >$.

Define a fuzzy subset ρ of G as follows:

$$\rho(a) = \begin{cases} 0.3, & \text{if } a \in < b >, \\ 0.1, & \text{otherwise.} \end{cases}$$

Take $\eta = 0.6$ then we have $\rho^{\eta}(a) = 1 - \eta\ \ for\ all\ a \in G$.

$$a_1\rho^{\eta}(g) = \min\{1 - \rho\left(a_1{}^{-1}g\right), 1 - \eta\} = 1 - \eta$$

$$= \min\{1 - \rho\left(ga_1{}^{-1}\right), 1 - \eta\} = \rho^{\eta}a_1(g).$$

Then $a_1\rho^{\eta}(g) = \rho^{\eta}a_1(g)$ which implies that ρ is η-$FNS(G)$. But it can be seen that ρ is not $FNS(G)$. This is because $\rho\left((a^2)(ab)\right) = 0.3$ and $\rho\left((ab)(a^2)\right) = 0.1$. i.e. $\rho\left(a^{-1}g\right) = \rho\left(ga^{-1}\right)$ not hold.

Proposition 8. Let ρ be η-FNS(G). Then $\rho^{\eta}\left(b^{-1}a_1b\right) = \rho^{\eta}(a_1)$ or $\rho^{\eta}(a_1a_2) = \rho^{\eta}(a_2a_1)$ hold, $for\ all\ a_1, a_2 \in G$.

Proof: Since, we have ρ be $\eta - FNS(G)$ then we have $a_1\rho^{\eta} = \rho^{\eta}a_1\ for\ all\ a_1 \in G$.

This implies that $a_1\rho^{\eta}\left(a_2{}^{-1}\right) = \rho^{\eta}a_1\left(a_2{}^{-1}\right)\ for\ all\ a_2{}^{-1} \in G$.

$$= \min\{1 - \rho\left(a_1{}^{-1}a_2{}^{-1}\right), 1 - \eta\} = \min\{1 - \rho\left(a_2{}^{-1}a_1{}^{-1}\right), 1 - \eta\} = \rho^{\eta}\left(a_1{}^{-1}a_2{}^{-1}\right) = \rho^{\eta}\left(a_2{}^{-1}a_1{}^{-1}\right).$$

Consequently, we have $\rho^{\eta}\left((a_2a_1)^{-1}\right) = \rho^{\eta}\left((a_1a_2)^{-1}\right)$.

Hence, $\rho^{\eta}(a_1a_2) = \rho^{\eta}(a_2a_1)$.

Theorem 3. Let ρ be η-$FS(G)$. Then following statements are equivalent:

I. $\rho^{\eta}(a_2a_1) = \rho^{\eta}(a_1a_2)$ $for\ all\ a_1, a_2 \in G$.
II. $\rho^{\eta}\left(a_1a_2a_1^{-1}\right) = \rho^{\eta}(a_2)$ $for\ all\ a_1, a_2 \in G$.
III. $\rho^{\eta}\left(a_1a_2a_1^{-1}\right) \geq \rho^{\eta}(a_2)$ $for\ all\ a_1, a_2 \in G$.
IV. $\rho^{\eta}\left(a_1a_2a_1^{-1}\right) \leq \rho^{\eta}(a_2)$ $for\ all\ a_1, a_2 \in G$.

Proposition 9. Let ρ be η-$FS(G)$. Let $\eta \geq m$, where $\eta \in [0,1]$ and $m = \sup\left\{\rho(a_1), a_1 \in G\right\}$, then ρ is also η-$FNS(G)$.

Proof: Since, we have $\eta \geq m$, so $\eta \geq sup\left\{\rho(a_1), a_1 \in G\right\}$, which implies that $\eta \geq \rho(a_1)$ $for\ all\ a_1 \in G$.

So, we have $\rho^{\eta}(a_1) = \min\left\{1 - \rho(a_1), 1 - \eta\right\} = 1 - \eta$,

$\rho^{\eta}a_1(g) \geq \min\left\{1 - \rho\left(ga_1^{-1}\right), 1 - \eta\right\} = 1 - \eta$.

Similarly, $a_1\rho^{\eta}(g) \geq \min\left\{1 - \rho\left(a_1^{-1}g\right), 1 - \eta\right\} = 1 - \eta$.

This implies that $a_1\rho^{\eta} = \rho^{\eta}a_1$ $for\ all\ a_1 \in G$.

Hence, ρ is also η-$FNS(G)$.

The following result illustrate that the set $G_{\rho^{\eta}}$ is infect a normal subgroup of G.

Proposition 10. Let ρ be η-$FNS(G)$. Then the set define as $G_{\rho^{\eta}} = \left\{a_1 \in G : \rho^{\eta}(a_1) = \rho^{\eta}(e)\right\} \trianglelefteq G$.

Proof: Since, $G_{\rho^{\eta}}$ is nonempty because $e \in G$. Let $a_1, a_2 \in G_{\rho^{\eta}}$,

$\rho^{\eta}\left(a_1a_2^{-1}\right) \geq \min\left\{\rho^{\eta}(a_1), \rho^{\eta}\left(a_2^{-1}\right)\right\}$ $for\ all\ a_1, a_2 \in G$

$= \min\left\{\rho^{\eta}(a_1), \rho^{\eta}(a_2)\right\}$ $for\ all\ a_1, a_2 \in G$

$= \min\left\{\rho^{\eta}(e), \rho^{\eta}(e)\right\}$ $for\ all\ a_1, a_2 \in G$

$= \rho^{\eta}(e)$.

This implies that $\rho^{\eta}\left(a_1a_2^{-1}\right) \geq \rho^{\eta}(e)$.

Since, ρ^{η} is $FS(G)$ which implies that $\rho^{\eta}\left(a_1a_2^{-1}\right) \leq \rho^{\eta}(e)$.

Hence, $\rho^{\eta}\left(a_1a_2^{-1}\right) = \rho^{\eta}(e)$ implies that $G_{\rho^{\eta}}$ is subgroup of G.

Now we prove it is normal subgroup of G. Let $a_1 \in G_{\rho^{\eta}}$ and $a_2 \in G$, then we have

$\rho^{\eta}\left(a_2^{-1}a_1a_2\right) = \rho^{\eta}(a_1) = \rho^{\eta}(e)$.

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This implies that $a_2^{-1}a_1a_2 \in G_{\rho^\eta}$.

Consequently, we have $G_{\rho^\eta} \trianglelefteq G$.

Proposition 11. Let ρ be η-$FNS(G)$ of a group G. Then the following statements hold

I. $a_1\rho^\eta = a_2\rho^\eta \iff a_1^{-1}a_2 \in G_{\rho^\eta}$.
II. $\rho^\eta a_1 = \rho^\eta a_2 \iff a_1a_2^{-1} \in G_{\rho^\eta}$.

Proof:

I. Suppose that $a_1\rho^\eta = a_2\rho^\eta$, then we have

$$\rho^\eta\left(a_1^{-1}a_2\right) = \min\left\{1-\rho\left(a_1^{-1}a_2\right), 1-\eta\right\}$$

$$= a_1\rho^\eta(a_2) = a_2\rho^\eta(a_2)$$

$$= \min\left\{1-\rho\left(a_2^{-1}a_2\right), 1-\eta\right\}$$

$$= \min\left\{1-\rho(e), 1-\eta\right\}$$

$$= \rho^\eta(e).$$

Thus $\rho^\eta\left(a_1^{-1}a_2\right) = \rho^\eta(e)$ implies that $a_1^{-1}a_2 \in G_{\rho^\eta}$.

Conversely,

$$a_1\rho^\eta(a_3) = \min\left\{1-\rho\left(a_1^{-1}a_3\right), 1-\eta\right\}.$$

So, $\rho^\eta\left(a_1^{-1}a_3\right) = \rho^\eta\left(a_1^{-1}a_2 . a_2^{-1}a_3\right) \geq \min\left\{\rho^\eta\left(a_1^{-1}a_2\right), \rho^\eta\left(a_2^{-1}a_3\right)\right\}$

$$= \min\left\{\rho^\eta(e), \rho^\eta\left(a_2^{-1}a_3\right)\right\}$$

$$= \rho^\eta\left(a_2^{-1}a_3\right) = a_2\rho^\eta(a_3).$$

By interchanging the a_1 and a_2, we have $a_1\rho^\eta(a_3) = a_2\rho^\eta(a_3)$ for all $a_3 \in G$.

Hence, $a_1\rho^\eta = a_2\rho^\eta$.

II. Similar as above proof.

Proposition 12. Let ρ be η-$FNS(G)$ of a group G and $a_1, a_2, x, y \in G$. If $a_1\rho^\eta = x\rho^\eta$ and $a_2\rho^\eta = y\rho^\eta$ then $a_1a_2\rho^\eta = xy\rho^\eta$.

Proof: Given that $a_1\rho^\eta = x\rho^\eta$ and $a_2\rho^\eta = y\rho^\eta$, which implies that $a_1^{-1}x,\ a_2^{-1}y \in G_{\rho^\eta}$.

Now $(a_1a_2)^{-1}xy = a_2^{-1}\left(a_1^{-1}x\right)y = a_2^{-1}\left(a_1^{-1}x\right)(a_2a_2^{-1})y$

$$= \left[a_2^{-1}\left(a_1^{-1}x\right)a_2\right]\left(a_2^{-1}y\right) \in G_{\rho^\eta} \quad \because G_{\rho^\eta} \trianglelefteq G.$$

This implies that $(a_1a_2)^{-1}xy \in G_{\rho^\eta}$. Hence, $a_1a_2\rho^\eta = xy\rho^\eta$.

Proposition 13. Let $\frac{G}{\rho^{\eta}}$ be the collection of all η-fuzzy cosets of a $\eta - FS(G)$. This form a group under the binary operation $\circledast$ define on the set $\frac{G}{\rho^{\eta}}$ as follows:

$\rho^{\eta}a_1 \circledast \rho^{\eta}a_2 = \rho^{\eta}a_1a_2 \; for \; all \; a_1, a_2 \in G.$

Proof: As we know that $\frac{G}{\rho^{\eta}} = \left\{ \rho^{\eta}a_1 : a_1 \in G \right\}$.

Let $\rho^{\eta}a_1 = \rho^{\eta}a_1$ and $\rho^{\eta}a_2 = \rho^{\eta}a_2{'} \; for \; all \; a_1, a_1{'}, a_2, a_2{'} \in G.$

Let $g \in G$ then $(\rho^{\eta}a_1 \circledast \rho^{\eta}a_2)(g) = \rho^{\eta}a_1a_2(g) = \min\left\{1 - \rho\left(g\left((a_1a_2)^{-1}\right), 1 - \eta\right.\right\}$

$= \min\left\{1 - \rho(\left(ga_2{}^{-1}\right)a_1{}^{-1}), 1 - \eta\right\}$

$= \rho^{\eta}a_1\left(ga_2{}^{-1}\right) = \rho^{\eta}a_1{'}\left(ga_2{}^{-1}\right)$

$= \min\left\{1 - \rho(\left(ga_2{}^{-1}\right)a_1{}^{-1}), 1 - \eta\right\}$

$= \min\left\{1 - \rho(\left(a'^{-1}g\right)b^{-1}\right), 1 - \eta\right\}$

$= \rho^{\eta}a_2\left(a_1{'}^{-1}g\right) = \rho^{\eta}a_2{'}\left(a_1{'}^{-1}g\right)$

$= \min\left\{1 - \rho\left(\left(a_1{'}^{-1}g\right)a_2{'}^{-1}\right), 1 - \eta\right\}$

$= \min\left\{1 - \rho\left(a_2{'}^{-1}\left(a_1{'}^{-1}g\right)\right), 1 - \eta\right\}$

$= \min\left\{1 - \rho\left((a_2{'}^{-1}a_1{'}^{-1})\right), 1 - \eta\right\}$

$= \min\left\{1 - \rho\left((a_1{'}a_2{'})^{-1}g)\right), 1 - \eta\right\}$

$= \min\left\{1 - \rho\left(g(a_1{'}a_2{'})^{-1})\right), 1 - \eta\right\}$

$= \rho^{\eta}a_1{'}a_2{'}.$

Hence, $\circledast$ is well define operation on the set $\frac{G}{\rho^{\eta}}$.

The set $\frac{G}{\rho^{\eta}}$ under this binary operation admits the associative law. The element $\rho^{\eta}e$ of $\frac{G}{\rho^{\eta}}$ is the identity element and the inverse of an element $\rho^{\eta}a_1$ is $\rho^{\eta}a_1{}^{-1}$.

Example 10. In view of *Example 7* consider ρ as η- $FNS(G)$.

The set $\frac{G}{\rho^{\eta}} = \{ \rho^{\eta}, 2 + \rho^{\eta} \}$ forms a group under the following binary operation defined on $\frac{G}{\rho^{\eta}}$ as $(a_1 + \rho^{\eta}) + (a_2 + \rho^{\eta}) = ((a_1 + a_2) + \rho^{\eta})$.

Note that $\rho^{\eta}(a_1)$ is identity element of this group and inverse of $a_1 + \rho^{\eta}$ is $(-a_1) + \rho^{\eta}$.

Definition 13. The group $\frac{G}{\rho^{\eta}}$ of η-fuzzy cosets of a $\eta - FNS(G)$ is called the quotient group of G by ρ^{η}.

Theorem 14. Let G be a group and $\frac{G}{\rho^{\eta}}$ be quotient group with respect to $\eta - FNS(G)$. There exist a natural epimorphism from G to $\frac{G}{\rho^{\eta}}$ which is defined as $\xi(a_1) = \rho^{\eta}a_1$ with $Ker\ \xi = G_{\rho^{\eta}}$.

Proof: Let $a_1, a_2 \in G$ be any elements. Then $\xi(a_1 a_2) = \rho^{\eta}a_1 a_2 = \rho^{\eta}a_1\, \rho^{\eta}a_2 = \xi(a_1)\xi(a_2)$.

Therefore, ξ is homomorphism. For each $\rho^{\eta}a_1 \in G_{\rho^{\eta}}$ we have $a_1 \in G$ such that $\xi(a_1) = \rho^{\eta}a_1$.

This implies that ξ is onto homomorphism.

Now $Ker\ \xi = \{\, a_1 \in G :\ \xi(a_1) = \rho^{\eta}e \,\}$

$$= \{\, a_1 \in G :\ \rho^{\eta}a_1 = \rho^{\eta}e \,\}$$

$$= \{\, a_1 \in G :\ a_1 e^{-1} \in G_{\rho^{\eta}} \,\}$$

$$= \left\{ a_1 \in G :\ a_1 \in G_{\rho^{\eta}} \right\} = G_{\rho^{\eta}}.$$

5 | Homomorphism of η-Fuzzy Subgroups

Anthony and Sherwood [2] observed that using a minimum in the Rosenfeld [8] definition of a fuzzy subgroup constrains the concept, rendering it useless in a variety of fuzzy situations. They introduced the concept of an T-norm and redefined the fuzzy subgroup by substituting a T-norm for a minimum. They investigated the impact of a simple homomorphism on fuzzy subgroups. Here, we present the results of homomorphism in frame work of our proposed definition.

Theorem 4. Let $\xi : G \longrightarrow G'$ be a bijective homomorphism of a group G into a group G'. If ρ is η-$FS(G)$ then the homomorphic image $\xi(\rho)$ is η-$FS(G')$.

Proof: Given that ρ be η-$FS(G')$. Let $a'_1, a'_2 \in G'$ be any element then we have unique elements $a_1, a_2 \in G$, such that $\varphi(a_1) = a'_1$ and $\varphi(a_2) = a'_2$.

Further, $\left(\xi(\rho)\right)^{\eta}(a'_1 a'_2)$

$$= \min\left\{1 - \xi(\rho)(a'_1 a'_2),\ 1 - \eta \right\}$$

$$= \min\left\{1 - \xi(\rho)\left(\xi(a_1)\xi(a_2)\right),\ 1 - \eta \right\}$$

$$= \min\left\{1 - \xi(\rho)\left(\xi(a_1 a_2)\right),\ 1 - \eta \right\}$$

$$= \min\left\{1 - \xi(\rho)(a_1 a_2),\ 1 - \eta \right\}$$

$$= \rho^{\eta}(a_1 a_2)$$

$$\geq \min\left\{\rho^{\eta}(a_1), \rho^{\eta}(a_2) \right\} \mathit{for\ all}\ a_1, a_2 \in G$$

$$= \min\left\{\xi(\rho)^{\eta}\xi(a_1), \xi(\rho)^{\eta}\xi(a_2) \right\}$$

$$= \min\left\{\xi(\rho)^{\eta}(a'_1), \xi(\rho)^{\eta}(a'_2) \right\}.$$

Consequently,

$(\xi(\rho))^\eta(a'_1 a'_2) \geq \min\{(\xi(\rho))^\eta(a'_1),\ (\xi(\rho))^\eta(a'_2)\}.$

Also, $\left(\xi(\rho)\right)^\eta(a'^{-1}) = \xi(\rho^\eta)(a'^{-1}) = \xi(\rho^\eta)\left(\varphi\left(a^{-1}\right)\right) = \rho^\eta\left(a^{-1}\right) = \rho^\eta(a)$

$= \xi(\rho^\eta)\left(\xi(a)\right) = \left(\xi(\rho)\right)^\eta(a')$

Thus, $\left(\xi(\rho)\right)^\eta(a'^{-1}) = \left(\xi(\rho)\right)^\eta(a')$.

Consequently, $\xi(\rho)$ is η-$FS(G')$.

Theorem 5. Let $\xi : G \longrightarrow G'$ be a bijective homomorphism of a group G into G'. If ρ is η-$FNS(G)$ then the homomorphic image $\xi(\rho)$ is η-$FNS(G')$.

Proof: Given that ρ be η- $FNS(G)$. Let $a'_1, a'_2 \in G'$ be any element then we have unique elements $a_1, a_2 \in G$, such that $\xi(a_1) = a'_1$ and $\xi(a_2) = a'_2$.

$\left(\xi(\rho)\right)^\eta(a'_1 a'_2) = \min\{1 - \xi(\rho)(a'_1 a'_2),\ 1 - \eta\}$

$= \min\{1 - \xi(\rho)\left(\xi(a_1)\xi(a_2)\right),\ 1 - \eta\}$

$= \min\{1 - \xi(\rho)\left(\xi(a_1 a_2)\right),\ 1 - \eta\}$

$= \min\{1 - \xi(\rho)\left(\xi(a_2 a_1)\right),\ 1 - \eta\}$

$= \min\{1 - \xi(\rho)\left(\xi(a_2)\xi(a_1)\right),\ 1 - \eta\}$

$= \min\{1 - \xi(\rho)\left(a'_2 a'_1\right),\ 1 - \eta\}$

$= \left(\xi(\rho)\right)^\eta(a'_1 a'_2)$.

Consequently, $\xi(\rho)$ is $\eta - FNS(G')$.

Theorem 6. Let $\xi : G \longrightarrow G'$ be a homomorphism of a group G into G'. If σ is $\eta - FS(G')$ then the pre-image $\xi^{-1}(\sigma)$ is η-$FS(G)$.

Proof: Given that σ be η-$FS(G')$. Let $a_1, a_2 \in G$ be any element then we have

$(\xi^{-1}(\sigma))^\eta(a_1 a_2) = \xi^{-1}(\sigma^\eta)(a_1 a_2) = \sigma^\eta\left(\xi(a_1 a_2)\right) = \sigma^\eta\left(\xi(a_1)\xi(a_2)\right)$

$\geq \min\{\sigma^\eta\left(\xi(a_1)\right),\ \sigma^\eta\left(\xi(a_2)\right)\}$

$= \min\{\xi^{-1}(\sigma^\eta)(a_1),\ \xi^{-1}(\sigma^\eta)(a_1)\}.$

Thus,

$(\xi^{-1}(\sigma))^\eta(a_1 a_2) \geq \min\{(\xi^{-1}\sigma)^\eta)(a_1), (\xi^{-1}\sigma)^\eta)(a_2)\}.$

Also $(\xi^{-1}(\sigma))^\eta\left(a^{-1}\right) = \xi^{-1}(\sigma^\eta)\left(a^{-1}\right) = \sigma^\eta\left(\xi\left(a^{-1}\right)\right) = \sigma^\eta\left(\xi(a)\right) = \xi^{-1}(\sigma^\eta)(a).$

Thus, $(\xi^{-1}(\sigma))^{\eta}(a^{-1}) = (\xi^{-1}(\sigma))^{\eta}(a)$.

Hence, $\xi^{-1}(\sigma)$ is η-$FS(G)$.

Theorem 7. Let $\xi : G \longrightarrow G'$ be a homomorphism of a group G into G'. If σ is η-$FNS(G')$ then the pre-image $\xi^{-1}(\sigma)$ is η-$FNS(G)$.

Proof: Let σ be $\eta - FNS(G')$. Let $a_1, a_2 \in G$ be any element then we have

$$(\xi^{-1}(\sigma))^{\eta}(a_1a_2) = \xi^{-1}(\sigma^{\eta})(a_1a_2) = \sigma^{\eta}\big(\xi(a_1a_2)\big) = \sigma^{\eta}\big(\xi(a_1)\xi(a_2)\big) = \sigma^{\eta}\big(\xi(a_2a_1)\big)$$

$$= (\xi^{-1}(\sigma))^{\eta}(a_2a_1).$$

Thus,

$$(\xi^{-1}(\sigma))^{\eta}(a_1a_2) = (\xi^{-1}(\sigma))^{\eta}(a_2a_1).$$

Hence $\varphi^{-1}(\sigma)$ is η- $FNS(G)$.

Theorem 8. Let ρ be a η-$FNS(G)$ and $a_1, a_2 \in G$ be any element. If $a_1\rho^{\eta} = a_2\rho^{\eta}$ then $\rho^{\eta}(a_1) = \rho^{\eta}(a_2)$.

Proof: Suppose that $a_1\rho^{\eta} = a_2\rho^{\eta}$ then by *Proposition 8* we have ${a_1}^{-1}a_2 \in G_{\rho^{\eta}}$ and ${a_2}^{-1}a_1 \in G_{\rho^{\eta}}$. Since, η-$FNS(G)$, this implies that

$$\rho^{\eta}(a_1) = \rho^{\eta}\big({a_2}^{-1}a_1a_2\big) \geq \min\left\{\rho^{\eta}\big({a_2}^{-1}a_1\big),\ \rho^{\eta}(a_2)\right\} = \min\left\{\rho^{\eta}(e),\ \rho^{\eta}(a_2)\right\} = \rho^{\eta}(a_2).$$

Therefore, we have $\rho^{\eta}(a_1) \geq \rho^{\eta}(a_2)$. Similarly, we have $\rho^{\eta}(a_1) \leq \rho^{\eta}(a_2)$.

Hence, $\rho^{\eta}(a_1) = \rho^{\eta}(a_2)$.

Theorem 9. Let ρ be a η-$FNS(G)$. Then $\frac{G}{\rho^{\eta}} \cong \frac{G}{G_{\rho^{\eta}}}$.

Proof: Define a map $\varphi : \frac{G}{\rho^{\eta}} \longrightarrow \frac{G}{G_{\rho^{\eta}}}$ by the rule

$\xi(a_1\rho^{\eta}) = a_1G_{\rho^{\eta}}$, for all $a_1 \in G$. In view of *Proposition 8* ξ is well define.

The application of *Proposition 8* leads to note that ξ is injective. ξ is obviously surjective.

Now consider, for $a_1\rho^{\eta}, a_2\rho^{\eta} \in \frac{G}{\rho^{\eta}}$ we have $\xi\big((a_1\rho^{\eta})(a_2\rho^{\eta})\big) = \varphi(a_1a_2\rho^{\eta}) = a_1a_2G_{\rho^{\eta}} = a_1G_{\rho^{\eta}}.a_2G_{\rho^{\eta}} = \xi(a_1\rho^{\eta})\xi(a_2\rho^{\eta})$.

So, ξ is homomorphism. Since ξ is a bijective mapping, which implies this is an isomorphism. Hence, $\frac{G}{\rho^{\eta}} \cong \frac{G}{G_{\rho^{\eta}}}$.

6 | Conclusion

In this paper, we introduced the idea of $\eta - FS(G)$ and η-fuzzy cosets for a given group. We used the concept of η-$FNS(G)$ and discussed various related results and properties. We also studied the effect on the image and inverse image of $\eta - FS(G)$ $(FNS(G))$ under group homomorphism. We shall extend this concept to intuitionistic fuzzy sets in the upcoming studies and look into itsnumerous algebraic features. Moreover, we used the concept of η −fuzzy subset in classical field theory.

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A Cubic Set Discussed in Incline Algebraic Sub–Structure

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Abstract

An effective and flexible method for encoding ambiguous data is using cubic sets. The concept of incline algebraic sub-structure is considered and is interlinked with the notation of the cubic set to define cubic subincline. The sense of cubic sub incline of algebra is established with relevant results. Additionally, the results such as homomorphic image, preimage, cartesian product and level sets of cubic sub incline are worked out in this study, and several of its associated findings were looked into.

Keywords: Incline algebra, Cubic sets, Cubic subincline, Image and preimage, Cartesian product.

1 | Introduction

Zadeh [16] worked on a new set called as fuzzy set where the concept of uncertainty in many real applications are analyzed and this fuzzy set plays a vital role in the recent research. And the concept of interval valued fuzzy set were brought out that is membership function in terms of a collection of closed sub-interval of [0, 1]. Later, Atanassov [18] extended the fuzzy set into an intuitionistic fuzzy set by adding non membership for every element. Further, Jun et al. [4] explored a notation in which the first term is of interval valued fuzzy and the second term as fuzzy and is named as a cubic set. Many research is moving a long way with these types of sets and from those work this paper is motivated to work on this topic [1], [2], [8]-[15].

All the structure of incline algebra is introduced by Coa et al. [3] which is a generalization of both Boolean and Fuzzy algebras, which is associative, commutative under addition and multiplication is distributive over addition with $x_0 + x_0 = x_0$, $x_0 + x_0\, y_0 = x_0, y_0 + x_0\ \ y_0 = y_0$ $for\ all\ x_0, y_0$. It has both a semiring structure and a poset structure and this incline algebra deals with different fields such as the graph theory, decision making, matrices etc.

Many algebraic structures are fused with fuzzy set was started by Rosenfeld [17]. Further many algebras like BF, BCK, BCI and B are successfully correlated with various kinds of fuzzy sets and the

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incline algebra was merged with fuzzy and its generalization by Jun et al. [5]-[7]. Here, this paper is also merging the concept of incline algebra with cubic set and introduces the notion of cubic subincline. Moreover, it deals about the cartesian product, homomorphic image and some related results of the structure introduced.

This work is characterized as the following sections: Section 1 introduces about the work. Section 2 defines the needed definition of this paper and Section 3 deals with the definition of cubic subincline and at last Section 4 concludes the work.

2 | Preliminaries

This part provides some essential definitions on incline and fuzzy sets.

Definition 1. A non-empty set $(\urcorner, +, *)$ is an incline algebra if for all $x_0, y_0, z_0 \in \urcorner$ the following hold:

- $+$ *is commutative and associative.*
- $*$ *is associative and distributive (both left and right) under* $+$.
- $x_0 + x_0 = x_0$.
- $x_0 + (x_0 * y_0) = x_0$.
- $y_0 + (x_0 * y_0) = y_0$.

Definition 2. A subincline of an incline $\urcorner$ is a non-empty subset M of $\urcorner$ which is closed under addition and multiplication.

Definition 3. A fuzzy set in a universal set χ is defined as $\gamma : \chi \to [0,1]$.

Definition 4. An interval valued fuzzy set on χ is defined by $C = \{x_0, \overline{\gamma}_C(x_0)\}$ *for all* $x_0 \in \chi$ where $\overline{\gamma}_C : \chi \to D[0,1]$; $D[0,1]$ denotes the family of all closed subintervals of [0, 1]. Here $\overline{\gamma}_C(x_0) = [\gamma_C^L(x_0), \gamma_C^U(x_0)]$ *for all* $x_0 \in \chi$ *with* $\gamma_C^L \leq \gamma_C^U$ *and* γ_C^L, γ_C^U are fuzzy sets.

Definition 5. Let χ be a non-empty set and a cubic fuzzy set in χ is of the form $C = \{x_0, \overline{\gamma}_C(x_0), \nu_C(x_0)\}$ *for all* $x_0 \in \chi$ where $\overline{\gamma}_C : \chi \to D[0,1]$ and $\nu_C : \chi \to [0,1]$, where $\overline{\gamma}_C(x_0)$ is an interval valued fuzzy set and $\nu_C(x_0)$ is a fuzzy set.

Definition 6. A non-empty set C in an incline is called a fuzzy subincline if $\wedge\big(C(x_0 + y_0), C(x_0 * y_0)\big) \geq \wedge\big(C(x_0), C(y_0)\big)$ *for all* $x_0, y_0 \in \urcorner$.

Here the notation for min is represented as $\wedge$, rmin is represented as $\overline{\wedge}$ and max as $\vee$.

3 | Cubic Subincline of Incline Algebra

This section deals the structure of cubic subincline and some related results.

Definition 7. Let $\urcorner$ be an incline algebra and a cubic set C in $\urcorner$ is said to be a cubic subincline of incline algebra if it satisfies:

- $\overline{\wedge}\big(\overline{\gamma}(x_0 + y_0), \overline{\gamma}(x_0 * y_0)\big) \geq \overline{\wedge}\big(\overline{\gamma}(x_0), \overline{\gamma}(y_0)\big)$,
- $\vee\,(\nu(x_0 + y_0), \nu(x_0 * y_0)) \leq \vee\,(\nu(x_0), \nu(y_0))$ *for all* $x_0, y_0 \in \urcorner$.

Theorem 1. Let C_1 and C_2 be two cubic subinclines of $\urcorner$, then so is $C_1 \cap C_2$.

Proof: Let C_1 and C_2 be two cubic subinclines of $\urcorner$, *for all* $x_0, y_0 \in \urcorner$.

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Since $\overline{\gamma}_{(C_1)}$ & $\overline{\gamma}_{(C_2)}$ are interval valued fuzzy sets on C_1 and C_2

$$\overline{\wedge}\ (\overline{\wedge}\ (\overline{\gamma}_{(C_1\cap C_2)}\ (x_0+y_0\ ), \overline{\gamma}_{(C_1\cap C_2)}\ (x_0*y_0\ )))$$

$$= \overline{\wedge}\ (\overline{\wedge}\ (\overline{\gamma}_{(C_1)}\ (x_0+y_0\ ), \overline{\gamma}_{(C_2)}(x_0+y_0)), \overline{\wedge}\ (\overline{\gamma}_{(C_1)}\ (x_0*y_0), \overline{\gamma}_{(C_2)}\ (x_0*y_0\ )))$$

$$= \overline{\wedge}\ (\overline{\wedge}\ (\overline{\gamma}_{(C_1)}(x_0+y_0\ ), \overline{\gamma}_{(C_1)}\ (x_0*y_0\ )),\ \overline{\wedge}\ (\overline{\gamma}_{(C_2)}\ (x_0+y_0\ ), \overline{\gamma}_{(C_2)}\ (x_0*y_0\ )))$$

$$\geq \overline{\wedge}\ \left(\overline{\wedge}\left(\overline{\gamma}_{(C_1)}\ (x_0\ ), \overline{\gamma}_{(C_1)}\ (y_0\ )\right), \overline{\wedge}\left(\ \overline{\gamma}_{(C_2)}\ (x_0\ ), \overline{\gamma}_{(C_2)}\ (y_0\ )\right)\right)$$

$$\geq \overline{\wedge}\ \left(\overline{\wedge}\left(\overline{\gamma}_{(C_1)}\ (x_0), \overline{\gamma}_{(C_2)}\ (x_0)\right),\ \overline{\wedge}\left(\ \overline{\gamma}_{(C_1)}\ (y_0)\ , \overline{\gamma}_{(C_2)}(y_0)\right)\right)$$

$$= \overline{\wedge}\ (\ \overline{\gamma}_{(C_1\cap C_2)}\ (x_0\ ), \overline{\gamma}_{(C_1\cap C_2)}\ (y_0\ ))$$

$$\overline{\wedge}\ (\overline{\wedge}\ (\ \overline{\gamma}_{(C_1\cap C_2)}\ (x_0+y_0\ ), \overline{\gamma}_{(C_1\cap C_2)}\ (x_0*y_0\ ))) \geq \overline{\wedge}\ (\ \overline{\gamma}_{(C_1\cap C_2)}\ (x_0\ ), \overline{\gamma}_{(C_1\cap C_2)}\ (y_0)).$$

Similarly, $\upsilon_{(C_1)}$ *and* $\upsilon_{(C_2)}$ *are fuzzy sets on* C_1 *and* C_2.

$$\curlyvee\ (\curlyvee\ (\ \upsilon_{(C_1\cap C_2)}\ (x_0+y_0\ ), \upsilon_{(C_1\cap C_2)}\ (x_0*y_0\ )))$$

$$= \curlyvee\ (\curlyvee\ \left(\upsilon_{(C_1)}\ (x_0+y_0\ ), \upsilon_{(C_2)}(x_0+y_0)\right), \curlyvee\ (\upsilon_{(C_1)}\ (x_0*y_0), \upsilon_{(C_2)}\ (x_0*y_0\ )))$$

$$= \curlyvee\ (\curlyvee\ (\upsilon_{(C_1)}(x_0+y_0\ ), \upsilon_{(C_1)}\ (x_0*y_0\ )),\ \curlyvee\ (\upsilon_{(C_2)}\ (x_0+y_0\ ), \upsilon_{(C_2)}\ (x_0*y_0\ )))$$

$$\leq \curlyvee\ \left(\curlyvee\left(\upsilon_{(C_1)}\ (x_0\ ), \upsilon_{(C_1)}\ (y_0\ )\right), \curlyvee\left(\ \upsilon_{(C_2)}\ (x_0\ ), \upsilon_{(C_2)}\ (y_0\ )\right)\right)$$

$$\leq \curlyvee\ \left(\curlyvee\left(\upsilon_{(C_1)}\ (x_0), \upsilon_{(C_2)}\ (x_0)\right), \curlyvee\left(\ \upsilon_{(C_1)}\ (y_0)\ , \upsilon_{(C_2)}(y_0)\right)\right)$$

$$= \curlyvee\ (\ \upsilon_{(C_1\cap C_2)}\ (x_0\ ), \upsilon_{(C_1\cap C_2)}\ (y_0\ ))$$

$$\curlyvee\left(\curlyvee\left(\ \upsilon_{(C_1\cap C_2)}\ (x_0+y_0\ ), \upsilon_{(C_1\cap C_2)}\ (x_0*y_0\ )\right)\right) \leq \curlyvee\left(\ \upsilon_{(C_1\cap C_2)}\ (x_0\ ), \upsilon_{(C_1\cap C_2)}\ (y_0)\right).$$

Thus $C_1 \cap C_2$ is a cubic subincline of ٦ .

Theorem 2. Let the cubic set $\mathscr{C} = \{\overline{\gamma}_{\mathscr{C}},\ \nu_{\mathscr{C}}\}$ be a cubic subincline of ٦ $\Leftrightarrow$ $\overline{\gamma}_{\mathscr{C}}$ is an interval valued fuzzy and $\nu_{\mathscr{C}}$ are anti fuzzy subincline of ٦.

Proof: Let the cubic set $\mathscr{C} = \{\ \overline{\gamma}_{\mathscr{C}},\ \nu_{\mathscr{C}}\}$ be a cubic subincline of ٦ and take $x_0\ ,\ y_0\ \in$ ٦.

Then to prove $\overline{\wedge}\left(\overline{\gamma}_{\mathscr{C}}(x_0+y_0), \overline{\gamma}_{\mathscr{C}}(x_0*y_0)\right) \geq \overline{\wedge}\left(\overline{\gamma}_{\mathscr{C}}(x_0), \overline{\gamma}_{\mathscr{C}}(y_0)\right)$:

$$\left(\overline{\gamma}_{\mathscr{C}}(x_0+y_0)\right) = \left[\ \gamma^L_{\mathscr{C}}(x_0+y_0)\ , \gamma^U_{\mathscr{C}}(x_0+y_0)\right]$$

$$\geq [\ \wedge [\gamma^L_{\mathscr{C}}(x_0), \gamma^L_{\mathscr{C}}(y_0)],\ \wedge \left[\gamma^U_{\mathscr{C}}(x_0), \gamma^U_{\mathscr{C}}(y_0)\right]]$$

$$= [\ \wedge [\gamma^L_{\mathscr{C}}(x_0), \gamma^U_{\mathscr{C}}(x_0)],\ \wedge \left[\gamma^L_{\mathscr{C}}(y_0), \gamma^U_{\mathscr{C}}(y_0)\right]]$$

$$\geq \overline{\wedge}\left(\overline{\gamma}_{\mathscr{C}}(x_0), \overline{\gamma}_{\mathscr{C}}(y_0)\right)$$

$\barwedge\left(\bar{\gamma}_{\mathscr{C}}(x_0+y_0)\right) \geq \barwedge\left(\bar{\gamma}_{\mathscr{C}}(x_0), \bar{\gamma}_{\mathscr{C}}(y_0)\right)$.

Similarly, $\barwedge\left(\bar{\gamma}_{\mathscr{C}}(x_0 * y_0)\right) \geq \barwedge\left(\bar{\gamma}_{\mathscr{C}}(x_0), \bar{\gamma}_{\mathscr{C}}(y_0)\right)$

$$\barwedge\left(\bar{\gamma}_{\mathscr{C}}(x_0+y_0), \bar{\gamma}_{\mathscr{C}}(x_0 * y_0)\right) \geq \barwedge\left(\barwedge\left(\bar{\gamma}_{\mathscr{C}}(x_0), \bar{\gamma}_{\mathscr{C}}(y_0)\right), \barwedge\left(\bar{\gamma}_{\mathscr{C}}(x_0), \bar{\gamma}_{\mathscr{C}}(y_0)\right)\right)$$

$$\barwedge\left(\bar{\gamma}_{\mathscr{C}}(x_0+y_0), \bar{\gamma}_{\mathscr{C}}(x_0 * y_0)\right) \geq \barwedge\left(\bar{\gamma}_{\mathscr{C}}(x_0), \bar{\gamma}_{\mathscr{C}}(y_0)\right).$$

To prove $v_{\mathscr{C}}$ is an anti-fuzzy subincline.

$$\vee\left(v_{\mathscr{C}}(x_0+y_0), v_{\mathscr{C}}(x_0 * y_0)\right) = \vee\left((1- v'_{\mathscr{C}}(x_0+y_0)), (1- v'_{\mathscr{C}}(x_0 * y_0))\right)$$

$$= \vee\left((1-(v'_{\mathscr{C}}(x_0+y_0), v'_{\mathscr{C}}(x_0 * y_0))\right)$$

$$= 1-\vee\left((v'_{\mathscr{C}}(x_0+y_0), v'_{\mathscr{C}}(x_0 * y_0))\right)$$

$$\geq 1-\vee\left((v'_{\mathscr{C}}(x_0), v'_{\mathscr{C}}(y_0))\right)$$

$$\geq \vee\left((1-v'_{\mathscr{C}}(x_0), 1- v'_{\mathscr{C}}(y_0))\right)$$

$$= \vee\left(v_{\mathscr{C}}(x_0), v_{\mathscr{C}}(y_0)\right).$$

Therefore, $v_{\mathscr{C}}$ is an anti-fuzzy subincline of $\daleth$.

Conversely, $\bar{\gamma}_{\mathscr{C}}$ is aninterval valued fuzzy subincline of $\daleth$ and $v_{\mathscr{C}}$ is an anti-fuzzy subincline of $\daleth$.

To prove $(\bar{\gamma}_{\mathscr{C}}, v_{\mathscr{C}})$ is a cubic subincline of $\daleth$.

Therefore, the proof is obvious by the definition of a cubic subincline.

$\mathscr{C} = (\bar{\gamma}_{\mathscr{C}}, v_{\mathscr{C}})$ is a cubic subincline of $\daleth$.

Definition 8. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two cubic fuzzy sets of $\daleth$ and $\beth$ respectively. The direct product of $\mathscr{C}_1 \times \mathscr{C}_2 : \daleth \times \beth \rightarrow [0,1]$ of $\mathscr{C}_1$ and $\mathscr{C}_2$ is defined by $\bar{\gamma}_{\mathscr{C}_1 \times \mathscr{C}_2}(x_0, y_0) = \barwedge\ (\bar{\gamma}_{\mathscr{C}_1}(x_0), \bar{\gamma}_{\mathscr{C}_2}(y_0))$ and $v_{\mathscr{C}_1 \times \mathscr{C}_2}(x_0, y_0) = \vee\ (v_{\mathscr{C}_1}(x_0), v_{\mathscr{C}_2}(y_0))$.

Lemma 1. Let $(\daleth, +, *)$ and $(\beth, +, *)$ be two incline algebra, for all $(x_1, y_1), (x_2, y_2) \in \daleth \times \beth$, define $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2), (y_1 + y_2)$; $(x_1, y_1) * (x_2, y_2) = (x_1 * x_2), (y_1 * y_2)$. Then $(\daleth \times \beth, +, *)$ is also an incline algebra.

Theorem 3. Let $\mathscr{C}_1$ & $\mathscr{C}_2$ be two cubic subincline of $\daleth$ and $\beth$ respectively, then $\mathscr{C}_1 \times \mathscr{C}_2$ is a cubic subincline of $\daleth \times \beth$.

Proof: Let $\bar{\gamma}_{\mathscr{C}_1}, \bar{\gamma}_{\mathscr{C}_2}$ be two interval valued fuzzy cubic subincline of $\daleth$ and $\beth$ respectively.

Now, $\barwedge\left(\bar{\gamma}_{\mathscr{C}_1 \times \mathscr{C}_2}(x_1, y_1) + (x_2, y_2), \bar{\gamma}_{\mathscr{C}_1 \times \mathscr{C}_2}(x_1, y_1) * (x_2, y_2)\right)$

$$= \barwedge\left(\bar{\gamma}_{\mathscr{C}_1 \times \mathscr{C}_2}(x_1 + x_2), (y_1 + y_2), \bar{\gamma}_{\mathscr{C}_1 \times \mathscr{C}_2}(x_1 * x_2), (y_1 * y_2)\right)$$

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$$= \barwedge \Big(\big(\overline{\gamma}_{\mathscr{C}_1}(x_1+x_2),\ \overline{\gamma}_{\mathscr{C}_2}(y_1+y_2)\big),\big(\overline{\gamma}_{\mathscr{C}_1}(x_1*x_2),\ \overline{\gamma}_{\mathscr{C}_2}(y_1*y_2)\big)\Big)$$

$$= \barwedge \Big(\barwedge \big(\overline{\gamma}_{\mathscr{C}_1}(x_1+x_2),\ \overline{\gamma}_{\mathscr{C}_1}(x_1*x_2)\big), \barwedge \big(\overline{\gamma}_{\mathscr{C}_2}(y_1+y_2),\ \overline{\gamma}_{\mathscr{C}_2}(y_1*y_2)\big)\Big)$$

$$\geq \barwedge \Big(\barwedge \big(\overline{\gamma}_{\mathscr{C}_1}(x_1),\ \overline{\gamma}_{\mathscr{C}_1}(x_2)\big),\ \barwedge \big(\overline{\gamma}_{\mathscr{C}_2}(y_1),\ \overline{\gamma}_{\mathscr{C}_2}(y_2)\big)\Big)$$

$$= \barwedge \Big(\barwedge \big(\overline{\gamma}_{\mathscr{C}_1}(x_1), \overline{\gamma}_{\mathscr{C}_2}(y_1)\big),\ \barwedge \big(\overline{\gamma}_{\mathscr{C}_1}(x_2),\ \overline{\gamma}_{\mathscr{C}_2}(y_2)\big)\Big)$$

$$= \barwedge \ \big(\overline{\gamma}_{\mathscr{C}_1\times\mathscr{C}_2}(x_1,y_1),\ \overline{\gamma}_{\mathscr{C}_1\times\mathscr{C}_2}(x_2,y_2)\big)$$

$$\barwedge \Big(\overline{\gamma}_{\mathscr{C}_1\times\mathscr{C}_2}(x_1,y_1)+\ (x_2,y_2),\overline{\gamma}_{\mathscr{C}_1\times\mathscr{C}_2}(x_1,y_1)*\ (x_2,y_2)\Big) \geq \barwedge\ \big(\overline{\gamma}_{\mathscr{C}_1\times\mathscr{C}_2}(x_1,y_1),\ \overline{\gamma}_{\mathscr{C}_1\times\mathscr{C}_2}(x_2,y_2)\big).$$

Let $\upsilon_{\mathscr{C}_1}$ and $\upsilon_{\mathscr{C}_2}$be fuzzy subincline of $\daleth$ and $\beth$ respectively.

$$\curlyvee \Big(\upsilon_{\mathscr{C}_1\times\mathscr{C}_2}(x_1,y_1)+\ (x_2,y_2),\upsilon_{\mathscr{C}_1\times\mathscr{C}_2}(x_1,y_1)*\ (x_2,y_2)\Big)$$

$$= \curlyvee \Big(\upsilon_{\mathscr{C}_1\times\mathscr{C}_2}(x_1+x_2),\ (y_1+y_2),\upsilon_{\mathscr{C}_1\times\mathscr{C}_2}(x_1*x_2),(y_1*y_2)\Big)$$

$$=\curlyvee \Big(\big(\upsilon_{\mathscr{C}_1}(x_1+x_2),\ \upsilon_{\mathscr{C}_2}(y_1+y_2)\big),\big(\upsilon_{\mathscr{C}_1}(x_1*x_2),\upsilon_{\mathscr{C}_2}(y_1*y_2)\big)\Big)$$

$$= \curlyvee \Big(\barwedge \big(\upsilon_{\mathscr{C}_1}(x_1+x_2),\upsilon_{\mathscr{C}_1}(x_1*x_2)\big), \curlyvee \big(\upsilon_{\mathscr{C}_2}(y_1+y_2),\upsilon_{\mathscr{C}_2}(y_1*y_2)\big)\Big)$$

$$\leq \curlyvee \Big(\curlyvee \big(\upsilon_{\mathscr{C}_1}(x_1),\ \upsilon_{\mathscr{C}_1}(x_2)\big),\ \curlyvee \big(\upsilon_{\mathscr{C}_2}(y_1),\upsilon_{\mathscr{C}_2}(y_2)\big)\Big)$$

$$=\curlyvee \Big(\curlyvee \big(\upsilon_{\mathscr{C}_1}(x_1),\upsilon_{\mathscr{C}_2}(y_1)\big),\ \curlyvee \big(\upsilon_{\mathscr{C}_1}(x_2),\upsilon_{\mathscr{C}_2}(y_2)\big)\Big)$$

$$= \curlyvee \big(\upsilon_{\mathscr{C}_1\times\mathscr{C}_2}(x_1,y_1),\ \upsilon_{\mathscr{C}_1\times\mathscr{C}_2}(x_2,y_2)\big)$$

$$\curlyvee \Big(\upsilon_{\mathscr{C}_1\times\mathscr{C}_2}(x_1,y_1)+(x_2,y_2),\upsilon_{\mathscr{C}_1\times\mathscr{C}_2}(x_1,y_1)*\ (x_2,y_2)\Big) \leq \curlyvee\ \big(\upsilon_{\mathscr{C}_1\times\mathscr{C}_2}(x_1,y_1),\ \upsilon_{\mathscr{C}_1\times\mathscr{C}_2}(x_2,y_2)\big).$$

Thus, $\mathscr{C}_1 \times \mathscr{C}_2$ is also a cubic subincline of $\daleth \times \beth$.

Definition 9. A mapping $f : \daleth \to \beth$. Let $\mathscr{C}$ is a cubic subincline of $\daleth$ then the image of a cubic subincline of $\beth$ is defined as $f(\mathscr{C}) = (\overline{\gamma}_f, \upsilon_f)$ where $\overline{\gamma}_f(y_0) = \barwedge\ \{\overline{\gamma}_{\mathscr{C}}(x_0)/ f(x_0) = y_0\}$, $\upsilon_f(y_0) = \curlyvee\ \{\upsilon_{\mathscr{C}}(x_0)/ f(x_0) = y_0\}$.

Theorem 4. Let $f : \daleth \to \beth$ be a mapping and $\mathscr{C} = \{\overline{\gamma}_{\mathscr{C}}, \nu_{\mathscr{C}}\}$ is an cubic subincline of $\daleth$, then $f(\mathscr{C})$ is also an cubic subincline of $\beth$ where $f(\mathscr{C}) = (\overline{\gamma}_f, \upsilon_f)$ satisfies $\overline{\gamma}_f(y_0) = \barwedge\ \{\overline{\gamma}_{\mathscr{C}}(x_0)/ f(x_0) = y_0\}$ and $\upsilon_f(y_0) = \curlyvee\ \{\upsilon_{\mathscr{C}}(x_0)/ f(x_0) = y_0\}$.

Proof: Given $f : \daleth \to \beth$ be a mapping and $x_1, x_2 \in \beth$ such that $f(x_1) = y_1$, $f(x_2) = y_2$ *for all* $y_1, y_2 \in \beth$.

Since $\mathscr{C}$ is a cubic subincline of $\daleth$, $\barwedge\big(\overline{\gamma}(x_1+x_2), \overline{\gamma}(x_1*x_2)\big) \geq \barwedge\big(\overline{\gamma}(x_1), \overline{\gamma}(x_2)\big)$.

$\barwedge = \barwedge\ \{\overline{\gamma}_{\mathscr{C}}(x_0)/ f(x_0) = y_1 + y_2\}$ where $x_0 = x_1 + x_2$,

$\overline{\gamma}_f(y_1 + y_2) = \barwedge\ \{\overline{\gamma}_{\mathscr{C}}(x_1 + x_2)/ f(x_1 + x_2) = y_1 + y_2\}$,

$\overline{\gamma}_f(y_1 + y_2) = \barwedge\ \{\overline{\gamma}_{\mathscr{C}}(x_1 + x_2)/ f(x_1) + f(x_2) = y_1 + y_2\}$,

$\overline{\gamma}_f(y_1 * y_2) = \barwedge\ \{\overline{\gamma}_{\mathscr{C}}(x_0)/ f(x_0) = y_1 * y_2\}$ where $x_0 = x_1 * x_2$,

$\overline{\gamma}_f(y_1 * y_2) = \barwedge\ \{\overline{\gamma}_{\mathscr{C}}(x_1 * x_2)/ f(x_1 * x_2) = y_1 * y_2\}$,

$\overline{\gamma}_f(y_1 * y_2) = \barwedge\ \{\overline{\gamma}_{\mathscr{C}}(x_1 * x_2)/ f(x_1) * f(x_2) = y_1 * y_2\}$,

$\barwedge\ (\overline{\gamma}_f(y_1 + y_2), \overline{\gamma}_f(y_1 * y_2)) \geq \barwedge\ (\barwedge\ \{\overline{\gamma}_{\mathscr{C}}(x_1 + x_2)/ f(x_1) + f(x_2) = y_1 + y_2\}$,

$\barwedge\ \{\overline{\gamma}_{\mathscr{C}}(x_1 * x_2)/ f(x_1) * f(x_2) = y_1 * y_2\})$

$= \barwedge\ (\barwedge\ \{\overline{\gamma}_{\mathscr{C}}(x_1 + x_2), \overline{\gamma}_{\mathscr{C}}(x_1 * x_2)/ f(x_1) + f(x_2) = y_1 + y_2, f(x_1) * f(x_2) = y_1 * y_2\})$

$= \barwedge\ (\barwedge\ \{\overline{\gamma}_{\mathscr{C}}(x_1), \overline{\gamma}_{\mathscr{C}}(x_2)/ f(x_1) + f(x_2) = y_1 + y_2, f(x_1) * f(x_2) = y_1 * y_2\})$

$= \barwedge\ (\barwedge\ \{\overline{\gamma}_{\mathscr{C}}(x_1), \overline{\gamma}_{\mathscr{C}}(x_2)/ f(x_1) = y_1, f(x_2) = y_2\})$

$= \barwedge\ (\barwedge\ \{\overline{\gamma}_{\mathscr{C}}(x_1)/ f(x_1) = y_1\}, \barwedge\ \{\overline{\gamma}_{\mathscr{C}}(x_2)/, f(x_2) = y_2\})$

$= \barwedge\ \{\overline{\gamma}_f(y_1), \overline{\gamma}_f(y_2)\}$

$\barwedge\ (\overline{\gamma}_f(y_1 + y_2), \overline{\gamma}_f(y_1 * y_2)) \geq \barwedge\ \{\overline{\gamma}_f(y_1), \overline{\gamma}_f(y_2)\}$.

The same procedure is followed for the falsity membership function.

Definition 10. A mapping f on $\daleth$, if $\mathscr{C}_0 = \left\{\overline{\gamma}_{\mathscr{C}_0}, \nu_{\mathscr{C}_0}\right\}$ is a cubic fuzzy set in $f(\daleth)$, $\mathscr{C} = \{\overline{\gamma}_{\mathscr{C}}, \nu_{\mathscr{C}}\}$ is a cubic fuzzy set in $\daleth$ then cubic fuzzy set $\mathscr{C} = \{\overline{\gamma}_{\mathscr{C}}, \nu_{\mathscr{C}}\} = \mathscr{C}_0 \circ f$, $\overline{\gamma}_{\mathscr{C}}(x_0) = (\overline{\gamma}_{\mathscr{C}_0} \circ f)(x_0) = \overline{\gamma}_{\mathscr{C}_0}(f(x_0))$ and $\nu_{\mathscr{C}}(x_0) = (\nu_{\mathscr{C}_0} \circ f)(x_0) = \nu_{\mathscr{C}_0}(f(x_0))$ in $\daleth$ is called preimage of $\mathscr{C} = \{\overline{\gamma}_{\mathscr{C}}, \nu_{\mathscr{C}}\}$ under f.

Theorem 5. An epimorphism preimage of a cubic subincline of $\daleth$ is a cubic subincline.

Proof: Let $f: \daleth \to \beth$ be an epimorphism ($\daleth, \beth$ are inclines) $\mathscr{C}_0 = \left\{\overline{\gamma}_{\mathscr{C}_0}, \nu_{\mathscr{C}_0}\right\}$ is a cubic fuzzy set in $f(\daleth)$ and $\mathscr{C} = \{\overline{\gamma}_{\mathscr{C}}, \nu_{\mathscr{C}}\}$ be an inverse image of $\beth$ under f. For $x_1, x_2 \in \daleth$,

$\overline{\gamma}_{\mathscr{C}}(x_1 + x_2) = (\overline{\gamma}_{\mathscr{C}_0} \circ f)(x_1 + x_2)$

$= \overline{\gamma}_{\mathscr{C}_0}(f(x_1 + x_2))$

$= \overline{\gamma}_{\mathscr{C}_0}(f(x_1) + f(x_2))$

$\overline{\gamma}_{\mathscr{C}}(x_1 * x_2) = (\overline{\gamma}_{\mathscr{C}_0} \circ f)(x_1 * x_2)$

$= \overline{\gamma}_{\mathscr{C}_0}(f(x_1 * x_2))$

$= \overline{\gamma}_{\mathscr{C}_0}(f(x_1) * f(x_2))$

$\barwedge\ (\overline{\gamma}_{\mathscr{C}}(x_1 + x_2), \overline{\gamma}_{\mathscr{C}}(x_1 * x_2)) = \barwedge\ (\overline{\gamma}_{\mathscr{C}_0}(f(x_1) + f(x_2)), \overline{\gamma}_{\mathscr{C}_0}(f(x_1) * f(x_2)))$

$\geq \barwedge\ (\overline{\gamma}_{\mathscr{C}_0}(f(x_1)), \overline{\gamma}_{\mathscr{C}_0}(f(x_2)))$

$\geq \barwedge\ (\overline{\gamma}_{\mathscr{C}}(x_1), \overline{\gamma}_{\mathscr{C}}(x_2))$

$\nu_{\mathscr{C}}(x_1 + x_2) = (\nu_{\mathscr{C}_0} \circ f)(x_1 + x_2)$

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$= \nu_{\mathscr{C}_0} (f\,(x_1 + x_2))$

$= \nu_{\mathscr{C}_0} (f\,(x_1) + f\,(x_2))$

$\nu_{\mathscr{C}}\,(x_1 * x_2) = (\nu_{\mathscr{C}_0} \circ f)(x_1 * x_2)$

$= \nu_{\mathscr{C}_0} (f\,(x_1 * x_2))$

$= \nu_{\mathscr{C}_0} (f\,(x_1) * f\,(x_2))$

$\vee\,(\nu_{\mathscr{C}}\,(x_1 + x_2), \nu_{\mathscr{C}}\,(x_1 * x_2)) = \vee\,(\nu_{\mathscr{C}_0} (f\,(x_1) + f\,(x_2)), \nu_{\mathscr{C}_0} (f\,(x_1) * f\,(x_2)))$

$\leq \vee\,(\nu_{\mathscr{C}_0} (f\,(x_1)), \nu_{\mathscr{C}_0} (f\,(x_2)))$

$\leq \vee\,(\nu_{\mathscr{C}}\,(x_1), \overline{\gamma}_{\mathscr{C}}\,(x_2))$.

Definition 11. Let $\mathscr{C} = \{\overline{\gamma}_{\mathscr{C}}, \nu_{\mathscr{C}}\}$ be cubic set of $\daleth$ and the cubic level subset is defined as $\mathscr{C}_{(\overline{\alpha},\beta)} = \{x_0 \in \daleth \,/\, \overline{\gamma}_{\mathscr{C}}\,(x_0) \geq \overline{\alpha}, \nu_{\mathscr{C}}\,(x_0) \leq \beta\}$.

Theorem 6. For a cubic set $\mathscr{C} = \{\overline{\gamma}_{\mathscr{C}}, \nu_{\mathscr{C}}\} \in \mathscr{C}\,(\daleth)$, the following are equivalent ($\mathscr{C}\,(\daleth)$ is the family of cubic sets in a set $\daleth$).

I. $\mathscr{C} = \{\overline{\gamma}_{\mathscr{C}}, \nu_{\mathscr{C}}\}$ is a cubic subincline of $\daleth$.
II. The non-empty cubic level set of $\mathscr{C} = \{\overline{\gamma}_{\mathscr{C}}, \nu_{\mathscr{C}}\}$ is a subincline of $\daleth$.

Proof: Assume that $\mathscr{C} = \{\overline{\gamma}_{\mathscr{C}}, \nu_{\mathscr{C}}\}$ is a cubic subincline of $\daleth$.

Let $x_0, y_0 \in \mathscr{C}\,(\mathscr{C}, (\overline{\alpha}, \beta))$ for all $\overline{\alpha} \in$ D[0, 1] ; $\beta \in [0, 1]$.

Then $\overline{\gamma}_{\mathscr{C}}\,(x_0) \geq \overline{\alpha}$, $\overline{\gamma}_{\mathscr{C}}\,(y_0) \geq \overline{\alpha}$; $\nu_{\mathscr{C}}\,(x_0) \leq \beta$, $\nu_{\mathscr{C}}\,(y_0) \leq \beta$.

By the definition of subincline

$\barwedge\,(\overline{\gamma}_{\mathscr{C}}\,(x_0 + y_0), \overline{\gamma}_{\mathscr{C}}\,(x_0 * y_0)) \geq \barwedge\,(\overline{\gamma}_{\mathscr{C}}\,(x_0), \overline{\gamma}_{\mathscr{C}}\,(y_0))$

$= \barwedge\,(\overline{\alpha}, \overline{\alpha}) = \overline{\alpha}$

$\vee\,(\nu_{\mathscr{C}}\,(x_0 + y_0), \nu_{\mathscr{C}}\,(x_0 * y_0)) \leq \vee\,(\nu_{\mathscr{C}}\,(x_0), \nu_{\mathscr{C}}\,(y_0))$

$= \vee\,(\beta, \beta) = \beta$.

So that $x_0 + y_0$ & $x_0 * y_0 \in \mathscr{C}\left(\mathscr{C}, (\overline{\alpha}, \beta)\right)$.

Therefore, the non-empty cubic level subset of $\mathscr{C} = \{\overline{\gamma}_{\mathscr{C}}, \nu_{\mathscr{C}}\}$ is a subincline of $\daleth$.

Conversely, assume that $\mathscr{C}\left(\mathscr{C}, (\overline{\alpha}, \beta)\right)$ is a subincline of $\daleth$ for all $\beta \in [0, 1]$; $\overline{\alpha} \in$ D[0, 1], with $\mathscr{C}\left(\mathscr{C}, (\overline{\alpha}, \beta)\right) \neq \emptyset$:

I. Suppose that $\mathscr{C}$ is not a cubic set and to prove $\mathscr{C}$ is a cubic subincline of $\daleth$. There exists $\overline{\alpha}' \in$ D[0, 1] and $x_0, y_0 \in \daleth$ such that

$\barwedge\,(\overline{\gamma}_{\mathscr{C}}\,(x_0 + y_0), \overline{\gamma}_{\mathscr{C}}\,(x_0 * y_0)) < \overline{\alpha}'$

$\leq \barwedge\ (\bar{\gamma}_{\mathscr{C}}\ (x_0)\ ,\ \bar{\gamma}_{\mathscr{C}}\ (y_0))$

$\vee\ (\nu_{\mathscr{C}}\ (x_0 +\ y_0\ )\ , \nu_{\mathscr{C}}\ (\ x_0 *\ y_0)) \leq \vee\ (\nu_{\mathscr{C}}\ (x_0)\ ,\ \nu_{\mathscr{C}}\ (y_0)),$

which implies that $x_0\ , y_0 \in \mathscr{C}\ \ (\mathscr{C}\ ; \bar{\alpha}', \vee\ (\nu_{\mathscr{C}}(x_0)\ , \nu_{\mathscr{C}}\ (y_0))),$

but $(x_0 +\ y_0)\ , (x_0 {*} y_0)\ \notin \mathscr{C}(\mathscr{C}\ ; \bar{\alpha}', \vee\ (\nu_{\mathscr{C}}\ (x_0\ )\ , \nu_{\mathscr{C}}\ (y_0)))$.

This is a contradiction.

II. Now assume that $\mathscr{C}$ is a cubic set but $\mathscr{C}$ is not a cubic subincline, then

$\barwedge\ (\bar{\gamma}_{\mathscr{C}}\ (x_0 +\ y_0\ )\ , \bar{\gamma}_{\mathscr{C}}\ (x_0 *\ y_0)) \leq \barwedge\ (\bar{\gamma}_{\mathscr{C}}\ (x_0)\ ,\ \bar{\gamma}_{\mathscr{C}}\ (y_0))$ and $\vee\ (\nu_{\mathscr{C}}\ (x_0 +\ y_0\ )\ , \nu_{\mathscr{C}}\ (x_0 *\ y_0)) >\ \beta' >$ $\vee\ (\nu_{\mathscr{C}}\ (x_0)\ ,\ \nu_{\mathscr{C}}\ (y_0))$ for some $\beta'\ \in [0,1]$ and $x_0\ , y_0 \in \urcorner$.

Thus, $x_0\ , y_0 \in \mathscr{C}\ \ (\mathscr{C}\ ;\ (\ \bar{\gamma}_{\mathscr{C}}\ (x_0)\ ,\ \bar{\gamma}_{\mathscr{C}}\ (y_0)), \beta'\ ),$

but $(x_0 +\ y_0)\ , (x_0 {*} y_0)\ \notin \mathscr{C}\ \ (\mathscr{C}\ ;\ (\ \bar{\gamma}_{\mathscr{C}}\ (x_0)\ ,\ \bar{\gamma}_{\mathscr{C}}\ (y_0)), \beta'\ )$ which is a contradiction.

III. Assume that there exists $\bar{\alpha}' \in$ D[0, 1] , $\beta'\ \in [0,1]$ and $x_0\ , y_0 \in \urcorner$ such that

$\barwedge\ (\bar{\gamma}_{\mathscr{C}}\ (x_0 +\ y_0\ )\ , \bar{\gamma}_{\mathscr{C}}\ (\ x_0 *\ y_0))\ < \bar{\alpha}'\ \leq \barwedge\ (\bar{\gamma}_{\mathscr{C}}\ (x_0)\ ,\ \bar{\gamma}_{\mathscr{C}}\ (y_0))$ and

$\vee\ (\nu_{\mathscr{C}}\ (x_0 +\ y_0\ )\ , \nu_{\mathscr{C}}\ (x_0 *\ y_0)) >\ \beta'\ > \vee\ (\nu_{\mathscr{C}}\ (x_0)\ ,\ \nu_{\mathscr{C}}\ (y_0)).$

Then $x_0\ , y_0 \in \mathscr{C}\ \ (\mathscr{C}\ ; \bar{\alpha}', \beta')$ but $(x_0 +\ y_0)\ , (x_0 {*} y_0)\ \notin \mathscr{C}\ \ (\mathscr{C}\ ; \bar{\alpha}', \beta')$.

This is a contradiction.

Hence $\mathscr{C}$ is a cubic set and $\mathscr{C}$ is a cubic subincline of $\urcorner$.

Therefore, $\mathscr{C}\ = \{\ \bar{\gamma}_{\mathscr{C}},\ \nu_{\mathscr{C}}\}$ is a cubic subincline of $\urcorner$.

Theorem 7. For a subset $\mathscr{M}$ of $\urcorner$, let $\mathscr{C}\ = \{\ \bar{\gamma}_{\mathscr{C}},\ \nu_{\mathscr{C}}\}\ \in \mathscr{C}\ (\urcorner)$ be defined by

$$\bar{\gamma}_{\mathscr{C}}\ (x_0) = \begin{cases} \bar{\alpha}, & \text{if } x_0 +\ y_0, \quad x_0 * y_0\ \in\ \mathscr{M}, \\ \bar{0}, & \text{otherwise,} \end{cases}$$

and

$$\nu_{\mathscr{C}}\ (x_0) = \begin{cases} 1, & \text{if } x_0 +\ y_0, \quad x_0 * y_0\ \in\ \mathscr{M}, \\ \beta, & \text{otherwise,} \end{cases}$$

where $\bar{\alpha} \in$ D [0, 1] , $\beta\ \in$ [0, 1] with $<\ .\ \alpha_2$ Then

I. If $\mathscr{M}$ is a subincline of $\urcorner$ then $\mathscr{C}\ = \{\ \bar{\gamma}_{\mathscr{C}},\ \nu_{\mathscr{C}}\}$ is a cubic subincline of $\urcorner$ and $\mathscr{C}\ \left(\mathscr{C}\ , (\bar{\alpha}\ , \beta)\right) = \mathscr{M}$.

II. If $\mathscr{C}\ = \{\ \bar{\gamma}_{\mathscr{C}},\ \nu_{\mathscr{C}}\}$ is a cubic subincline of $\urcorner$, then $\mathscr{M}$ is a subincline of $\urcorner$.

Proof: Assume $\mathscr{M}$ is a subincline of $\urcorner$.

Obviously, $\mathscr{C}\ \left(\mathscr{C}\ , (\bar{\alpha}\ , \beta)\right) = \mathscr{M}$.

Let $x_0, y_0\ \in\ \urcorner$ if $x_0\ ,\ y_0\ \in\ \mathscr{M}$ then $x_0 + y_0\ ,\ x_0 * y_0\ \in\ \mathscr{M}$ and so

$\barwedge\ (\bar{\gamma}_{\mathscr{C}}\ (x_0 +\ y_0\ )\ , \bar{\gamma}_{\mathscr{C}}\ (x_0 *\ y_0)) =\ \bar{\alpha}$

$= \barwedge \{\bar{\alpha}, \bar{\alpha}\}$

$\leq \barwedge (\bar{\gamma}_{\mathscr{C}}(x_0), \bar{\gamma}_{\mathscr{C}}(y_0))$

$\curlyvee (\nu_{\mathscr{C}}(x_0 + y_0), \nu_{\mathscr{C}}(x_0 * y_0)) = 1 = \curlyvee (\nu_{\mathscr{C}}(x_0), \nu_{\mathscr{C}}(y_0)).$

I. If $x_0 + y_0$, $x_0 * y_0 \notin \mathscr{M}$ then

$\bar{\gamma}_{\mathscr{C}}(x_0 + y_0) = \bar{0} = \bar{\gamma}_{\mathscr{C}}(x_0 * y_0)$ and

$(\nu_{\mathscr{C}}(x_0 + y_0) = \beta = \nu_{\mathscr{C}}(x_0 * y_0)).$

Hence, $\barwedge(\bar{\gamma}_{\mathscr{C}}(x_0 + y_0), \bar{\gamma}_{\mathscr{C}}(x_0 * y_0)) \geq \bar{0}$

$= \barwedge(\bar{0}, \bar{0})$

$= \barwedge (\bar{\gamma}_{\mathscr{C}}(x_0), \bar{\gamma}_{\mathscr{C}}(y_0))$

$\curlyvee(\nu_{\mathscr{C}}(x_0 + y_0), \nu_{\mathscr{C}}(x_0 * y_0)) \geq \beta$

$= \curlyvee(\beta, \beta)$

$= \curlyvee (\nu_{\mathscr{C}}(x_0), \nu_{\mathscr{C}}(y_0)).$

II. If $x_0 + y_0$, $x_0 * y_0 \notin \mathscr{M}$ then

$\bar{\gamma}_{\mathscr{C}}(x_0 + y_0) = \bar{\alpha}$, $\bar{\gamma}_{\mathscr{C}}(x_0 * y_0) = \bar{0}$ and

$(\nu_{\mathscr{C}}(x_0 + y_0) = 1;\ \nu_{\mathscr{C}}(x_0 * y_0)) = \beta$

$\barwedge(\bar{\gamma}_{\mathscr{C}}(x_0 + y_0), \bar{\gamma}_{\mathscr{C}}(x_0 * y_0)) \geq \bar{0}$

$= \barwedge(\bar{0}, \bar{0})$

$= \barwedge (\bar{\gamma}_{\mathscr{C}}(x_0), \bar{\gamma}_{\mathscr{C}}(y_0))$

$\curlyvee(\nu_{\mathscr{C}}(x_0 + y_0), \nu_{\mathscr{C}}(x_0 * y_0)) \leq \beta$

$= \curlyvee(1, \beta)$

$= \curlyvee (\nu_{\mathscr{C}}(x_0), \nu_{\mathscr{C}}(y_0)).$

III. If $x_0 + y_0$, $x_0 * y_0 \notin \mathscr{M}$ then

$\bar{\gamma}_{\mathscr{C}}(x_0 + y_0) = \bar{0}$, $\bar{\gamma}_{\mathscr{C}}(x_0 * y_0) = \bar{\alpha}$ and

$(\nu_{\mathscr{C}}(x_0 + y_0) = \beta;\ \nu_{\mathscr{C}}(x_0 * y_0)) = 1$

$\barwedge(\bar{\gamma}_{\mathscr{C}}(x_0 + y_0), \bar{\gamma}_{\mathscr{C}}(x_0 * y_0)) \geq \bar{0}$

$= \barwedge(\bar{0}, \bar{0})$

$\leq \barwedge (\bar{\gamma}_{\mathscr{C}}(x_0), \bar{\gamma}_{\mathscr{C}}(y_0))$

$\curlyvee(\nu_{\mathscr{C}}(x_0 + y_0), \nu_{\mathscr{C}}(x_0 * y_0)) \leq \beta$

$= \curlyvee (1, \beta)$

$= \curlyvee \; (\nu_{\mathscr{C}}\,(x_0)\,, \nu_{\mathscr{C}}\,(y_0)\,).$

Therefore, $\mathscr{C} = \{\; \overline{\gamma}_{\mathscr{C}},\; \nu_{\mathscr{C}}\}$ is a cubic subincline of $\urcorner$.

Suppose that $\mathscr{C} = \{\; \overline{\gamma}_{\mathscr{C}},\; \nu_{\mathscr{C}}\}$ is a cubic subincline of $\urcorner$.

$x_0 + \; y_0\,,\; x_0 * y_0 \;\notin\; \mathscr{M}$ then $\overline{\gamma}_{\mathscr{C}}\;(x_0 + y_0\,) = \;\overline{\alpha}\; = \;\overline{\gamma}_{\mathscr{C}}\;(x_0 * \;y_0\,),$

And $\;(\,\nu_{\mathscr{C}}\;(\,x_0 + \;y_0\,) = 1 = \;\nu_{\mathscr{C}}\;(x_0 * \;y_0))$

$\overline{\curlywedge}\,(\,\overline{\gamma}_{\mathscr{C}}\;(x_0 + \;y_0\,)\,,\;\overline{\gamma}_{\mathscr{C}}\;(\,x_0 * \;y_0\,)\,) \geq \;(\overline{\gamma}_{\mathscr{C}}\;(\,x_0)\,, \overline{\gamma}_{\mathscr{C}}\;(\,y_0)\,)$

$= \overline{\curlywedge}\;\{\overline{\alpha}\,, \overline{\alpha}\;\}\;\; \geq \;\overline{\alpha}$

$\curlyvee\,(\,\nu_{\mathscr{C}}\;(\,x_0 + \;y_0\,), \nu_{\mathscr{C}}\;(\,x_0 * \;y_0\,)\,) \leq \;\; \curlyvee\,(\,\nu_{\mathscr{C}}\;(\,x_0\,), \nu_{\mathscr{C}}\;(\;y_0\,)\,)$

$= \curlyvee\,(1{,}1)\; = 1.$

Thus, $x_0 + \; y_0\,,\; x_0 * y_0 \;\notin\; \mathscr{M}$ and therefore, $\mathscr{M}$ is a subincline of $\urcorner$.

4 | Conclusion

The structure of cubic subincline was introduced in this paper as an extension of the interval valued fuzzy subincline of incline algebra and analyzed the study of cubic subincline using homomorphic image, preimage, cartesian product and the level subset. The same idea can also be applied and extended to many other substructures like regular, filter of an incline algebra for a future scope.

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Intuitionistic Fuzzy Complex Subgroups with Respect to Norms (T,S)

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Abstract

In our work in this paper, we define intuitionistic fuzzy complex subgroups with respect to t-norm T and s-norm S and investigate some properties of them in detail. Next, we obtain some results about them and give some relationships between them. Later, we introduce the inverse, composition, intersection and normality of them and we prove some basic new results and present some properties of them such that the inverse and composition of two intuitionistic fuzzy complex subgroups with respect to t-norm T and s-norm S will be intuitionistic complex fuzzy subgroups with respect to t-norm T and s-norm S. Also we consider and give some characterizations of them. Finally, we discuss them under group homomorphisms and investigate some related properties such that the image and preimage of two intuitionistic fuzzy complex subgroups with respect to t-norm T and s-norm S will be intuitionistic complex fuzzy subgroups with respect to t-norm T and s-norm S.

Keywords: Group theory, Theory of fuzzy sets, Intuitionistic fuzzy complex groups, Norms, Homomorphisms, Intersection.

1 | Introduction

In mathematics, fuzzy sets (uncertain sets) are somewhat like sets whose elements have degrees of membership. The concept of fuzzy sets was introduced by Zadeh [1] in 1965. Atanassov [2] innovated the theory of Intuitionistic Fuzzy Sets (IFS) as a powerful extension of classical fuzzy sets. This particular theory has been a great source of inspiration for many mathematicians in various scientific fields like decision making problems [3] and medical diagnosis determination [4]. Roenfeld [5] started the investigation of fuzzy subgroups and found numerous essential properties of this concept. Biswas [6] started the conception of intuitionistic fuzzy subgroups in 1997. A new concept of complex fuzzy sets was presented by Ramot et al. [7]. The extension of fuzzy sets to complex fuzzy sets is comparable to the extension of real numbers to complex numbers. The more development of complex fuzzy sets can be viewed in [8]. Alkouri and Salleh [9] gave the idea of complex intuitionistic fuzzy subsets and enlarge the basic properties of this phenomena. This concept became more effective and useful in scientific field because it deals with degree of membership and non-membership in complex plane. They also initiated the concept of complex intuitionistic fuzzy relation and developed fundamental

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operation of complex IFSs in [10]. Al-Husban and Salleh [11] introduced the concept of complex fuzzy subgroups in 2016. Ali and Tamir [12] innovated the notion complex intuitionistic fuzzy classes in 2016. The author by using norms, investigated some properties of fuzzy algebraic structures [13–15]. In Section 2, we recall the elementary notions which will be needed in the sequel. Next, in Section 3, we define intuitionistic fuzzy complex subgroups with respect to t-norm T and t-conorm S (in short, *IFCN(G)*) of G and investigate properties of them as *Propositions 2* and *3*. Later, in Section 4, we introduce composition, inverse and intersection of two elements $A, B \in IFCN(G)$ and we prove that $AoB \in IFCN(G)$ and $A \cap B \in IFCN(G)$ under some conditions. Also in Section 5, we define normality of two elements $A,B \in IFCN(G)$ and discuss some properties of them. Finally, in Section 6, we investigate image and pre image of them under group homomorphisms.

2 | Preliminaries

We recall first the elementary notions which play a key role for our further analysis. This section contains some basic definitions and preliminary results which will be needed in the sequel. For details we refer to [2, 7, 9], [16–21].

Definition 1. A group is a non-empty set G on which there is a binary operation (a,b) as ab such that

I. If a and b belong to G then ab is also in G (closure).
II. $a(bc)=(ab)c$ for all $a, b, c \in G$ (associativity).
III. There is an element $e_G \in G$ such that $ae_G=e_Ga=a$ for all $a \in G$ (identity).
IV. If $a \in G$, then there is an element $a^{-1} \in G$ such that $aa^{-1}=a^{-1}a=e_G$ (inverse).

One can easily check that this implies the unicity of the identity and of the inverse. A group G is called abelian if the binary operation is commutative, i.e., $ab = ba$ for all $a,b \in G$.

Remark 1: There are two standard notations for the binary group operation: either the additive notation, that is $a, b)=a+b$ in which case the identity is denoted by 0, or the multiplicative notation, that is $a, b)=ab$ for which the identity is denoted by e.

Definition 2. Let G be an arbitrary group with a multiplicative binary operation and identity e. A fuzzy subset of G, we mean a function from G into $[0,1]$.

Definition 3. For sets X, Y and Z, $f=(f_1, f_2) :X \to Y \times Z$ is called a complex mapping if $f_1:X \to Y$ and $f_2 :X \to Z$ are mappings.

Definition 4. Let X be a nonempty set. A complex mapping $A=(\mu_A, \vartheta_A) :X \to [0,1] \times [0,1]$ is called an IFS in X if $\mu_A + \vartheta_A \leq 1$ where the mappings $\mu_A :X \to [0,1]$ and $\vartheta_A :X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\vartheta_A(x)$) for each $x \in X$ to A, respectively. In particular $0_\sim$ and $1_\sim$ denote the intuitionistic fuzzy empty set and intuitionistic fuzzy whole set in X defined by $0_\sim(x) = (0,1)$ and $1_\sim(x)=(0,1)$, respectively. We will denote the set of all *IFSs* in X as *IFS(X)*.

Definition 5. Let X be a nonempty set and let $A=(\mu_A, \vartheta_A)$ and $A=(\mu_B, \vartheta_B)$ be *IFSs* in X. Then

I. $A \subset B$ iff $\mu_A \leq \mu_B$ and $\vartheta_A \geq \vartheta_B$.
II. $A=B$ iff $A \subset B$ and $B \subset A$.

Definition 6. Let X be a nonempty set. A complex fuzzy set A on X is an object having the form $A = \{(x, \mu_A(x)) | x \in X\}$, where μ_A denotes the degree of membership function that assigns each element $x \in X$ a complex number $\mu_A(x)$ lies within the unit circle in the complex plane. We shall assume that is $\mu_A(x)$ will be represented by $r_{A(x)}e^{iw_A(x)}$ where $i=\sqrt{-1}$, and $r : X \to [0,1]$ and $w : X \to [0,2\pi]$. Note that by setting

$w(x) = 0$ in the definition above, we return back to the traditional fuzzy subset. Let $\mu_1=r_1e^{w_1}$, and $\mu_2=r_2e^{w_2}$ be two complex numbers lie within the unit circle in the complex plane. By $\mu_1 \leq \mu_2$, we mean $r_1 \leq r_2$ and $w_1 \leq w_2$.

Definition 7. A complex IFS A, defined on a universe of discourse U, is characterized by membership and non-membership functions $\mu_A\ x)$ and $\gamma_A\ x)$, respectively, that assign any element $x \in U$ a complex-valued grade of both membership and non-membership in S. By definition, the values of $\mu_A\ x)$ and $\gamma_A\ x)$ and their sum may receive all lying within the unit circle in the complex plane, and are on the form $\mu_A\ x)=r_A\ x)e^{iw_{\mu_A}\ x)}$ for membership function in S and $\gamma_A\ x)=k_A\ x)e^{iw_{\gamma_A}\ x)}$ for non-membership function in A, where $i=\sqrt{-1}$, each of $r_A\ x)$ and $k_A\ x)$ are real-valued and both belong to the interval $[0,1]$ such that $0 \leq r_A\ x)+ k_A\ x) \leq 1$ and $iw_{\mu_A}\ x)$ and $iw_{\gamma_A}\ x)$ are real-valued.

Definition 8. A t-norm T is a function $T:[0, 1]\times[0, 1]\rightarrow[0, 1]$ having the following four properties:

I. $T\ x,1)= x$ (neutral element).
II. $T\ x, y)\leq T\ x, z)$ if $y \leq z$ (monotonicity).
III. $T\ x, y)= T\ y, x)$ (commutativity).
IV. $T(x,T\ y, z))= T\ T\ x, y), z)$ (associativity).

For all $x, y, z \in [0, 1]$.

It is clear that if $x_1\geq x_2$ and $y_1\geq y_2$, then $T\ \ x_1, y_1)\geq T\ \ x_2, y_2)$.

Example 1.

I. Standard intersection T-norm $T_m\ x, y)=min\{x, y\}$
II. Bounded sum T-norm $T_b\ x, y)=max\{0, x + y- 1\}$.
III. Algebraic product T-norm $T_p\ x, y)= xy$.

IV. Drastic T-norm.

$$T_D\ x,y)=\begin{cases} y, & \text{if } x=1,\\ x, & \text{if } y=1,\\ 0, & \text{otherwise.}\end{cases}$$

V. Nilpotent minimum T-norm.

$$T_{nM}\ x,y)=\begin{cases} \min\{x, y\}, & \text{if } x+y>1,\\ 0, & \text{otherwise.}\end{cases}$$

VI. Hamacher product T-norm.

$$T_{H_0}\ x,y)=\begin{cases} 0, & \text{if } x=y=0,\\ \dfrac{xy}{x+y-xy}, & \text{otherwise.}\end{cases}$$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm: $T_D\ x,y)\leq T\ x, y)\leq T_m\ x, y)$ for all $x, y\in[0, 1]$.

Recall that t-norm T will be idempotent if for all $x \in [0,1]$, we have $T\ x, x)= x$.

Lemma 1. Let T be a t-norm. Then

$$T(T\ x, y), T\ w, z))= T(T\ x, w), T\ y, z)), \textit{ for all } x, y, w, z \in[0, 1].$$

Definition 9. An s-norm S is a function $S:[0, 1]\times[0, 1]\rightarrow[0, 1]$ having the following four properties:

I. $S\ x, 0)=x$.
II. $S\ x, y)\leq S\ x, z)$ if $y\leq z$.

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III. $S(x, y)=S(y, x)$.
IV. $S(x, S(y, z))=S(S(x, y), z)$.

For all $x, y, z\in[0,1]$.

We say that S is idempotent if for all $x\in[0,1]$, $S(x, x) = x$.

Example 2. The basic S-norms are

$$S_m(x, y) = \max\{x, y\},$$
$$S_b(x, y) = \min\{1, x + y\},$$

and

$$S_p\ x, y)= x+y-xy,$$

For all $x, y\in[0,1]$.

S_m is standard union, S_b is bounded sum, S_p is algebraic sum.

Lemma 2. Let S be a s-norm. Then $S\big(S\ x, y), S\ w, z)\big)= S\big(S\ x, w), S\ y, z)\big)$, for all $x, y, w, z\in[0,1]$.

Proposition 1. Let G be a group. Let H be a non-empty subset of G. The following are equivalent:

I. H is a subgroup of G.
II. $x, y\in H$ implies $xy^{-1}\in H$ for all x, y.

Definition 9. Let H be subgroup of group G. Then we say that H is normal subgroup of G if for all $g\in G$ and $h\in H$, we have that $ghg^{-1}\in H$.

Definition 10. Let G and H be any two groups and $f{:}G\rightarrow H$ be a function. Then f is called a homomorphism if $f\ xy)=f\ x)f\ y)$ for all $x, y\in G$.

3 | Intuitionistic Fuzzy Complex Subgroups with Respect to Norms (t-Norm T and s-Norm S)

Definition 11. Let G be a group such that $\mu_A = r_A e^{iw_A}$ and $\vartheta_A = r_A e^{iw_A}$ be two complex fuzzy sets on G. An $A= \mu_A, \vartheta_A)\in IFS\ G)$ is said to be intuitionistic complex fuzzy subgroup with respect to norms(t-norm T and s-norm S) (in short, $IFCN(G)$) of G if

I. $R_A\ xy)\geq T\big(r_A\ x), r_A\ y)\big)$.
II. $r_A\big(x^{-1}\big)\geq r_A\ x)$.
III. $W_A\ xy)\geq min\big\{w_A\ x), w_A\ y)\big\}$.
IV. $W_A\big(x^{-1}\big)\geq w_A\ x)$.
V. $R_A\ xy)\leq S\big(r_A\ x), r_A\ y)\big)$.
VI. $R_A\big(x^{-1}\big)\leq r_A\ x)$.
VII. $W_A\ xy)\leq max\big\{w_A\ x), w_A\ y)\big\}$.
VIII. $W_A\big(x^{-1}\big)\leq w_A\ x)$.

For all $x, y\in G$.

Example 3. Let $G = \{0, a, b, c\}$ be the Klein's group. Every element is its own inverse, and the product of any two distinct non-identity elements is the remaining non-identity element. Thus the Klein 4-group admits the elegant presentation $a^2 = b^2 = c^2 = abc = 0$. Define $r_A : G \to [0, 1]$ by

$$r_A(x)=\begin{cases}0.75, & \text{if } x = a,\\ 0.65, & \text{if } x = b,\\ 0.55, & \text{if } x = c,\\ 0.45, & \text{if } x = 0,\end{cases}$$

and $w_A : G \to [0, 2\pi]$ by

$$w_A(x)=\begin{cases}0.45\pi, & \text{if } x = a,\\ 0.45\pi, & \text{if } x = b,\\ 0.55\pi, & \text{if } x = c,\\ 0.65\pi, & \text{if } x = 0.\end{cases}$$

$r_A : G \to [0, 1]$ by

$$r_A(x)=\begin{cases}0.25, & \text{if } x = a,\\ 0.35, & \text{if } x = b,\\ 0.45, & \text{if } x = c,\\ 0.55, & \text{if } x = 0,\end{cases}$$

and $w_A : G \to [0, 2\pi]$ by

$$w_A=\begin{cases}0.55\pi, & \text{if } x = a,\\ 0.55\pi, & \text{if } x = b,\\ 0.45\pi, & \text{if } x = c,\\ 0.35\pi, & \text{if } x = 0.\end{cases}$$

Let $T(a, b) = T_p(a, b) = ab$ and $S(a, b) = S_p(a, b) = a+b-ab$ for all $a, b \in [0, 1]$, then $A = (\mu_A, \vartheta_A) \in IFCN(G)$.

Proposition 2. Let $A = (\mu_A, \vartheta_A) \in IFCN(G)$ and T and S be idempotent. Then for all $x \in G$, and $n \geq 1$,

I. $A(e) \supseteq A(x)$.
II. $A(x^n) \supseteq A(x)$.
III. $A(x) = A(x^{-1})$.

Proof: As $\mu_A = r_A e^{iw_A} \in ICFN(G)$ so

I.

$$r_A(e) = r_A(xx^{-1}) \geq T(r_A(x), r_A(x^{-1})) \geq T(r_A(x), r_A(x)) = r_A(x),$$

and

$$w_A(e) = w_A(xx^{-1}) \geq \min\{w_A(x), w_A(x^{-1})\} \geq \min\{w_A(x), w_A(x)\} = w_A(x),$$

and then

$$\mu_A(e) = r_A(e)e^{iw_A(e)} \geq r_A(x)e^{iw_A(x)} = \mu_A(x). \quad \text{(a)}$$

Also

$$w_A(e) = w_A(xx^{-1}) \leq \max\{w_A(x), w_A(x^{-1})\} \leq \max\{w_A(x), w_A(x)\} = w_A(x),$$

and so

$$\vartheta_A(e) = r_A(e)e^{iw_A(e)} \leq r_A(x)e^{iw_A(x)} = \vartheta_A(x). \quad \text{(b)}$$

Now from *Eqs. (a)* and *(b)* we obtain that

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$$A(e) = (\mu_A(e), \vartheta_A(e)) \supseteq (\mu_A(x), \vartheta_A(x)) = A(x).$$

II.

$$r_A(x^n) = r_A\left(\underset{n}{x.x...x}\right) \geq T(\underbrace{r_A\ x), r_A\ x), ... r_A(x}_{n})) = r_A(x),$$

and

$$w_A(x^n) = w_A\left(\underset{n}{x.x...x}\right) \geq \min\{\underbrace{w_A\ x), w_A\ x), ... w_A(x}_{n})\} = w_A(x),$$

and so

$$\mu_A\ x^n) = r_A\ x^n)e^{iw_A\ x^n)} \geq r_A\ x)e^{iw_A\ x)} = \mu_A\ x). \quad \text{(a)}$$

Also

$$r_A(x^n) = r_A\left(\underset{n}{x.x...x}\right) \leq S\{\underbrace{r_A\ x), r_A\ x), ..., r_A(x}_{n})\} = r_A(x),$$

and

$$w_A(x^n) = w_A\left(\underset{n}{x.x...x}\right) \leq \max\{\underbrace{w_A\ x), w_A\ x), ..., w_A(x}_{n})\} = w_A(x),$$

and

$$\vartheta_A\ x^n) = r_A\ x^n)e^{iw_A\ x^n)} \leq r_A\ x)e^{iw_A\ x)} = \vartheta_A\ x). \quad \text{(b)}$$

Now using *Eqs. (a)* and *(b)* give us

$$A\ x^n) = \left(\mu_A\ x^n), \vartheta_A\ x^n)\right) \supseteq \left(\mu_A\ x), \vartheta_A\ x)\right) = A\ x).$$

III. As

$$r_A\ x) = r_A\left(x^{-1}\right)^{-1} \geq r_A\left(x^{-1}\right) \geq r_A\ x),$$

and so $r_A\ x) = r_A\left(x^{-1}\right)$ and as

$$w_A\ x) = w_A\left(x^{-1}\right)^{-1} \geq w_A\left(x^{-1}\right) \geq w_A\ x).$$

Then $w_A\ x) = w_A\left(x^{-1}\right)$. Then

$$\mu_A\left(x^{-1}\right) = r_A\left(x^{-1}\right)e^{iw_A\left(x^{-1}\right)} = r_A\ x)e^{iw_A\ x)} = \mu_A\ x). \quad \text{(a)}$$

Now

$$r_A\ x) = r_A\left(x^{-1}\right)^{-1} \leq r_A\left(x^{-1}\right) \leq r_A\ x),$$

so $r_A\ x) = r_A\left(x^{-1}\right)$ and as

$$w_A\ x) = w_A\left(x^{-1}\right)^{-1} \leq w_A\left(x^{-1}\right) \leq w_A\ x),$$

so $w_A(x) = w_A(x^{-1})$. Then

$$\vartheta_A(x^{-1}) = r_A(x^{-1})e^{iw_A(x^{-1})} = r_A(x)e^{iw_A(x)} = \vartheta_A(x). \quad \text{(b)}$$

Thus from *Eqs. (a)* and *(b)* we give that

$$A(x^{-1}) = (\mu_A(x^{-1}), \vartheta_A(x^{-1})) = (\mu_A(x), \vartheta_A(x)) = A(x).$$

Proposition 3. Let $A = (\mu_A, \vartheta_A) \in IFCN(G)$ and T and S be idempotent. Then $A(xy)=A(y)$ if and only if $A(x)=A(e)$ for all $x, y \in G$.

Proof: As $A(xy)=A(y)$ for all $x, y \in G$ so if we let $y=e$, then we get that $A(x)= A(e)$.

Conversely, suppose that $A(x)=A(e)$ so from *Proposition 2*, we get that $A(x) \supseteq A(y)$ and $A(x) \supseteq A(xy)$. Then $r_A(x) \geq r_A(y)$ and $r_A(x) \geq r_A(xy)$ and $w_A(x) \geq w_A(y), w_A(xy)$. Also $r_A(x) \leq r_A(y)$ and $r_A(x) \leq r_A(xy)$ and $w_A(x) \leq w_A(y)$ and $w_A(x) \leq w_A(xy)$.

Now

$$\begin{aligned} r_A(xy) &\geq T(r_A(x), r_A(y)) \\ &\geq T(r_A(y), r_A(y)) \\ &= r_A(y) = r_A(x^{-1}xy) \\ &\geq T(r_A(x), r_A(xy)) \\ &\geq T(r_A(xy), r_A(xy)) \\ &= r_A(xy), \end{aligned}$$

and then

$$r_A(xy) = r_A(y). \quad \text{(a)}$$

Also

$$\begin{aligned} w_A(xy) &\geq \min\{w_A(x), w_A(y)\} \\ &\geq \min\{w_A(y), w_A(y)\} = w_A(y) = w_A(x^{-1}xy) \\ &\geq \min\{w_A(x), w_A(xy)\} \\ &\geq \min\{w_A(xy), w_A(xy)\} \\ &= w_A(xy), \end{aligned}$$

and then

$$w_A(xy) = w_A(y). \quad \text{(b)}$$

Therefore from *Eqs. (a)* and *(b)* we obtain that

$$\mu_A(xy) = r_A(xy)e^{iw_A(xy)} = r_A(y)e^{iw_A(y)} = \mu_A(y). \quad \text{(c)}$$

Also

$$\begin{aligned} r_A(xy) &\leq S(r_A(x), r_A(y)) \\ &\leq S(r_A(y), r_A(y)) \\ &= r_A(y) = r_A(x^{-1}xy) \\ &\leq S(r_A(x), r_A(xy)) \\ &\leq S(r_A(xy), r_A(xy)) \\ &= r_A(xy), \end{aligned}$$

Then

$$r_A(xy) = r_A(y). \quad \text{(d)}$$

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Also

$$\begin{aligned} w_A\ xy) &\leq \max\{w_A\ x), w_A\ y)\} \\ &\leq \max\{w_A\ y), w_A\ y)\} \\ &= w_A\ y) = w_A(x^{-1}xy) \\ &\leq \max\{w_A\ x), w_A\ xy)\} \\ &\leq \max\{w_A\ xy), w_A\ xy)\} \\ &= w_A\ xy), \end{aligned}$$

and then

$$w_A\ xy) = w_A\ y). \tag{e}$$

Therefore from *Eqs. (d)* and *(e)* we obtain that

$$\vartheta_A\ xy) = r_A\ xy)e^{iw_A\ xy)} = r_A\ y)e^{iw_A\ y)} = \vartheta_A\ y). \tag{f}$$

Now as *Eqs. (c)* and *(f)* we get that

$$A(xy) = (\mu_A(xy), \vartheta_A(xy)) = (\mu_A(y), \vartheta_A(y)) = A(y).$$

4 | Composition, Inverse and Intersection of IFCN(G)

Definition 12. Let $A = \mu_A, \vartheta_A) \in IFCN\ G)$ and $B = \mu_B, \vartheta_B) \in IFCN\ G)$ such that $\mu_A = r_A e^{iw_A} \in IFCN\ G)$ and $\vartheta_A = r_A e^{iw_A}$ and $\mu_B = r_B e^{iw_B}$ and $\vartheta_B = r_B e^{iw_B}$. We define the composion of A and B as $A\ oB$ and for all $x \in G$ we have

$$A \text{ o } B)\ x) = \big(\ \mu_A, \vartheta_A) \text{ o } \ \mu_B, \vartheta_B)\big)\ x) = \big(\mu_{AoB}\ x), \vartheta_{AoB}\ x)\big) =$$
$$((r_A o r_B)(x)e^{i(w_A \text{ o } w_B)(x)}, (r_A o r_B)(x)e^{i(w_A \text{ o } w_B)(x)}.$$

Such that $r_A\ o\ r_B : G \to [0, 1]$ and $w_A o w_B : G \to [0, 2\pi]$ and $r_A\ o\ r_B : G \to [0, 1]$ and $w_A o\ w_B : G \to [0, 2\pi]$. Now define

$$r_A \text{ o } r_B)\ x) = \begin{cases} \sup\limits_{x=ab} T\big(r_A\ a), r_B\ b)\big), & \text{if } \ x = ab, \\ 0, & \text{if } \ x \neq ab, \end{cases}$$

and

$$(w_A \text{ o } w_B)\ x) = \begin{cases} \min\limits_{x=ab}\{w_A\ a), w_B\ b)\}, & \text{if } \ x = ab, \\ 0, & \text{if } \ x \neq ab, \end{cases}$$

and

$$(r_A \text{ o } r_B)\ x) = \begin{cases} \inf\limits_{x=ab} S\big(r_A\ a), r_B\ b)\big), & \text{if } \ x = ab, \\ 0, & \text{if } \ x \neq ab, \end{cases}$$

and

$$(w_A \text{ o } w_B)\ x) = \begin{cases} \max\limits_{x=ab}\{w_A\ a), w_A\ b)\}, & \text{if } \ x = ab, \\ 0, & \text{if } \ x \neq ab, \end{cases}$$

For all $x \in G$.

Proposition 4. Let $A^{-1} = (\mu_A^{-1}, \vartheta_A^{-1}) \in IFS(G)$ be the inverse of $A = \mu_A, \vartheta_A) \in IFCN(G)$ such that for all $x \in G$:

$$A^{-1}(x) = (\mu_A^{-1}(x), \vartheta_A^{-1}(x)) = (\mu_A(x^{-1}), \nu A(x^{-1})) = A(x^{-1}).$$

If T and S be idempotent then $A = \mu_A, \vartheta_A) \in IFCN\ G)$ if and only if A satisfies the following conditions:

I. $A \supseteq A \text{ o } A$.
II. $A^{-1} = A$.

Proof: Let $x, y, z \in G$ with $x = yz$ and $A \in ICF\ N(G)$. Then

I.

$$r_A(x) = r_A(yz) \geq T(r_A(y), r_A(z)) = (r_A \text{ o } r_A)(x),$$

and

$$w_A(x) = w_A(yz) \geq \min\{w_A(y), w_A(z)\} = (w_A \text{ o } w_A)(x),$$

then

$$\mu_A\ x) = r_A\ x)e^{iw_A\ x)} \geq\ r_A \text{ o } r_A)\ x)e^{i(w_A o w_A)(x)} =\ \mu_{AoA})\ x). \tag{a}$$

Also

$$r_A(x) = r_A(yz) \leq S(r_A(y), r_A(z)) = (r_A \text{ o } r_A)(x),$$

and

$$w_A\ x) = w_A\ yz) \geq \min\{w_A\ y), w_A\ z)\} =\ w_A \text{ o } w_A)\ x),$$

then

$$\vartheta_A\ x) = r_A\ x)e^{iw_A\ x)} \geq\ r_A \text{ o } r_A)\ x)e^{i(w_A o w_A)(x)} =\ \vartheta_{AoA})\ x). \tag{b}$$

Thus from *Eqs. (a)* and *(b)* we get that

$$A(x) = (\mu_A(x), \vartheta_A(x)) \supseteq (\mu_{AoA})(x), \vartheta_{AoA})(x)) = (A \text{ o } A)(x),$$

and then $A \supseteq A\ oA$.

II. As *Proposition 2*, we have that $A^{-1}(x) = A(x^{-1}) = A(x)$ for all $x \in G$. Thus $A^{-1} = A$.

Conversely, let $A \supseteq A\ o\ A$ and $A^{-1} = A$ and $x, y, z \in G$ with $x = yz$. Since $A \supseteq A\ o\ A$ so $r_A(x) \geq (r_A\ o\ r_A)(x)$ and then

$$r_A\ yz) =\ r_A\ x) \geq\ r_A o r_A\ x) = \sup_{x=ab} T\left(r_A\ y), r_B\ z)\right) \geq T\left(r_A\ y), r_A\ z)\right). \tag{a}$$

$w_A(x) \geq (w_A \text{ o } w_A)(x)$ and thus,

$$w_A\ yz) = w_A\ x) \geq\ w_A \text{ o } w_A)\ x) = \min_{x=yz}\left\{w_A\ y), w_B\ z)\right\}\ \geq \left\{w_A\ y), w_A\ z)\right\}. \tag{b}$$

$r_A(x) \leq (r_A\ o\ r_A)(x)$ and then,

$$r_A\ yz) = r_A\ x) \leq r_A o r_A\ x) = \inf_{x=yz} S\left(r_A\ y), r_B\ z)\right)\ \leq S\left(r_A\ y), r_A\ z)\right). \tag{c}$$

$w_A(x) \leq (\ w_A\ o\ w_A)(x)$ and so

$$w_A\ yz) = w_A\ x) \leq\ w_A o w_A)\ x) = \max_{x=yz}\left\{w_A\ y), w_A\ z)\right\}\ \leq \left\{w_A\ y), w_A\ z)\right\}. \tag{d}$$

As $A^{-1} = A$ so,

$$r_A\left(x^{-1}\right) = r_A^{\ -1}\ x) = r_A\ x). \tag{e}$$
$$r_A\left(x^{-1}\right) = r_A^{\ -1}\ x) = r_A\ x). \tag{f}$$
$$w_A\left(x^{-1}\right) = w_A^{\ -1}\ x) = w_A\ x). \tag{g}$$
$$w_A\left(x^{-1}\right) = w_A^{\ -1}\ x) = w_A\ x). \tag{h}$$

Thus from *Eqs. (a)-(h)*, we get that $A \in IFCN(G)$.

Corollary 1. Let $A= \mu_A, \vartheta_A)\in IFCN\ G)$ and $B= \mu_B, \vartheta_B)\in IFCN\ G)$ and G be commutative group. Then $A\ o\ B\in IFCN(G)$ if and only if $A\ o\ B = B\ o\ A$.

Proof: If $A, B, A\ o\ B\in IFCN(G)$, then from *Proposition 4* we get that $A^{-1} = A$, $B^{-1} = B$ and $(BoA)^{-1} = B\ o\ A$. Now $A\ o\ B= A^{-1}\ o\ B^{-1} =(BoA)^{-1} =B\ o\ A$. conversely, since $A\ o\ B = B\ o\ A$ we have

$$(AoB)^{-1} =(BoA)^{-1} =A^{-1}\ o\ B^{-1} =A\ o\ B.$$

Also

$$(A\ o\ B)\ o\ (A\ o\ B) =A\ o\ (B\ o\ A)\ o\ B =A\ o\ (A\ o\ B)\ o\ B =(A\ o\ A)\ o\ (B\ o\ B)\subseteq A\ o\ B.$$

Now *Proposition 4* gives us that $A\ o\ B\in IFCN(G)$.

Definition 13. Let $A= \mu_A, \vartheta_A)\in IFCN\ G)$ and $B= \mu_B, \vartheta_B)\in IFCN\ G)$ such that, $\mu_A= r_A e^{iw_A}$ and $\vartheta_A\ x)= r_A e^{iw_A}$ and $\mu_B= r_B e^{iw_B}$ and $\vartheta_B\ x)=r_B e^{iw_B}$. define the intersection of A and B as $A\cap B$ such that for all $x\in G$:

$$\begin{aligned}&(A\cap B)(x) = (\ \mu_A, \vartheta_A) \cap\ \ \mu_B, \vartheta_B))(x)\\ &=(\mu_{A\cap B}(x), \nu A\cap B(x))\\ &=((r_A\cap r_B)(x)e^{i(w_A\cap w_B)(x)}, ((r_A\cap r_B)(x)e^{i\ w_A\cap\ w_B)(x)}.\end{aligned}$$

Such that $r_A\cap r_B\colon G\to [0,1]$ and $w_A\cap w_B\colon G\to [0,2\pi]$ and $r_A\cap r_B\colon G\to [0,1]$ and $w_A\cap w_A : G\to [0,2\pi]$ define:

$$\begin{aligned}&r_A\cap r_B)\ x)=T\big(r_A\ x), r_A\ x)\big),\\ &w_A\cap w_B)\ x)=\min\{w_A\ \ x), w_A\ \ x)\},\\ &r_A\cap r_B)\ x)= S\big(r_A\ x), r_B\ x)\big),\\ &w_A\cap w_B)\ x)=\max\{w_A\ \ x), w_B\ \ x)\},\end{aligned}$$

for all $x\in G$.

Proposition 5. Let $A= \mu_A, \vartheta_A)\in IFCN\ G)$ and $B= \mu_B, \vartheta_B)\in IFCN\ G)$. Then $A\cap B\in IFCN\ G)$.

Proof: Let $A= \mu_A, \vartheta_A)\in IFCN\ G)$ and $B= \mu_B, \vartheta_B)\in IFCN\ G)$ such that $\mu_A= r_A e^{iw_A}$ and $\vartheta_A\ x)= r_A e^{iw_A}$ and $\mu_B= r_B e^{iw_B}$ and $\vartheta_B\ x)= r_B e^{iw_B}$.

I. Let $g_1,g_2\epsilon G$. then

$$\begin{aligned}&(r_A\cap r_B)(g_1g_2) = T\ (r_A(g_1g_2), r_B(g_1g_2))\\ &\geq T\ (T\ (r_A(g_1), r_A(g_2)), T\ (r_B(g_1), r_B(g_2)))\\ &= T\ (T\ (r_A(g_1), r_B(g_1)), T\ (r_A(g_2), r_B(g_2)))\ \text{(Lemma 1)}\\ &= T\ ((r_A\cap r_B)(g_1), (r_A\cap r_B)(g_2)),\end{aligned}$$

and thus

$$(r_A\cap r_B)(g_1g_2)\geq T\ ((r_A\cap r_B)(g_1), (r_A\cap r_B)(g_2)).$$

II. If $g\in G$, then

$$(r_A\cap r_B)(g^{-1})=T\ (r_A(g^{-1}), r_B(g^{-1}))\geq T\ (r_A(g), r_B(g))=(r_A\cap r_B)(g),$$

and so $r_A\cap r_B)\big(g^{-1}\big)\geq r_A\cap r_B)\ g)$.

III. If $g\in G$, then

$$\begin{aligned}&w_A\cap w_B)\ g_1g_2)= \min\ \{w_A\ g_1g_2), w_B\ g_1g_2)\}\\ &\geq \min\{\min\{w_A\ g_1),w_A\ g_2)\}, \min\{w_B\ g_1),w_B\ g_2)\}\}\\ &=\min\ \{\min\ \{w_A\ g_1),w_B\ g_1)\}, \min\ \{w_B\ g_2),w_B\ g_2)\}\}\\ &=\min\{\ w_A\cap w_B)\ g_1),\ \ w_A\cap w_B)\ g_2)\},\end{aligned}$$

and so $w_A \cap w_B)\ g_1g_2) \geq min\{\ w_A \cap w_B)\ g_1),\ \ w_A \cap w_B)\ g_2)\}$.

IV. Let g ∈ G, so

$$(w_A \cap w_B)(g^{-1}) = min\{w_A(g^{-1}), w_B(g^{-1})\} \geq min\{w_A\ g), w_B\ g)\} = (w_A \cap w_B)(g),$$

and so $(w_A \cap w_B)(g^{-1}) \geq (w_A \cap w_B)\ g)$.

V. Let $g_1, g_2 \epsilon G$. then

$$r_A \cap r_B)\ g_1g_2) = S\left(r_A\ g_1g_2), r_B\ g_1g_2)\right)$$
$$\leq S\left(S\left(r_A\ g_1), r_A\ g_2)\right), S\left(r_B\ g_1), r_B\ g_2)\right)\right)$$
$$= S\left(S\left(r_A\ g_1), r_B\ g_1)\right), S\left(r_A\ g_2), r_B\ g_2)\right)\right) \text{ Lemma 1)}$$
$$= S\left(\ r_A \cap r_B)\ g_1),\ \ r_A \cap r_B)\ g_2)\right),$$

and thus

$$(r_A \cap r_B)(g_1g_2) \leq S((r_A \cap r_B)(g_1), (r_A \cap r_B)(g_2)).$$

VI. If $g \in G$, then

$$(r_A \cap r_B)(g^{-1}) = S\ (r_A(g^{-1}), r_B(g^{-1})) \leq S\ (r_A(g), r_B(g)) = (r_A \cap r_B)(g),$$

and so $\ r_A \cap r_B)(g^{-1}) \leq\ r_A \cap r_B)\ g)$.

VII. Let $g_1, g_2 \epsilon G$. Then

$$w_A \cap w_B)\ g_1g_2) = max\ \{w_A\ g_1g_2), w_B\ g_1g_2)\}$$
$$\leq max\{max\{w_A\ g_1), w_A\ g_2)\}, max\{w_B\ g_1), w_B\ g_2)\}\}$$
$$= max\ \{max\ \{w_A\ g_1), w_B\ g_1)\}, max\ \{w_A\ g_2), w_B\ g_2)\}\}$$
$$= max\{\ w_A \cap w_B)\ g_1),\ \ w_A \cap w_B)\ g_2)\},$$

and so $\ w_A \cap w_B)\ g_1g_2) \leq max\{\ w_A \cap w_B)\ g_1),\ \ w_A \cap w_B)\ g_2)\}$.

VIII. Let g ∈ G, so

$$w_A \cap w_B)(g^{-1}) = max\{w_A(g^{-1}), w_B(g^{-1})\} \leq max\{w_A\ g), w_B\ g)\} = \ w_A \cap w_B)\ g),$$

and so $(\dot{w}_A \cap \dot{w}_B)(g^{-1}) \leq (\dot{w}_A \cap \dot{w}_B)\ g)$.

Then above steps give us that $A \cap B \epsilon IFCN(G)$.

Corollary 2. Let $I_n = \{1, 2, ..., n\}$. If $\{A_i = (\mu_{A_i}, \vartheta_{A_i}) \mid i \in I_n\} \subseteq IFCN(G)$.

Then $A = \bigcap_{i \in I_n} A_i\ \epsilon IFCN(G)$.

5 | Normality of ICFN(G)

Definition 14. Let $A = \ \mu_A, \vartheta_A) \epsilon IFCN\ G)$ such that $\mu_A = r_A e^{iw_A}$ and $\vartheta_A\ x) = r_A e^{iw_A}$. We say that $A = \ \mu_A, \vartheta_A)$ is normal if for all $x, y \epsilon G$, we have that $A(xyx^{-1}) = A(y)$ which means that $r_A(xyx^{-1}) = r_A(y)$ and $w_A(xyx^{-1}) = w_A(y)$ and $r_A(xyx^{-1}) = r_A(y)$ and $w_A(xyx^{-1}) = w_A\ y)$. We denote by $NIFCN(G)$ the set of all normal intuitionistic fuzzy complex subgroups with respect to norms (t-norm T and s-norm S).

Proposition 6. Let $A = \ \mu_A, \vartheta_A) \epsilon NIFCN\ G)$ and $B = \ \mu_B, \vartheta_B) \epsilon NIFCN\ G)$ such that $\mu_A = r_A e^{iw_A}$ and $\vartheta_A\ x) = r_A e^{iw_A}$ and $\mu_B = r_B e^{iw_B}$ and $\vartheta_B\ x) = r_B e^{iw_B}$. Then $A \cap B \epsilon NICFN\ G)$.

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Proof: As *Proposition 5* we have that $A \cap B \in IFCN\ G)$. Let $x, y, \in G$ then

I. $(r_A \cap r_B)(xyx^{-1}) = T(r_A(xyx^{-1}), r_B(xyx^{-1})) = T(r_A(y), r_B(y)) = (r_A \cap r_B)(y)$.
II. $(w_A \cap w_B)(xyx^{-1}) = \min\{w_A(xyx^{-1}), w_B(xyx^{-1})\} = \min\{w_A\ y), w_B\ y) = (w_A \cap w_B)(y)$.
III. $(r'_A(\cap r'_B)(xyx^{-1}) = S(r'_A((xyx^{-1}), r'_B(xyx^{-1})) = S(r'_A((y), r'_B(y)) = (r'_A(\cap r'_B)(y)$.
IV. $(w'_A \cap w'_B)(xyx^{-1}) = \max\{w'_A(xyx^{-1}), w'_B(xyx^{-1})\} = \max\{w'_A\ y), w'_B\ y)\} = (w'_A\ w'_B)(y)$.

Then from above steps, we get that

$$(A \cap B)(xyx^{-1}) = (\mu_{A\cap B}(xyx^{-1}), \vartheta_{A\cap B}(xyx^{-1})) = (\mu_{A\cap B}(y), \vartheta_{A\cap B}(y)) = (A \cap B)(y).$$

And so $A \cap B \in NIFCN(G)$.

Corollary 3. Let $I_n = \{1, 2, \ldots, n\}$. If $\{A_i = (\mu_{A_i}, \vartheta_{A_i}) \mid i \in I_n\} \subseteq NIFCN(G)$. Then $A = \bigcap_{i \in I_n} A_i \in NIFCN(G)$.

Definition 15. Let $A = \mu_A, \vartheta_A) \in NIFCN\ G)$ and $B = \mu_B, \vartheta_B) \in IFCN\ G)$ such that $A \subseteq B$. Then A is called normal of B, written $A \trianglelefteq B$, if

I. $r_A(xyx^{-1}) \geq T(r_A\ y), r_B\ x))$.
II. $w_A(xyx^{-1}) \geq \min\{w_A(y), w_B(x)\}$.
III. $r_A(xyx^{-1}) \leq S\{r_A\ y), r_B\ x)\}$.
IV. $w_A(xyx^{-1}) \leq \max\{w_A(y), w_B(x)\}$.

For all $x, y \in G$.

Proposition 7. If T and S be idempotent and $A = \mu_A, \vartheta_A) \in IFCN\ G)$, then $A \trianglelefteq A$.

Proof: Let $x, y \in G$ and $A = \mu_A, \vartheta_A) \in IFCN\ G)$. Then

$$r_A(xyx^{-1}) \geq T\left(r_A\ xy), r_A(x^{-1})\right)$$
$$\geq T(r_A\ xy), r_A\ x)) \geq T\left(T(r_A\ x), r_A\ y)), r_A\ x)\right)$$
$$= T\left(T(r_A\ x), r_A\ x)), r_A\ y)\right) = T(r_A\ x), r_A\ y)) = T(r_A(y), r_A(x)),$$

and so

$$r_A(xyx^{-1}) \geq T(r_A\ y), r_A\ x)). \quad (1)$$

also

$$w_A(xyx^{-1}) \geq \min\{w_A\ xy), w_A(x^{-1})\}$$
$$= \min\{w_A\ xy), w_A\ x)\}$$
$$\geq \min\{\min\{w_A\ x), w_A\ y)\}, w_A\ x)\}$$
$$= \min\{\min\{w_A\ x), w_A\ x)\}, w_A\ y)\}$$
$$= \min\{w_A\ x), w_A\ y)\}$$
$$= \min\{w_A\ y), w_A\ x)\},$$

Then

$$w_A(xyx^{-1}) \geq \min\{w_A\ y), w_A\ x)\}. \quad (2)$$

Now

$$\begin{aligned}
&r_A(xyx^{-1}) \leq S\left(r_A\ xy), r_A(x^{-1})\right) \\
&\leq S(r_A\ xy), r_A\ x)) \\
&\leq S\left(S(r_A\ x), r_A\ y)), r_A\ x)\right) \\
&= S\left(S(r_A\ x), r_A\ x)), r_A\ y)\right) \\
&= S(r_A\ x), r_A\ y)) \\
&= S(r_A(y), r_A(x)),
\end{aligned}$$

thus

$$r_A(xyx^{-1}) \leq S(r_A\ y), r_A\ x)). \quad (3)$$

Finally,

$$\begin{aligned}
&w_A(xyx^{-1}) \leq \max\{w_A\ xy), w_A(x^{-1})\} \\
&\leq \max\{w_A\ xy), w_A\ x)\} \\
&\leq \max\{\min\{w_A\ x), w_A\ y)\}, w_A\ x)\} \\
&= \max\{\min\{w_A\ x), w_A\ x)\}, w_A\ y) \\
&= \max\{w_A\ x), w_A\ y)\} \\
&= \max\{w_A\ y), w_A\ x)\},
\end{aligned}$$

then

$$w_A(xyx^{-1}) \leq \min\{w_A\ y), w_A\ x)\}. \quad (4)$$

Then *Eqs. (1)-(4)* give us that $A \sqsubseteq A$.

Proposition 8. Let $A= \mu_A, \vartheta_A) \in NIFCN\ G)$ and $B= \mu_B, \vartheta_B) \in IFCN\ G)$ such that $\mu_A = r_A e^{iw_A}$ and $\vartheta_A\ x) = r_A e^{iw_A}$ and $\mu_B = r_B e^{iw_B}$ and $\vartheta_B\ x) = r_B e^{iw_B}$. If T and S be idempotent, then $A \cap B \sqsubseteq B$.

Proof: As *Proposition 6* we have that $A \cap B \in NIFCN(G)$. Let $x, y \in G$ then

$$\begin{aligned}
&(r_A \cap r_B)(xyx^{-1}) = T((r_A(xyx^{-1}), r_B(xyx^{-1})) \\
&= T((r_A(y), r_B(xyx^{-1})) \\
&\geq T(rA(y), T(r_B(xy), r_B(x^{-1})) \\
&\geq T(r_A(y), T(T\ (r_B(x), r_B(y)), r_B(x))) \\
&= T(r_A(y), T(r_B(y), T\ (r_B(x), r_B(x)))) \\
&= T\left(r_A\ y), T(r_B\ y), r_B\ x))\right) \\
&= T\left(T(r_A\ y), r_B\ y)), r_B\ x)\right) \\
&= T((r_A \cap r_B)(y), r_B(x)),
\end{aligned}$$

and then

$$(r_A \cap r_B)(xyx^{-1}) \geq T((r_A \cap r_B)(y), r_B(x)). \quad (5)$$

Also

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$$
\begin{aligned}
&(w_A \cap w_B)\left(xyx^{-1}\right) = \min\{(w_A(xyx^{-1}), w_B(xyx^{-1})\} \\
&= \min\{(w_A(y), w_B(xyx^{-1})\} \\
&\geq \min\{w_A(y), \min\{w_B(xy), w_B(x^{-1})\}\} \\
&\geq \min\{w_A(y), \min\{\min\{w_B(x), w_B(y)\}, w_B(x)\}\} \\
&= \min\left\{w_A\ y), \min\left\{w_B\ y), \mathrm{Min}\{w_B\ x), w_B\ x)\}\right\}\right\} \\
&= \min\left\{w_A\ y), \min\left\{w_B\ y), w_B\ x)\right\}\right\} \\
&= \min\{\min\{w_A(y), w_B(y)\}, w_B(x)\}\} \\
&= \min\{(w_A \cap w_B)(y), w_B(x)\},
\end{aligned}
$$

then

$$(w_A \cap w_B)\left(xyx^{-1}\right) \geq \min\{(w_A \cap w_B)(y), w_B(x)\}. \tag{6}$$

Now

$$
\begin{aligned}
&(r_A \cap r_B)\left(xyx^{-1}\right) = S((r_A(xyx^{-1}), r_B(xyx^{-1})) \\
&= S((r_A(y), r_B(xyx^{-1})) \\
&\leq S(r_A(y), S(r_B(xy), r_B(x^{-1})) \\
&\leq S\ (r_A(y), S(S(r_B(x), r_B(y)), r_B(x))) \\
&= S(r_A(y), S(r_B(y), S(r_B(x), r_B(x)))) \\
&= S\left(r_A\ y), S\left(r_B\ y), r_B\ x)\right)\right) \\
&= S\left(S\left(r_A\ y), r_B\ y)\right), r_B\ x)\right) \\
&= S((r_A \cap r_B)(y), r_B(x)),
\end{aligned}
$$

and then

$$(r_A \cap r_B)\left(xyx^{-1}\right) \leq S((r_A \cap r_B)(y), r_B(x)). \tag{7}$$

As

$$
\begin{aligned}
&(w_A \cap w_B)\left(xyx^{-1}\right) = \min\{w_A(xyx^{-1}), w_B(xyx^{-1})\} \\
&= \min\{w_A(y), w_B(xyx^{-1})\} \\
&\geq \min\{w_A(y), \min\{w_B(xy), w_B(x^{-1})\}\} \\
&\geq \min\{w_A(y), \min\{\min\{w_B(x), w_B(y)\}, w_B(x)\}\} \\
&= \min\left\{w_A\ y), \min\left\{w_B\ y), \min\{w_B\ x), w_B\ x)\}\right\}\right\} \\
&= \min\left\{w_A\ y), \min\left\{w_B\ y), w_B\ x)\right\}\right\} \\
&= \min\{\min\{w_A(y), w_B(y)\}, w_B(x)\}\} \\
&= \min\{(w_A \cap w_B)(y), w_B(x)\},
\end{aligned}
$$

then

$$(w_A \cap w_B)\left(xyx^{-1}\right) \geq \min\{(w_A \cap w_B)(y), w_B(x)\}. \tag{8}$$

Then *Eqs. (5)-(8)* mean that $A \cap B \sqsubseteq B$.

Proposition 9. Let $A = \mu_A, \vartheta_A) \in IFCN\ G)$ and $B = \mu_B, \vartheta_B) \in IFCN\ G)$ and $C = \mu_C, \vartheta_C) \in IFCN\ G)$ such that $\mu_A = r_A e^{iw_A}$ and $\vartheta_A\ x) = r_A e^{iw_A}$ and $\mu_B = r_B e^{iw_B}$ and $\vartheta_B\ x) = r_B e^{iw_B}$ and $\mu_C = r_C e^{iw_C}$ and $\vartheta_C\ x) = r_C e^{iw_C}$ Let T and S be idempotent and $A \sqsubseteq C$ and $B \sqsubseteq C$. Then $A \cap B \sqsubseteq C$.

Proof: From *Proposition 6* we get that $A \cap B \in ICFN(G)$. Now for all $x, y \in G$ we get that

$$(r_A \cap r_B)(xyx^{-1}) = T(r_A(xyx^{-1}), r_B(xyx^{-1}))$$
$$\geq T(T(r_A(y), r_C(x)), T(r_B(y), r_C(x)))$$
$$= T(T(rA(y), r_B(y)), T(r_C(x), r_C(x)))$$
$$= T(T(r_A(y), r_B(y)), r_C(x))$$
$$= T((r_A \cap r_B)(y), r_C(x)),$$

and then

$$r_A \cap r_B)\left(xyx^{-1}\right) \geq T\left(r_A \cap r_B) y), r_C x)\right). \tag{9}$$

Also

$$w_A \cap w_B)\left(xyx^{-1}\right) = \min\left\{w_A\left(xyx^{-1}\right), w_B\left(xyx^{-1}\right)\right\}$$
$$\geq \min\left\{\min\left\{w_A y), w_C x)\right\}, \min\left\{w_B y), w_C x)\right\}\right\}$$
$$= \min\left\{\min\left\{w_A y), w_B y)\right\}, \min\{w_C x), w_C x)\}\right\}$$
$$= \min\left\{\min\left\{w_A y), w_B y)\right\}, w_C x)\right\}$$
$$= \min\{(w_A \cap w_B)(y), w_C(x)),$$

then

$$(w_A \cap w_B)\left(xyx^{-1}\right) \geq \min\{(w_A \cap w_B)(y), w_C(x)). \tag{10}$$

As

$$(r_A \cap r_B)(xyx^{-1}) = S(r_A(xyx^{-1}), r_B(xyx^{-1}))$$
$$\leq S(S(r_A(y), r_C(x)), S(r_B(y), r_C(x)))$$
$$= S(S(r_A(y), r_B(y)), S(r_C(x), r_C(x)))$$
$$= S(S(r_A(y), r_B(y)), r_C(x))$$
$$= S((r_A \cap r_B)(y), r_C(x)),$$

so

$$r_A \cap r_B)\left(xyx^{-1}\right) \leq S\left(r_A \cap r_B) y), r_C x)\right). \tag{11}$$

Since

$$(w_A \cap w_B)(xyx^{-1}) = \max\{w_A\left(xyx^{-1}\right), w_B\left(xyx^{-1}\right)\}$$
$$\leq \max\{\max\{w_A y), w_C x)\}, \max\{w_B y), w_C x)\}\}$$
$$= \max\{\max\{w_A y), w_B y)\}, \max\{w_C x), w_C x)\}\} = \max\{ \max\left\{w_A y), w_B y)\right\}, w_C x)\} \tag{12}$$
$$= \max\{(w_A \cap w_B)(y), w_C(x)).$$

then

$$(w_A \cap w_B)\left(xyx^{-1}\right) \leq \max\{(w_A \cap w_B)(y), w_C(x)). \tag{13}$$

Then as *Eqs. (9)-(13)* we get that $A \cap B \sqsubseteq C$.

Corollary 4. Let $I_n=\{1, 2, ..., n\}$. If $\{A_i=(\mu_{A_i}, \vartheta_{A_i}) \mid i\in I_n\} \subseteq IFCN(G)$ such that $\{A_i=(\mu_{A_i}, \vartheta_{A_i}) \mid i\in I_n\} \sqsubseteq B= \mu_B, \vartheta_B)$. Then $A=\bigcap_{i\in I_n} A_i \sqsubseteq B= \mu_B, \vartheta_B)$.

6 | Group Homomorphisms and IFCN(G)

Definition 16. Let $A= \mu_A, \vartheta_A)\in IFCN\ G)$ and $B= \mu_B, \vartheta_B)\in IFCN\ H)$ such that $\mu_A= r_A e^{iw_A}$ and $\vartheta_A\ x)=r_A e^{iw_A}$ and $\mu_B=r_B e^{iw_B}$ and $\vartheta_B\ x)= r_B e^{iw_B}$.

Let $\varphi :G\rightarrow H$ be a group homomorphism. Define:

$$\varphi\ A)=\varphi(\mu_A),\varphi(\vartheta_A))=\left(\varphi(r_A e^{iw_A}),\varphi(r_A e^{iw_A})\right)=\left(\varphi(r_A)e^{i\varphi(w_A)},\varphi(r_A)e^{i\varphi(w_A)})\right).$$

For all $h\in H$ define:

$$\varphi(r_A):H\rightarrow[0,1] \text{ as } \varphi(r_A)(h)=\sup\{r_A(g)\mid g\in G,\varphi(g)=h\},$$
$$\varphi(w_A):H\rightarrow[0,2\pi] \text{ as } \varphi(w_A)(h)=\sup\{w_A(g)\mid g\in G,\varphi(g)=h\},$$
$$\varphi(r_A):H\rightarrow[0,1] \text{ as } \varphi(r_A)(h)=\inf\{r_A(g)\mid g\in G,\varphi(g)=h\},$$

nd

$$\varphi(w_A):H\rightarrow[0,2\pi] \text{ as } \varphi(w_A)(h)=\inf\{w_A(g)\mid g\in G,\varphi(g)=h\}.$$

Also define

$$\varphi^{-1}\ B)=\left(\varphi^{-1}(\mu_B),\varphi^{-1}(\vartheta_B)\right)=\left(\varphi^{-1}(r_B e^{iw_B}),\varphi^{-1}(r_B e^{iw_B})\right)=$$
$$\left(\varphi^{-1}(r_B)e^{i\varphi^{-1}(w_B)}),\varphi^{-1}(r_B)e^{i\varphi^{-1}(w_B)})\right),$$

such that for all $g\in G$:

$$\varphi^{-1}(r_B):G\rightarrow[0,1] \text{ as } \varphi^{-1}(r_B)(g)=r_B(\varphi\ g)),$$
$$\varphi^{-1}(r_B):G\rightarrow[0,1] \text{ as } \varphi^{-1}(r_B)(g)=r_B(\varphi\ g)),$$
$$\varphi^{-1}(w_B):G\rightarrow[0,2\pi] \text{ as } \varphi^{-1}(w_B)(g)=w_B(\varphi\ g)),$$
$$\varphi^{-1}(w_B):G\rightarrow[0,2\pi] \text{ as } \varphi^{-1}\ w_B)\ g)=w_B\left(\varphi\ g)\right).$$

Proposition 10. Let $A=\ \mu_A,\vartheta_A)\in IFCN\ G)$ and H be a group. Suppose that $\varphi:G\rightarrow H$ is a group homomorphism. Then $\varphi(A)\in IFCN(H)$.

Proof: Let $\varphi\ A)=\varphi(\mu_A),\varphi(\vartheta_A))=\left(\varphi(r_A)e^{i\varphi(w_A)}),\varphi(r_A^{'})e^{i\varphi(w_A^{'})})\right)$ and $h_1,h_2\in H$ and $g_1,g_2\in G$ such that $\varphi(g_1)=h_1$ and $\varphi(g_2)=h_2$. Then

$$\varphi(r_A)(h_1h_2)=\sup\{r_A(g_1g_2)\mid g_1=\varphi(h_1),g_2=\varphi(h_2)\}$$
$$\geq\sup\{T(r_A(g_1),rA(g_2))\mid g_1=\varphi(h_1),g_2=\varphi(h_2)\}$$
$$=T(\sup\{r_A(g_1)\mid g_1=\varphi(h_1)\},\sup\{r_A(g_2)\mid g_2=\varphi(h_2)\})$$
$$=T(\varphi(r_A)(h_1),\varphi(r_A)(h_2)),$$

and so

$$\varphi\ r_A)\ h_1h_2)\geq T\left(\varphi\ r_A)\ h_1),\varphi\ r_A)\ h_2)\right). \tag{14}$$

Let $g\in G$ and $h\in H$ such that $\varphi(g)=h$. Then

$$\varphi(r_A)(h^{-1})=\sup\{r_A(g^{-1})\mid g^{-1}\in G,\varphi(g^{-1})=h^{-1}\}$$
$$\geq\sup\{rA(g)\mid g^{-1}\in G,\varphi^{-1}(g)=h^{-1}\}$$
$$=\sup\{rA(g)\mid g\in G,\varphi(g)=h\}$$
$$=\varphi(rA)(h),$$

and then

$$\varphi\ r_A)\left(h^{-1}\right)\geq\varphi\ r_A)\ h). \tag{15}$$

Let $h_1,h_2\in H$ and $g_1,g_2\in G$ with $\varphi(g_1)=h_1$ and $\varphi(g_2)=h_2$. Then

$$\varphi(w_A)(h_1h_2)=\sup\{w_A(g_1g_2)\mid g_1=\varphi(h_1),g_2=\varphi(h_2)\}$$
$$\geq\sup\{\min\{w_A\ g_1),w_A\ g_2)\}\mid g_1=\varphi(h_1),g_2=\varphi(h_2)\}$$
$$=\min\{\sup\{w_A(g_1)\mid g_1=\varphi(h_1)\},\sup\{w_A(g_2)\mid g_2=\varphi(h_2)\}\}$$
$$=\min\{\varphi\ w_A)\ h_1),\varphi\ w_A)\ h_2)\},$$

and so

$$\varphi\ w_A)\ h_1h_2)\geq\min\{\varphi\ w_A)\ h_1),\varphi\ w_A)\ h_2)\}. \tag{16}$$

Let $g \in G$ and $h \in H$ such that $\varphi(g)=h$. Then

$$\begin{aligned}
&\varphi(w_A)(h^{-1})=\sup\{w_A(g^{-1}) \mid g^{-1} \in G, \varphi(g^{-1})=h^{-1}\} \\
&\geq \sup\{w_A(g) \mid g^{-1} \in G, \varphi^{-1}(g)=h^{-1}\} \\
&= \sup\{w_A(g) \mid g \in G, \varphi(g) = h\}= \varphi(w_A)(h),
\end{aligned}$$

then

$$\varphi\ w_A)\ h_1h_2) \geq \min\{\varphi\ w_A)\ h_1), \varphi\ w_A)\ h_2)\}. \tag{17}$$

et $h_1, h_2 \in H$ and $g_1, g_2 \in G$ with $\varphi(g_1) = h_1$ and $\varphi(g_2) = h_2$. Then

$$\begin{aligned}
&\varphi(r_A)(h_1h_2)=\inf\{r_A(g_1g_2) \mid g_1= \varphi(h_1), g_2 =\varphi(h_2)\} \\
&\leq \inf\{T(r_A(g_1), r_A(g_2)) \mid g_1=\varphi(h_1), g_2 = \varphi(h_2)\} \\
&=S(\inf\{r_A(g_1) \mid g_1 =\varphi(h_1)\},\inf\{r_A(g_2) \mid g_2 =\varphi(h_2)\}) \\
&=S(\varphi(r_A)(h_1), \varphi(r_A)(h_2)),
\end{aligned}$$

then

$$\varphi\ w_A)\ h_1h_2) \geq \min\{\varphi\ w_A)\ h_1), \varphi\ w_A)\ h_2)\}. \tag{18}$$

Let $h_1, h_2 \in H$ and $g_1, g_2 \in G$ with $\varphi(g_1) = h_1$ and $\varphi(g_2) = h_2$. Then

$$\begin{aligned}
&\varphi(r_A)(h_1h_2)=\inf\{r_A(g_1g_2) \mid g_1= \varphi(h_1), g_2 =\varphi(h_2)\} \\
&\leq \inf\{T(r_A(g_1), r_A(g_2)) \mid g_1=\varphi(h_1), g_2 = \varphi(h_2)\} \\
&=S(\inf\{r_A(g_1) \mid g_1 =\varphi(h_1)\},\inf\{r_A(g_2) \mid g_2 =\varphi(h_2)\}) \\
&=S(\varphi(r_A)(h_1), \varphi(r_A)(h_2)),
\end{aligned}$$

and so

$$\varphi\ w_A)\ h_1h_2) \geq \min\{\varphi\ w_A)\ h_1), \varphi\ w_A)\ h_2)\}. \tag{19}$$

Let $\in G$ and $h \in H$ such that $\varphi(g)=h$. Then

$$\begin{aligned}
&\varphi(r_A)(h^{-1})=\inf\{r_A(g^{-1}) \mid g^{-1} \in G, \varphi(g^{-1})=h^{-1}\} \\
&\leq \inf\{r_A(g) \mid g^{-1} \in G, \varphi^{-1}(g)=h^{-1}\} \\
&=\inf\{r_A(g) \mid g \in G, \varphi(g) = h\} \\
&=\varphi(r_A)(h),
\end{aligned}$$

and then

$$\varphi\ r_A)\left(h^{-1}\right) \leq \varphi\ r_A)\ h). \tag{20}$$

Let $h_1, h_2 \in H$ and $g_1, g_2 \in G$ with $\varphi(g_1) = h_1$ and $\varphi(g_2) = h_2$. Then

$$\begin{aligned}
&\varphi(w_A)(h_1h_2)=\inf\{w_A(g_1g_2) \mid g_1= \varphi(h_1), g_2 =\varphi(h_2)\} \\
&\leq \inf\{\max\{w_A\ g_1),w_A\ g_2)\} \mid g_1=\varphi(h_1), g_2 = \varphi(h_2)\} \\
&= \max\{\inf\{w_A(g_1) \mid g_1=\varphi(h_1)\},\inf\{w_A(g_2) \mid g_2 =\varphi(h_2)\}\} \\
&= \max\{\varphi\ w_A)\ h_1), \varphi\ w_A)\ h_2)\},
\end{aligned}$$

and so

$$\varphi\ w_A)\ h_1h_2) \leq \max\{\varphi\ w_A)\ h_1), \varphi\ w_A)\ h_2)\}. \tag{21}$$

Let $g \in G$ and $h \in H$ such that $\varphi(g)=h$. Then

$$\begin{aligned}
&\varphi(w_A)(h^{-1})=\inf\{w_A(g^{-1}) \mid g^{-1} \in G, \varphi(g^{-1})=h^{-1}\} \\
&\leq \inf\{w_A(g) \mid g^{-1} \in G, \varphi^{-1}(g)=h^{-1}\} \\
&= \inf\{w_A(g) \mid g \in G, \varphi(g) = h\} \\
&= \varphi(w_A)(h),
\end{aligned}$$

then

$$\varphi\ w_A)\left(h^{-1}\right) \leq \varphi\ w_A)\ h). \tag{22}$$

Therefore from *Eqs. (14)-(22)* we get that $\varphi(A) \in ICFN(H)$.

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Proposition 11. Let H be a group and $B=\ \mu_B,\ \vartheta_B)\in IFCN\ H)$ and $\varphi : G \to H$ is a group homomorphism. Then $\varphi^{-1}\ B)\in IFCN\ G)$.

Proof: Let $B=\ \mu_B,\ \vartheta_B)\in IFCN\ H)$ such that $\mu_B = r_B e^{iw_B}$ and $\vartheta_B\ x) = r_B e^{iw_B}$ and $\varphi^{-1}\ B) = \left(\varphi^{-1}(r_B)e^{i\varphi^{-1}(w_B)}\right),$ $\varphi^{-1}(r_B)e^{i\varphi^{-1}(w_B)})\big)$. Let $g_1, g_2 \in G$. Then

$$\begin{aligned}
&\varphi^{-1}(r_B)(g_1g_2) = r_B(\varphi(g_1g_2))\\
&= r_B(\varphi(g_1)\varphi(g_2))\\
&\geq T(r_B(\varphi(g_1)), r_B(\varphi(g_2)))\\
&= T(\varphi^{-1}(r_B)(g_1), \varphi^{-1}(r_B)(g_2)),
\end{aligned} \tag{23}$$

and so $\varphi^{-1}\ r_B)\ g_1g_2) \geq T\left(\varphi^{-1}\ r_B)\ g_1), \varphi^{-1}\ r_B)\ g_2)\right)$.

$$\begin{aligned}
&\varphi^{-1}(r_B)(g_1g_2) = r_B(\varphi(g_1g_2))\\
&= r_B(\varphi(g_1)\varphi(g_2))\\
&\leq T(r_B(\varphi(g_1)), r_B(\varphi(g_2)))\\
&= T(\varphi^{-1}(r_B)(g_1), \varphi^{-1}(r_B)(g_2)),
\end{aligned} \tag{24}$$

and so $\varphi^{-1}\ r_B)\ g_1g_2) \leq S\left(\varphi^{-1}\ r_B)\ g_1), \varphi^{-1}\ r_B)\ g_2)\right)$.

$$\begin{aligned}
&\varphi^{-1}(w_B)(g_1g_2) = w_B(\varphi(g_1g_2))\\
&= w_B(\varphi(g_1)\varphi(g_2))\\
&\geq \min\{w_B(\varphi\ g_1)), w_B(\varphi\ g_2))\}\\
&= \min\{\varphi^{-1}\ w_B)\ g_1), \varphi^{-1}\ w_B)\ g_2)\}
\end{aligned} \tag{25}$$

and so $\varphi^{-1}\ w_B)\ g_1g_2) \geq \min\{\varphi^{-1}\ w_B)\ g_1), \varphi^{-1}\ w_B)\ g_2)\}$.

$$\begin{aligned}
&\varphi^{-1}(w'_B)(g_1g_2) = w'_B(\varphi(g_1g_2))\\
&= w'_B(\varphi(g_1)\varphi(g_2))\\
&\leq maz\{w'_B\left(\varphi(g_1)\right), w'_B\left(\varphi(g_2)\right)\}\\
&= \max\{\varphi^{-1}(w'_B)(g_1), \varphi^{-1}(w'_B)(g_2)\},
\end{aligned} \tag{26}$$

so $\varphi^{-1}\ w_B)\ g_1g_2) \leq max\{\varphi^{-1}\ w_B)\ g_1), \varphi^{-1}\ w_B)\ g_2)\}$.

$$\varphi^{-1}\ r_B)(g^{-1}) = r_B\left(\varphi(g^{-1})\right) = r_B(\varphi^{-1}\ g)) \geq r_B(\varphi\ g)) = \varphi^{-1}\ r_B)\ g), \tag{27}$$

$$\varphi^{-1}\ r_B)(g^{-1}) = r_B\left(\varphi(g^{-1})\right) = r_B(\varphi^{-1}\ g)) \leq r_B(\varphi\ g)) = \varphi^{-1}\ r_B)\ g), \tag{28}$$

$$\varphi^{-1}\ w_B)(g^{-1}) = w_B\left(\varphi(g^{-1})\right) = w_B(\varphi^{-1}\ g)) \geq w_B(\varphi\ g)) = \varphi^{-1}\ w_B)\ g), \tag{29}$$

$$\varphi^{-1}\ w_B)(g^{-1}) = w_B\left(\varphi(g^{-1})\right) = w_B(\varphi^{-1}\ g)) \leq w_B(\varphi\ g)) = \varphi^{-1}\ w_B)\ g). \tag{30}$$

Let $g\in G$.

Thus *Eqs. (23)-(30)* give us that $\varphi^{-1}(B)\in ICFN(G)$.

Proposition 12. Let $A=\ \mu_A,\ \vartheta_A)\in NIFCN\ G)$ and H be a group. Suppose that $\varphi : G \to H$ is a homomorphism. Then $\varphi(A)\in NIFCN(H)$.

Proof: Using *Proposition 10*, we give that $\varphi(A)\in IFCN(H)$. Let $x, y\in H$ such that $\varphi(u)=x$ and $\varphi(w)=y$ with $u, w\in G$. Then

$$\begin{aligned}
&\varphi(r_A(xyx^{-1}))=\sup\{r_A(w) \mid w\in G, \varphi(w)= xyx^{-1}\}\\
&=\sup\{r_A(w) \mid w\in G, \varphi(w)=\varphi(u)\varphi(w)\varphi(u^{-1})\}\\
&=\sup\{r_A(w) \mid w \in G, \varphi(w) = \varphi(uwu^{-1})\}\\
&=\sup\{r_A(uwu^{-1}) \mid w \in G, \varphi(uwu^{-1})=y\}\\
&=\sup\{r_A(w) \mid w \in G, \varphi(w)=y\}\\
&=\varphi(r_A(y)),
\end{aligned} \tag{31}$$

so $\varphi\left(r_A\left(xyx^{-1}\right)\right)=\varphi\left(r_A\ y\right))$.

$$\begin{aligned}
&\varphi(w_A(xyx^{-1}))=\sup\{w_A(w) \mid w\in G, \varphi(w)= xyx^{-1}\}\\
&=\sup\{w_A(w) \mid w\in G, \varphi(w)=\varphi(u)\varphi(w)\varphi(u^{-1})\}\\
&=\sup\{w_A(w) \mid w \in G, \varphi(w) = \varphi(uwu^{-1})\}\\
&=\sup\{w_A(uwu^{-1}) \mid w \in G, \varphi(uwu^{-1})=y\}\\
&=\sup\{w_A(w) \mid w \in G, \varphi(w)=y\}\\
&=\varphi(w_A(y)),
\end{aligned} \tag{32}$$

then $\varphi\left(w_A\left(xyx^{-1}\right)\right)=\varphi\left(w_A\ y\right))$.

$$\begin{aligned}
&\varphi(r_A(xyx^{-1}))=\inf\{r_A(w) \mid w\in G, \varphi(w)= xyx^{-1}\}\\
&=\inf\{r_A(w) \mid w\in G, \varphi(w)=\varphi(u)\varphi(w)\varphi(u^{-1})\}\\
&=\inf\{r_A(w) \mid w \in G, \varphi(w) = \varphi(uwu^{-1})\}\\
&=\inf\{r_A(uwu^{-1}) \mid w \in G, \varphi(uwu^{-1})=y\}\\
&=\inf\{r_A(w) \mid w \in G, \varphi(w)=y\}\\
&=\varphi(r_A(y)),
\end{aligned} \tag{33}$$

then $\varphi(r'_A(xyx^{-1}))=\varphi(r'_A(y))$.

$$\begin{aligned}
&\varphi(w_A(xyx^{-1}))=\inf\{w_A(w) \mid w\in G, \varphi(w)= xyx^{-1}\}\\
&=\inf\{w_A(w) \mid w\in G, \varphi(w)=\varphi(u)\varphi(w)\varphi(u^{-1})\}\\
&=\inf\{w_A(w) \mid w \in G, \varphi(w) = \varphi(uwu^{-1})\}\\
&=\inf\{w_A(uwu^{-1}) \mid w \in G, \varphi(uwu^{-1})=y\}\\
&=\inf\{w_A(w) \mid w \in G, \varphi(w)=y\}\\
&=\varphi(w_A(y)),
\end{aligned} \tag{34}$$

then $\varphi\left(w_A\ xyx^{-1}\right))=\varphi\left(w_A\ y\right))$.

Thus for all $x, y\in H$ and from *Eqs. (31)-(34)* we get that

$$\begin{aligned}
&\varphi(A)(xyx^{-1})=(\varphi(\mu_A)(xyx^{-1}), \varphi(\vartheta_A)(xyx^{-1}))\\
&=(\varphi(r_A)(xyx^{-1}))e^{i\varphi(w_A)(xyx^{-1})}, \varphi(r_A)(xyx^{-1})e^{i\varphi(w_A)(xyx^{-1})})\\
&=(\varphi(r_A)(y))e^{i\varphi(w_A)(y)}, \varphi(r_A)(y)e^{i\varphi(w_A)(y)})\\
&=(\varphi(\mu_A)(y), \varphi(\vartheta_A)(y))\\
&=\varphi(A)(y),
\end{aligned}$$

Then $\varphi(A)\in NICFN(H)$.

Proposition 13. Let H be a commutative group and $B= \mu_B, \vartheta_B)\in NIFCN(H)$. If $\varphi :G \to H$ be a group homomorphism, then $\varphi^{-1}(B)\in NIFCN(G)$.

Proof: From *Proposition 11*, we get that φ^{-1} B)∈IFCN G). Let x, y∈G then

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$$
\begin{aligned}
\varphi^{-1}(r_B)(xyx^{-1}) &= r_B(\varphi(xyx^{-1})) \\
&= r_B(\varphi(x)\varphi(y)\varphi(x^{-1})) \\
&= r_B(\varphi(x)\varphi(y)\varphi^{-1}(x)) \\
&= r_B(\varphi(y)) \\
&= \varphi^{-1}(r_B)(y),
\end{aligned} \tag{35}
$$

and thus $\varphi^{-1}\ r_B)\left(xyx^{-1}\right)=\varphi^{-1}\ r_B)\ y)$.

$$
\begin{aligned}
\varphi^{-1}(w_B)(xyx^{-1}) &= w_B(\varphi(xyx^{-1})) \\
&= w_B(\varphi(x)\varphi(y)\varphi(x^{-1})) \\
&= w_B(\varphi(x)\varphi(y)\varphi^{-1}(x)) \\
&= w_B(\varphi(y)) \\
&= \varphi^{-1}(w_B)(y),
\end{aligned} \tag{36}
$$

so $\varphi^{-1}\ w_B)\left(xyx^{-1}\right)=\varphi^{-1}\ w_B)\ y)$.

$$
\begin{aligned}
\varphi^{-1}(r_B)(xyx^{-1}) &= r_B(\varphi(xyx^{-1})) \\
&= r_B(\varphi(x)\varphi(y)\varphi(x^{-1})) \\
&= r_B(\varphi(x)\varphi(y)\varphi^{-1}(x)) \\
&= r_B(\varphi(y)) \\
&= \varphi^{-1}(r_B)(y),
\end{aligned} \tag{37}
$$

then $\varphi^{-1}\ r_B)\left(xyx^{-1}\right)=\varphi^{-1}\ r_B)\ y)$.

$$
\begin{aligned}
\varphi^{-1}(w_B)(xyx^{-1}) &= w_B(\varphi(xyx^{-1})) \\
&= w_B(\varphi(x)\varphi(y)\varphi(x^{-1})) \\
&= w_B(\varphi(x)\varphi(y)\varphi^{-1}(x)) \\
&= w_B(\varphi(y)) \\
&= \varphi^{-1}(w_B)(y).
\end{aligned} \tag{38}
$$

thus $w_B)\left(xyx^{-1}\right)=\varphi^{-1}\ w_B)\ y)$. Therefore *Eqs. (35)-(38)* give us that

$$
\begin{aligned}
\varphi^{-1}(r_B)(xyx^{-1}) &= r_B(\varphi(xyx^{-1})) \\
&= r_B(\varphi(x)\varphi(y)\varphi(x^{-1})) \\
&= r_B(\varphi(x)\varphi(y)\varphi^{-1}(x)) \\
&= r_B(\varphi(y)) \\
&= \varphi^{-1}(r_B)(y),
\end{aligned}
$$

Thus $\varphi^{-1}(B)\in NIFCN(G)$.

Proposition 14. Let $A=\ \mu_A,\ \vartheta_A)\in$ ***IFCN(G)*** and $B=\ \mu_B,\ \vartheta_B)\in$ ***IFCN(G)*** such that $A\sqsubseteq B$.

If $\varphi:G\rightarrow H$ is a group homomorphism, then $\varphi(A)\sqsubseteq\varphi(B)$.

Proof: Let $A=\ \mu_A,\ \vartheta_A)\in IFCN\ G)$ and $B=\ \mu_B,\ \vartheta_B)\in IFCN\ G)$ such that $\mu_A=r_Ae^{iw_A}$ and $\vartheta_A\ x)=r_Ae^{iw_A}$ and $\mu_B=r_Be^{iw_B}$ and $\vartheta_B\ x)=r_Be^{iw_B}$. Using *Proposition 10.* we will have that

$$\varphi\ A)=\ \varphi(\mu_A),\varphi(\vartheta_A))=\left(\varphi(r_A)e^{i\varphi(w_A)}),\ \varphi(r_A)e^{i\varphi(w_A)})\right)\in ICFN\ H),$$

And

$$\varphi\ B)=\ \varphi(\mu_B),\varphi(\vartheta_B))=\left(\varphi(r_B)e^{i\varphi(w_B)}),\ \varphi(r_B)e^{i\varphi(w_B)})\right)\in ICFN\ H).$$

Let $x,\ y\in H$ and $u,\ v\in G$ then

$$\begin{aligned}
&\varphi(r_A)(xyx^{-1})=\sup\{r_A(z) \mid z\in G, \varphi(z)=xyx^{-1}\}\\
&=\sup\{r_A(uvu^{-1}) \mid u, v\in G, \varphi(u)\ x, \varphi(v)=y\}\\
&\geq\sup\{T(rA(v), r_B(u)) \mid \varphi(u)\ x, \varphi(v)=y\}\\
&=T(\sup\{r_A(v) \mid y=\varphi(v)\},\sup\{r_B(u) \mid x=\varphi(u)\})\\
&=T(\varphi(r_A)(y), \varphi(r_B)(x)),
\end{aligned} \tag{39}$$

and so $\varphi\ r_A)\left(xyx^{-1}\right)\geq T\left(\varphi\ r_A)\ y), \varphi\ r_B)\ x)\right)$.

$$\begin{aligned}
&\varphi(w_A)(xyx^{-1})=\sup\{w_A(z) \mid z\in G, \varphi(z)=xyx^{-1}\}\\
&=\sup\{w_A(uvu^{-1}) \mid u, v\in G, \varphi(u)\ x, \varphi(v)=y\}\\
&\geq\sup\{\min\{w_A\ v), w_B\ u)\} \mid \varphi(u)\ x, \varphi(v)=y\}\\
&=\min\{\sup\{w_A(v) \mid y=\varphi(v)\},\sup\{w_B(u) \mid x=\varphi(u)\}\}\\
&=\min\{\varphi\ w_A)\ y), \varphi\ w_B)\ x)\},
\end{aligned} \tag{40}$$

and so $\varphi\ w_A)\left(xyx^{-1}\right)\geq\min\{\varphi\ w_A)\ y), \varphi\ w_B)\ x)\}$,

$$\begin{aligned}
&\varphi(r_A)(xyx^{-1})=\sup\{r_A(z) \mid z\in G, \varphi(z)=xyx^{-1}\}\\
&=\inf\{r_A(uvu^{-1}) \mid u, v\in G, \varphi(u)\ x, \varphi(v)=y\}\\
&\leq\inf\{S(r_A(v), r_B(u)) \mid \varphi(u)\ x, \varphi(v)=y\}\\
&=S(\inf\{r_A(v) \mid y=\varphi(v)\},\inf\{r_B(u) \mid x=\varphi(u)\})\\
&=S(\varphi(r_A)(y), \varphi(r_B)(x)),
\end{aligned} \tag{41}$$

thus $\varphi\ r_A)\left(xyx^{-1}\right)\leq S(\varphi(r_A')(y), \varphi(r_B')(x))$.

$$\begin{aligned}
&\varphi(w_A)(xyx^{-1})=\inf\{w_A(z) \mid z\in G, \varphi(z)=xyx^{-1}\}\\
&=\inf\{w_A(uvu^{-1}) \mid u, v\in G, \varphi(u)\ x, \varphi(v)=y\}\\
&\leq\inf\{\max\{w_A\ v), w_B\ u)\} \mid \varphi(u)\ x, \varphi(v)=y\}\\
&=\max\{\inf\{w_A(v) \mid y=\varphi(v)\},\inf\{w_B(u) \mid x=\varphi(u)\}\}\\
&= \max\{\varphi\ w_A)\ y), \varphi\ w_B)\ x)\},
\end{aligned} \tag{42}$$

and so $\varphi\ w_A')\left(xyx^{-1}\right)\leq\max\{\varphi\ w_A')\ y), \varphi\ w_B')\ x)\}$.

Thus using *Eqs. (39)-(42)* we will have that $\varphi(A)\sqsubseteq\varphi(B)$.

Proposition 15. Let $A=\ \mu_A, \vartheta_A)\in IFCN(H)$ and $B=\ \mu_B, \vartheta_B)\in IFCN(H)$ such that $A\sqsubseteq B$.

If $\varphi :G\rightarrow H$ is a group homomorphism, then $\varphi^{-1}(A)\sqsubseteq\varphi^{-1}(B)$.

Proof: Let $A=\ \mu_A, \vartheta_A)\in IFCN\ H)$ and $B=\ \mu_B, \vartheta_B)\in IFCN\ H)$ such that $\mu_A= r_A e^{iw_A}$ and $\vartheta_A\ x)=r_A e^{iw_A}$ and $\mu_B=r_B e^{iw_B}$ and $\vartheta_B(x)= r_B' e^{iw_B'}$. Using *Proposition 11.* we will have that

$$\varphi^{-1}\ A)=\left(\varphi^{-1}(\mu_A), \varphi^{-1}(\vartheta_A)\right)=\left(\varphi^{-1}(r_A)e^{i\varphi^{-1}(w_A)}), \varphi^{-1}(r_A)e^{i\varphi^{-1}(w_A)})\right)\in ICFN\ G).$$

And

$$\varphi^{-1}\ B)=\left(\varphi^{-1}(\mu_B), \varphi^{-1}(\vartheta_B)\right)=\left(\varphi^{-1}(r_B)e^{i\varphi^{-1}(w_B)}), \varphi^{-1}(r_B)e^{i\varphi^{-1}(w_B)})\right)\in ICFN\ G).$$

Let $x, y\in G$, then

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$$\begin{aligned}\varphi^{-1}(r_A)(xyx^{-1}) &= r_A(\varphi(xyx^{-1}))\\ &=r_A(\varphi(x)\varphi(y)\varphi(x^{-1}))\\ &= r_A(\varphi(x)\varphi(y)\varphi^{-1}(x))\\ &\geq T(r_A(\varphi(y)), r_B(\varphi(x)))\\ &=T(\varphi^{-1}(r_A)(y), \varphi^{-1}(r_B)(x)),\end{aligned} \tag{43}$$

Then $\varphi^{-1}\ r_A)\left(xyx^{-1}\right)\geq T\left(\varphi^{-1}\ r_A)\ y), \varphi^{-1}\ r_B)\ x)\right)$.

$$\begin{aligned}\varphi^{-1}(w_A)(xyx^{-1}) &= w_A(\varphi(xyx^{-1}))\\ &=w_A(\varphi(x)\varphi(y)\varphi(x^{-1}))\\ &=w_A(\varphi(x)\varphi(y)\varphi^{-1}(x))\\ &\geq \min\{w_A\left(\varphi\ y)\right), w_B\left(\varphi\ x)\right)\}\\ &=\min\{\varphi^{-1}\ w_A)\ y), \varphi^{-1}\ w_B)\ x)\},\end{aligned} \tag{44}$$

thus $\varphi^{-1}\ w_A)\left(xyx^{-1}\right)\geq \min\{\varphi^{-1}\ w_A)\ y), \varphi^{-1}\ w_B)\ x)\}$.

$$\begin{aligned}\varphi^{-1}(r_A)(xyx^{-1}) &=r_A(\varphi(xyx^{-1}))\\ &=r_A(\varphi(x)\varphi(y)\varphi(x^{-1}))\\ &=r_A(\varphi(x)\varphi(y)\varphi^{-1}(x))\\ &\leq S(r_A(\varphi(y)), r_B(\varphi(x)))\\ &=S(\varphi^{-1}(r_A)(y), \varphi^{-1}(r_B)(x)),\end{aligned} \tag{45}$$

so $\varphi^{-1}\ r_A)\left(xyx^{-1}\right)\leq S\left(\varphi^{-1}\ r_A)\ y), \varphi^{-1}\ r_B)\ x)\right)$.

$$\begin{aligned}\varphi^{-1}(w_A)(xyx^{-1})&=w_A(\varphi(xyx^{-1}))\\ &=w_A(\varphi(x)\varphi(y)\varphi(x^{-1}))\\ &=w_A(\varphi(x)\varphi(y)\varphi^{-1}(x))\\ &\leq \max\{w_A\left(\varphi\ y)\right), w_B\left(\varphi\ x)\right)\}\\ &= \max\{\varphi^{-1}\ w_A)\ y), \varphi^{-1}\ w_B)\ x)\},\end{aligned} \tag{46}$$

thus $\varphi^{-1}\ w_A)\left(xyx^{-1}\right)\leq \max\left\{\varphi^{-1}\ w_A)\ y), \varphi^{-1}\ w_B)\ x)\right\}$.

Thus *Eqs. (43)-(46)* give us that $\varphi^{-1}(A)\sqsubseteq\varphi^{-1}(B)$.

7 | Conclusion and Open Problem

In this study, intuitionistic fuzzy complex subgroups with respect to t-norm T and s-norm sare defined and investigated some properties of them. Later, the inverse, composition, intersection and normality of them are introduced and we proved some basic new results and present some properties of them. Now one can investigate intuitionistic fuzzy complex submodules with respect to t-norm T and s-norm S as we did and this can be an open problem. We would like to thank the reviewers for carefully reading the manuscript and making several helpful comments to increase the quality of the paper.

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Conflicts of Interest

The author has seen and agreed with the contents of the manuscript and there is no financial interest to report and certifies that the submission is original work and is not under review at any other publication.

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Picture Fuzzy Semi-Prime Ideals

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Abstract

Picture Fuzzy Sets (PFSs) are expanded to include Intuitionistic Fuzzy Sets (IFSs), with the extra advantage of avoiding underlying limitations. PFS based models may be adequate in situations when we face opinions involving more answer of types: yes, abstain and no. In this paper, the concepts of semi-prime ideals of PFS are explained. We also discussed how to construct picture fuzzy regular and intra-regular ideals and represents certain fundamental facts.

Keywords: Intuitionistic fuzzy set, Picture fuzzy set, Picture fuzzy ideals, Picture fuzzy semi-prime ideals, Picture fuzzy regular ideals.

1 | Introduction

Zadeh [21] developed the fuzzy set methodology, that assigns a number from the unit range [0, 1] to each element of the discursive multiverse to indicate the degree of sense of belonging to the set under consideration using a degree of membership μ. Fuzzy sets are a subset of set theory that allows for states halfway between entire and nothing. A membership function is employed in a fuzzy set to represent the extent to which an element belongs to a class. The membership value can be anything between 0 and 1, with 0 indicating that the element is not a member of a class, 1 indicating that it is, and other values indicating the degree of membership. The membership function in fuzzy sets replaced the characteristic function in crisp sets. Fuzzy set theory has been applied to a variety of domains since Zadeh's seminal work, including artificial intelligence, management sciences, engineering, mathematics, statistics, signal processing, automata theory, social sciences, medical sciences, and biological sciences.

Because of the absence of nonmembership functions and the disregard for the potential of hesitation margin, the idea of fuzzy sets theory appears to be inconclusive. Atanassov [9] examined these flaws and created the concept of Intuitionistic Fuzzy Sets (IFSs) to address them. The construct (that is IFSs) combines the membership function, with the nonmembership function, ν, and the hesitation margin, π (that is neither membership nor nonmembership functions), resulting in $\mu+\nu\leq 1$ and

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$\mu+\nu+\tau\leq 1$. IFSs give a versatile framework for elaborating uncertainty and ambiguity. IFS overcomes the defects of fuzzy set and can deal with fuzzy, uncertainty and incomplete information. There are lots work done in the field of IFSs [1], [2], [5], [11].

Although IFS has been successfully applied in many domains, it cannot handle inconsistent information in real life. Such as voting questions, all voting results can be divided into four groups that are "vote for", "abstain", "vote against" and "refuse to vote". In order to solve this type of issue, Picture Fuzzy Set (PFS) was proposed by Cuong [23]. PFS consists of three functions: positive membership function, neutral membership function and negative membership function. The PFS solved the voting problem successfully, and is applied to clustering, fuzzy inference, and decision-making.

Algebraic structures are important in mathematics. The concept of intuitionistic fuzzification of various semigroup ideals was introduced by Jun et al. [12]-[14]. Kim and Lee [15] gave the notion of intuitionistic fuzzy bi-ideals of semigroups. Manna et al. [3] have discussed on R-subgroup of near-rings. Adak et al. [4], [6]-[8], [22] present some results on pythagorean fuzzy ideal and Q-fuzzy ideals of near rings. Biswas [10] gives some properties of fuzzy subgroups. Yun [20] discussed on fuzzy ideal of ordered semi-group. Sardar et al. [18] gave the concept of intuitionistic fuzzy prime ideals, semi-prime ideals and also intuitionistic fuzzy ideal extension in a Γ -semigroup in [16], [17], [19].

In this paper, we introduce the notion of picture fuzzy subsemigroup, picture fuzzy left and right ideals of ordered semigroup. Also, we define picture fuzzy semi-prime ideals and picture fuzzy prime ideals. We investigate some important results picture fuzzysemi-prime ideals. The concepts of picture fuzzy left regular ideal and picture fuzzy right ideals are presented. Also, discussed important properties of these regular ideals.

The remainder of the paper is laid out as follows: preliminaries and definitions such as ordered set, ordered subgroups, IFSs, and PFSs are given in Section 2. In Section 3, we introduced some aspects of picture fuzzy prime ideals and semi-prime ideals as well as some of the important properties of picture fuzzy prime ideals. Section 4 concludes with a conclusion.

2 | Preliminaries and Definitions

We will review the related concepts of fuzzy sets, IFSs, and PFSs in this section. The definition of oredered set, ordered semigroup, prime ideal, semi-prime ideal are represented.

Definition 1 (Ordered Semigroup). A non empty set M is called an ordered semigroup if it is both an ordered set and a semigroup that meets the following criteria:

$a\leq b\Rightarrow xa\leq xb$ and $ax\leq bx$ *for all* $a,b,x\in M$.

Definition 2. Consider $(M,.,\leq)$ be an ordered semigroup. A non-empty subset G of M is called a subsemigroup of M if $G^2\subseteq G$.

Definition 3. Let P be a subset of an ordered semigroup M, that isn't empty. Then P is called a left (resp. right) ideal of M if it satisfies:

I. $MP\subseteq P$ (resp. $PM\subseteq P$).
II. $(\text{for all } p\in P)(\text{for all } q\in M),(q\leq p\Rightarrow q\in P)$.

P will be ideal of M if it is both left and right ideal of M.

Definition 4. Let $(M,.,\leq)$ be an ordered semigroup and N be a non-empty subset of M. Then N is called prime if $pq \in N \Rightarrow p \in N$ or $q \in N$ for all $p,q \in M.$

Let N be an ideal of M, , if N is prime subset of M, then N is called prime ideal.

Definition 5. Let $(M,.,\leq)$ be an ordered semigroup and N be a non-empty subset of $M.$ Then N is called semi-prime if $p^2 \in N \Rightarrow p \in N$ for all $p \in M.$ Let N be an ideal of $M.$ If N is a semi-prime subset of $M,$ then N is called semi-prime ideal.

Definition 6. A fuzzy set F in a universal set X is defined as

$$F = \{< x, \mu_F(x) >: x \in X\},$$

where $\mu_F : X \to [0,1]$ is a mapping that is known as the fuzzy set's membership function.

The complement of μ is defined by $\bar{\mu}(x) = 1 - \mu(x)$ for all $x \in X$ and denoted by $\bar{\mu}.$

Definition 7. Let $(M,.,\leq)$ be an ordered semigroup. A fuzzy subset μ of M is called a fuzzy ideal of M, , if the following axioms are satisfied:

I. If $p \leq q$ then $\mu(p) \geq \mu(q).$
II. $\mu(pq) \geq max\{\mu(p), \mu(q)\}$ for all $(p.q) \in M.$

Definition 8. Let X be a fixed set. An IFS A in X is an expression having the form

$$A = \{< x, \mu_A(x), w_A(x) >: x \in X\},$$

where the $\alpha_A(x)$ is the memebership grade and $\beta_A(x)$ is the non-membership grade of the element $x \in X$ respectively.

Also $u_A : X \to [0,1]$, $w_A : X \to [0,1]$ and satisfy the condition $0 \leq u_A(x) + w_A(x) \leq 1$ for all $x \in X$.

The degree of indeterminacy is $h_A(x) = 1 - u_A(x) - w_A(x)$.

Definition 9. Let $A = (u_A, v_A, w_A)$ be a PFS in $M.$ Then $A = (u_A, v_A, w_A)$ is called picture fuzzy subsemigroup of M if it satisfies the following axioms:

I. $u_A(pq) \geq min\{u_A(p), u_A(q)\}.$
II. $v_A(pq) \leq max\{v_A(p), v_A(q)\}.$
III. $w_A(pq) \leq max\{w_A(p), w_a(q)\}$ for all $p,q \in M.$

Definition 10. A PFS $A = (u_A, v_A, w_A)$ in M is said to be picture fuzzy left ideal of M if following axioms are satisfied:

I. $p \leq q$ implies $u_A(p) \geq u_A(q)$ and $u_A(pq) \geq u_A(q).$
II. $p \leq q$ implies $v_A(p) \leq v_A(q)$ and $v_A(p) \leq v_A(q).$
III. $p \leq q$ implies $w_A(p) \leq w_A(q)$ and $w_A(pq) \leq w_A(q)$ for all $p,q \in M.$

Definition 11. A PFS $A=(u_A,v_A,w_A)$ in M is said to be picture fuzzy right ideal of M if following axioms are satisfied:

I. $p\le q$ implies $u_A(p)\ge u_A(q)$ and $u_A(pq)\ge u_A(p)$.

II. $p\le q$ implies $v_A(p)\le v_A(p)$ and $v_A(pq)\le v_A(q)$.

III. $p\le q$ implies $w_A(p)\le w_A(p)$ and $w_A(pq)\le w_A(q)$ *for all* $p,q\in M$.

A PFS $A=(u_A,v_A,w_A)$ is called a picture fuzzy ideal of M if it is left ideal as well as right ideal.

3 | Some Results on Picture Fuzzy Semi-Prime Ideals

This section introduces the notion of picture fuzzy prime ideal, picture fuzzy semi-prime ideal, picture fuzzy regular ideals and picture fuzzy intra-regular ideals of ordered semigroups. Also, prove some important results utilizing characteristic function of a non-empty subset of ordered semigroups.

Definition 12. A fuzzy subset μ of M is called prime, if

$$\mu(pq)=\max\{\mu(p),\mu(q)\} \text{ for all } p,q\in M,$$

where $(M,.\le)$ be an ordered semigroup.

A fuzzy ideal μ of M is called a fuzzy prime ideal of M if μ is a prime fuzzy subset of M.

Definition 13. Let $A=(u_A,v_A,w_A)$ be a PFS in M. Then $A=(u_A,v_A,w_A)$ is called picture fuzzy prime of M if it satisfies the following axioms:

I. $u_A(pq)=max\{u_A(p),u_A(q)\}$.

II. $v_a(pq)=min\{v_A(p),v_A(q)\}$.

III. $w_A(pq)=min\{w_a(p),w_A(q)\}$ *for all* $p,q\in M$.

Definition 14. Let us consider μ be a fuzzy subset of an ordered semigroup M. If $\mu(p)\ge\mu(p^2)$ for all $p\in M$, then μ is called semi-prime. A fuzzy ideal μ of M is called a fuzzy semi-prime ideal of M if μ is a fuzzy semi-prime subset of M.

Definition 15. Let $A=(u_A,v_A,w_A)$ be a PFS in M. Then $A=(u_A,v_A,w_A)$ is called picture fuzzy semi-prime of M if following criterias are satisfied:

I. $u_A(p)\ge u_A(p^2)$.

II. $v_a(p)\le v_A(p^2)$.

III. $w_A(p)\le w_A(p^2)$ *for all* $p\in M$.

Theorem 1. For any picture fuzzy subsemigroup $A=(u_A,v_A,w_A)$ of M, if $A=(u_A,v_A,w_A)$ is picture fuzzy semi-prime, then $A(p)=A(p^2)$ holds.

Proof: Let p be an element of M. Since u_A is a fuzzy subsemigroup of M, then and so we have $u_A(p)=u_A(p^2)$.

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Also

$$u_A(p) \geq u_A(p^2) \equiv \min\{u_A(p), u_A(p)\} = u_A(p),$$

and

$$v_A(p) \leq v_A(p^2) \equiv \max\{v_A(p), v_A(p)\} = v_A(p),$$

thus $v_A(p) = v_A(p^2)$.

Also, we have

$$w_A(p) \leq w_A(p^2) \equiv \max\{w_A(p), w_A(p)\} = w_A(p),$$

thus $w_A(p) = w_A(p^2)$.

This proves the theorem. □

Definition 16. An ordered semigroup M is called left (resp. right) regular if, for each element a of M, there exists an element x in M such that $a \leq xa^2 (resp.\ a \leq a^2 x)$.

Theorem 2. Let M be left regular. Then, for every picture fuzzy left ideal $A = (u_A, v_A, w_A)$ of M, $P(p) = P(p^2)$ holds for all $p \in M$.

Proof: Let p be any element of M. Since M is left regular, there exists an element x in M such that $p \leq xp^2$.

Thus we have

$$u_A(p) \geq u_A(xp^2) \geq u_A(p^2) \geq u_A(p),$$

and so we have

$$u_A(p) = u_A(p^2).$$

Again

$$v_A(p) \leq v_A(xp^2) \leq v_A(p^2) \leq v_A(p),$$

thus

$$v_A(p) = v_A(xp^2).$$

Also, we have

$$w_A(p) \leq w_A(xp^2) \leq w_A(p^2) \leq w_A(p),$$

thus

$$w_A(p) = w_A(xp^2),$$

so, $P(p) = P(p^2)$.

This completes the proof. □

Theorem 3. Let M be left regular. Then, every picture fuzzy left ideal of M is picture fuzzy semi-prime.

Proof: Let $A = (u_A, v_A, w_A)$ be a picture fuzzy left ideal of M and let $p \in M$. Then, there exists an element x in M such that $p \leq xp^2$ since M is left regular. So, we have

$$u_A(p) \geq u_A(xp^2) \geq u_A(p^2),$$
$$v_A(p) \leq v_A(xp^2) \leq v_A(p^2),$$

and

$$w_A(p) \leq w_A(xp^2) \leq w_A(p^2).$$

This completes the proof. □

Definition 17. An ordered semigroup M is called intra-regular if, for each element p of M, there exist elements x and y in M such that $p \leq xp^2 y$.

Definition 18. Let $A = (u_A, v_A, w_A)$ be a PFS in M. Then $A = (u_A, v_A, w_A)$ is called a picture fuzzy interior ideal of M if it satisfies axioms:

I. $x \leq y$ implies $u_A(x) \geq u_A(y)$ and $u_A(xsy) \geq u_A(s)$.
II. $x \leq y$ implies $v_A(x) \leq v_A(y)$ and $v_A(xsy) \leq v_A(s)$.
III. $x \leq y$ implies $w_A(x) \leq w_a(y)$ and $w_a(xsy) \leq w_A(s)$ *for all* $x, y \in M$.

Theorem 4. Let $A = (u_A, v_A, w_A)$ be a PFS in an intra-regular ordered semigroup M. Then, $A = (u_A, v_A, w_A)$ is a picture fuzzy interior ideal of M if and only if $A = (u_A, v_A, w_A)$ is an picture fuzzy ideal of M.

Proof: Let p, q be any elements of M, and let $A = (u_A, v_A, w_A)$ be a picture fuzzy interior ideal of M.

Then, since M is intra-regular, there exist elements x, y, p and in M such that $q \leq uq^2 v$. Then, since u_A is a fuzzy interior ideal of M, we have

$$u_A(pq) \geq u_A((xp^2y)q) = u_A((xp)p(yq)) \geq u_a(p),$$

and

$$u_A(pq) \geq u_A(p(xq^2y)) = u_A((px)q(qy)) \geq u_A(q).$$

Again

$$v_A(pq) \geq v_A((xp^2y)q) = v_A((xp)p(yq)) \geq v_A(p),$$

and

$$v_A(pq) \geq v_A(p(xq^2y)) = v_A((px)q(qy)) \geq v_A(q).$$

Also, we have

$$w_A(pq) \geq w_A((xp^2y)q) = w_A((xp)p(yq)) \geq w_A(p),$$

and

$$w_A(pq) \geq w_A(p(xq^2y)) = w_A((px)q(qy)) \geq w_A(q).$$

On the other hand, let $A = (u_A, v_A, w_A)$ be a picture fuzzy ideal of M. Then, since u_A is a fuzzy ideal of M, we have

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$$u_a(xpy) = u_A(x(py)) \geq u_A(py) \geq u_A(p),$$
$$v_a(xpy) = v_A(x(py)) \leq v_A(py) \leq v_A(p),$$

and

$$w_a(xpy) = w_A(x(py)) \leq w_A(py) \leq w_A(p).$$

For all x, a and $y \in M$.

This completes the proof. □

Theorem 5. Let $A = (u_A, v_A, w_A)$ be a picture fuzzy ideal of M. If M is intra-regular,then $A = (u_A, v_A, w_A)$ is picture fuzzy semi-prime.

Proof: Let p be any element of M. Then since M is intra-regular, there exist x and y in M such that $p \leq xp^2y$. So, we have

$$u_A(p) \geq u_A(xp^2y) \geq u_A(p^2y) \geq u_A(p^2),$$
$$v_A(p) \leq v_A(xp^2y) \leq v_A(p^2y) \leq v_A(p^2),$$

and

$$w_A(p) \leq w_A(xp^2y) \leq w_A(p^2y) \leq w_A(p^2).$$

This proves the theorem. □

Theorem 6. Let $A = (u_A, v_A, w_A)$ be a picture fuzzy interior ideal of M. If M is an intra-regular, then $A = (u_A, v_A, w_A)$ is a picture fuzzy semi-prime.

Proof: Let p be any element of M. Then since M is intra-regular, there exist x and y in M such that $p \leq xp^2y$.

$$u_A(p) \geq u_A(xp^2y) \geq u_A(p^2),$$
$$v_A(p) \leq v_A(xp^2y) \leq v_A(p^2y) \leq v_A(p^2),$$

and

$$w_A(p) \leq w_A(xp^2y) \leq w_A(p^2y) \leq w_A(p^2).$$

This proves the theorem. □

Theorem 7. Let M be intra-regular. Then, for all picture fuzzy interior ideal $A = (u_A, v_A, w_A)$ and for all $p \in M$, $A(p) = A(p^2)$ holds.

Proof: Let p be any element of M. Then since M is intra-regular, there exist x and y in M such that $p \leq xp^2y$. So, we have

$$u_A(p) \geq u_A(xp^2y) \geq u_A(p^2) \geq u_a((xp^2y)(xp^2y)) = u_A((xp)p(yxp^2y) \geq u_A(p)),$$
$$v_A(p) \leq v_A(xp^2y) \geq v_A(p^2) \geq v_a((xp^2y)(xp^2y)) = v_A((xp)p(yxp^2y) \leq v_A(p)),$$

and

$$w_A(p) \leq w_A(xp^2y) \geq w_A(p^2) \geq w_a((xp^2y)(xp^2y)) = w_A((xp)p(yxp^2y) \leq w_A(p)).$$

So, we have $A(p)=A(p^2)$.

This completes the proof. □

Theorem 8. Let M be intra-regular. Then, for all picture fuzzy interior ideal $A=(u_A,v_A,w_A)$ and for all $p,q\in M$, $A(pq)=A(qp)$ holds.

Proof: Let p be any element of M. Then since M is intra-regular, there exist x and y in M such that $p\le xp^2y$. So, we have

$$u_A(pq)=u_A((pq)^2)=u_A(p(qp)q)\ge u_A(qp)=u_A((qp)^2)=u_A(q(pq)p)\ge u_A(pq),$$
$$v_A(pq)=v_A((pq)^2)=v_A(p(qp)q)\le v_A(qp)=v_A((qp)^2)=v_A(q(pq)p)\le v_A(pq),$$

and

$$w_A(pq)=w_A((pq)^2)=w_A(p(qp)q)\le w_A(qp)=w_A((qp)^2)=w_A(q(pq)p)\le w_A(pq).$$

So, we have $P(pq)=P(qp)$.

This proves the theorem. □

Definition 19. An ordered semigroup M is called archimedean if, for any elements p,q there exists a positive integer n such that $p^2\in M\ q\ M$.

Theorem 9. Suppose M be an ordered archimedean semigroup. Then, each picture fuzzy semi-prime fuzzy ideal of M is a constant function.

Proof: Let $A=(u_A,v_A,w_A)$ be any picture fuzzy semi-prime fuzzy ideal of M and $p,q\in M$. Then since M is archimedean, there exist x and y in M such that $p^n=xqy$ for some integer n.

Then, we have

$$u_A(p)=u_A(p^n)=u_A(xqy)\ge u_A(q),$$

and

$$u_A(q)=u_A(q^n)=u_A(xpy)\ge u_A(p),$$

and

$$v_A(p)=v_A(p^n)=v_A(xqy)\le v_A(q),$$

and

$$v_A(q)=v_A(q^n)=v_A(xpy)\le v_A(p),$$

also, we have

$$w_A(p)=w_A(p^n)=w_A(xqy)\le w_A(q),$$

and

$$w_A(q)=w_A(q^n)=w_A(xpy)\le w_A(p).$$

Therefore, we have

$$P(p)=P(q), \text{ for all } p,q \in M.$$

This proves the theorem. □

4 | Conclusion

The PFS is an effective expansion of the IFS for dealing with knowledge uncertainty. In this context, we present the concepts of picture fuzzy prime ideals and semi-prime ideals of ordered semigroups in this study. Several of its appealing characteristics have also been studied. We also explore various findings on picture fuzzy regular ideals and intraregular ideals of ordered semigroups, along with promote the implementation of picture fuzzy regular ideals.

We'll look into the decision-making process more in the future. Interval-valued PFSs are being used to solve difficulties with uncertain data. An investigation of the interval-valued picture fuzzy will be conducted oredered semigroups, near-rings and interval-valued picture prime and semi-prime ideals, as well as their algebraic features.

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Conflicts of Interest

The authors declare that there is no competing of interests.

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Pythagorean and Fermatean Fuzzy Sub-Group Redefined in Context of $\mathscr{T}$-Norm and $\mathscr{S}$-Conorm

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Abstract

This paper aims to study Pythagorean and Fermatean Fuzzy Subgroups (FFSG) in the context of $\mathscr{T}$-norm and $\mathscr{S}$-conorm functions. The paper examines the extensions of fuzzy subgroups, specifically "Pythagorean Fuzzy Subgroups (PFSG)" and "FFSG", along with their properties. In the existing literature on Pythagorean and FFSG, the standard properties for membership and non-membership functions are based on the "min" and "max" operations, respectively. However, in this work, we develop a theory that utilizes the $\mathscr{T}$-norm for "min" and the $\mathscr{S}$-conorm for "max", providing definitions of Pythagorean and FFSG with these functions, along with relevant examples. By incorporating this approach, we introduce multiple options for selecting the minimum and maximum values. Additionally, we prove several results related to Pythagorean and FFSG using the $\mathscr{T}$-norm and $\mathscr{S}$-conorm, and discuss important properties associated with them.

Keywords: Fuzzy sets, Pythagorean fuzzy subgroups, Fermatean fuzzy subgroups, $\mathscr{T}$-Norm, $\mathscr{S}$-Conorm.

1 | Introduction

In the fields of mathematical analysis and information sciences, it's crucial to investigate and manage uncertainty. This inherent uncertainty extends even to symmetries in various objects, requiring the conceptual tools of fuzzy logic and group theory for its characterization. Zadeh [1] ignited the spark of this exploration with the introduction of fuzzy set theory, opening up avenues for dealing with nuanced classifications and gradations in sets. Rosenfeld [2] brought this concept into the domain of group theory, significantly contributing to our understanding of fuzzy groups. As the field matured, Anthony and Sherwood [3] added another layer of depth to fuzzy sets by redefining fuzzy groups with respect to t-norm. Meanwhile, Bhattacharya and Mukherjee [4] expanded upon the interaction between sets and groups by introducing the concept of fuzzy relations and fuzzy groups. Atanassov

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[5] proposed Intuitionistic Fuzzy Sets (IFSs), further refining the concept of fuzzy sets. Around the same time, Das [6] introduced the notion of fuzzy groups and level subgroups, paving a new direction for fuzzy set theory. Ajmal and Prajapati [7] and Ajmal and Thomas [8] contributed significantly to the evolution of this field. Their introduction of fuzzy cosets, fuzzy normal subgroups, and the lattice structure of these subgroups provided a fresh perspective on the structure of fuzzy groups. Adding complexity to fuzzy set theory, Dixit et al. [9] delved into level subgroups and the union of fuzzy subgroups. Complementing this work, Gau and Buehrer [10] presented vague sets, adding a new dimension to the understanding of fuzziness. Fast forwarding to the new millennium, Khan et al. [11] brought the vague sets concept into the fold of groups, contributing to the formation of vague groups. Yager [12] then stretched the boundaries of classical fuzzy set theory with the introduction of Pythagorean fuzzy sets. Rasuli [13–15] expanded the application of IFSs to intuitionistic fuzzy subgroups with respect to norms and introduced fuzzy equivalence relation, fuzzy congruence relation, and fuzzy normal subgroups on group G over t-norms.

Simultaneously, the 21st century has seen the advent of innovative fuzzy set categories. Gayen et al. [16] proposed the interval-valued neutrosophic subgroup based on interval-valued triple t-norm, while Senapati and Yager [17] introduced Fermatean fuzzy sets. In a similar vein, Bejines et al. [18] and Ardanza-Trevijano et al. [19] explored the aggregation of fuzzy and T-subgroups respectively. Kumar et al. [20] redefined vague groups with respect to t-norms, while Bhunia et al. [21] and Razaq et al. [22] investigated the characterization of Pythagorean Fuzzy Subgroups (PFSG) and normal subgroups. Adding to the complexity of the fuzzy logic framework, Boixader and Recasens [23] presented the concepts of vague and fuzzy t-norms and t-conorms. Moreover, Silambarasan [24] applied the Fermatean fuzzy sets concept to subgroups, leading to the exploration of Fermatean Fuzzy Subgroups (FFSG).

These works demonstrate the application of fuzzy logic and its extensions in the realm of crisp abstract algebra, equipping researchers with powerful tools to address uncertainty in group theory and related fields. The $\mathscr{T}$-norm function, serving as a key component, can be defined in various ways, including standard intersection, algebraic product, bounded difference, and drastic intersection. Similarly, the $\mathscr{S}$-conorm function, another crucial element, offers multiple definitions, such as standard union, algebraic sum, bounded sum, and drastic union. In this paper, we take a significant step by replacing the traditional "minimum" and "maximum" properties in the definitions of Pythagorean and FFSG, respectively, with the $\mathscr{T}$-norm and $\mathscr{S}$-conorm functions. By doing so, we establish a novel framework and explore its implications. Furthermore, we provide proof for several propositions associated with these modified definitions, deepening our understanding of Pythagorean and FFSG in the context of the $\mathscr{T}$-norm and $\mathscr{S}$-conorm functions.

This research paper is structured into six sections, each contributing to the exploration of Pythagorean and FFSG. Section 1 serves as an introduction, providing an overview of previous work in the field. It establishes the foundation upon which our research builds. In Section 2, we present the definitions of key notions and concepts that underpin our study. These fundamental definitions lay the groundwork for our subsequent analyses. Section 3 focuses on the core definitions, delving into the specifics of PFSG. We establish their formal definitions, employing the $\mathscr{T}$ norm and $\mathscr{S}$-conorm functions. Moving forward, Section 4 unveils significant findings derived from our exploration of PFSG. We present and discuss the outcomes, shedding light on their implications. Section 5 introduces the Fermatean fuzzy subgroup, defined in relation to the $\mathscr{T}$-norm and $\mathscr{S}$-conorm functions. Additionally, we provide a collection of results that are closely associated with this new concept. Finally, the research paper concludes in the sixth section, summarizing our findings and providing insights into the significance of our contributions.

2 | Notations

$R = non-empty\ set.$

$R_f = Fuzzy\ set.$

$* = binary\ operation\ defined\ on\ set\ R.$

$(R,*) = A\ group\ equipped\ with\ binary\ operation\ *.$

3 | Preliminaries

3.1 | Fuzzy Set

Assume that R is a set that is not void. A fuzzy set R_f of R is described as $R_f = \{(v, \gamma_{R_f}\ v)) : v \in R\}$. Where γ_{R_f} is a membership function defined as $\gamma_{R_f}: R \to [0,1]\quad for\ all\ v \in R$.

3.2 | Fuzzy Subgroup

Suppose $(R,*)$ be a group. If a fuzzy set R_f of $(R,*)$ satisfies the requirements listed below, then R_f is a fuzzy subgroup.

I. $\gamma_{R_f}\ v_1 * v_2) \geq min(\gamma_{R_f}\ v_1), \gamma_{R_f}\ v_2))\quad for\ all\ v_1, v_2\ \in R.$
II. $\gamma_{\tilde{R}_f}(\tilde{v}^{-1}) \geq \gamma_{\tilde{R}_f}(\tilde{v})\ \ for\ all\ \tilde{v}\ \in \tilde{R}.$

Where γ_{R_f} is a membership function defined as $\gamma_{R_f}: R \to [0,1]\ for\ all\ v \in R$.

3.3 | Intuitionistic Fuzzy Set

A definition of IFS $\widetilde{IR}_f$ of a non-void set R is $\widetilde{IR}_f = \{(v, \gamma_{\widetilde{IR}_f}\ v), \vartheta_{\widetilde{IR}_f}\ v)) : v \in R\}$. Where $\gamma_{\widetilde{IR}_f}$ is a membership function defined as $\gamma_{\widetilde{IR}_f}: R \to [0,1]$ and where $\vartheta_{\widetilde{IR}_f}$ is a non-membership function defined as $\vartheta_{\widetilde{IR}_f}: R \to [0,1]\ \ for\ all\ v \in R$. Also, $0 \leq \gamma_{\widetilde{IR}_f}\ v) + \vartheta_{\widetilde{IR}_f}\ v) \leq 1$ and the degree of hesitation is given by $1 - \gamma_{\widetilde{IR}_f}\ v) - \vartheta_{\widetilde{IR}_f}\ v)\ for\ all\ v \in R$.

3.4 | Intuitionistic Fuzzy Subgroup

Suppose $(R,*)$ be a group. If IFS $\widetilde{IR}_f$ of $(R,*)$ satisfies the characteristics listed below, then $\widetilde{IR}_f$ is an IFSG of $(R,*)$:

I.

$$\gamma_{\widetilde{IR}_f}\ v_1 * v_2) \geq \min(\gamma_{\widetilde{IR}_f}\ v_1), \gamma_{\widetilde{IR}_f}\ v_2))\text{ and }\vartheta_{\widetilde{IR}_f}\ v_1 * v_2) \leq \max(\vartheta_{\widetilde{IR}_f}\ v_1), \vartheta_{\widetilde{IR}_f}\ v_2))\text{ for all }v_1, v_2\ \in R.$$

II.

$$\gamma_{\widetilde{IR}_f}(v^{-1}) \geq \gamma_{\widetilde{IR}_f}(v) \text{ and } \vartheta_{\widetilde{IR}_f}(v^{-1}) \leq \vartheta_{\widetilde{IR}_f}(v) \text{ for all } v \in R,$$

where $\gamma_{\widetilde{IR}_f}$ is a membership function defined as $\gamma_{\widetilde{IR}_f}: R \to [0,1]$ and $\vartheta_{\widetilde{IR}_f}$ is a non-membership function defined as $\vartheta_{\widetilde{IR}_f}: R \to [0,1]$ *for all* $v \in R$.

3.5 | Pythagorean Fuzzy Set

Let R be a non-void set. The definition of Pythagorean fuzzy set $\widetilde{PR}_f$ of $R$is: $\widetilde{PR}_f = \left\{\left(v, \gamma_{\widetilde{PR}_f}(v), \vartheta_{\widetilde{PR}_f}(v)\right): v \in R\right\}$. Where $\gamma_{\widetilde{PR}_f}$ is a membership function defined as $\gamma_{\widetilde{PR}_f}: R \to [0,1]$ and, $\vartheta_{\widetilde{PR}_f}$ is a non-membership function defined as $\vartheta_{\widetilde{PR}_f}: R \to [0,1]$ *for all* $v \in R$. Also, $0 \leq \left(\gamma_{\widetilde{PR}_f}(v)\right)^2 + \left(\vartheta_{\widetilde{PR}_f}(v)\right)^2 \leq 1$ *for all* $v \in R$.

3.6 | Pythagorean Fuzzy Subgroup

If a Pythagorean fuzzy set $\widetilde{PR}_f$ of group $(R,*)$ satisfies the following criteria, it supposedly is a Pythagorean fuzzy subgroup.

I.

$$\gamma^2_{\widetilde{PR}_f}(v_1 * v_2) \geq \min\left(\gamma^2_{\widetilde{PR}_f}(v_1), \gamma^2_{\widetilde{PR}_f}(v_2)\right) \text{ and } \vartheta^2_{\widetilde{PR}_f}(v_1 * v_2) \leq \max\left(\vartheta^2_{\widetilde{PR}_f}(v_1), \vartheta^2_{\widetilde{PR}_f}(v_2)\right) \text{ for all } v_1, v_2 \in R.$$

II.

$$\gamma^2_{\widetilde{PR}_f}(v^{-1}) \geq \gamma^2_{\widetilde{PR}_f}(v) \text{ and } \vartheta^2_{\widetilde{PR}_f}(v^{-1}) \leq \vartheta^2_{\widetilde{PR}_f}(v) \text{ for all } v \in R,$$

where γ_{R_f} is a membership function defined as $\gamma_{R_f}: R \to [0,1]$ and $\vartheta_{\widetilde{IR}_f}$ is a non-membership function defined as $\vartheta_{\widetilde{IR}_f}: R \to [0,1]$ *for all* $v \in R$ and $\gamma^2_{\widetilde{PR}_f}(v) = \left(\gamma_{\widetilde{IR}_f}(v)\right)^2$ and $\vartheta^2_{\widetilde{PR}_f}(v) = \left(\vartheta_{\widetilde{IR}_f}(v)\right)^2$ *for all* $v \in R$.

3.7 | Fermatean Fuzzy Set

Let R be a non-void set. The definition of Fermatean fuzzy set $\widetilde{FR}_f$ of R is $\widetilde{FR}_f = \left\{\left(v, \gamma_{\widetilde{FR}_f}(v), \vartheta_{\widetilde{FR}_f}(v)\right): v \in R\right\}$. Where $\gamma_{\widetilde{FR}_f}$ is a membership function defined as $\gamma_{\widetilde{FR}_f}: R \to [0,1]$ and, $\vartheta_{\widetilde{FR}_f}$ is a non-membership function defined as $\vartheta_{\widetilde{FR}_f}: R \to [0,1]$ *for all* $v \in R$. Also, $0 \leq \left(\gamma_{\widetilde{FR}_f}(v)\right)^3 + \left(\vartheta_{\widetilde{FR}_f}(v)\right)^3 \leq 1$ *for all* $v \in R$.

3.8 | Fermatean Fuzzy Subgroup

If a Fermatean fuzzy set $\widetilde{FR}_f$ of group $(R,*)$ satisfies the following criteria, then it is Fermatean fuzzy subgroup.

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I.

$$\gamma^3_{\widetilde{FR}_f}\ v_1 * v_2) \geq \min\left(\gamma^3_{\widetilde{FR}_f}\ v_1), \gamma^3_{\widetilde{FR}_f}\ v_2)\right) \text{ and } \vartheta^3_{\widetilde{FR}_f}\ v_1 * v_2) \leq$$
$$\max\left(\vartheta^3_{\widetilde{FR}_f}\ v_1), \vartheta^3_{\widetilde{FR}_f}\ v_2)\right) \text{ for all } v_1, v_2\ \in \widetilde{R}.$$

II.

$$\gamma^3_{\widetilde{FR}_f}\left(v^{-1}\right) \geq \gamma^3_{\widetilde{FR}_f}\ v) \text{ and } \vartheta^3_{\widetilde{FR}_f}\left(v^{-1}\right) \leq \vartheta^3_{\widetilde{FR}_f}\ v) \text{ for all } v\ \in \widetilde{R},$$

where γ_{R_f} is a membership function defined as $\gamma_{R_f}: R \to [0,1]$ and $\vartheta_{\widetilde{IR}_f}$ is a non-membership function defined as $\vartheta_{\widetilde{IR}_f}: R \to [0,1]$ $\ for\ all\ v \in R$ and $\gamma^3_{\widetilde{FR}_f}\ v) = \left(\gamma_{\widetilde{FR}_f}\ v)\right)^3$ and $\vartheta^3_{\widetilde{PR}_f}\ v) = \left(\vartheta_{\widetilde{IR}_f}\ v)\right)^3$ $for\ all\ v \in R.$

3.9 | $\mathcal{T}$ -Norm

Definition 1. A function $\mathcal{T}$: [0,1] × [0,1] → [0,1] is referred to as $\mathcal{T}$ - norm if "$\mathcal{T}$" meets the criteria listed below:

I. $\mathcal{T}\ 0{,}0) = 0, \mathcal{T}\ \rho', 1) = \gamma' = \mathcal{T}\ 1, \rho').$
II. $\mathcal{T}\ \mathfrak{a}', \mathfrak{b}') \leq \mathcal{T}\ \rho', \tau')\ if\ \mathfrak{a}' \leq \rho'\ and\ \mathfrak{b}' \leq \tau'.$
III. $\mathcal{T}(\mathfrak{a}', \tau') = \mathcal{T}(\tau', \mathfrak{a}').$
IV. $\mathcal{T}(\mathfrak{a}', \mathcal{T}(\tau', \rho')) = \mathcal{T}(\mathcal{T}(\mathfrak{a}', \tau'), \rho')\ for\ all\ \mathfrak{a}', \mathfrak{b}', \tau', \rho' \in R.$

Some various types of $\mathcal{T}$ -norm function are as follows:

I. Standard intersection: $\mathcal{T}\ \mathfrak{a}', \mathfrak{b}') = min\ \mathfrak{a}', \mathfrak{b}').$
II. Algebraic product: $\mathcal{T}\ \mathfrak{a}', \mathfrak{b}') = \mathfrak{a}'.\mathfrak{b}'.$
III. Bounded difference: $\mathcal{T}\ \mathfrak{a}', \mathfrak{b}') = max(0, \mathfrak{a}' + \mathfrak{b}' - 1).$
IV. Drastic intersection: $\mathcal{T}\ \mathfrak{a}', \mathfrak{b}') = \begin{cases} \mathfrak{a}', & when\ \mathfrak{b}' = 1, \\ \mathfrak{b}', & when\ \mathfrak{a}' = 1, \\ 0, & otherwise. \end{cases}$

The relation between these four are $T_{min}\ \mathfrak{a}', \mathfrak{b}') \leq max(0, \mathfrak{a}' + \mathfrak{b}' - 1) \leq \mathfrak{a}'.\mathfrak{b}' \leq min\ \mathfrak{a}', \mathfrak{b}').$

3.10 | $\widetilde{\mathfrak{S}}$-Conorm

Definition 2. A function $\mathcal{S}$: [0,1] × [0,1] → [0,1] is referred to as $\mathcal{S}$-conorm if "$\mathcal{S}$" meets the criteria listed below:

I. $\mathcal{S}(\mathfrak{a}', 0) = \mathfrak{a}'.$
II. $\mathfrak{b}' \leq \rho'\ implies\ \mathcal{S}\ \mathfrak{a}', \mathfrak{b}') \leq \mathcal{S}\ \mathfrak{a}', \rho').$
III. $\mathcal{S}(\mathfrak{a}', \tau') = \mathcal{S}(\tau', \mathfrak{a}').$
IV. $\mathcal{S}(\mathfrak{a}', S(\tau', \rho')) = \mathcal{S}(\mathcal{S}(\mathfrak{a}', \tau'), \rho')\ for\ all\ \mathfrak{a}', \mathfrak{b}', \rho', \tau' \in R.$

Some various types of $\mathcal{S}$-conorm function are as follows:

I. Standard union: $\mathcal{S}\ \mathfrak{a}', \mathfrak{b}') = max\ \mathfrak{a}', \mathfrak{b}').$
II. Algebraic sum: $\mathcal{S}\ \mathfrak{a}', \mathfrak{b}') = \mathfrak{a}' + \mathfrak{b}' - \mathfrak{a}'\mathfrak{b}'.$
III. Bounded sum: $\mathcal{S}\ \mathfrak{a}', \mathfrak{b}') = min(1, \mathfrak{a}' + \mathfrak{b}').$
IV. Drastic union: $\mathcal{S}\ \mathfrak{a}', \mathfrak{b}') = \begin{cases} \mathfrak{a}', & when\ \mathfrak{b}' = 0, \\ \mathfrak{b}', & when\ \mathfrak{a}' = 0, \\ 1, & otherwise. \end{cases}$

The relation between these four are:

$$max\ \mathfrak{a}', \mathfrak{b}') \leq \mathfrak{a}' + \mathfrak{b}' - \mathfrak{a}'\mathfrak{b}' \leq \min 1, \mathfrak{a}' + \mathfrak{b}') \leq S_{max}\ \mathfrak{a}', \mathfrak{b}').$$

Definition 3. A fuzzy set R_f of a group $(R,*)$ is a Fuzzy Subgroup (FSG) in terms of the t-norm "$\mathscr{T}$" if R_f meets the criteria listed below:

I. $\gamma_{R_f}\ v_1 * v_2) \geq T\left(\gamma_{R_f}\ v_1), \gamma_{R_f}\ v_2)\right)\ for\ all\ v_1, v_2\ \in R.$

II. $\gamma_{R_f}\left(v^{-1}\right) \geq \gamma_{R_f}\ v)\ for\ all\ v\ \in R.$

Where γ_{R_f} is a membership function defined as $\gamma_{R_f}: R \rightarrow\ [0,1]\ for\ all\ v \in R$.

4 | Pythagorean Fuzzy Subgroup in Context of $\widetilde{\mathcal{T}}$-Norm and $\widetilde{\mathcal{S}}$-Conorm

In this section, we redefine the concept of PFSG within the framework of the $\mathscr{T}$-norm and $\mathscr{S}$-conorm, providing a fresh perspective and enhancing our understanding of this fundamental concept. We delve into the intricate properties associated with these redefined PFSGs, shedding light on their unique characteristics and implications.

Bhunia et al. [21] made significant contributions to the study of PFSG. His work was instrumental in defining the concept and establishing crucial conditions for the membership and non-membership functions. The conditions, as provided below, play a vital role in characterizing PFSG. Bhunia's work laid the foundation for our exploration, paving the way for the redefinition and analysis of PFSGs within the $\mathscr{T}$-norm and $\mathscr{S}$-conorm framework. By leveraging Bhunia's insights, we build upon the existing knowledge to further deepen our understanding of PFSGs and their properties.

I. $\gamma^2_{\widetilde{PR}_f}\ v_1 * v_2) \geq min\left(\gamma^2_{\widetilde{PR}_f}\ v_1), \gamma^2_{\widetilde{PR}_f}\ v_2)\right)\ and\ \vartheta^2_{\widetilde{PR}_f}\ v_1 * v_2) \leq max\left(\vartheta^2_{\widetilde{PR}_f}\ v_1), \vartheta^2_{\widetilde{PR}_f}\ v_2)\right)$
$for\ all\ t_1, t_2\ \in R.$

II. $\gamma^2_{\widetilde{PR}_f}\left(v^{-1}\right) \geq \gamma^2_{\widetilde{PR}_f}\ v)\ and\ \vartheta^2_{\widetilde{PR}_f}\left(v^{-1}\right) \leq \vartheta^2_{\widetilde{PR}_f}\ v)\ \ for\ all\ v\ \in R.$

In Section 2, an in-depth exploration of the $\mathscr{T}$-norm and $\mathscr{S}$-conorm was undertaken, highlighting their diverse types and characteristics. Upon closer examination of the definition of a Pythagorean fuzzy subgroup, a compelling insight emerged: the traditional "minimum" and "maximum" operations can be effectively replaced by the $\mathscr{T}$-norm and $\mathscr{S}$-conorm, respectively. This revelation implies that we are no longer confined to a single option for selecting the minimum and maximum values. Harnessing this newfound flexibility, we proceed to define the Pythagorean fuzzy subgroup, leveraging the power of the $\mathscr{T}$-norm and $\mathscr{S}$-conorm. By adopting this novel approach, we expand the possibilities and refine our understanding of PFSG within the context of these operations.

Definition 4. Suppose $R,*)$ be a group. A Pythagorean fuzzy set $\widetilde{PR}_f$ of $(R,*)$ is a PFSG in context of t-norm '$\mathscr{T}$' and s-conorm '$\mathscr{S}$' if $\widetilde{PR}_f$ satisfies the following conditions:

I. $\gamma^2_{\widetilde{PR}_f}\ v_1 * v_2) \geq \mathscr{T}\left(\gamma^2_{\widetilde{PR}_f}\ v_1), \gamma^2_{\widetilde{PR}_f}\ v_2)\right)\ and\ \vartheta^2_{\widetilde{PR}_f}\ v_1 * v_2) \leq \mathscr{S}\left(\vartheta^2_{\widetilde{PR}_f}\ v_1), \vartheta^2_{\widetilde{PR}_f}\ v_2)\right)\ for\ all\ v_1, v_2\ \in R.$

II. $\gamma^2_{\widetilde{PR}_f}(\tilde{v}^{-1}) \geq \gamma^2_{\widetilde{PR}_f}(\tilde{v})\ and\ \vartheta^2_{\widetilde{PR}_f}(\tilde{v}^{-1}) \leq \vartheta^2_{\widetilde{PR}_f}(\tilde{v})\ \ for\ all\ \tilde{v}\ \in \tilde{R}.$

Where $\gamma_{\widetilde{PR}_f}$ is a membership function defined as $\gamma_{\widetilde{PR}_f}: R \to [0,1]$ and $\vartheta_{\widetilde{PR}_f}$ is a non-membership function defined as $\vartheta_{\widetilde{PR}_f}: R \to [0,1]$ $for\ all\ t \in R$ and $\gamma^2_{\widetilde{PR}_f}\ v) = \left(\gamma_{\widetilde{PR}_f}\ v)\right)^2$ and $\vartheta^2_{\widetilde{PR}_f}\ v) = \left(\vartheta_{\widetilde{PR}_f}\ v)\right)^2$ $for\ all\ v \in R$.

Example 1. Let $z_3,*)$ be a group of addition modulo 3 and let Pythagorean fuzzy set $\widetilde{PR}_f$ of $z_3,*)$ is $\widetilde{P\ z_3,*)}_f = \{<0,[0.8,0.5]>, <1,[0.7,0.6]>, <2,[0.7,0.6]>\}$. Then $\widetilde{P\ z_3,*)}_f$ is a Pythagorean fuzzy subgroup in context of t-norm '$\mathscr{T}$'and s-conorm '$\mathscr{S}$'. Where $\mathscr{T} = min\left(\gamma^2_{\widetilde{PR}_f}\ v_1), \gamma^2_{\widetilde{PR}_f}\ v_2)\right)$ and $\mathscr{S} = max\left(\vartheta^2_{\widetilde{PR}_f}\ v_1), \vartheta^2_{\widetilde{PR}_f}\ v_2)\right)$.

Proposition 1. If $\widetilde{PR}_f$ is a PFSG of a group $(R,*)$ in context of t-norm '$\mathscr{T}$' and s-conorm '$\mathscr{S}$', then $(R,*)_1 = \{v \in (R,*) : \widetilde{PR}_f(v) = 1\ i.e.\ \gamma^2_{\widetilde{PR}_f}\ v) = 1\ and\ \vartheta^2_{\widetilde{PR}_f}\ v) = 1\}$ is either empty or is a subgroup of $(R,*)$.

Proof: If $v_1, v_2 \in (R,*)_1$ then $\gamma^2_{\widetilde{PR}_f}(v_1 * v_2{}^{-1}) \geq \mathscr{T}\left(\gamma^2_{\widetilde{PR}_f}\ v_1), \gamma^2_{\widetilde{PR}_f}(v_2{}^{-1})\right) = \mathscr{T}\left(\gamma^2_{\widetilde{PR}_f}\ v_1), \gamma^2_{\widetilde{PR}_f}\ v_2)\right) = \mathscr{T}\ 1,1) = 1$. Therefore, $\gamma^2_{\widetilde{PR}_f}(v_1 * v_2{}^{-1}) = 1$ implies $v_1 * v_2{}^{-1} \in (R,*)_1$. And if $v_1, v_2 \in (R,*)_1$ then $\vartheta^2_{\widetilde{PR}_f}(v_1 * v_2{}^{-1}) \leq \mathscr{S}\left(\vartheta^2_{\widetilde{PR}_f}\ v_1), \vartheta^2_{\widetilde{PR}_f}(v_2{}^{-1})\right) = \mathscr{S}\left(\vartheta^2_{\widetilde{PR}_f}\ v_1), \vartheta^2_{\widetilde{PR}_f}\ v_2)\right) = \mathscr{S}\ 1,1) = 1$. Therefore, $\vartheta^2_{\widetilde{PR}_f}(v_1 * v_2{}^{-1}) = 1$ implies $v_1 * v_2{}^{-1} \in (R,*)_1$.

Consequently, $(R,*)_1$ is a subgroup of group $(R,*)$.

Proposition 2. Let $\widetilde{PR}_f$ be a PFSG of a group $(R,*)$ in context of t-norm '$\mathscr{T}$'and s-conorm '$\mathscr{S}$'. If $\gamma^2_{\widetilde{PR}_f}(v_1 * v_2{}^{-1}) = 1$ then $\gamma^2_{\widetilde{PR}_f}\ v_1) = \gamma^2_{\widetilde{PR}_f}\ v_2)$, and if $\vartheta^2_{\widetilde{PR}_f}(v_1 * v_2{}^{-1}) = 1$ then $\vartheta^2_{\widetilde{PR}_f}\ v_1) = \vartheta^2_{\widetilde{PR}_f}\ v_2)\ for\ all\ v \in (R,*)$.

Proof: Consider $for\ all\ v_1, v_2 \in (R,*)$.

$\gamma^2_{\widetilde{PR}_f}\ v_1) = \gamma^2_{\widetilde{PR}_f}((v_1 * v_2{}^{-1})v_2) \geq \mathscr{T}\left(\gamma^2_{\widetilde{PR}_f}(v_1 * v_2{}^{-1}), \gamma^2_{\widetilde{PR}_f}\ v_2)\right) = \mathscr{T}\left(1, \gamma^2_{\widetilde{PR}_f}\ v_2)\right) = \gamma^2_{\widetilde{PR}_f}\ v_2) = \gamma^2_{\widetilde{PR}_f}(v_2{}^{-1})$. [Since $\widetilde{PR}_f$ is a Pythagorean fuzzysubgroup]. Again consider, $\gamma^2_{\widetilde{PR}_f}\ v_2) = \gamma^2_{\widetilde{PR}_f}(v_1(v_1 * v_2{}^{-1})) \geq \mathscr{T}\left(\gamma^2_{\widetilde{PR}_f}\ v_1), \gamma^2_{\widetilde{PR}_f}(v_1 * v_2{}^{-1})\right) = \mathscr{T}\left(\gamma^2_{\widetilde{PR}_f}\ v_1), 1\right) = \gamma^2_{\widetilde{PR}_f}\ v_2) = \gamma^2_{\widetilde{PR}_f}(v_2{}^{-1})$. [Since $\widetilde{PR}_f$is a Pythagorean fuzzy subgroup].

$\Rightarrow \gamma^2_{\widetilde{PR}_f}\ v_1) = \gamma^2_{\widetilde{PR}_f}\ v_2)$.

In the same manner, we can prove that $\vartheta^2_{\widetilde{PR}_f}\ v_1) = \vartheta^2_{\widetilde{PR}_f}\ v_2)$.

Proposition 3. Let $\widetilde{PR}_f$ be a PFS of a group $(R,*)$ in context of thet-norm '$\mathscr{T}$'and s-conorm '$\mathscr{S}$'. If $\widetilde{PR}_f(e) = 1$ i.e., $\gamma^2_{\widetilde{PR}_f}\ e) = 1\ and\ \vartheta^2_{\widetilde{PR}_f}\ e) = 1$ and $\gamma^2_{\widetilde{PR}_f}(v_1 * v_2{}^{-1}) \geq \mathscr{T}\left(\gamma^2_{\widetilde{PR}_f}\ v_1), \gamma^2_{\widetilde{PR}_f}\ v_2)\right)\ for\ all\ v_1, v_2 in(R,*)$ and $\vartheta^2_{\widetilde{PR}_f}(v_1 * v_2{}^{-1}) \leq$

$S\left(\vartheta^2_{\widetilde{PR}_f}\ v_1), \vartheta^2_{\widetilde{PR}_f}\ v_2)\right)$ *for all* $v_1, v_2 in\left(R,*\right)$. Then $\widetilde{PR}_f$ is a PFSG of group $\left(R,*\right)$ in context of t-norm '$\mathscr{T}$' and s-conorm '$\mathscr{S}$'.

Proof: Consider, *for all* $v_1, v_2 \in \left(R,*\right), \gamma^2_{\widetilde{PR}_f}\left(v^{-1}\right) = \gamma^2_{\widetilde{PR}_f}\left(ev^{-1}\right) \geq \mathscr{T}\left(\gamma^2_{\widetilde{PR}_f}\ e), \gamma^2_{\widetilde{PR}_f}\ v)\right) = \mathscr{T}\left(1, \gamma^2_{\widetilde{PR}_f}\ v)\right) = \gamma^2_{\widetilde{PR}_f}(v)$. And similarly, $\gamma^2_{\widetilde{PR}_f}(v) \geq \gamma^2_{\widetilde{PR}_f}(v^{-1})$ so that $\gamma^2_{\widetilde{PR}_f}\ v) = \gamma^2_{\widetilde{PR}_f}(v^{-1})$. Moreover, $\gamma^2_{\widetilde{PR}_f}\ v_1 * v_2) \geq \gamma^2_{\widetilde{PR}_f}\left(v_1 * {v_2}^{-1}\right) \geq \mathscr{T}\left(\gamma^2_{\widetilde{PR}_f}\ v_1), \gamma^2_{\widetilde{PR}_f}\left({v_2}^{-1}\right)\right) = \mathscr{T}\left(\gamma^2_{\widetilde{PR}_f}\ v_1), \gamma^2_{\widetilde{PR}_f}\ v_2)\right)$. Thus $\gamma^2_{\widetilde{PR}_f}$ is a PFSG of group $\left(R,*\right)$ in context of t-norm '$\mathscr{T}$'.

Likewise, we can demonstrate $\vartheta^2_{\widetilde{PR}_f}$ is a PFSG of group $\left(R,*\right)$ in context of s-conorm '$\mathscr{S}$'. Consequently, $\widetilde{PR}_f$ is a PFSG of group $\left(R,*\right)$ in context of t-norm '$\mathscr{T}$'and s-conorm '$\mathscr{S}$'.

5 | Fermatean Fuzzy Subgroup in Context of $\mathscr{T}$-Norm and $\mathscr{S}$-Conorm

Silambarasan [24] introduced the concept of FFSG. Building upon this pioneering work, we further advance the field by redefining the notion of a "Fermatean fuzzy subgroup" in the context of the t-norm function denoted as '$\mathscr{T}$' and the s-conorm function denoted as '$\mathscr{S}$'. This redefinition provides a novel perspective and deepens our understanding of FFSG. Furthermore, we explore and present a collection of properties associated with these redefined subgroups, shedding light on their unique characteristics and implications within the broader context of fuzzy subgroup theory.

Definition 5. Suppose$\left(R,*\right)$ be a group. AFFS $\widetilde{FR}_f$ of $\left(R,*\right)$ is FFSG in context of t-norm '$\mathscr{T}$' and s-conorm '$\mathscr{S}$' if $\widetilde{FR}_f$ satisfies the following conditions:

I.

$$\gamma^3_{\widetilde{\mathrm{FR}}_\mathrm{f}}\ \mathrm{v}_1 * \mathrm{v}_2) \geq \mathscr{T}\left(\gamma^3_{\widetilde{\mathrm{FR}}_\mathrm{f}}\ \mathrm{v}_1), \gamma^3_{\widetilde{\mathrm{FR}}_\mathrm{f}}\ \mathrm{v}_2)\right) \text{ and } \vartheta^3_{\widetilde{\mathrm{FR}}_\mathrm{f}}\ \mathrm{v}_1 * \mathrm{v}_2) \leq \mathscr{S}\left(\vartheta^3_{\widetilde{\mathrm{FR}}_\mathrm{f}}\ \mathrm{v}_1), \vartheta^3_{\widetilde{\mathrm{FR}}_\mathrm{f}}\ \mathrm{v}_2)\right) \text{ for all } \mathrm{v}_1, \mathrm{v}_2 \in \mathrm{R}.$$

II.

$$\gamma^3_{\widetilde{\mathrm{FR}}_\mathrm{f}}\left(\mathrm{v}^{-1}\right) \geq \gamma^3_{\widetilde{\mathrm{FR}}_\mathrm{f}}\ \mathrm{v}) \text{ and } \vartheta^3_{\widetilde{\mathrm{FR}}_\mathrm{f}}\left(\mathrm{v}^{-1}\right) \leq \vartheta^3_{\widetilde{\mathrm{FR}}_\mathrm{f}}\ \mathrm{v}) \text{ for all } \mathrm{v} \in \mathrm{R},$$

where $\gamma_{\widetilde{FR}_f}$ is a membership function defined as $\gamma_{\widetilde{FR}_f}: R \to [0,1]$ and $\vartheta_{\widetilde{FR}_f}$ is a non-membership function defined as: $\vartheta_{\widetilde{FR}_f}: R \to [0,1]$ *for all* $v \in R$ and $\gamma^3_{\widetilde{FR}_f}\ v) = \left(\gamma_{\widetilde{FR}_f}\ v)\right)^3$ and $\vartheta^3_{\widetilde{FR}_f}\ v) = \left(\vartheta_{\widetilde{FR}_f}\ v)\right)^3$ *for all* $v \in R$.

Example 2. Let $z_3,*)$ be a group of addition modulo 3 and let FFS $\widetilde{FR}_f$ of $z_3,*)$ is $\widetilde{F\ z_3,*)}_f = \{< 0, [0.9,0.5] >, < 1, [0.9,0.6] >, < 2, [0.9,0.6] >\}$.Then $\widetilde{P\ z_3,*)}_f$ is a Fermatean fuzzy subgroupin context of t-norm '$\mathscr{T}$' and s-norm '$\mathscr{S}$'. Where $\mathscr{T} = min\left(\gamma^3_{\widetilde{PR}_f}\ v_1), \gamma^3_{\widetilde{PR}_f}\ v_2)\right)$ and $\mathscr{S} = max\left(\vartheta^3_{\widetilde{PR}_f}\ v_1), \vartheta^3_{\widetilde{PR}_f}\ v_2)\right)$.

Proposition 4. If $\widetilde{FR}_f$ is a FFSG of a group $\left(R,*\right)$ in context of t-norm '$\mathscr{T}$', and s-conorm '$\mathscr{S}$' then $\left(R,*\right)_1 = \{v \in \left(R,*\right) : \widetilde{FR}_f(v) = 1\ i.e. \gamma^3_{\widetilde{FR}_f}\ v) = 1\ and\ \vartheta^3_{\widetilde{FR}_f}\ v) = 1\}$ is either empty or is a subgroup of $\left(R,*\right)$.

Proof: If $v_1, v_2 \in \left(R,*\right)_1$ then $\gamma^3_{\widetilde{FR}_f}\left(v_1 * v_2^{-1}\right) \geq \mathcal{T}\left(\gamma^3_{\widetilde{FR}_f}\ v_1), \gamma^3_{\widetilde{FR}_f}\left(v_2^{-1}\right)\right) = \mathcal{T}\left(\gamma^3_{\widetilde{FR}_f}\ v_1), \gamma^3_{\widetilde{FR}_f}\ v_2)\right) = \mathcal{T}\ 1,1) = 1$. Therefore, $\gamma^3_{\widetilde{FR}_f}\left(v_1 * v_2^{-1}\right) = 1$ implies $v_1 * v_2^{-1} \in \left(R,*\right)_1$. And if $v_1, v_2 \in \left(R,*\right)_1$ then $\vartheta^3_{\widetilde{FR}_f}\left(v_1 * v_2^{-1}\right) \leq \mathcal{S}\left(\vartheta^3_{\widetilde{FR}_f}\ v_1), \vartheta^3_{\widetilde{FR}_f}\left(v_2^{-1}\right)\right) = \mathcal{S}\left(\vartheta^3_{\widetilde{FR}_f}\ v_1), \vartheta^3_{\widetilde{FR}_f}\ v_2)\right) = \mathcal{S}\ 1,1) = 1$. Therefore, $\vartheta^3_{\widetilde{FR}_f}\left(v_1 * v_2^{-1}\right) = 1$ implies $v_1 * v_2^{-1} \in \left(R,*\right)_1$.

Consequently, $\left(R,*\right)_1$ is a subgroup of group $\left(R,*\right)$.

Proposition 5. Let $\widetilde{FR}_f$ be a FFSG of a group $\left(R,*\right)$ in context of t-norm $\mathcal{T}$ "and s-conorm '$\mathcal{S}$'. If $\gamma^3_{\widetilde{FR}_f}\left(v_1 * v_2^{-1}\right) = 1$ then $\gamma^3_{\widetilde{FR}_f}\ v_1) = \gamma^3_{\widetilde{FR}_f}\ v_2)$, and if $\vartheta^3_{\widetilde{FR}_f}\left(v_1 * v_2^{-1}\right) = 1$ then $\vartheta^3_{\widetilde{FR}_f}\ v_1) = \vartheta^3_{\widetilde{FR}_f}\ v_2)\ for\ all\ v \in \left(R,*\right)$.

Proof: Consider $for\ all\ v_1, v_2 \in \left(R,*\right)$, $\gamma^3_{\widetilde{FR}_f}\ v_1) = \gamma^3_{\widetilde{FR}_f}((v_1 * v_2^{-1})v_2) \geq \mathcal{T}\left(\gamma^3_{\widetilde{FR}_f}\left(v_1 * v_2^{-1}\right), \gamma^3_{\widetilde{FR}_f}\ v_2)\right) = \mathcal{T}\left(1, \gamma^3_{\widetilde{FR}_f}\ v_2)\right) = \gamma^3_{\widetilde{FR}_f}\ v_2) = \gamma^3_{\widetilde{FR}_f}(v_2^{-1})$. [Since $\widetilde{FR}_f$ is a Fermatean fuzzy subgroup].

Again consider $\gamma^3_{\widetilde{FR}_f}\ v_2) = \gamma^3_{\widetilde{FR}_f}(v_1(v_1 * v_2^{-1})) \geq \mathcal{T}\left(\gamma^3_{\widetilde{FR}_f}\ v_1), \gamma^3_{\widetilde{FR}_f}\left(v_1 * v_2^{-1}\right)\right) = \mathcal{T}\left(\gamma^3_{\widetilde{FR}_f}\ v_1), 1\right) = \gamma^3_{\widetilde{FR}_f}\ v_2) = \gamma^3_{\widetilde{FR}_f}(v_2^{-1})$. [Since $\widetilde{FR}_f$ is a Fermatean fuzzy subgroup].

$\Rightarrow \gamma^3_{\widetilde{FR}_f}\ v_1) = \gamma^3_{\widetilde{FR}_f}\ v_2)$.

In a similar way, we may demonstrate that $\vartheta^3_{\widetilde{FR}_f}\ v_1) = \vartheta^3_{\widetilde{FR}_f}\ v_2)$.

Proposition 6. Let $\widetilde{FR}_f$ be a Fermatean fuzzy set of a group $\left(R,*\right)$ in context of t-norm '$\mathcal{T}$' and s-conorm '$\mathcal{S}$'. If $\widetilde{FR}_f(e) = 1$ i.e., $\gamma^3_{\widetilde{FR}_f}\ e) = 1\ and\ \vartheta^3_{\widetilde{FR}_f}\ e) = 1$ and $\gamma^3_{\widetilde{FR}_f}\left(v_1 * v_2^{-1}\right) \geq \mathcal{T}\left(\gamma^3_{\widetilde{FR}_f}\ v_1), \gamma^3_{\widetilde{FR}_f}\ v_2)\right)\ for\ all\ v_1, v_2\ in\ \left(R,*\right)$ and $\vartheta^3_{\widetilde{FR}_f}\left(v_1 * v_2^{-1}\right) \leq S\left(\vartheta^3_{\widetilde{FR}_f}\ v_1), \vartheta^3_{\widetilde{FR}_f}\ v_2)\right)\ for\ all\ v_1, v_2\ in\ \left(R,*\right)$. Then $\widetilde{PR}_f$ is a FFSG of group $\left(R,*\right)$ in context of t-norm '$\mathcal{T}$' and s-conorm '$\mathcal{S}$'.

Proof: Consider $for\ all\ v_1, v_2 \in \left(R,*\right)$, $\gamma^3_{\widetilde{FR}_f}\left(v^{-1}\right) = \gamma^3_{\widetilde{FR}_f}\left(ev^{-1}\right) \geq \mathcal{T}\left(\gamma^3_{\widetilde{FR}_f}\ e), \gamma^3_{\widetilde{FR}_f}\ v)\right) = \mathcal{T}\left(1, \gamma^3_{\widetilde{FR}_f}\ v)\right) = \gamma^3_{\widetilde{FR}_f}(v)$. And similarly, $\gamma^3_{\widetilde{FR}_f}(v) \geq \gamma^3_{\widetilde{FR}_f}(v^{-1})$ so that $\gamma^3_{\widetilde{FR}_f}\ v) = \gamma^3_{\widetilde{FR}_f}(v^{-1})$. Moreover, $\gamma^3_{\widetilde{FR}_f}\ v_1 * v_2) \geq \gamma^3_{\widetilde{FR}_f}\left(v_1 * v_2^{-1}\right) \geq \mathcal{T}\left(\gamma^3_{\widetilde{FR}_f}\ v_1), \gamma^3_{\widetilde{FR}_f}\left(v_2^{-1}\right)\right) = \mathcal{T}\left(\gamma^3_{\widetilde{FR}_f}\ v_1), \gamma^3_{\widetilde{FR}_f}\ v_2)\right)$. thus $\gamma^3_{\widetilde{FR}_f}$ is a FFSG of group $\left(R,*\right)$ in context of t-norm '$\mathcal{T}$'.

Likewise, we can demonstrate $\vartheta^3_{\widetilde{FR}_f}$ is a FFSG of group $\left(R,*\right)$ in context of s-conorm '$\mathcal{S}$'. Consequently, $\widetilde{FR}_f$ is a FFSG of group $\left(R,*\right)$ in context of '$\mathcal{T}$'and s-conorm '$\mathcal{S}$'.

6 | Conclusion

The presence of symmetry in our environment is undeniable, yet it is not always flawlessly precise. Symmetry often exhibits a degree of vagueness. To address this inherent vagueness, fuzzy logic has found its application in the field of group theory. Over time, extensions of fuzzy sets have emerged, introducing concepts such as Pythagorean and Fermatean fuzzy sets. This research paper has aimed to redefine PFSG and FFSG by leveraging the $\mathscr{T}$-norm and $\mathscr{S}$-conorm functions. The $\mathscr{T}$-norm function offers diverse definitions, including standard intersection, algebraic product, bounded difference, and drastic intersection. Similarly, the $\mathscr{S}$-conorm function encompasses various definitions, such as standard union, algebraic sum, bounded sum, and drastic union. In this study, we have utilized the $\mathscr{T}$-norm and $\mathscr{S}$-conorm in the existing definitions of Pythagorean and FFSG, resulting in their redefinition within the context of these functions. Concrete examples have been provided to elucidate these redefined subgroups. Moreover, we have established and proven several propositions pertinent to these newly defined subgroups.

In conclusion, the research conducted in this paper sets the stage for further promotion and dissemination of our findings. It paves the way for future investigations and advancements in the field of fuzzy subgroups, PFSG and Fermatean fuzzy subgroup in context of norm and conorm, fostering a deeper understanding of symmetry in the presence of uncertainty.

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Conflicts of Interests

There are no conflicts of interest for authors.

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On Refined Neutrosophic Finite p-Group

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Abstract

The neutrosophic automorphisms of a neutrosophic groups $G(I)$, denoted by Aut($G(I)$) is a neu-trosophic group under the usual mapping composition. It is a permutation of $G(I)$ which is also a neutrosophic homomorphism. Moreover, suppose that $X_1 = X(G(I))$ is the neutrosophic group of inner neutrosophic auto-morphisms of a neutrosophic group $G(I)$ and X_n the neutrosophic group of inner neutrosophic automorphisms of X_{n-1}. In this paper, we show that if any neutrosophic group of the sequence $G(I), X_1, X_2, \ldots$ is the identity, then $G(I)$ is nilpotent.

Keywords: Neutrosophic automorphism, Commutator subgroup, Neutrosophic subgroup, Minimal condition, Mapping composition, Nilpotency.

1 | Introduction

The concepts of refined neutrosophic algebraic structures and studies of refined neutrosophic groups in particular were introduced by Agboola [1]. After the successful feat, many other neutrosophic researchers have as well tried to establish more further studies on the refined neutrosophic algebraic structures [2]. Further studies on refined neutrosophic rings and refined neutrosophic subrings, their presentations and fundamentals were also worked upon.

Also, Agboola [3] has examined and as well studied the refined neutrosophic quotient groups, where more properties of re ned neutrosophic groups were presented and it was shown that the classical isomorphism theorems of groups do not hold in the refined neu-trosophic groups. The existence of classical morphisms between refined neutrosophic groups $G(I_1; I_2)$ and neutrosophic groups $G(I)$ were established. The readers can as well consult [4–7] in order to have detailed knowledge concerning the refined neutrosophic logic, neutrosophic groups, refined neutrosophic groups and neutrosophy, in general. Please note the following: throughout this paper, our binary operation is strictly the usual ordinary addition (as the operation of multiplication may not be de ned due to the fact that I_1 does not exist).

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Definition 1 ([3]). Suppose that (X (I_1; I_2); +; .) is any re ned neutro-sophic algebraic structure. Here, + and . are ordinary addition and multiplication respectively. Then I_1 and I_2 are the split components of the indeterminacy factor I that is $I = \alpha_1 I_1 + \alpha_2 I_2$ with α_i in C (the set of complex numbers); i = 1; 2.

Definition 2 ([3]). Suppose that (G; *) is any group. Then, the couple (G (I_1; I_2); *) can be referred to as the refined neutrosophic group. Furthermore, this group can be said to be generated by G, I_1 and I_2 and (G (I_1; I_2); *) is said to be commutative if for all x; y in for all G (I_1; I_2); we have x*y = y*x: otherwise, (G (I_1; I_2); *) can be referred to as a non-commutative refined neutrosophic group.

Here, I has been refined as I_1 and I_2: note that it is possible to refine T and F as well as T_1, T_2 and F as F_1, F_2 (see [8] for some details on this. We hope to make substantial contributions and relevant considerations in this regards as future possible studies.)

Theorem 1 ([3]). 1) every refined neutrosophic group is a semigroup but not a group, and 2) every refined neutrosophic group contains a group.

Corollary 1 ([3]). Every refined neutrosophic group (G (I_1; I_2); +) is a group.

Definition 3 ([3]). Let (G (I_1; I_2);*) be a refined neutrosophic group and let A (I_1; I_2) be a nonempty subset of G (I_1; I_2): A (I_1; I_2) is called a refined neutrosophic sub-group of G (I_1; I_2) if (A (I_1; I_2);*) is a refined neutrosophic group. It is essential that A (I_1; I_2) contains a proper subset which is a group. Otherwise, A (I_1; I_2) will be called a pseudo refined neutrosophic subgroup of G (I_1; I_2).

Definition 4 ([3]). Let H (I_1; I_2) be a refined neutrosophic subgroup of a refined neutrosophic group (G (I_1; I_2); .): define x = (a; bI_1; cI_2) in G (I_1; I_2).

Theorem 2 ([3]). Let (G (I_1; I_2); +) be a refined neutrosophic group and let (G (I); +) be a neutrosophic group such that where $I = xI_1 + yI_2$ with x; y in C. Let ϕ: G (I_1; I_2)⟶G (I)be a mapping defined by ((a; xI_1; yI_2))=(a; (x + y)I) for all (a; xI_1; yI_2) in (G (I_1; I_2) with a; x; y in G: then ϕ is a group homomorphism.

An interesting type of neutrosophic isomorphism of a neutrosophic groups G (I) would occur when the image neutrosophic group G (I) coincides with G (I). The classical group concepts as regards to this has been discussed by [9]. A neutrosophic isomorphism α: G (I)⟶G (I) of G (I) onto itself can be called a neutrosophic automorphism of G (I). In particular, permutes the elements of G(I). The collection of all neutrosophic automorphisms of G (I) forms a neutrosophic group under composition of maps.

If β: G (I)⟶G (I) is another neutrosophic automorphism, we denote the product of α and β by $\alpha\beta$. The group of all neutrosophic automorphisms of G (I) denoted Aut (G(I)) can be called the neutrosophic automorphism group of G(I). The unit element of G(I) is the neutrosophic identity automorphism i. This which leaves every element of G (I) fixed i.e.,

$$ix = x, \left(\left(a;\ bI_1;\ cI_2 \right) = x \in G(I) \right).$$

Definition 5. A neutrosophic group G (I) can be said to be nilpotent if it has a normal series of a finite length n. That is,

$$G(I) = G0(I) \geq G1(I) \geq G2(I) \geq \ldots \geq Gn(I) = \{e\},$$

where

$$G_i(I) / G_{i+1}(I) \leq Z(G(I) / G_{i+1}(I)).$$

By this notion, every finite neutrosophic p-group G(I) is nilpotent. The nilpotence property is an hereditary one. Thus

I. Any finite product of nilpotent neutrosophic group is nilpotent.
II. If G (I) is nilpotent of a class c, then, every neutrosophic subgroup as well as the neutrosophic quotient group of G (I) is nilpotent and of class $\leq$ c.

Definition 6. Suppose that (W (I); #) and (V (I); $\oplus$) are two neutrosophic groups. Define a neutrosophic homomorphism from α: W (I) to V (I) to be a mapping: W (I) $\oplus$ V (I) such that $\alpha(x\#y) = \alpha(x)\ \alpha(y)$ where $x = (a_1;\ b_1I_1;\ c_1I_2)$, and $y = (a_2;\ b_2I_1;\ c_2I_2)$. A neutrosophic homomorphism α which maps a neutrosophic group W (I) on itself is called a neutrosophic endomorphism. A bijective neutrosophic endomorphism is known as a neutrosophic automorphism.

Now, let $t = (a;\ bI_1;\ cI_2)$ be a fixed element of a group W (I). The mapping β_t: W (I)$\longrightarrow$W (I) which could be defined by $\beta_t(x)$ = txt-1 for all $(x_1;\ x_2I_1;\ x_3I_2) = x$ in W (I) is known as an inner neutrosophic automorphism of the group W (I).

Every other neutrosophic automorphism of W (I) is called outer neutrosophic automor-phism. (The classical group concepts on this was also discussed in [10] and [11].)

Theorem 3. A neutrosophic abelian group G (I) of order $p_1^{\alpha 1} p_2^{\alpha 2} \dots p_n^{\alpha n}$, where $p_1\ p_2 \dots p_n$ are distinct primes, is the direct product of groups $G_{p1}(I)$, $G_{p2}(I)$, $G_{p3}(I)$, … $G_{pn}(I)$ of respective orders $p_1^{\alpha 1}$, $p_2^{\alpha 2}$, … $p_n^{\alpha_n}$.

The subgroup $G_p(I)$ is formed of all the operations of G (I) whose orders are powers of p with the identical operation (see also [12] for the classical group concepts.)

2 | Statement of Proof of the Main Results

We are now about to prove the main results. Already, an inner neutrosophic automorphism of a neutrosophic group has been de ned. Now, given that $X_1 = X(G(I))$ is the neutrosophic group of inner neutrosophic automorphisms of a group W (I). Also X_n is the neutrosophic group of the inner neutrosophic automorphisms of X_{n-1}, n, an integer.

Definition 7. Suppose there exists the lower central series of a group G (I) given by:

$G(I)=G_{(0)}(I) \supseteq G_{(1)}(I) \supseteq G_{(2)}(I) \supseteq \dots$. Here, $G_{(i)}(I)=[G_{(i-1)}(I), G(I)]$, i>0. i.e., $G_{(1)}(I)=[G_{(0)}(I), G(I)]=[G(I), G(I)]=G''(I)$, the commutator subgroup of G (I) such that the lower central series terminates at {ε} after a finite number of steps (i.e. $G_{(n)}(I) = \{\varepsilon\}$, for some integer n). Then G (I) is said to be nilpotent.

Define $u^{-1}v^{-1}uv=[u; v]$, the commutator of u and v, in a group G (I).

And $u^v = v^{-1}uv$. Here, $u = (u_1;\ u_2I_1;\ u_3I_2)$, and $v = \left(v_1;\ v_2I_1;\ v_3I_2\right)u^{-1}v^{-1}uv$

$$
\begin{aligned}
&= \left(u_1^{-1};\ u_2^{-1}I_1;\ u_3^{-1}I_2\right)\left(v_1^{-1};\ v_2^{-1}I_1;\ v_3^{-1}I_2\right)\left(u_1;\ u_2I_1;\ u_3I_2\right)\left(v_1;\ v_2I_1;\ v_3I_2\right) \\
&= \left(u_1^{-1}v_1^{-1};\ u_2^{-1}v_2^{-1}I_1;\ u_3^{-1}v_3^{-1}I_2\right)\left(u_1v_1;\ u_2v_2I_1;\ u_3v_3I_2\right) \\
&= \left(u_1^{-1}v_1^{-1}u_1v_1;\ u_2^{-1}v_2^{-1}u_2v_2I_1;\ u_3^{-1}v_3^{-1}u_3v_3I_2\right) = \left[u;\ v\right].
\end{aligned}
$$

By the definition of inner neutrosophic automorphism,using induction on G (I),

$$
\begin{aligned}
X(G(I)) = X_1 &= \left\{x_1^{-1}gx_1 j(a_1;\ a_2I_1;\ a_3I_2) = g \text{ in } G(I)\right\} \\
&= \{g^{x1} \mid g \in G(I)\} = \{ gg^{-1}x_1^{-1}gx_1 j(a_1;\ a_2I_1;\ a_3I_2) = g \in G \text{ and } x_i;\ a_1;\ a_2;\ a_3; \in G\} \\
&= \{ g[g;\ x_1] j(a_1;\ a_2I_1;\ a_3I_2) = g \in G(I)\}.
\end{aligned}
$$

If n=2. Then X_2 is the neutrosophic group of the inner neutrosophic automorphisms of $X_1 = \{g[g;\ x_1] \,/\, (a_1;\ a_2I_1;\ a_3I_2) = gG(I)\}$.

Hence, there exists $x_2 \in G(I)$ such that

$$X_2 = \{x_2 -^1 (g^{x1})x_2 \mid g^{x1} \in X_1\} = \{(g^{x1})^{x2} \mid g^{x1} \in X_1\}.$$

Following similar trends, there exists $x_3 \in G(I)$ such that

$$X_3 = \{x_3 -^1 (g^{x_1x_2})x_3 | g^{x_1x_2} \in X_2\} = \{g^{x_1x_2x_3} \mid g^{x_1x_2} \in x_2\}.$$

Also, there exists $x_4 \in G(I)$, such that

$$X_4 = \{x_4^{-1}(g^{x_1x_2x_3})x_4 \mid g^{x_1x_2x_3}\ X_3\} = fg^{x_1x_2x_3x_4} \mid g^{x_1x_2x_3} \in X_3\}.$$

And for $n = k$, there exists $x_k \in G(I)$, $k \in N$, such that

$$X_k = \{X_k^{-1}(g^{x_1x_2x_{k-1}})X_{k+1} \mid g^{x_1x_2x_k} \in X_{k+1}.$$

If the truth of the last statement is assumed, there exists $x_{k+1} \in G(I)$, $k \in N$ such that

$$X_{k+1} = \{x_{k+1}^{-1}(g^{x1x2\ xk})x_{k+1} \mid g^{x1x2\ xk}\ X_{k+1}\}.$$

We have that

$$G(I) = X_0 \supseteq X_1 \supseteq X_2 \supseteq X_k \supseteq X_{k+1} \supseteq \ldots \supseteq X_n \supseteq \ldots.$$

Definition 8. A neutrosophic group A (I) is said to satisfy the Descending Chain Condition (DCC) for any neutrosophic subgroups if every descending chain, $A_1(I) \supseteq A_2(I) \supseteq$ i.e, of neutrosophic subgroups terminates, i.e., there exists t in N (the set of natural numbers) such that for all $n \geq t$, $A_n(I) = A_t(I)$. Hence, every non-empty subset of the neutrosophic subgroups of A (I) has a minimal element. By the original hypothesis, let X_{n+1} be the identity $\{\varepsilon\}$ of the sequence $G(I), X_1, X_2, \ldots$.

Then, the minimal condition implies that

$$G(I) = X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots \supseteq X_n \supseteq X_{n+1} = \{\varepsilon\} \text{ and } X_i = X_{i-1}(G(I)).$$

This actually shows the nilpotence of G (I) (for more and extensive discussion regarding to the classical group concepts, please see [10] and [13].)

3 | Applications

This findings can be fully applicable to every other nite group in general, most especially those nite groups that are nilpotent.

4 | Conclusion

Finally, the nilpotent characteristics of every nite p-group has been observed to be highly hereditary and so, any other neutrosophic product groups formed which have origin from nite p-group would de nitely display neutrosopiy.

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Conflicts of Interest

The authors declare that there is no competing of interests.

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n-Cylindrical Fuzzy Neutrosophic Topological Spaces

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Abstract

The objective of this study is to incorporate topological space into the realm of n-Cylindrical Fuzzy Neutrosophic Sets (n-CyFNS), which are the most novel type of fuzzy neutrosophic sets. In this paper, we introduce n-Cylindrical Fuzzy Neutrosophic Topological Spaces (n-CyFNTS), n-Cylindrical Fuzzy Neutrosophic (n-CyFN) open sets, and n-CyFN closed sets. We also defined the n-CyFN base, n-CyFN subbase, and some related theorems here.

Keywords: n-Cylindrical fuzzy neutrosophic sets, n-Cylindrical fuzzy neutrosophic open sets, n-Cylindrical fuzzy neutrosophic closed sets, n-Cylindrical fuzzy neutrosophic base.

1 | Introduction

Zadeh [1] laid the stepping stone to the field of uncertainties called fuzzy sets. The prime field of mathematics where the concepts and ideas of fuzzy sets drew a parallel was topology. Chang [2] enlivened the concept of fuzzy topological spaces using Zadeh's definition. Since then the various notions in classical topology have been extended to fuzzy topological spaces. Subsequently in the second half of 1970 and the beginning of 1980, many authors contributed a lot to this new field. Later Atanassov [3], [4] introduced a new set called Intuitionistic Fuzzy Set (IFS) in which the sum of both acceptance degree and rejection degree grades does not exceed 1. Later, intuitionistic fuzzy topological spaces via IFSs were obtained by Coker [5] in intuitionistic fuzzy topological spaces, Lee and Lee [6] discovered the properties of continuous, open, and closed maps. Yager [7] proposed the Pythagorean Fuzzy Set (PyFS) as a generalisation of IFS in 2013, which ensures that the value of the square sum of its membership degrees is less than or equal to 1. The concept of pythagorean fuzzy topological space was introduced by Olgun et al. [8]. Cuong [9] initiated the idea of the Picture Fuzzy Set (PFS). He utilized three indices (membership degree P (x), neutral-membership degree I (x), and non-membership degree N (x) in PFS with the condition that is $0 \leq P(x) + I(x) + N(x) \leq 1$. Obviously PFSs is more suitable than IFS and PyFS to deal with fuzziness and vagueness. The idea of picture fuzzy topological spaces was first initiated by Razaq et.al [10]. Later Spherical Fuzzy Sets (SFS) have been

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proposed by Kahraman and Gündogdu [11]. SFS should satisfy the condition that the squared sum of membership degree and non-membership degree and hesitancy degree should be equal to or less than one. Princy and Mohana [12] introduced spherical fuzzy topological spaces.

The neutrosophic set was introduced by Smarandache [13] and neutrosophic set is a generalization of IFS. Salama and Alblowi [14] introduced the concept of neutrosophic topological spaces. They introduced neutrosophic topological space as a generalization of intuitionistic fuzzy topological space and a neutrosophic set besides the degree of membership, the degree of indeterminacy and the degree of non-membership of each element. Smarandache [15] introduced the dependence degree of (also, the independence degree of) the fuzzy components, as well as the neutrosophic components, for the first time in 2006. Arockiarani and Jency [16] initiated the notion of fuzzy neutrosophic set as the sum of all the three membership functions does not exceed 3. Fuzzy neutrosophic topological space and basic operations on it was proposed by Veereswari [17]. Sarannya Kumari et al. [18] recently introduced n-Cylindrical Fuzzy Neutrosophic Sets (n-CyFNS), which have T and F as dependent components and I as independent components. Except for fuzzy neutrosophic sets, the n-CyFNS is the largest extension of fuzzy sets. In this case, the degree to which positive, neutral, and negative membership functions satisfy the condition, $0 \leq \beta A(x) \leq 1$ and $0 \leq \alpha A\ n(x) + \gamma An(x) \leq 1$, $n > 1$, is an integer. They also defined the distance between two n-CyFNS, as well as their properties and basic operations.

In this paper, we introduce topological space in n-CyFNS environment. This is a new type of fuzzy neutrosophic sets in which T and F are dependent components and I independent components. Here we defined n-CyFN topological space, n-CyFN open sets. We also initiated n-CyFN base, n-CyFN subbase and some related results.

2 | Preliminaries

This section covers some basic definitions and examples that will be useful in subsequent discussions. Throughout this paper, Ʋ denotes the universe of discourse.

Definition 1 ([1]). A fuzzy set A in Ʋ is defined by membership function μ_A: A⟶ [0, 1] whose membership value $\mu_A(x)$ shows the degree to which $x \in$ Ʋ includes in the fuzzy set A for all $x \in$ Ʋ.

Definition 2 ([2]). A fuzzy topological space is a pair (X, T), where X is any set and T is a family of fuzzy sets in X satisfying following axioms:

I. $\Phi, X \in T$.
II. If $A, B \in T$, then $A \cap B \in T$.
III. If $A_i \in T$ for each $i \in I$, then $\cup_i A_i \in T$.

Definition 3 ([3]). An IFS A on Ʋ is an object of the form $A = \{(x, \alpha_A(x), \gamma_A(x) \mid x \in$ Ʋ$)\}$ where $\alpha_A(x) \in [0,1]$ is called the degree of membership of x in A, $\gamma_A(x) \in [0, 1]$ is called the degree of non-membership of x in A, and where α_A and γ_A satisfy (for all $x \in$ Ʋ) $(\alpha_A(x) + \gamma_A(x) \leq 1)$ IFS (Ʋ) denote the set of all the IFSs on a universe Ʋ.

Definition 4 ([13]). A neutrosophic set A on Ʋ is $A = < x, T_A(x), I_A(x), F_A(x) >$; $x \in$ Ʋ, where T_A, I_A, F_A: A⟶ $]^-0,1^+[$ and $^-0 < T_A(x) + I_A(x) + F_A(x) < n\,3^+$.

Definition 5 ([16]). A fuzzy neutrosophic set A on Ʋ is $A = < x, T_A(x), I_A(x), F_A(x) >$; $x \in$ Ʋ, where T_A, I_A, F_A: A⟶ [0, 1] and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 6 ([13]). A neutrosophic set A on Ʋ is an object of the form $A = \{(x, u_A(x), \zeta_A(x), v_A(x))$: $x \in$ Ʋ $\}$, where $u_A(x), \zeta_A(x), v_A(x) \in [0,1]$, $0 \leq u_A(x) + \zeta_A(x) + v_A(x) \leq 3$ for all $x \in$ Ʋ. $u_A(x)$ is the degree of truth membership, $\zeta_A(x)$ is the degree of indeterminacy and $v_A(x)$ is the degree of non-membership. Here $u_A(x)$ and $v_A(x)$ are dependent components and $\zeta_A(x)$ is an independent component.

Definition 7 ([14]). A Neutrosophic Topology (NT) on a non-empty set X is a family τ of neutrosophic subsets in X satisfying the following axioms:

I. (NT1) $0_N, 1_N \in \tau$.
II. (NT2) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$.
III. (NT3) $\cup G_i \in \tau$ for all $\{G_i : i \in J\} \subseteq \tau$.

In this case the pair (X, τ) is called a Neutrosophic Topological Space (NTS) and any neutrosophic set in τ is known as Neutrosophic Open Set (NOS) in X. The elements of τ are called open neutrosophic sets. A neutrosophic set F is closed if and only if it C (F) is neutrosophic open.

Definition 8 ([17]). A Fuzzy Neutrosophic Topology (FNT) a non-empty set X is a family τ of fuzzy neutrosophic subsets in X satisfying the following axioms:

I. (FNT1) 0_N, $1_N \in \tau$.
II. (FNT2) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$.
III. (FNT3) $\cup G_i \in \tau$ for all $\{G_i : i \in J\} \subseteq \tau$.

In this case the pair (X, τ) is called a Fuzzy Neutrosophic Topological Space (FNTS) and any fuzzy neutrosophic set in τ is known as Fuzzy Neutrosophic Open Set (FNOS) in X. The elements of τ are called open fuzzy neutrosophic sets.

Definition 9 ([18]). An n-CyFNS A on Ʊ is an object of the form A= $\{\langle x, \alpha_A(x), \beta_A(x), \gamma_A(x) \rangle \mid x \in Ʊ\}$ where $\alpha_A(x) \in [0, 1]$, called the degree of positive membership of x in A, $\beta_A(x) \in [0,1]$, called the degree of neutral membership of x in A and $\gamma_A(x) \in [0, 1]$, called the degree of negative membership of x in A, which satisfies the condition, (for all x∈ Ʊ) $(0 \leq \beta_A(x) \leq 1$ and $0 \leq \alpha_A{}^n(x) + \gamma_A{}^n(x) \leq 1$, n>1, is an integer. Here T and F are dependent neutrosophic components and I is 100% independent.

For the convenience, $\langle \alpha_A(x), \beta_A(x), \gamma_A(x) \rangle$ is called as n-Cylindrical Fuzzy Neutrosophic Number (n-CyFNN) and is denoted as A= $\langle \alpha_A, \beta_A, \gamma_A \rangle$.

Definition 10 ([18]). (The Basic Connectives). Let $Ꞇ_N$ (Ʊ) denote the family of all n-CyFNS on Ʊ.

Definition 11. Inclusion: For every two A, B ∈ $Ꞇ_N$ (Ʊ), A ⊆ B iff (for all x ∈ Ʊ, $\alpha_A(x) \leq \alpha_B(x)$ and $\beta_A(x) \leq \beta_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$) and A = B iff (A ⊆ B and B ⊆ A).

Definition 12. Union: For every two A, B ∈ $Ꞇ_N$ (Ʊ), the union of two n-CyFNSs A and B is A∪B(x) = $\{\langle x, \max(\alpha_A(x), \alpha_B(x)), \max(\beta_A(x), \beta_B(x)), \min(\gamma_A(x), \gamma_B(x)) \rangle \mid x \in Ʊ\}$.

Definition 13. Intersection: For every two A, B ∈ $Ꞇ_N$ (Ʊ), the intersection of two n-CyFNSs A and B is A∩B(x) = $\{\langle x, \min(\alpha_A(x), \alpha_B(x)), \min(\beta_A(x), \beta_B(x)), \max(\gamma_A(x), \gamma_B(x)) \rangle \mid x \in Ʊ\}$.

Definition 14. Complementation: For every A ∈ $Ꞇ_N$ (Ʊ), the complement of an n-CyFNS A is $A^c = \{\langle x, \gamma_A(x), \beta_A(x), \alpha_A(x) \rangle \mid x \in Ʊ\}$.

Definition 15. Sum: For every two A, B ∈ $Ꞇ_N$ (Ʊ), the sum of two n-CyFNSs A and B is A⊕B(x) = $\{\langle x, (\frac{\alpha_A(x).\alpha_B(x)}{\alpha_A\ x)+\alpha_B(x)}, \max(\beta_A(x), \beta_B(x)), \min(\gamma_A(x), \gamma B(x)) \rangle \mid x \in Ʊ\}$.

Definition 16. Difference: For every two A, B ∈ $\mathcal{C}_N$ (U), the difference of two n-CyFNSs A and B is A⊖B(x) = {⟨x, max ($\alpha_A(x)$, $\alpha_B(x)$), min ($\beta_A(x)$, $\beta_B(x)$), $\frac{\gamma_A(x).\,\gamma_B(x)}{\gamma_A(x)+\gamma_B(x)}$ ⟩ | x ∈ U}.

Definition 17. Product: For every two A, B ∈ $\mathcal{C}_N$ (U), the product of two n-CyFNSs A and B is A⊗B (x) = {⟨x, ($\alpha_A(x).\alpha_B(x)$, $\beta_A(x).\beta_B(x)$, $\gamma_A(x).\gamma_B(x)$) ⟩ | x ∈ U}.

Definition 18. Division: For every two A, B ∈ $\mathcal{C}_N$ (U), A⊘B is A⊘ B(x) = {⟨x, min ($\alpha_A(x)$, $\alpha_B(x)$), $\beta_A(x)$. $\beta_B(x)$, max ($\gamma_A(x)$. $\gamma_B(x)$) ⟩ | x ∈ U}.

Results ([18]):

I. If A ⊆ B and B ⊆ C then A ⊆ C.
II. A∪B = B∪A & A∩B = B∩A.
III. (A∪B) ∪C = A∪ (B∪C) & (A∩B) ∩C = A∩ (B∩C).
IV. (A∪B) ∩C = (A∩C) ∪ (B∩C) & (A∩B) ∪C = (A∪C) ∩ (B∪C).
V. A∩A = A & A ∪ A = A.
VI. De Morgan's Law for A & B ie, $(A∪B)^c = A^c∩B^c$ & $(A∩B)^c = A^c ∪ B^c$.
VII. (A⊕B) = (B⊕A).
VIII. (A⊗ B) = (B ⊗A).

3 | n-Cylindrical Fuzzy Neutrosophic Topological Spaces

Definition 18. Let $\{A_i : i \in I\}$ be an arbitrary family of n-CyFNS in U.

Then $\cap A_i$ = {⟨x, inf ($\alpha_{Ai}(x)$), inf ($\beta_{Ai}(x)$), sup (($\gamma_{Ai}(x)$) ⟩ | x ∈ U}.

$\cup A_i$ = {⟨x, sup ($\alpha_{Ai}(x)$), sup ($\beta_{Ai}(x)$), inf (($\gamma_{Ai}(x)$) ⟩ | x ∈ U}.

Definition 19. 0_{CyN} = {⟨x, 0, 0, 1⟩ | x ∈ U} and 1_{CyN} = {⟨x, 1, 1, 0⟩ | x ∈ U}.

3.1 | n-Cylindrical Fuzzy Neutrosophic Topological Spaces

In this part, we give a definition of n-Cylindrical Fuzzy Neutrosophic Topology (n-CyFNT) and its related properties according to Chang's FTS.

Definition 20. An n-CyFNT on a non-empty set X is a family, τ_X, of n-CyFNS in X which satisfies the following conditions:

I. 0_{cyN}, $1_{cyN} \in \tau_X$.
II. $A_1 \cap A_2 \in \tau_X$.
III. $\cup A_i \in \tau_X$, for any arbitrary family $A_i \in \tau_X$, i ∈ I.

The pair (X, τ_X) is called an n-cylindrical fuzzy neutrosophic topological space n-Cylindrical Fuzzy Neutrosophic Topological Spaces (n-CyFNTS) and any n-CyFNS belongs to τ_X is called an n-Cylindrical Fuzzy Neutrosophic Open Set (n-CyFNOS) and the complement of n-CyFNOS is called n-Cylindrical Fuzzy Neutrosophic Closed Set (n-CyFNCS) in X. Like classical topological spaces and fuzzy topological spaces, the family {0_{cyN}, 1_{cyN}} is called indiscrete n-CyFNTS and the topology containing all the n-CyFN subsets is called discrete n-CyFNTs.

Remark: Obviously any fuzzy topological space or intuitionistic fuzzy topological space or pythagorean fuzzy topological space is an n-CyFN topological space as any subsets of the fuzzy space, intuitionistic

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fuzzy space, and pythagorean fuzzy space can be viewed as n-CyFN subsets. But the converse of the above doesn't follow and it can be evident from the following example:

Example 1. Let X ={ p, q} and τ_X ={ 1_{cyN} , 0_{cyN}, A, B, C, D}, where,

A = {<p; 0.5, 0.5, 0.7>, <q; 0.2, 05, 0.4>}, B = {<p; 0.6, 0.5, 0.5>, <q; 0.3, 0.5, 0.9>}, C = {<p; 0.6, 0.5, 0.5>, <q; 0.3, 0.5, 0.4>}, D = {<p; 0.5, 0.5, 0.7>, <q; 0.2, 0.5, 0.9>}, is clearly an n-CyFNTS.

Definition 21. Let (X, τ_{X1}) and (X, τ_{X2}) be n-CyFNTSs.

I. τ_{X2} is finer than τ_{X1} if $\tau_{X2} \supseteq \tau_{X1}$.
II. τ_{X2} is strictly finer than τ_{X1} if $\tau_{X2} \supset \tau_{X1}$.
III. τ_{X2} *and* τ_{X1} are said to be comparable if it holds $\tau_{X2} \supseteq \tau_{X1}$ or $\tau_{X1} \supseteq \tau_{X2}$.

Example 2. Consider the *Example 1*.

X = {p, q}, τ_X = {1_{cyN}, 0_{cyN}, A, B, C, D} and τ_{X1} = {1_{cyN}, 0_{cyN}, A} are two n-CyFN topologies on X. Clearly we can see that $\tau_X \supset \tau_{X1}$.

Definition 22. Let (X, τ_X) be a CyFNTS on X.

$\mathscr{B} \subseteq \tau_X$, a sub family of τ_X is called an n-CyFN base for (X, τ_X), if each member of τ_X may be expressed as the union of members in $\mathscr{B}$.

$\mathscr{S} \subseteq \tau_X$, a sub family of τ_X is called a n-CyFN sub-base for (X, τ_X), if the family of all finite intersections of $\mathscr{S}$ forms a base for (X, τ_X). Here it can be said that $\mathscr{S}$ generates (X, τ_X).

Theorem 1. Let (X, τ_X) be an n-CyFNTS and $\mathscr{B} \subseteq \tau_X$, be a n-cylindrical fuzzy neutrosophic base for τ_X. Then τ_X is the collection of all union of members of $\mathscr{B}$.

Proof: The definition of the base of an n-CyFNTS clearly proves the theorem.

Theorem 2. Let (X, τ_X) be an n-CyFNTS and $\mathscr{B} \subseteq \tau_X$. Then $\mathscr{B}$ is an n-cylindrical fuzzy neutrosophic base for τ_X if and only if for any x∈X and any G∈ τ_X containing x, there exists B∈ $\mathscr{B}$ such that x∈B⊆G.

Proof: Suppose $\mathscr{B}$ is an n-cylindrical fuzzy neutrosophic base for τ_X.

Let G∈ τ_X and x∈G. Now G = $\bigcup_i B_i$, $B_i \in \mathscr{B}$, x∈G ⇒x∈ $\bigcup_i B_i$ ⇒x∈ B_i for some B_i and let $B_i = B$.

That is, x∈B = $B_i \subseteq \bigcup_i B_i \subseteq$ G, hence x ∈ ⊆ B G.

Conversely suppose the given condition holds, ie, Let G ∈ τ_X. For each x∈G, there exists $B_x \in \mathscr{B}$ such that x ∈ B_x ⊆ G.

B_x ⊆ G for all x. Then,

$$\cup_{x\in G} \text{ Bx} \subseteq \text{G}. \tag{1}$$

But from the assumption G∈ τ_X and, x ∈ B_x for all x∈G and B_x ⊆ G Since G is n-CyFNOS in X, G can be expressed as:

$$\text{G} \subseteq \cup_{x\in G} \text{ Bx} \quad \text{where Bx} \in \mathscr{B} \subseteq \tau_X. \tag{2}$$

From *Eqs. (1)* and *(2)*; G = $\cup_{x\in G}$ B_x ; $B_x \in \mathscr{B}$ thus $\mathscr{B}$ is an n-CyFN base for τ_X.

Definition 23. Let (X, τ_X) be an n-CyFNTS and Y⊆ X. Then the collection $\tau_Y = \{X_i \cap Y: X_i \in \tau_X, i \in I\}$ is called n-cylindrical fuzzy neutrosophic subspace topology on Y. Hence (Y, τ_Y) is called n-cylindrical fuzzy neutrosophic topological subspace of (X, τ_X).

Theorem 3. Let (X, τ_X) be an n-CyFNTS and Y⊆ X, then τ_Y, an n-CyFN subspace topology on Y is an n-CyFNTS.

Proof: Certainly 0_{cy}, $1_{cy} \in \tau_Y$ since $0_{cx} \cap Y = 0_{cy}$ and $1_{cx} \cap Y = 1_{cy}$.

Also $\tau_X = \{X_i \subseteq X, \ i \in I\}$.

Hence it is closed under arbitrary n-cylindrical fuzzy neutrosophic union.

$$\bigcup_i (X_i \cup Y) = (\bigcup_i X_i) \cup Y. \tag{3}$$

Also it is closed under finite n-cylindrical fuzzy neutrosophic intersection.

Hence the theorem follows.

$$\bigcap_{i=1}^{n} (X_i \cap Y) = (\bigcap_{i=1}^{n} X_i) \cap Y. \tag{4}$$

Example 3. Let X be the set of all integers. Consider f ∈ n-CyFNS such that f(x) = $< 1, \frac{1}{x}, 0 >$; x≥ 1 and x ∈ X= $< 0, -\frac{1}{x}, 1 >$; x≤ −1 = $< 1, 1, 0 >$; x = 0, then (X, τ_X) is an n-CyFNTS with $\tau_X = \{1_{cyN}, 0_{cyN}, f\}$.

Let Y denote set of all even integers ie, y = 2x ∈Yg(y) = $< 1, \frac{1}{y}, 0 >$; y≥ 1 = $< 0, -\frac{1}{y}, 1 >$; y≤ −1 = $< 1, 1, 0 >$; y = 0. Clearly (Y, τ_Y) is a sub space topology, $\tau_Y = \{1_{cyN}, 0_{cyN}, g\}$.

Theorem 4. If $\mathscr{B}$ is an n-CyFN base for (X, τ_X) and Y⊆ X, then $\mathscr{B}_Y = \{B \cap Y \mid B \in \mathscr{B}\}$ is an n-CyFN base for (Y, τ_Y).

Proof: Let G is n-CyFN open in X and y∈G∩Y. Now choose $B \in \mathscr{B}$ such that y∈B⊆G.

Thus y∈B∩ Y ⊆G∩Y. Hence $\mathscr{B}_Y$ an n-CyFN base for (Y, τ_Y) by *Theorem 2*.

Theorem 5. Let (X, τ_X) be a CyFNTS and (Y, τ_Y) be an n-cylindrical fuzzy neutrosophic topological subspace. If $\subseteq Y$ Zis n-cylindrical fuzzy neutrosophic open in Y then Z is n-cylindrical fuzzy neutrosophic open in X.

Proof: It is evident from the definition of n-cylindrical fuzzy topological subspace.

4 | Conclusion

Our goal with this paper is to broaden the scope of n-CyFNS to topological spaces. Here we introduce the fundamental definitions of n-CyFNTS, n-CyFN open sets, and n-CyFN closed sets, as well as examples. The terms n-CyFN base, n-CyFN sub base, and related theorems were also defined. This paper is the first to investigate n-CyFNTS. This research will undoubtedly be the basis for the further development of n-CyFNTS and their applications in various fields. Evidently, these ideas have the potential to inspire additional research in the future.

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Conflicts of Interest

The authors declare no conflict of interest.

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